# Topological Models for Arithmetic

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Abstract. We find explicit models for the etale topological type of certain arithmetic rings $A$, and in some cases use these models to compute the $\ell$–adic topological $K$-theory of $A$. Through a comparison map, this gives information about the algebraic $K$-theory of $A$. For example, we are able to compute the mod $\ell$ cohomology of certain "unstable" topological $K$-theory spaces and verify that it injects into the cohomology of the corresponding unstable algebraic $K$-theory spaces. This gives an explicit lower bound for $H^*(\mathrm{GL}(n, A), \mathbf{Z}/\ell)$. In some cases we also find explicit geometric or cohomological reformulations of the Lichtenbaum-Quillen conjectures.

## §0. Introduction

The *etale topological type* $A_{\text{et}}$ of a commutative ring $A$ is an interesting topological object (technically a pro-space) associated to $A$. In [**4**] we introduced the "etale $K$-groups" of $A$ as local coefficient $\ell$-adic topological $K$-theory groups of $A_{\text{et}}$, where $\ell$ is a prime invertible in $A$. Building upon work of C. Soulé [**19**], we described a natural map from the algebraic $K$-groups of $A$ (completed at $\ell$) to the etale $K$-groups, and showed that in general this map is surjective for arithmetic rings. The well-known Lichtenbaum-Quillen Conjectures essentially assert that this map is an isomorphism.

The basic results of class field theory and arithmetic duality can be interpreted as statements about the etale cohomology of arithmetic rings $A$ [**10**] or equivalently as statements about the ordinary cohomology of $A_{\text{et}}$. In [**5**] we exploited this fact to find, for some specific arithmetic rings $A$, an elementary topological space $X^A$ mapping to $A_{\text{et}}$ by a cohomology isomorphism. Finding $X^A$ allowed us not only to compute the etale $K$-groups of $A$ explicitly, but also to identify the underlying local coefficient topological $K$-theory space and compute *its* cohomology (which, by the Lichtenbaum-Quillen conjecture, should be isomorphic to the cohomology of $\mathrm{GL}(A)$).

We do two things in the present paper. First of all, we extend the above program of finding "models" for $A_{\text{et}}$ to some other rings. In

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each case, for a specific prime $\ell$ invertible in $A$, we find a "good mod $\ell$ model" $X^A$ for $A$; this is an explicit space or pro-space $X^A$ which in an appropriate sense captures the mod $\ell$ cohomology of $A_{\mathrm{et}}$. Here are the main examples:

(1) If $A = \mathbf{Z}[1/\ell]$ for an odd regular prime $\ell$, then $X^A$ is the one-point union $\mathbf{R}P^\infty \vee S^1$ (2.1), where $S^1$ is the circle.
(2) If $A$ is the coordinate ring of a suitable affine curve over $\mathbf{F}_q$ then $X^A$ is a fibration over $S^1$ with fibre the $p$-completion of a finite wedge of circles (3.3).
(3) If $A$ is a generalized local field of transcendence degree $r$ over $\mathbf{F}_q$ then $X^A$ is a fibration over $S^1$ with fibre the $p$-completion of a product of $r$ circles (3.5, see also 3.6 for the case $r = 1$).
(4) If $A$ is a suitable finite extension of $\mathbf{Q}_\ell$ then $X^A = K(G, 1)$, where $G$ is a one-relator group very similar to a surface group (3.7).

Secondly, in case (1) above we identify the etale $K$-theory space of the model $X^A$ (2.3) and show that its cohomology injects into the cohomology of the algebraic $K$-theory space of $A$ (6.4, 6.3 is a related unstable statement). This proves in particular that if the Lichtenbaum-Quillen conjecture is true for $\mathbf{Z}$ at an odd regular prime $\ell$, then $\mathrm{BGL}(\mathbf{Z})^+$ is equivalent at $\ell$ to $\mathrm{F}\,\Psi^p \times SU/O$ (2.4), where $p$ is as in §2. We also show that if the Lichtenbaum-Quillen conjecture is true in this case then $H^*(\mathrm{BGL}(\mathbf{Z}[1/\ell]), \mathbf{Z}/\ell)$ is detected on certain abelian subgroups (6.6).

The organization of our paper is as follows. Section 1 motivates and presents the definition of a good mod $\ell$ model of an arithmetic ring $A$. Such a model captures the etale cohomology of $A$ with coefficients in "Tate twists" of $\mathbf{Z}/\ell$. In §2, we provide a particularly simple model for $A = \mathbf{Z}[1/\ell]$ whenever $\ell$ is an odd regular prime. Verifying this model is more delicate than verifying the models for affine curves and local fields described in §3. The remainder of the paper is concerned with analyzing the cohomology of (unstable) $K$-theory spaces associated with our good mod $\ell$ models and studying the relationship between this cohomology and that of the general linear groups of $A$. In §4, we discuss the Eilenberg-Moore spectral sequence and apply it to our topological $K$-theory spaces. Our comparison of cohomology of algebraic and topological $K$-theory spaces is achieved by restricting to maximal tori: the actual comparison is carried out in §6, following a check in §5 that corresponding algebraic and topological tori have the same cohomology. We conclude the paper by mentioning a few open problems in §7.

0.1 Notation: We fix a prime number $\ell$ and let $R$ denote the ring $\mathbf{Z}[1/\ell]$, $\mathbf{F}$ the finite field $\mathbf{Z}/\ell$, $\zeta_{\ell^n}$ the $\ell^n$'th root of unity $e^{2\pi i/\ell^n}$ and $\mu_{\ell^n}$ the multiplicative group of $\ell^n$'th roots of unity. The symbol $\zeta$ without a subscript denotes $\zeta_\ell$.

Our fundamental reference for homotopy theoretic completions is [**2**]. If $X$ is a space or pro-space the $\ell$–adic completion tower of $X$ is a pro-space denoted $\mathbf{F}_\omega(X)$; if $X$ is a space then $\mathbf{F}_\omega(X)$ is a tower in the usual sense whose inverse limit is the $\ell$–adic completion $\mathbf{F}_\infty(X)$ of $X$. If $E \to B$ is a fibration the fibrewise $\ell$–adic completion tower of $E$ over $B$ [**4**, p. 250] is denoted $\mathbf{F}^\bullet_\omega(E)$; if $E$ and $B$ are spaces, the inverse limit of this tower is the fibrewise $\ell$–adic completion $\mathbf{F}^\bullet_\infty(E)$ of $E$ over $B$.

## §1. Good mod $\ell$ models

We recall that $X_{\mathrm{et}}$ is a "space" which reflects the etale cohomology of a noetherian simplicial scheme $X$. The reader can consult [**11**] for a coherent discussion of the etale cohomology $H^*_{\mathrm{et}}(X, F)$ of $X$ with coefficients in a sheaf $F$ on the "etale site"; endurance now assured, he or she can then peruse [**1**] and [**7**] for the construction and properties of the pro-space (i.e., inverse system of simplicial sets) $X_{\mathrm{et}}$.

For any finitely generated $R$-algebra $A$, let $A_{\mathrm{et}}$ denote the etale topological type $(\operatorname{Spec} A)_{\mathrm{et}}$. For any connected, noetherian simplicial scheme $X$ (e.g., $\operatorname{Spec} A$ for a noetherian domain $A$), $X_{\mathrm{et}}$ is an inverse system of connected simplicial sets [**7**, 4.4]. If $X$ is normal in each simplicial degree, as is always the case in our arithmetic examples, $X_{\mathrm{et}}$ is homotopy equivalent to an inverse system in the homotopy category of simplicial sets each of which has finite homotopy groups [**7**, 7.3].

Two key properties of $A_{\mathrm{et}}$ are given in the following

1.1 Proposition. *Let $\bar{a} : \operatorname{Spec} k \to \operatorname{Spec} A$ be a geometric point of the finitely generated normal $R$-algebra $A$, corresponding to a homomorphism from $A$ to a separably closed field $k$ and determining a base point of $A_{\mathrm{et}}$. Then*

(1) $\pi_1(A_{\mathrm{et}}, \bar{a}) = \pi_1^{\mathrm{et}}(A, \bar{a})$ , the (profinite) Grothendieck fundamental group of $A$ pointed by $\bar{a}$ (this group classifies finite etale covering spaces of $\operatorname{Spec} A$).

(2) $H^*(A_{\mathrm{et}}, C) = H^*_{\mathrm{et}}(\operatorname{Spec} A, C)$, the etale cohomology of $\operatorname{Spec} A$ with coefficients in the local coefficient system $C$ (such a local coefficient system is an abelian group $C$ provided with a continuous action of $\pi_1^{\mathrm{et}}(A, \bar{a})$).

The etale $K$-theory space associated to $A_{\mathrm{et}}$ is constructed using function complexes. Namely, for each integer $n$, we consider the group scheme $\mathrm{GL}_{n,R}$ whose coordinate algebra is the Hopf algebra

$$R[x_{i,j}, t]/\det(x_{i,j})t = 1\,.$$

The usual bar construction [**7**, 1.2, §2] (over $R$) determines a simplicial scheme $\mathrm{BGL}_{n,R}$ with etale topological type $(\mathrm{BGL}_{n,R})_{\mathrm{et}}$.

1.2 Definition: The space $\mathrm{BGL}_n(A_{\mathrm{et}})$ is defined to be (see [**4**, 2.4, 4.5])

$$\mathrm{BGL}_n(A_{\mathrm{et}}) \equiv \mathrm{Hom}^0_\ell(A_{\mathrm{et}}, (\mathrm{BGL}_{n,R})_{\mathrm{et}})_{R_{\mathrm{et}}}$$

the connected component of the function complex of maps over $R_{\mathrm{et}}$ from $A_{\mathrm{et}}$ to the pro-space fibrewise mod $\ell$ completion over $R_{\mathrm{et}}$ of $(\mathrm{BGL}_{n,R})_{\mathrm{et}}$.

More generally, if $G_R$ is a group scheme over $R$, we define $\mathrm{B}G(A_{\mathrm{et}})$ to be

$$\mathrm{B}G(A_{\mathrm{et}}) \;\equiv\; \mathrm{Hom}^0_\ell(A_{\mathrm{et}}, (\mathrm{B}G_R)_{\mathrm{et}})_{R_{\mathrm{et}}}\,.$$

Remark: The curious reader might wonder how in general terms to obtain a single space as a function complex of pro-spaces. The answer is relatively simple: take the homotopy colimit indexed by the indexing category of the domain and then the homotopy limit indexed by the indexing category of the range.

1.3 Definition: Let $A$ be a finitely generated $R$-algebra of finite mod $\ell$ etale cohomological dimension. The (connected) etale $K$-theory space $\mathrm{BGL}(A_{\mathrm{et}})$ is defined to be

$$\mathrm{BGL}(A_{\mathrm{et}}) \equiv \mathrm{colim}_n\{(\mathrm{BGL}_n(A_{\mathrm{et}})\}$$

and the $i$-th mod $\ell^\nu$ etale $K$-group of $A$ is defined to be

$$K_i^{\mathrm{et}}(A, \mathbf{Z}/\ell^\nu) \equiv \; \pi_i(\mathrm{BGL}(A_{\mathrm{et}}), \mathbf{Z}/\ell^\nu) \quad i > 1.$$

Remark: The homotopy fibre of the map $(\mathrm{BGL}_{n,R})_{\mathrm{et}} \to R_{\mathrm{et}}$ (see 1.2) is $\ell$-equivalent to the completion $\mathbf{F}_\omega(\mathrm{BU}_n)$ [**7**, §8]; this leads to an interpretation of $\pi_* \mathrm{BGL}(A_{\mathrm{et}})$ as the "local coefficient topological $K$-theory" of the pro-space $A_{\mathrm{et}}$. There is an Atiyah-Hirzebruch spectral sequence to support this interpretation [**4**, 5.1]. Using the fact that both algebraic $K$-theory and etale $K$-theory are the homotopy groups of infinite loop spaces and that the map between these spaces is an infinite loop map [**4**, 4.4], one can extend the above definition to include $K_0$ and $K_1$. Moreover, one can extend this definition to certain rings of virtually finite etale cohomological dimension (e.g., $\mathbf{Z}[1/2]$ at the prime 2) by a "descent procedure" that involves taking homotopy fixed point spaces [**5**, p. 140].

1.4 Remark: For any group scheme $G_R$ over $R$ there is a natural map

$$\mathrm{B}G(A) \;\to \mathrm{B}G(A_{\mathrm{et}})$$

constructed as follows. One views a $t$-simplex of $\mathrm{B}G(A)$ as a morphism of simplicial schemes $\operatorname{Spec} A \otimes \Delta[t] \to \mathrm{B}G_R$ over $\operatorname{Spec} R$; by functorality this determines a map of pro-spaces $(\operatorname{Spec} A)_{\mathrm{et}} \times \Delta[t] \to (\mathrm{B}G_R)_{\mathrm{et}}$ over $R_{\mathrm{et}}$, and thus a $t$-simplex of $\mathrm{B}G(A_{\mathrm{et}})$. In particular, there are maps $\mathrm{BGL}_n(A) \to \mathrm{BGL}_n(A_{\mathrm{et}})$; these pass to a map

$$\mathrm{BGL}(A)^+ \to \mathrm{BGL}(A_{\mathrm{et}})$$

which induces homomorphisms $K_i(A, \mathbf{Z}/\ell^\nu) \to K_i^{\mathrm{et}}(A, \mathbf{Z}/\ell^\nu)$.

Before proceeding to new results concerning the maps of 1.4, we briefly summarize some earlier results from [**4**].

1.5 Theorem. *Denote the map* $K_i(A, \mathbf{Z}/\ell^\nu) \to K_i^{\mathrm{et}}(A, \mathbf{Z}/\ell^\nu)$ *by* $\kappa_i^\nu(A)$.

(1) *If $A$ is a finite field of order $q$ where $\ell$ does not divide $q$, then $\kappa_i^\nu(A)$ is an isomorphism for all $\nu > 0$.*
(2) *If $A$ is the ring of $S$-integers in a global field and contains a primitive $4$'th root of unity if $\ell = 2$, then for $i > 0$ and $\nu > 0$ the map $\kappa_i^\nu(A)$ is surjective.*
(3) *The well-known "Lichtenbaum-Quillen Conjecture" is equivalent to the conjecture that the maps in (2) are isomorphisms.*
(4) *More generally, if the* mod $\ell$ *etale cohomological dimension of $A$ is $\leq 2$ (e.g., if $A$ is a local field or the coordinate algebra of a smooth affine curve over a finite field), then for $i > 0$ and $\nu > 0$ the map $\kappa_i^\nu(A)$ is surjective.*

The preceding discussion should provide motivation for our efforts to obtain computable models for $\mathrm{BGL}_n(A_{\mathrm{et}})$. Such models will be function spaces with domain $X^A$, a "good mod $\ell$ model for $A$", which cover a structure map reflecting the action of $\pi_1^{\mathrm{et}}(A)$ on the $\ell$-primary roots of unity.

Let $R_\infty$ denote the ring obtained from $R$ by adjoining all $\ell$–primary roots of unity,

$$R_\infty \equiv R[\mu_{\ell^\infty}] \equiv \mathbf{Z}[1/\ell, \mu_{\ell^\infty}]\,,$$

let $\Lambda$ denote the pro-group

$$\Lambda \equiv \mathrm{Gal}(R_\infty, R) \equiv \{\mathrm{Aut}(\mu_{\ell^\nu}), \nu > 0\} \cong \{(\mathbf{Z}/\ell^\nu)^*, \nu \geq 1\}\,,$$

and let $\lambda$ denote the quotient $\mathrm{Aut}(\mu_\ell) \cong (\mathbf{Z}/\ell)^*$ of $\Lambda$. Observe that $R_{\mathrm{et}}$ is provided with the natural structure map

$$R_{\mathrm{et}} \to K(\Lambda, 1)$$

which "classifies" the finite etale extensions $R \to R[\mu_{\ell^\nu}]$. More generally, if $A$ is any $R$-algebra, then $A_{\mathrm{et}}$ is provided with a natural structure map to $A_{\mathrm{et}} \to R_{\mathrm{et}} \to K(\Lambda, 1)$.

We denote by $\mathbf{Z}/\ell^n(i)$ the local system on $K(\Lambda, 1)$ given by the $i$-fold tensor power of the natural action of $\Lambda$ on $\mu_{\ell^n}$. For any pro-space $X$ provided with a structure map $X \to K(\Lambda, 1)$, we also use $\mathbf{Z}/\ell^n(i)$ to denote the induced local system on $X$.

The next definition formalizes a modelling technique implicit in [**5**].

1.6 Definition: Let $A$ be a finitely generated $R$-algebra. Then a map of pro-spaces

$$f^A : X^A \to K(\Lambda, 1)$$

is said to be a *good* mod $\ell$ *model for* $A$ (or for $A_{\mathrm{et}}$) if there exist pro-spaces $X_i$ over $K(\Lambda, 1)$, each of finite mod $\ell$ cohomological dimension, and a chain of maps

$$X^A = X_m \leftarrow \ X_{n-1} \to \cdots \to X_0 \ = A_{\mathrm{et}}$$

over $K(\Lambda, 1)$ satisfying the condition that $X_j \to X_{j\pm 1}$ induces isomorphisms

$$H^*(X_{j\pm 1}, \mathbf{Z}/\ell(i)) \cong H^*(X_j, \mathbf{Z}/\ell(i)), \quad i = 1, \ldots, \ \ell - 1\,.$$

Remark: For the purposes of this paper it would be enough to work over $K(\lambda, 1)$ instead of over $K(\Lambda, 1)$ in defining a good mod $\ell$ model (cf. proof of 1.8). There are other situations in which this would not be sufficient.

1.7 Remark: The condition of 1.6 is equivalent to the condition that the maps $X_j \to X_{j\pm 1}$ induce mod $\ell$ homology equivalences

$$\tilde{X}_j \to \tilde{X}_{j\pm 1}$$

on covering spaces determined by $X_{j\pm 1} \to K(\Lambda, 1) \to K(\lambda, 1)$; this, in turn, is equivalent to the condition that $\tilde{X}_j \to \tilde{X}_{j\pm 1}$ induces isomorphisms

$$H^*(\tilde{X}_{j\pm 1}, \mathbf{Z}/\ell^n(i)) \cong H^*(\tilde{X}_j, \mathbf{Z}/\ell^n(i)), \quad \text{all } n,\ i > 0,$$

or, in fact, to the condition that $X_j \to X_{j\pm 1}$ induces isomorphisms

$$H^*(X_{j\pm 1}, \mathbf{Z}/\ell^n(i)) \cong H^*(X_j, \mathbf{Z}/\ell^n(i)), \quad \text{all } n,\ i > \ 0.$$

The significance for us of a good mod $\ell$ model is given by the following proposition.

1.8 Proposition. *Let* $f^A : X^A \to K(\Lambda, 1)$ *be a good* mod $\ell$ *model for* $A_{\mathrm{et}}$. *Then the chain of maps* $X^A = X_m \leftarrow \cdots \to X_0 = A_{\mathrm{et}}$ *extends to a chain of pairs (of pro-spaces)*

$$\begin{array}{ccccccc}
\mathrm{BGL}_{n,X^A} & \longleftarrow & \mathrm{BGL}_{n,X_{m-1}} & \longrightarrow & \cdots & \longrightarrow & \mathbf{F}^{\bullet}_{\omega}(\mathrm{BGL}_{n,A})_{\mathrm{et}} \\
\downarrow & & \downarrow & & & & \downarrow \\
X^A & \longleftarrow & X_{m-1} & \longrightarrow & \cdots & \longrightarrow & A_{\mathrm{et}}
\end{array}$$

*with each map between pairs inducing a homotopy equivalence on homotopy fibres (i.e., with each map between pairs determining a homotopy cartesian square). As a consequence, the space* $\mathrm{BGL}_n(A_{\mathrm{et}})$ *is homotopy equivalent to the space of sections* $\Gamma(X^A, \mathrm{BGL}_{n,X^A})$ *of* $\mathrm{BGL}_{n,X^A} \to X^A$.

Proof: Given $X_i \to X_{i-1}$ and $\mathrm{BGL}_{n,X_{i-1}} \to X_{i-1}$, we define

$$\mathrm{BGL}_{n,X_i} \to X_i$$

as the evident pullback.

Given $X_i \to X_{i+1}$ and $\mathrm{BGL}_{n,X_i} \to X_i$, we proceed as follows to define $BGL_{n,X_{i+1}} \to X_{i+1}$. Suppose first that the map $\pi_1(X_i) \to \Lambda \to \lambda$ is trivial. The homotopy fibre $F$ of the map $\mathrm{BGL}_{n,X_i} \to X_i$ is $\ell$-equivalent to $\mathbf{F}_\omega(\mathrm{BU}_n)$ and $\pi_1 X_i$ acts trivially on the mod $\ell$ (co)-homology of this pro-space (cf. [**4**, p. 260]). It follows from the fibre lemma [**2**] that $F$ is equivalent to the homotopy fibre of the completed map $\mathbf{F}_\omega(\mathrm{BGL}_{n,X_i}) \to \mathbf{F}_\omega(X_i)$, or to the homotopy fibre of the composite map $\mathbf{F}_\omega(\mathrm{BGL}_{n,X_i}) \to \mathbf{F}_\omega(X_i) \to \mathbf{F}_\omega(X_{i+1})$. (Observe that $X_i \to X_{i+1}$ gives an isomorphism on homology and so induces an equivalence $\mathbf{F}_\omega(X_i) \to \mathbf{F}_\omega(X_{i+1})$). The pro-space $\mathrm{BGL}_{n,X_{i+1}}$ is then the homotopy pullback of $\mathbf{F}_\omega(\mathrm{BGL}_{n,X_i}) \to \mathbf{F}_\omega(X_{i+1})$ over the completion map $X_{i+1} \to \mathbf{F}_\omega(X_{i+1})$.

In the general case, let $\tilde{X}_j$, $j = i,\ i+1$ be the connected covering space determined by the composition

$$X_j \to K(\Lambda, 1) \to K(\lambda, 1)$$

and let $\tau_j \subset \lambda$ denote the corresponding group of covering transformations. The conditions of 1.7 imply that $\tau_i = \tau_{i+1}$; let $\tau$ denote this common group. Let $\mathrm{BGL}_{n,\tilde{X}_i}$ denote the corresponding covering space of $\mathrm{BGL}_{n,X_i}$. Construct a pro-space "$\mathrm{BGL}_{n,\tilde{X}_{i+1}}$" as above. By naturality this pro-space has an action of the group $\tau$, and we define $\mathrm{BGL}_{n,X_{i+1}}$ as $E\tau \times_\tau \mathrm{BGL}_{n,\tilde{X}_{i+1}}$; the desired map $\mathrm{BGL}_{n,X_i} \to \mathrm{BGL}_{n,X_{i+1}}$ can be obtained by combining the $\tau$-equivariant map $\mathrm{BGL}_{n,\tilde{X}_i} \to \mathrm{BGL}_{n,\tilde{X}_{i+1}}$

with some chosen $\tau$-equivariant map $\mathrm{BGL}_{n,\tilde{X}_i} \to E\tau$ and then passing to $\tau$ orbits.

Finally, the homotopy equivalence of spaces of sections is proved by comparing homotopy spectral sequences [**5**, 3.2]. The space $\mathrm{BGL}_n(A_{\mathrm{et}})$ is homotopy equivalent to $\Gamma(A, \mathbf{F}^{\bullet}_{\omega}(\mathrm{BGL}_{n,A})_{\mathrm{et}})$ because

$$\begin{array}{ccc} \mathbf{F}^{\bullet}_{\omega}(\mathrm{BGL}_{n,A})_{\mathrm{et}} & \longrightarrow & \mathbf{F}^{\bullet}_{\omega}(\mathrm{BGL}_{n,R})_{\mathrm{et}} \\ \Big\downarrow & & \Big\downarrow \\ A_{\mathrm{et}} & \longrightarrow & R_{\mathrm{et}} \end{array}$$

is homotopy cartesian. ∎

We conclude this section with the following example from [**5**, §4]. Its verification entails interpreting certain cohomological calculations arising from class field theory.

1.9 EXAMPLE: Let $A$ denote the ring $R[\zeta]$, and assume that $\ell$ is odd and *regular* in the sense of number theory; this last means that $\ell$ does not divide the order of the ideal class group of $A$ or equivalently that $\ell$ does not divide the order of the ideal class group of $\mathbf{Z}[\zeta]$. Choose a prime $p$ which is congruent to 1 mod $\ell$ but is not congruent to 1 mod $\ell^2$. Let $\vee S^1$ denote a bouquet of $(\ell+1)/2$ circles, and construct a map

$$f^A : (\vee S^1) \to K(\Lambda, 1)$$

by sending a generator of the fundamental group of the first circle to $p \in (\mathbf{Z}/\ell^\nu)^*$ and mapping the other circles trivially. Then there exists a map

$$\vee S^1 \to A_{\mathrm{et}}$$

over $K(\Lambda, 1)$ which satisfies the conditions of 1.7. Consequently, $f^A$ is a good mod $\ell$ model for $A$.

## §2. $\mathbf{Z}[1/\ell]$ WITH $\ell$ AN ODD REGULAR PRIME

In this section we will assume that $\ell$ is an odd regular prime (see 1.9); recall that $R$ denotes the ring $\mathbf{Z}[1/\ell]$. Fix a prime $p$ which generates the multiplicative group of units in $\mathbf{Z}/\ell^2$, let $X^R$ denote the space $\mathbf{R}P^\infty \vee S^1$ (the wedge of real infinite projective space and a circle), and let

$$f^R : X^R \to K(\Lambda, 1)$$

be a map with sends the generator of $\pi_1(\mathbf{R}P^\infty) \cong \mathbf{Z}/2$ to $(-1) \in (\mathbf{Z}/\ell^\nu)^*$ and the generator of $\pi_1(S^1) \cong \mathbf{Z}$ to $p \in (\mathbf{Z}/\ell^\nu)^*$.

2.1 THEOREM. *The above map* $f^R : X^R \to K(\Lambda, 1)$ *is a good* mod $\ell$ *model for* $R$.

REMARK: In [**5**] there is a description of a parallel good mod 2 model for the ring $\mathbf{Z}[1/2]$.

Let $e^{\mathbf{R}} : \mathbf{R}P^\infty \to \mathbf{R}_{\text{et}}$ be a homotopy equivalence and $e^{\mathbf{Z}_p} : S^1 \to (\mathbf{Z}_p)_{\text{et}}$ a mod $\ell$ equivalence which sends the generator of $\pi_1(S^1)$ to the Frobenius element in $\pi_1^{\text{et}}\mathbf{Z}_p$ (3.2, 3.4). Given an embedding $\gamma : \mathbf{Z}_p \to \mathbf{C}$, we construct a commutative diagram of rings

$$\begin{array}{ccc} R & \longrightarrow & \mathbf{R} \\ \downarrow & & \downarrow \\ \mathbf{Z}_p & \xrightarrow{\gamma} & \mathbf{C} \end{array} ,$$

choose a basepoint (in an essentially unique way) in the contractible pro-space $\mathbf{C}_{\text{et}}$, and use this basepoint together with the commutative diagram to obtain a map $\epsilon^R(\gamma) : \mathbf{R}_{\text{et}} \vee (\mathbf{Z}_p)_{\text{et}} \to R_{\text{et}}$. Let $e^R(\gamma)$ denote the composite map

$$e^R(\gamma) : \mathbf{R}P^\infty \vee S^1 \xrightarrow{e^{\mathbf{R}} \vee e^{\mathbf{Z}_p}} \mathbf{R}_{\text{et}} \vee (\mathbf{Z}_p)_{\text{et}} \xrightarrow{\epsilon^R(\gamma)} R_{\text{et}} \ .$$

Theorem 2.1 is an immediate consequence of the following result.

2.2 PROPOSITION. *There exists an embedding* $\gamma : \mathbf{Z}_p \to \mathbf{C}$ *such that the above maps* $\epsilon^R(\gamma)$ *and* $e^R(\gamma)$ *each satisfy the conditions of 1.6.*

REMARK: The reader might be surprised to learn that one can always choose an embedding $\gamma' : \mathbf{Z}_p \to \mathbf{C}$ such that the conditions of 1.6 are *not* satisfied by either $\epsilon^R(\gamma')$ or $e^R(\gamma')$. Namely, the proof of Proposition 2.2 shows that choices of elements $\sigma$, $\tau \in G$ as in Remark 2.10 will determine such a "bad" embedding.

Before proving 2.2, we point out the following corollary. In the statement, $\mathrm{F}\,\Psi^p$ refers to the space studied by Quillen [**14**].

2.3 COROLLARY. *The etale* $K$*-theory space* $\mathrm{BGL}(R_{\text{et}})$ *of* $R$ *is homotopy equivalent to*

$$\mathbf{F}_\infty(\mathrm{F}\,\Psi^p \times \mathrm{U/O})\,.$$

PROOF OF 2.3: By Proposition 1.8, we conclude that $\mathrm{BGL}(R_{\text{et}})$ is homotopy equivalent to

$$\mathrm{colim}_n\{\Gamma((\mathbf{Z}_p)_{\text{et}} \vee \mathbf{R}_{\text{et}}, \mathrm{BGL}_{n,(\mathbf{Z}_p)_{\text{et}} \vee \mathbf{R}_{\text{et}}})\}\,.$$

We interpret each space in the colimit as a fibre product of

$$\Gamma((\mathbf{Z}_p)_{\mathrm{et}}, \mathbf{F}^{\bullet}_{\omega}\,\mathrm{BGL}_{n,(\mathbf{Z}_p)_{\mathrm{et}}}) \text{ and } \Gamma(\mathbf{R}_{\mathrm{et}}, \mathbf{F}^{\bullet}_{\omega}\,\mathrm{BGL}_{n,\mathbf{R}_{\mathrm{et}}})$$

over

$$\Gamma(\mathrm{pt}, \mathbf{F}^{\bullet}_{\omega}\,\mathrm{BGL}_{n,\mathrm{pt}})\,.$$

In any given dimension the homotopy groups of the spaces involved in these fibre products stabilize as $n$ gets large [**4**, 4.5], and we identify the colimit of these cartesian squares as

$$\begin{array}{ccc} \mathrm{BGL}(R_{\mathrm{et}}) & \longrightarrow & \mathbf{F}_\infty\,\mathrm{BO} \\ \downarrow & & \downarrow \\ \mathbf{F}_\infty(\mathrm{F}\,\Psi^p) & \longrightarrow & \mathbf{F}_\infty\,\mathrm{BU} \end{array}$$

(see [**5**, 4.1 ff.]) so that $\mathrm{BGL}(R_{\mathrm{et}})$ fits into a fibration sequence

$$\mathbf{F}_\infty(\mathrm{U/O}) \to \mathrm{BGL}(R_{\mathrm{et}}) \to \mathbf{F}_\infty(\mathrm{F}\,\Psi^p)\,.$$

The fibration $\mathrm{BGL}(R_{\mathrm{et}}) \to \mathbf{F}_\infty(\mathrm{F}\,\Psi^p)$ has a section given by the map $\mathbf{F}_\infty(\mathrm{F}\,\Psi^p) = \mathbf{F}_\infty(\mathrm{im}\,J_p) \to \mathbf{F}_\infty(B\Sigma^+_\infty) \to \mathrm{BGL}(R_{\mathrm{et}})$ (see for instance [**12**, §4]) and, since this is a fibration of infinite loop spaces, the section gives the desired product decomposition for $BGL(R_{\mathrm{et}})$. (A more delicate argument would show that the product decomposition holds even in the category of infinite loop spaces). ∎

2.4 Remark: Let $C$ be the infinite cyclic subgroup of $R^*$ generated by $\ell$. Consider the maps

$$\mathrm{BGL}(\mathbf{Z}) \times S^1 \xrightarrow{f} \mathrm{BGL}(R) \xrightarrow{g} \mathbf{F}_\infty(\mathrm{F}\,\Psi^p \times U/O)$$

obtained on the one hand by passing to the limit with the evident direct sum maps $\mathrm{GL}_n(\mathbf{Z}) \times C \to \mathrm{GL}_{n+1}(R)$ and on the other by combining 1.4 with 2.3. It follows from the localization theorem and the triviality of $K_*(\mathbf{Z}/\ell)$ at $\ell$ that the map $f$ is an isomorphism on mod $\ell$ homology. The Lichtenbaum-Quillen conjecture for the ring $\mathbf{Z}$ at the prime $\ell$ is thus [**5**, 3.1] equivalent to the conjecture that the composite $g \cdot f$ is an isomorphism on mod $\ell$ homology.

The rest of this section is concerned with proving 2.2. For simplicity we will denote the space $X^R = \mathbf{R}P^\infty \vee S^1$ by $X$, and we will let $X_\lambda$ be the covering space of $X$ corresponding to the kernel of the composite homomorphism

$$\pi_1(X^R) \xrightarrow{\pi_1(f^R)} \Lambda \to \lambda\,.$$

We will denote the ring $R[\zeta]$ by $R_\lambda$.

Let $\mu$ denote the one dimensional representation of $\mathbf{F}[\lambda]$ given by the action of $\lambda$ on $\mu_\ell \subset R[\zeta]^*$. The ring $\mathbf{F}[\lambda]$ is semisimple (by Maschke's theorem) and

$$\mu, \mu^{\otimes 2}, \dots, \mu^{\otimes(\ell-2)}, \mu^{\otimes(\ell-1)} = \mathbf{Z}/\ell$$

is a complete list of the (isomorphism classes) of irreducible $\mathbf{F}[\lambda]$ modules. If $M$ is an $\mathbf{F}[\lambda]$ module, we denote by $M[\mathrm{odd}]$ (resp., $M[\mathrm{even}]$) the submodule of $M$ generated by irreducible summands isomorphic to odd (resp., even) powers of $\mu$. We shall have frequent occasion to consider the $\mathbf{F}[\lambda]$ module $\mathcal{L}$ given by

$$\mathcal{L} = (\mathbf{F}[\lambda])[\mathrm{odd}] = \mu \oplus \mu^{\otimes 3} \oplus \cdots \oplus \mu^{\otimes(\ell-4)} \oplus \mu^{\otimes(\ell-2)}\,.$$

The module $\mathcal{L}$ is isomorphic as an $\mathbf{F}[\lambda]$ module to its $\mathbf{F}$ dual.

2.5 Lemma. *There are isomorphisms of $\mathbf{F}[\lambda]$ modules*

$$H^1(X_\lambda, \mathbf{Z}/\ell) \cong \mathbf{Z}/\ell \oplus \mathcal{L}$$
$$H_1(X_\lambda, \mathbf{Z}/\ell) \cong \mathbf{Z}/\ell \oplus \mathcal{L}\,.$$

*The groups $H^i(X_\lambda, \mathbf{Z}/\ell)$ vanish for $i \geq 2$.*

Proof: We construct $X_\lambda$ geometrically as follows. Above the circle of $X = \mathbf{R}P^\infty \vee S^1$, we place a circle (constituting part of $X_\lambda$) mapping to $X$ as an $(\ell-1)$ fold covering. We mark on this upper circle the $(\ell-1)$ points given by the inverse image of the base point of $X$. The space $X_\lambda$ is then defined to be the union of this circle with $(\ell-1)/2$ copies of the infinite sphere $S^\infty$, where each $S^\infty$ maps via a 2-fold covering map to $\mathbf{R}P^\infty$ and where the two points of a given $S^\infty$ mapping to the base point of $\mathbf{R}P^\infty$ are identifed with antipodal marked points on the circle of $X_\lambda$ in such a way that each antipodal marked pair belongs to exactly one $S^\infty$. The resulting space $X_\lambda$ is a cyclic covering space of $X$ of degree $\ell-1$ and has the homotopy type of a wedge of $(\ell+1)/2$ circles. The action of the generator

$$\sigma \in \pi_1 S^1 \subset \pi_1 X$$

on $X_\lambda$ rotates the circle of $X_\lambda$ through an angle $2\pi/(\ell-1)$ and operates as continuity requires on the copies of $S^\infty$. The action of the generator

$$\tau \in \pi_1(\mathbf{R}P^\infty) \subset \pi_1 X$$

on $X_\lambda$ rotates the circle through an angle $\pi$ and operates as the antipodal map on each $S^\infty$. Thus, the action of $\tau$ is that of the $\sigma^{(\ell-1)/2}$. Since $\lambda$

is a cyclic group of order $\ell - 1$ generated by $p$ there is a unique action of $\lambda$ on $X_\lambda$ under which $p$ acts as $\sigma$ and $p^{(\ell-1)/2} = -1$ as $\tau$.

The group $\lambda$ acts trivially on the homology class of the circle in $X_\lambda$, and so this circle determines a copy of the trivial $\mathbf{F}[\lambda]$ module $\mathbf{Z}/\ell$ in $H_1(X_\lambda, \mathbf{Z}/\ell)$. Collapsing this circle to a point and replacing each $S^\infty$ by an arc yields a bouquet of $(\ell-1)/2$ circles and it is is clear that the action of $\lambda$ on the dimension one $\mathbf{Z}/\ell$ homology of this bouquet gives the representation $\mathcal{L}$. The desired identification of $H_1(X_\lambda, \mathbf{Z}/\ell)$ follows from the long exact homology sequence of a pair and the fact that exact sequences of $\mathbf{F}[\lambda]$ modules split; the identification of $H^1(X_\lambda, \mathbf{Z}/\ell)$ is by duality. The fact that the higher cohomology groups of $X_\lambda$ vanish is clear. ∎

Recall that $R_\lambda$ denotes $R[\zeta]$.

2.6 Lemma. *There are isomorphisms of* $\mathbf{F}[\lambda]$ *modules*

$$H^1_{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell) \cong \mathbf{Z}/\ell \oplus \mathcal{L}$$
$$H_1^{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell) \cong \mathbf{Z}/\ell \oplus \mathcal{L}\,.$$

*The groups* $H^i_{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell)$ *vanish for* $i \geq 2$.

Proof: Since $R_\lambda$ contains the $\ell$'th roots of unity $\mu_\ell$, there is an isomorphism of $\mathbf{F}[\lambda]$ modules

$$H^1_{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell) \cong H^1_{\mathrm{et}}(R_\lambda, \mu_\ell) \otimes \mu^{\otimes(-1)}$$

where $\mu^{\otimes(-1)} \cong \mu^{\otimes(\ell-2)}$. Let $E$ be the group of units in $R_\lambda$ and $E_\ell$ the quotient group $E/E^\ell$. There is for general reasons a short exact sequence of $\mathbf{F}[\lambda]$ modules (cf. proof of 5.2)

$$0 \to E_\ell \to H^1_{\mathrm{et}}(R_\lambda, \mu_\ell) \to {}_\ell\operatorname{Pic}(R_\lambda) \to 0$$

and in this case the group ${}_\ell\operatorname{Pic}(R_\lambda)$ (i.e., the group of elements of exponent $\ell$ in the ideal class group of $R_\lambda$) is trivial because $\ell$ is regular. Let $E'$ be the group of units in $\mathbf{Z}[\zeta]$ and $E'_\ell$ the quotient $E'/(E')^\ell$. According to [**20**, §8.3] there is an isomorphism of $\mathbf{F}[\lambda]$ modules

$$E'_\ell \cong \mu \oplus \bigoplus_{i=1}^{(\ell-3)/2} \mu^{\otimes 2i}$$

and so, since $E_\ell$ is the direct product of $E'_\ell$ and a cyclic group generated by the image of $\ell \in R^*$, there is an isomorphism

$$E_\ell \cong \mu \oplus \bigoplus_{i=0}^{(\ell-3)/2} \mu^{\otimes 2i} \cong (\mathbf{Z}/\ell \oplus \mathcal{L}) \otimes \mu\,.$$

The desired isomorphisms follow directly. The fact that the higher (co)homology groups vanish is proved in [**5**, 4.4] (cf. 1.9). ∎

We shall now let G denote the semidirect product of $\lambda$ and the $\mathbf{F}[\lambda]$ module $\mathcal{L}$ above. Up to isomorphism the group $G$ is the only extension of $\lambda$ by $\mathcal{L}$. Let $X_G \to X_\lambda$ denote the covering space corresponding to the kernel of the composition

$$\pi_1(X_\lambda) \to H_1(X_\lambda, \mathbf{Z}/\ell) \to H_1(X_\lambda, \mathbf{Z}/\ell)[\text{odd}] \cong \mathcal{L}\,.$$

Similarly, let $R_\lambda \to R_G$ denote the finite etale map corresponding to the kernel of the composition

$$\pi_1(R_\lambda) \to H_1^{\text{et}}(R_\lambda, \mathbf{Z}/\ell) \to H_1^{\text{et}}(R_\lambda, \mathbf{Z}/\ell)[\text{odd}] \cong \mathcal{L}\,.$$

In what follows we will assume that we have chosen an embedding $R_G \subset \mathbf{C}$ which extends the inclusion $R_\lambda \subset \mathbf{C}$. In view of 2.5 and 2.6, the following lemma is elementary.

2.7 Lemma. *The composition $X_G \to X_\lambda \to X$ is a normal covering map with group $G$. The composition $R \to R_\lambda \to R_G$ is a Galois extension with Galois group $G$.*

Let $p_\lambda$ be the unique prime of $R_\lambda$ which lies above the rational prime $p$.

2.8 Lemma. *The prime $p_\lambda$ splits completely in $R_G$.*

Proof: By Kummer theory, $R_G$ can be obtained from $R_\lambda$ by adjoining an element $u^{1/\ell}$ for each $u \in R_\lambda^*$ such that $\bar{u}u^{-1}$ is an $\ell$'th power. (Here $\bar{u}$ is the complex conjugate of $u$). For such a $u$, adjoining $u^{1/\ell}$ is the same as adjoining $(u\bar{u})^{1/\ell}$, so we can obtain $R_G$ from $R_\lambda$ by adjoining $\ell$'th roots of real units. Let $R_\lambda^+ = R_\lambda \cap \mathbf{R}$, let $p_\lambda^+$ be the prime of $R_\lambda^+$ below $p_\lambda$, and choose $u \in (R_\lambda^+)^*$. The residue class field $R_\lambda^+/p_\lambda^+$ has no $\ell$'th roots of unity, so raising to the $\ell$'th power is an automorphism of its multiplicative group and the image of $u$ in $R_\lambda^+/p_\lambda^+$ has at least one $\ell$'th root. By Hensel's lemma $u$ has at least one $\ell$'th root in the completion of $R_\lambda^+$ at $p_\lambda^+$. Since $\mu_\ell \subset R_\lambda^*$, all of the $\ell$'th roots of $u$ lie in the completion of $R_\lambda$ at $p_\lambda$. The lemma follows. ∎

We will have to use some properties of the group $G$ .

2.9 Lemma. *The group $G$ can be generated by two elements $\sigma$, $\tau \in G$ with $\sigma$ having order $\ell - 1$ and $\tau$ having order 2. Moreover, any non-trivial element of $G$ of order 2 is conjugate to $\tau$ and any subgroup of $G$ of order $\ell - 1$ is conjugate to the subgroup generated by $\sigma$.*

2.10 Remark: There certainly exist elements $\sigma$, $\tau \in G$ of order $(\ell - 1)$ and 2 respectively which do *not* generate $G$; for example, choose any $\sigma \in G$ of order $(\ell - 1)$ and let $\tau = \sigma^{(\ell-1)/2}$.

Proof of 2.9: We employ the description of $G$ as the group of deck transformations of $X_G$ over $X$. Since the group

$$\pi_1 X = \pi_1(\mathbf{R}P^\infty \vee S^1) = \mathbf{Z}/2 * \mathbf{Z}$$

is generated by two elements $\tilde{\sigma}$ and $\tilde{\tau}$ with $\tilde{\sigma}$ of infinite order and $\tilde{\tau}$ of order 2, it suffices to prove that the image $\sigma \in G$ of $\tilde{\sigma}$ has order $\ell - 1$. This follows from the fact that the element

$$\tilde{\sigma}^{\ell-1} \in \ker\{\pi_1(X) \to \lambda\} = \pi_1(X_\lambda).$$

corresponds to the circle of $X_\lambda$ (see proof of 2.5); this circle represents a homology class invariant under the action of $\lambda$ and so by construction lifts to a closed curve in $X_G$. The assertions involving conjugacy follow from combining the fact that $H^1(\lambda', \mathcal{L}) = 0$ for any subgroup $\lambda' \subset \lambda$ with the fact that $H^1(\lambda', \mathcal{L})$ can be interpreted as the set of $\mathcal{L}$-conjugacy classes of splittings of the extension

$$1 \to \mathcal{L} \to G' \to \lambda' \to 1$$

obtained by restricting the extension $G$ of $\lambda$ to $\lambda'$. ∎

Proof of 2.2: We will fix two elements $\sigma$, $\tau \in G$ as in 2.9; by the uniqueness provision of 2.9, we can assume after perhaps changing $\sigma$ up to conjugacy that $\tau$ is the restriction to $R_G \subset \mathbf{C}$ of complex conjugation. Denote by $R_\sigma$ the ring fixed by the action of $\sigma$ on $R_G$. According to 2.8 the subgroup $\mathcal{L} \subset G$ acts simply transitively on the set of all primes of $R_G$ which like above $p$. By 2.9, the isotropy (=inertia) subgroups of these primes correspond exactly to the subgroups of $G$ of order $\ell - 1$. Let $p_G$ be such a prime which is fixed by $\sigma$, and $p_\sigma$ the prime below $p_G$ in $R_\sigma$. The completion of $R_\sigma$ at $p_\sigma$ is isomorphic to $\mathbf{Z}_p$, so we can view the completion map as a map $\gamma' : R_\sigma \to \mathbf{Z}_p$.

Let $\gamma : \mathbf{Z}_p \to \mathbf{C}$ be an embedding such that the composite $\gamma \cdot \gamma'$ is the inclusion $R_\sigma \subset \mathbf{C}$. We will show that $e^R(\gamma)$ and $\epsilon^R(\gamma)$ satisfy the conditions of 1.7. All of the rings in question come with embeddings in $\mathbf{C}$ and we will use the chosen basepoint of $\mathbf{C}_{\mathrm{et}}$ to give basepoints for all etale fundamental groups. The pullback of the diagram

$$(\mathbf{Z}_p)_{\mathrm{et}} \xrightarrow{\gamma'_{\mathrm{et}}} (R_\sigma)_{\mathrm{et}} \leftarrow (R_G)_{\mathrm{et}}$$

is $(\mathbf{Z}_p \otimes_{R_\sigma} R_G)_{\mathrm{et}}$, which is connected because ($p_\sigma$ being inert in $R_G$) the ring $\mathbf{Z}_p \otimes_{R_\sigma} R_G$ is the completion of $R_G$ at $p_G$ and hence a domain. An elementary covering space argument now shows that the image of the composite map

$$\pi_1^{\mathrm{et}}\mathbf{Z}_p \to \pi_1^{\mathrm{et}}R_\sigma \to \pi_1^{\mathrm{et}}R \to G$$

contains (and is in fact equal to) the subgroup generated by $\sigma$; since the image under $e^{\mathbf{Z}_p}$ of $\pi_1(S^1)$ is dense in the profinite group $\pi_1^{\mathrm{et}}\mathbf{Z}_p$, the image of the map

$$\pi_1 S^1 \to \pi_1^{\mathrm{et}}\mathbf{Z}_p \to G$$

also contains $\sigma$. In the same way the image of the map $\pi_1\mathbf{R}P^\infty \to \pi_1^{\mathrm{et}}R$ is equal to the subgroup generated by the element $\tau \in G$ representing complex conjugation. By 2.9 the composite map

$$\pi_1 X = \pi_1(\mathbf{R}P^\infty \vee S^1) \xrightarrow{\pi_1(e^R(\gamma))} \pi_1^{\mathrm{et}}R \to G$$

is surjective, so that the map $\pi_1(X_\lambda) \to (R_\lambda)_{\mathrm{et}}$ induced by $e^R(\gamma)$ induces a surjection $H_1(X_\lambda, \mathbf{Z}/\ell)[\mathrm{odd}] \to H_1^{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell)[\mathrm{odd}]$. By 2.5. 2.6 and a descent spectral sequence argument, the map $H_1(X_\lambda, \mathbf{Z}/\ell)[\mathrm{even}] \to H_1^{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell)[\mathrm{even}]$ induced by $e^R(\gamma)$ is the same as the induced map

$$H_1(X, \mathbf{Z}/\ell) \to H_1^{\mathrm{et}}(R, \mathbf{Z}/\ell) \cong \mathbf{Z}/\ell\,.$$

This map is surjective because $p$ is inert in the (unique) unramified Galois $\mathbf{Z}/\ell$ extension of $R$; in fact, this extension is the maximal real subring of the ring obtained from $R$ by adjoining a primitive $\ell^2$ root of unity. (Here is the one place in which we use the fact that $p$ is a multiplicative generator mod $\ell^2$ and not just mod $\ell$.) Consequently, the map $H_1(X_\lambda; \mathbf{Z}/\ell) \to H_1^{\mathrm{et}}(R_\lambda, \mathbf{Z}/\ell)$ induced by $e^R(\gamma)$ is an epimorphism and hence by counting (2.5, 2.6) an isomorphism. This shows that $e^R(\gamma)$ satisfies the conditions of 1.7. Since the map $e^{\mathbf{R}} \vee e^{\mathbf{Z}_p}$ also satisfies the conditions of 1.7, the desired result for $\epsilon^R(\gamma)$ follows at once. ∎

## §3. Affine curves and local fields

Let $p$ be a prime number different from $\ell$. In this section, we construct good mod $\ell$ models for

(1) affine curves over the algebraic closure $\bar{\mathbf{F}}_p$ of $\mathbf{F}_p$,
(2) affine curves over a finite field $\mathbf{F}_q$ where $q = p^d$,
(3) higher $p$–adic local fields, and
(4) certain $\ell$–adic local fields.

Our first example, that of a smooth affine curve over $\bar{\mathbf{F}}_p$ , is implicit in [**9**].

3.1 Proposition. *Let $\bar{Y}$ denote a smooth complete curve over $\bar{\mathbf{F}}_p$ of genus $g$ and let $\bar{A}$ denote the coordinate algebra of the complement $\bar{U} \subset \bar{Y}$ of $s \geq 1$ points. Let $X^{\bar{A}} \equiv \mathbf{F}_\omega(\vee S^1)$ denote the $\ell$–adic completion of the wedge of $2g+s-1$ circles. Then there exists a* mod $\ell$ *equivalence*

$$\bar{A}_{\mathrm{et}} \to X^{\bar{A}} .$$

*Furthermore, the structure map $\bar{A}_{\mathrm{et}} \to K(\Lambda, 1)$ is (homotopically) trivial, so that $X^{\bar{A}}$ (with the trivial structure map to $K(\Lambda,1)$) is a good* mod $\ell$ *model for $\bar{A}$.*

Proof: Let $T \equiv \mathrm{Witt}(\bar{\mathbf{F}}_p)$ denote the Witt vectors of $\bar{\mathbf{F}}_p$. Then the "liftability of smooth curves to characteristic 0" asserts that there is a proper, smooth map $Y_T \to \operatorname{Spec} T$ with (geometric) fibre $\bar{Y}$ over the residue field of $T$. Moreover, we may find a closed subscheme $Z_T \subset Y_T$ with the properties that $Z_T$ is finite, etale over $\operatorname{Spec} T$ and that the fibre over the residue field of $U_T \equiv Y_T \backslash Z_T$ is $\bar{U}$. So constructed, $U_T \to \operatorname{Spec} T$ is a "geometric fibration", so that the $\ell$–adic completions of its geometric and homotopy theoretic fibres are equivalent [**8**], [**7**, 11.5].

Let $\gamma : \operatorname{Spec} \mathbf{C} \to \operatorname{Spec} T$ correspond to an embedding $T \subset \mathbf{C}$. The geometric fibre $Y_\gamma$ is a Riemann surface of genus $g$ with $s$ punctures, thereby having the homotopy type of a wedge of $2g+s-1$ circles. Since $T_{\mathrm{et}}$ is contractible, we conclude that the $\ell$–adic completion of the homotopy theoretic fibre of $U_T \to \operatorname{Spec} T$ is equivalent to $X^{\bar{A}}$. On the other hand, $\operatorname{Spec} \bar{A}$ is the geometric fibre of $U_T \to \operatorname{Spec} T$ above the residue field $\operatorname{Spec} \bar{\mathbf{F}}_p \to \operatorname{Spec} T$. Thus, $\bar{A}_{\mathrm{et}}$ has $\ell$–adic completion equivalent to $X_{\bar{A}}$. The triviality of the structure map $\bar{A}_{\mathrm{et}} \to K(\Lambda,1)$ follows from the fact that $\bar{A}$ contains all of the $\ell$–primary roots of unity. ∎

3.2 Remark: The fact that the finite field $\mathbf{F}_q$ $(q = p^d)$ has a unique extension inside $\bar{\mathbf{F}}_p$ of degree $s$ for any $s > 0$ and that each such extension is Galois and cyclic with a canonical generator (the Frobenius) for its Galois group implies that

$$(\mathbf{F}_q)_{\mathrm{et}} \cong \{K(\mathbf{Z}/s, 1)\}_{s>0} .$$

In fact, there is a natural mod $\ell$ equivalence $S^1 \to (\mathbf{F}_q)_{\mathrm{et}}$ which sends the generator of $\pi_1 S^1$ to the Frobenius map. The composite of this mod $\ell$ equivalence with the structure map $(\mathbf{F}_q)_{\mathrm{et}} \to K(\Lambda,1)$ is the map $\psi^q : S^1 \to K(\Lambda, 1)$ which sends the generator of $\pi_1 S^1$ to $q \in \mathrm{Aut}(\mu_{\ell^\nu}) \cong (\mathbf{Z}/\ell^\nu)^*$.

3.3 Proposition. *Let $Y$ denote a smooth, complete curve over $\mathbf{F}_q$ of genus $g$, let $A$ denote the coordinate algebra of the complement $U \subset Y$*

*of $s \geq 1$ $\mathbf{F}_q$–rational points, and let $\bar{A}$ denote $A \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p$. Let $X^{\bar{A}}$ be as in 3.1. Then there exists an equivalence*

$$\phi : X^{\bar{A}} \equiv \mathbf{F}_\omega(\vee S^1) \to \mathbf{F}_\omega(\vee S^1) \equiv X^{\bar{A}}$$

*such that the homotopy quotient $X^A \equiv X^{\bar{A}} \times_{\mathbf{Z}} E\mathbf{Z}$ of the action of $\phi$ on $X^{\bar{A}}$ is a good* mod $\ell$ *model for $A$; the structure map $f^A$ is the composition*

$$f^A = \psi^q \circ \mathrm{pr} : X^A \to B\mathbf{Z} = S^1 \to K(\Lambda, 1).$$

*Moreover, if $g = 0$ then $\phi$ is a map which induces multiplication by $q$ on $H_1(X^{\bar{A}}, \mathbf{Z}/\ell^\nu)$ for each $\nu > 0$.*

Proof: We view $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{F}_q$ as a geometric fibration with geometric fibre $\operatorname{Spec} \bar{A}$ (cf. proof of Proposition 3.1). Thus, the homotopy theoretic fibre of $A_{\mathrm{et}} \to (\mathbf{F}_q)_{\mathrm{et}}$ is mod $\ell$ equivalent to $X^{\bar{A}}$ [**8**]. Equivalently, the mod $\ell$ fibrewise completion $\mathbf{F}^\bullet_\omega(A_{\mathrm{et}})$ of $A_{\mathrm{et}}$ fits in a fibration sequence

$$X^{\bar{A}} \to \mathbf{F}^\bullet_\omega(A_{\mathrm{et}}) \to (\mathbf{F}_q)_{\mathrm{et}}$$

Let $S^1 \to (\mathbf{F}_q)_{\mathrm{et}}$ be given by sending a generator of $\pi_1(S^1) = \mathbf{Z}$ to the Frobenius map (as in 3.2). Then the pullback of the above fibration sequence by $S^1 \to (\mathbf{F}_q)_{\mathrm{et}}$ is a fibration sequence of the form

$$X^{\bar{A}} \to E \to S^1$$

This implies that $E$ is the homotopy quotient of $\phi$ , where $\phi : X^{\bar{A}} \to X^{\bar{A}}$ is the "deck transformation" associated to a generator of $\pi_1(S^1)$, the group of the covering space $X^{\bar{A}} \to X^A$. Let $X^A = E$.

By construction we have a commutative diagram of prospaces

$$\begin{tikzcd}
A_{\mathrm{et}} \arrow[r] \arrow[d] & \mathbf{F}^\bullet_\omega(A_{\mathrm{et}}) \arrow[d] & X^A \arrow[l] \arrow[d] \\
(\mathbf{F}_q)_{\mathrm{et}} \arrow[r, "="] & (\mathbf{F}_q)_{\mathrm{et}} & S^1 \arrow[l]
\end{tikzcd}$$

in which the horizontal arrows are (fibrewise) mod $\ell$ equivalences. Since the structure map $A_{\mathrm{et}} \to K(\Lambda, 1)$ factors through the structure map $(\mathbf{F}_q)_{\mathrm{et}} \to K(\Lambda, 1)$ and since composition of this latter map with $S^1 \to (\mathbf{F}_q)_{\mathrm{et}}$ was checked above to be $\psi^q$ we conclude that the indicated map $X^A \to K(\Lambda, 1)$ is a good mod $\ell$ model for $A$.

Finally, if $g = 0$ and $s = 2$ (so that $A \cong \mathbf{F}_q[t, t^{-1}]$), then the fact that $\mathrm{Pic}\,\bar{A} = \mathbf{Z}$ in this case implies that there is a natural isomorphism $H^1_{\mathrm{et}}(\bar{A}, \mu_{\ell^\nu}) \cong \bar{A}^* \otimes \mathbf{Z}/\ell^\nu$ (cf. the exact sequence in the proof of 2.6). Observe that $\bar{A}^* \otimes \mathbf{Z}/\ell^\nu$ is invariant under $\mathrm{Gal}(\bar{\mathbf{F}}_p, \mathbf{F}_q)$ so that $\mathrm{Gal}(\bar{\mathbf{F}}_p, \mathbf{F}_q)$ acts on $H^1_{\mathrm{et}}(\bar{A}, \mathbf{Z}/\ell^\nu) \equiv H^1_{\mathrm{et}}(\bar{A}, \mu_{\ell^\nu}) \otimes \mu_{\ell^\nu}^{-1}$ by multiplication by $q^{-1}$ and dually on $H_1^{\mathrm{et}}(\bar{A}, \mathbf{Z}/\ell^\nu)$ by multiplication by $q$. ∎

Remark: The "Riemann Hypothesis for Curves" provides further information about the self-equivalence $\phi$ of $X^{\bar{A}}$ in Proposition 3.3. Namely, we conclude that the induced map on $\ell$–adic homology is the sum of two maps: the first is multiplication by $q$ on $(\mathbf{Z}_\ell)^{s-1}$ (as in Proposition 3.3) and the second has eigenvalues of absolute value $q^{1/2}$ on $(\mathbf{Z}_\ell)^{2g}$.

3.4 Remark: If $A$ is a hensel local ring (e.g., a complete discrete valuation ring) with residue field $k$ and quotient field $K$, then the map $k_{\mathrm{et}} \to A_{\mathrm{et}}$ is an equivalence, but the relationship between $K_{\mathrm{et}}$ and $A_{\mathrm{et}}$ is considerably more subtle. The special case $r = 1$ of the following proposition treats this question in a special case, as does 3.6.

As considered for example in [**6**] a generalized local field $K$ of "transcendental degree" $r$ over a field $k$ is a field for which there exists a finite chain of homomorphisms

$$k = K_0 \leftarrow O_1 \rightarrow K_1 \leftarrow \cdots \leftarrow O_r \rightarrow K_r = K$$

such that each $O_j$ is a hensel discrete valuation ring with residue field $K_{j-1}$ and quotient field $K_j$.

3.5 Proposition. *Let $K$ be a generalized local field of transcendental degree $r$ over the finite field $\mathbf{F}_q$ and let $S = S^1$ denote the circle. Consider the space*

$$X^K \equiv \mathbf{F}_\omega(S^{\times r}) \times_{\mathbf{Z}} E\mathbf{Z}$$

*which is the homotopy quotient of the map given by multiplication by $q$ on the completion of the $r$–fold cartesian power of $S$. If $X^K$ is provided with the structure map*

$$f^K = \psi^q \circ \mathrm{pr} : X^K \to B\mathbf{Z} \to K(\Lambda, 1)\,,$$

*then it is a good* mod $\ell$ *model for $K$.*

Proof: The proof of [**6**, 3.4] establishes a chain of ring homomorphisms, each inducing an isomorphism in etale cohomology with $\mathbf{Z}/\ell(i)$ coefficients ($i = 1, \ldots, \ell - 1$), relating $A \equiv \mathbf{F}_q[t_1, ..., t_r, t_1^{-1}, ..., t_r^{-1}]$ and $K$. Thus, a good mod $\ell$ model for $A$ will also be a good mod $\ell$ model for $K$

(ring homomorphisms are always compatible with the structure maps to $K(\Lambda,1)$).

We view $\operatorname{Spec} A$ as the complement in $(P^1_{\mathbf{F}_q})^{\times r}$ of a divisor with normal crossings rational over $\mathbf{F}_q$. Replacing $\mathbf{F}_q$ by $T \equiv \operatorname{Witt}(\mathbf{F}_q)$, the Witt vectors of $\mathbf{F}_q$, we obtain $\tilde{U} = \operatorname{Spec} \tilde{A} \subset (P^1_T)^{\times r}$ (i.e., the $r$-fold fibre product of $P^1_T$ with itself over $T$) with the property that $\tilde{U} \to \operatorname{Spec} T$ is a geometric fibration. Letting $\gamma : \operatorname{Spec} \mathbf{C} \to \operatorname{Spec} T$ be given by a complex embedding of $T$, we observe that the geometric fibre of $U_\gamma$ is mod $\ell$ equivalent to $S^{\times r}$. As in the proof of Proposition 3.3, this implies the mod $\ell$ equivalences over $K(\Lambda,1)$

$$A_{\mathrm{et}} \to \mathbf{F}^{\bullet}_{\omega}(A_{\mathrm{et}}) \leftarrow \mathbf{F}_\omega(S^{\times r}) \times_{\mathbf{Z}} E\mathbf{Z}\,.$$

Since the action of $\operatorname{Gal}(\bar{\mathbf{F}}_p, \mathbf{F}_p)$ on $H^1_{\mathrm{et}}(A, \mu_{\ell^\nu}) \cong A^* \otimes \mathbf{Z}/\ell^\nu$ is trivial, we conclude as in the proof of Proposition 3.3 that the action of $\mathbf{Z}$ on $\mathbf{F}_\omega(S^{\times r})$ is given by multiplication by $q$. ∎

In the next proposition, we present a good mod $\ell$ model for a $p$–adic field which looks a bit different from the one provided by the special case $r = 1$ in Proposition 3.5.

3.6 Proposition. *Let $K$ be a $p$–adic field with residue field $\mathbf{F}_q$. Consider the* 1*-relator group $\pi_K$ given by*

$$\pi_K = <x, y : yxy^{-1} = x^q>\,.$$

*Then the map*

$$\tilde{f}^K : K(\pi_K,1) \to K(\Lambda,1)$$

*induced by the homomorphism $\pi_K \to \Lambda$ sending $x$ to the identity and $y$ to $q \in \operatorname{Aut}(\mu_{\ell^\nu}) = (\mathbf{Z}/\ell^\nu)^*$ is a good* mod $\ell$ *model for $K$. Furthermore, $K(\pi_K,1)$ is homotopy equivalent to the (homotopy) pushout $\Pi_K$ of the diagram*

$$(S^1 \vee S^1) \leftarrow S^1 \to pt$$

*in which the left map sends a generator of $\pi_1(S^1)$ to $x^{-q}yx^{-1}y^{-1}$, where* x *and $y$ are the generators of $\pi_1(S^1 \vee S^1)$ associated to the wedge summands.*

Proof: Using the relationship $yxy^{-1} = x^q$, we readily verify that the normal closure of the cyclic subgroup generated by $x$ consists of fractional powers of $x$ with denominators a multiple of $q$. Hence, $\pi_K$ lies in an extension $\mathbf{Z}[1/p] \to \pi_K \to \mathbf{Z}$ and $K(\pi_K,1)$ fits into a fibration sequence

$$(S^1)_{1/p} \to K(\pi_K,1) \to S^1\,.$$

Let $X^K$ be the good mod $\ell$ model for $K$ resulting from the case $r = 1$ of 3.5. There is evidently a commutative square

$$\begin{array}{ccc} K(\pi_K, 1) & \longrightarrow & X^K \\ \downarrow & & \downarrow \\ S^1 & \xrightarrow{=} & S^1 \end{array}$$

in which the induced map on fibres is a mod $\ell$ equivalence. We conclude that $\tilde{f}^K : K(\pi_K, 1) \to K(\Lambda, 1)$ is a good mod $\ell$ model for $K$.

The equivalence $\Pi_K \simeq K(\pi_K, 1)$ follows from a theorem of Lyndon (cf. [**3**, p. 37]) in view of the fact that the relator of $\pi_K$ is not a power. ∎

Our last example is that of an $\ell$–adic field containing a primitive $\ell$'th root of unity. The proposition below is based upon a theorem of Demuskin [**17**].

3.7 Proposition. *Let $L$ be an $\ell$–adic local field. i.e., the completion of a number field $F$ at a prime with residue field of characteristic $\ell$. Assume that $\mu_{\ell^s} \subset L$, where $\ell^s > 2$, that $\mu_{\ell^{s+1}} \not\subset L$, and that $2g$ is the degree of $L$ over $\mathbf{Q}_\ell$. Let $G_L$ denote the 1-relator group generated by $2g + 2$ elements $x, y, x_1, \ldots, x_{2g}$ subject to the defining relation*

$$x^{\ell^s+1} y x^{-1} y^{-1} = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \ .$$

*Then there is a homomorphism $G_L \to \Lambda$ such that the classifying map $f^L : K(G_L, 1) \to K(\Lambda, 1)$ is a good* mod $\ell$ *model for $L$. Moreover, $K(G_L, 1)$ is homotopy equivalent to the (homotopy) pushout $\Pi_L$ of the diagram*

$$(S^1 \vee S^1) \leftarrow S^1 \to (\vee_{2g} S^1)$$

*in which the left map sends a generator $t$ of $\pi_1(S^1)$ to $x^{\ell^s+1} y x^{-1} y$ and the right map sends $t$ to the product of commutators $[x_1, x_2] \ldots . [x_{2g-1}, x_{2g}]$ (here $x, y, x_1, \ldots, x_{2g}$ are generators of corresponding fundamental groups associated to wedge summands).*

Proof: The second assertion follows from [**3**, p. 50]; it implies that $G_L$ has cohomological dimension 2.

Let $\Gamma_L$ denote the Galois group $\mathrm{Gal}(\bar{L}, L)$, where $\bar{L}$ is the algebraic closure of $L$, and let "$\widehat{\ }$" denote profinite $\ell$–completion. The completion map $L_{\mathrm{et}} \cong K(\Gamma_L, 1) \to K(\hat{\Gamma}_L, 1)$ induces an isomorphism on $H_*(-, \mathbf{Z}/\ell)$ by [**16**, Ch. II, Prop. 20] and hence in particular [**16**, Ch. II, Prop. 15] both $\Gamma_L$ and $\hat{\Gamma}_L$ have cohomological dimension $\leq 2$.

By [**17**] (cf. [**16**, Ch. II, §5.6]) there is an isomorphism $\hat{G}_L \cong \hat{\Gamma}_L$. The map $G_L \to \hat{G}_L$ induces a map

$$H_i(K(G_L,1),\mathbf{Z}/\ell) \to H_i(K(\hat{G}_L,1),\mathbf{Z}/\ell)$$

which for general reasons is an isomorphism when $i \le 1$ and an epimorphism for $i = 2$. Since $H_2(K(G_L,1),\mathbf{Z}/\ell)$ is isomorphic to $\mathbf{Z}/\ell$ by the above construction of $K(G_L,1)$, $H_2(K(\hat{G}_L,1),\mathbf{Z}/\ell)$ is isomorphic to $\mathbf{Z}/\ell$ by [**16**, Ch. II, Th. 4], and the higher $\mathbf{Z}/\ell$ homology groups of these pro-spaces vanish, we conclude that in the diagram

$$L_{\text{et}} \cong K(\Gamma_L,1) \xrightarrow{\alpha} K(\hat{\Gamma}_L,1) \cong K(\hat{G}_L,1) \xleftarrow{\beta} K(G_L,1)$$

both of the maps $\alpha$ and $\beta$ induce isomorphisms on $\mathbf{Z}/\ell$ homology. Since $L$ contains $\mu_\ell$ the local systems $\mathbf{Z}/\ell(i)$ are all isomorphic to the trivial system $\mathbf{Z}/\ell$. Thus, if we define $f^L : K(G_L,1) \to K(\Lambda,1)$ to be given by the composite

$$G_L \to \hat{G}_L \cong \hat{\Gamma}_L \to \Lambda\,,$$

the map $f^L$ is a good mod $\ell$ model for $L$. ∎

## §4. The Eilenberg–Moore spectral sequence

In this section, we describe the machinery from algebraic topology which will be used to prove the cohomological injectivity statements in §6. Our main tool will be the Eilenberg–Moore spectral sequence, denoted EMSS; we will use it to study the cohomology of function complexes and in §6 to investigate the maps in cohomology induced by continuous maps between function complexes. This resembles Quillen's use of the EMSS in [**14**, §1–5].

Throughout this section we will consider graded algebras over the field $\mathbf{F}$, in particular, we shall use $H^*X$ to denote the mod $\ell$ cohomology $H^*(X,\mathbf{Z}/\ell)$ of a space $X$. We begin with a brief discussion of the EMSS; further details may be found in [**18**]. We consider a cartesian square of spaces with $f$ (and thus $f'$) a fibration:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \longrightarrow & B \end{array}$$

where $B$ is assumed to be simply connected and the cohomology (i.e., mod $\ell$ cohomology) of each space is assumed finite in each dimension. Then there is a strongly convergent second quadrant spectral sequnce

$$E_2^{-s,t} = \operatorname{Tor}_{H^*B}^{-s,t}(H^*(B'),H^*E) \Rightarrow H^*(E')$$

whose $r$'th differential takes the form

$$d_r : E_r^{-s,t} \to E_r^{-s+r,t-r+1}.$$

This is a spectral sequence of algebras; each differential is a derivation.

Recall that if $S$ is a (positively) graded commutative algebra and $M$, $N$ are graded modules over $S$, then $\mathrm{Tor}_S^{-s,t}(M,N)$, where $s$ is the homological degree and $t$ is the internal degree, is explained as follows. If

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_0 \to M$$

is a projective resolution of $M$ (as a graded right $S$-module) and $F'_n = F_n \otimes_S N$, then $\mathrm{Tor}_S^{-s,t}(M,N)$ denotes the degree $t$ component of the graded module

$$\ker\{F'_s \to F'_{s-1}\ \}/\operatorname{im}\{F'_{s+1} \to F'_s\}\,.$$

Let $S$ be a graded commutative algebra (e.g., $S = H^*B$). One of the most convenient contexts in which to compute $\mathrm{Tor}_S(M,N)$ is that in which $M$ is a quotient of $S$ by an ideal $I$ generated by a regular sequence $x_1, \ldots, x_n$ of homogeneous elements. Then one obtains a very efficient free resolution of $M$, the *Koszul resolution*, which takes the following form:

$$\cdots \to S \otimes \Lambda^n V \to S \otimes \Lambda^{n-1} V \to \cdots \to S \otimes V \to S \to M$$

where $V$ is a vector space spanned by elements $y_1, \ldots, y_n$ and where the tensor products are taken over $\mathbf{F}$.

The following proposition, an almost immediate consequence of the existence of the Koszul resolution, is essentially in [**18**, 2.10].

4.1 PROPOSITION. *Let $S$ be the polynomial algebra $\mathbf{F}[x_1, ..., x_n]$ graded so that each $x_i$ is homogeneous of degree $d_i$. Let $S(r)$ denote the $r$–fold tensor power of $S$ over $\mathbf{F}$, $S(r) \equiv S^{\otimes r}$, and give $S$ the structure of an $S(r)$ module by means of the multiplication map. Then $\mathrm{Tor}^*_{S(r)}(S,S)$ is naturally isomorphic as an $S$ algebra to the $(r-1)$–fold tensor power of the DeRham complex of $S$:*

$$\mathrm{Tor}^*_{S(r)}(S,S) \cong \Omega^*_{S/\mathbf{F}} \otimes_S \cdots \otimes_S \Omega^*_{S/\mathbf{F}}\ .$$

REMARK: The *DeRham complex* $\Omega^*_{S/\mathbf{F}}$ is an exterior algebra over $S$ generated by elements $dx_i$ ($i = 1, \ldots, n$) of homological degree 1 and internal degree $d_i$.

Proof of 4.1: The kernel of the multiplication map $S(r) \to S$ is generated by the regular sequence

$$\{x_{i,j} - x_{i,r};\ 1 \le i \le n,\ 1 \le j < r\}$$

in

$$S(r) \cong \mathbf{F}[x_1, ..., x_n]^{\otimes r} \cong \mathbf{F}[x_{i,j};\ 1 \le i \le n,\ 1 \le j \le r].$$

We obtain the asserted computation by taking the Koszul resolution of $S$ as an $S(r)$ module using this regular sequence, tensoring this resolution over $S(r)$ with $S$, and observing that the resulting complex has trivial differentials. ∎

With the aid of Proposition 4.1, the EMSS easily provides the following computation.

4.2 Proposition. *Let $X$ be a simply connected space with $H^*X$ isomorphic to the finitely generated polynomial algebra $S = \mathbf{F}[x_1, ..., x_n]$. Let $Y$ denote the space*

$$Y \equiv \mathrm{Map}(\vee S^1, X)$$

*of unpointed maps from a bouquet of $r$ circles to $X$. Then $H^*Y$ admits a filtration associated to the EMSS whose associated graded module is naturally isomorphic to the $r$-fold tensor power of the DeRham complex of $S$:*

$$\mathrm{gr}\ H^*Y \cong \Omega^*_{S/\mathbf{F}} \otimes_S \cdots \otimes_S \Omega^*_{S/\mathbf{F}}.$$

Proof: Let $\vee D^1$ be the one-point union of $r$ copies of the unit interval $D^1$ formed by glueing these copies together at $0 \in D^1$. The space $\vee S^1$ can be obtained from $\vee D^1$ by further identifying the $r$ images $1_i$ $(i = 1, \dots, r)$ of $1 \in D^1$ with the common image of 0. This implies that the space $Y$ fits into a cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Map}(\vee D^1, X) \simeq X \\ \big\downarrow & & \big\downarrow e \\ X & \xrightarrow{\Delta} & X^{r+1} \end{array}$$

where $\Delta$ is the diagonal and $e$ is the fibration given by evaluation at various interval endpoints. Under the basepoint evaluation equivalence $\mathrm{Map}(\vee D^1, X) \simeq X$ the map $e$ is homotopic to the diagonal. Using Proposition 4.1, we identify the $E_2$ term of the EMSS for this square as

$$E_2^{-s,*} = \bigoplus_{s = s_1 + \cdots + s_r} \Omega^{s_1}_{S/\mathbf{F}} \otimes_S \cdots \otimes_S \Omega^{s_r}_{S/\mathbf{F}}.$$

The shape of this spectral sequence implies that $E_2^{0,*}$ and $E_2^{-1,*}$ consist of permanent cycles; since these cycles generate $E_2^{*,*}$ multiplicatively, we conclude that $E_2^{*,*}$ consists of permanent cycles, or, in other words, that $E_2 = E_\infty$. ∎

The following generalization of Proposition 4.2 will be useful; it is proved in the same way as 4.2. Let $\vee S^1$ be a bouquet of $r$ circles, and $f : E \to \vee S^1$ a fibration with homotopy fibre $X = f^{-1}(*)$. Let $\vee D^1$ be the space in the proof of 4.2 and $f' : E' \to \vee D^1$ the (trivial) fibration obtained by pulling $E$ back over the quotient map $\vee D^1 \to \vee S^1$. As in the proof of 4.2, there is a homotopy fibre square for the space of sections $\Gamma(f)$ of $f$

$$\begin{array}{ccc} \Gamma(f) & \longrightarrow & \Gamma(f') \simeq X \\ \Big\downarrow & & \Big\downarrow e \\ X & \xrightarrow{\Delta} & X^{r+1} \end{array}$$

where again $\Delta$ is the diagonal and $e$ is given by evaluation at the points $0$ and $1_i$ $(i = 1, \ldots, r)$ in $\vee D^1$. Under the basepoint evaluation equivalence $\Gamma(f') \simeq X$ the map $e$ is homotopic to a product $(\phi_0, \ldots, \phi_r)$, where $\phi_0$ is the identity map of $X$ and each $\phi_i$, $i = 1, \ldots, r$ is a self-equivalence of $X$. We will call the self-equivalences $\phi_i$ the *monodromy maps* of the fibration $f$. The induced cohomology maps $\phi_i^*$ generate the usual monodromy action of the free group $\pi_1(\vee S^1)$ on $H^*X$.

4.3 PROPOSITION. *Let $f : E \to \vee S^1$ be a fibration with fibre $X$ over a bouquet $\vee S^1$ of $r$ circles. Assume that $X$ is simply connected. Let $\phi_i$, $i = 1, \ldots, r$ be the monodromy maps of $f$. Let $e : X \to X^{r+1}$ be the product $(\mathrm{id}, \phi_1, \ldots, \phi_r)$, and ${}_eH^*X$ the module over $H^*(X^{r+1})$ obtained using $e^*$. Then there is a strongly convergent EMSS*

$$\mathrm{Tor}^{-s,t}_{H^*(X^{r+1})}(H^*X,\, {}_eH^*X) \Rightarrow H^*(\Gamma(f)) \ .$$

*Finally, if $H^*X$ is isomorphic to a finitely generated polynomial algebra $S$ over $\mathbf{F}$ and $\pi_1(\vee S^1)$ acts trivially on $H^*X$ then this EMSS collapses to an isomorphism*

$$\mathrm{gr}\ H^*(\Gamma(f)) \cong \Omega^*_{S/\mathbf{F}} \otimes_S \cdots \otimes_S \Omega^*_{S/\mathbf{F}}.$$

## §5. ALGEBRAIC AND TOPOLOGICAL TORI

In this section, we verify the injectivity of the map on mod $\ell$ cohomology induced by the natural map

$$BT_n(A) \to BT_n(A_{\mathrm{et}})\,,$$

where $A$ is an $R$ algebra and $T_{n,R}$ is the rank $n$, $R$–split torus given by

$$T_{n,R} = \operatorname{Spec} R[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}].$$

The map in question is the $n$-fold cartesian power of the map $BT_1(A) \to BT_1(A_{\mathrm{et}})$ associated to the special case $n = 1$, and so it is this special case that we will concentrate on. This discussion prepares the way for the consideration in §6 of the more delicate map $\mathrm{BGL}_n(A) \to \mathrm{BGL}_n(A_{\mathrm{et}})$. We begin with the construction of non-connected versions of $BT_1(A)$ and $BT_1(A_{\mathrm{et}})$.

5.1 Lemma. *Let $A$ be a noetherian $R$ algebra. There exists infinite loop spaces*

$$B\tilde{T}_1(A) \equiv \operatorname{Hom}_g(A, BT_{1,R}) \text{ and } B\tilde{T}_1(A_{\mathrm{et}}) \equiv \operatorname{Hom}_\ell(A_{\mathrm{et}}, (BT_{1,R})_{\mathrm{et}})_{R_{\mathrm{et}}}$$

*with the following properties:*

1. (1) *$BT_1(A)$ and $BT_1(A_{\mathrm{et}})$ are respectively the identity components of $B\tilde{T}_1(A)$ and $B\tilde{T}_1(A_{\mathrm{et}})$.*
2. (2) $\pi_0(B\tilde{T}_1(A)) = \operatorname{Pic}(A)$.
3. (3) *The map $\phi_A^0 : BT_1(A) \to BT_1(A_{\mathrm{et}})$ is the restriction to identity components of a map of infinite loop spaces $\phi_A : B\tilde{T}_1(A) \to B\tilde{T}_1(A_{\mathrm{et}})$.*

Proof: The construction of $\operatorname{Hom}_g(A, BT_{1,R})$ and the natural map $\phi_A$ of (3) are provided in [**4**, §2]. One verifies that this map is the map on zero spaces of a map of spectra as in [**4**, §3, 4.4] using the multiplicative structure on $BT_{1,R}$ given by the commutative group scheme $T_{1,R}$. Now, (2) follows from [**4**, A.6] and (1) is implicit in the constructions just described. ∎

The following proposition is implicit in [**4**, p. 274].

5.2 Proposition. *Let $A$ be a noetherian $R$ algebra such that* $\operatorname{Pic}(A)$ *contains no infinitely $\ell$–divisible elements. Then the natural map (1.4) $BT_1(A) \to BT_1(A_{\mathrm{et}})$ induces an isomorphism*

$$H^*(BT_1(A_{\mathrm{et}}), \mathbf{Z}/\ell) \to H^*(BT_1(A), \mathbf{Z}/\ell)\ .$$

Proof: Let $F$ denote the homotopy fibre of $\phi_A$ (see 5.1); observe that $F$ has a natural basepoint. It is enough to show that $\pi_i(F)$ is uniquely $\ell$–divisible for $i \geq 1$ and that the cokernel of the map $\pi_1(\phi_A)$ is also uniquely $\ell$–divisible.

As in [**4**, p. 294], there is map of fibration sequences

$$\begin{array}{ccccc}
\mathrm{Hom}_g(A, B\mu_\ell)_R & \longrightarrow & B\tilde{T}_1(A) & \xrightarrow{\ell} & B\tilde{T}_1(A) \\
\cong\big\downarrow & & \phi_A\big\downarrow & & \phi_A\big\downarrow \\
\mathrm{Hom}_\ell(A, (B\mu_\ell)_{\mathrm{et}})_{R_{\mathrm{et}}} & \longrightarrow & B\tilde{T}_1(A_{\mathrm{et}}) & \xrightarrow{\ell} & B\tilde{T}_1(A_{\mathrm{et}})
\end{array}$$

The statement for $\pi_i(F)$, $i \geq 1$, follows from completing this map of fibration sequences to a $3 \times 3$ square by taking vertical fibres.

The map $\pi_1(\phi_A)$ can be identified as a natural map in etale cohomology

$$H^0(\mathrm{Spec}\, A, G_{\mathrm{m}}) \to H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$$

where $G_{\mathrm{m}} = T_{1,R}$ denotes the multiplicative group. This map appears as one of the rungs in a long exact etale cohomology ladder in which the top rail is the long exact sequence associated to

$$0 \to \mu_\ell \to G_{\mathrm{m}} \xrightarrow{\ell} G_{\mathrm{m}} \to 0$$

and the bottom rail is the long exact sequence assoicated to

$$0 \to \mathbf{Z}_\ell(1) \xrightarrow{\ell} \mathbf{Z}_\ell(1) \to \mu_\ell \to 0\,.$$

In this ladder, the rung maps involving $H^*(\mathrm{Spec}\, A, \mu_\ell)$ are the identity.

Let $C$ denote the cokernel of $\pi_1(\phi_A)$. We first show that $C$ has no $\ell$–torsion with the following straightforward diagram chase. Pick $x_0 \in C$ such that $\ell x_0 = 0$. Choose $x_1 \in H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$ projecting to $x_0$. By assumption, $\ell x_1$ is the image of an element $x_2 \in H^0(\mathrm{Spec}\, A, G_{\mathrm{m}})$. The image of $x_2$ in $H^1(\mathrm{Spec}\, A, \mu_\ell)$ is the same as the image of $x_1$, namely 0, so that $x_2 = \ell x_3$ for some $x_3 \in H^0(\mathrm{Spec}\, A, G_{\mathrm{m}})$. Let $x_4$ be the image of $x_3$ in $H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$. Clearly, $\ell(x_1 - x_4) = 0$, so that $x_1 - x_4$ is the image of an element $x_5 \in H^0(\mathrm{Spec}\, A, \mu_\ell)$. Let $x_6 = x_3 + x_5$. Then the image of $x_6$ in $H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$ is $x_1$ and hence $x_0 = 0$.

We finally verify that $C$ is $\ell$–divisible. Consider $x_0 \in C$. Pick $x_2 \in H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$ projecting to $x_0$, and let $x_3$ be the image of $x_2$ in $H^1(\mathrm{Spec}\, A, \mu_\ell)$. Consider first the case that $x_3$ is the image of an element $x_4$ from $H^0(\mathrm{Spec}\, A, G_{\mathrm{m}})$. Let $x_5$ be the image of $x_4$ in $H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$. It is clear that $x_2 - x_5$ has image 0 in $H^1(\mathrm{Spec}\, A, \mu_\ell)$ and so $x_2 - x_5 = \ell x_6$ for some $x_6 \in H^1(\mathrm{Spec}\, A, \mathbf{Z}_\ell(1))$. Then $x_0 = \ell x_1$, where $x_1$ is the image of $x_6$ in $C$. Assume on the contrary that $x_3$ in not the image of any such $x_4$. Let $x_7$ be the image of $x_3$ in $H^1(\mathrm{Spec}\, A, G_{\mathrm{m}})$;

it follows that $x_7 \neq 0$. Carrying out an inductive argument along similar lines with the long exact ladders associated to the sequences

$$0 \to \mathbf{Z}_\ell(1) \to \mathbf{Z}_\ell(1) \to \mu_{\ell^i} \to 0$$

and

$$0 \to \mu_{\ell^i} \to G_{\mathrm{m}} \to G_{\mathrm{m}} \to 0$$

shows that there are elements $x_7(i) \in H^1(\operatorname{Spec} A, G_{\mathrm{m}})$ such that $x_7(1) = x_7$ and $\ell x_7(i) = x_7(i-1)$. In view of the hypothesis on $\mathrm{Pic}(A)$, this is impossible. ∎

Proposition 5.2 has the following immediate corollary.

5.3 Corollary. *Let $A$ be a noetherian $R$ algebra such that* $\mathrm{Pic}(A)$ *contains no infinitely $\ell$–divisible elements. Then the natural map (1.4) $BT_n(A) \to BT_n(A_{\mathrm{et}})$ induces an isomorphism*

$$H^*(BT_n(A_{\mathrm{et}}), \mathbf{Z}/\ell) \to H^*(BT_n(A), \mathbf{Z}/\ell)\ .$$

## §6. Cohomological injectivity

In this section, we show that if $A = R[\zeta] = \mathbf{Z}[1/\ell, \zeta]$ and $\ell$ is a regular prime (1.9), then the natural cohomology map (1.4)

$$H^*(\mathrm{BGL}_n(A_{\mathrm{et}}), \mathbf{Z}/\ell) \to H^*(\mathrm{BGL}_n(A), \mathbf{Z}/\ell)$$

is a monomorphism. In this case, then, our good mod $\ell$ model for $A$ gives an explicit lower bound for $H^*(\mathrm{BGL}_n(A), \mathbf{Z}/\ell)$. For the ring $R$ itself with $\ell$ regular, we obtain an analogous, but stable, result. Our basic technique is to show that the natural map

$$BT_n(A_{\mathrm{et}}) \to \mathrm{BGL}_n(A_{\mathrm{et}})$$

induces an injection in mod $\ell$ cohomology and then apply Corollary 5.3.

We shall let $S$ denote the graded polynomial algebra

$$S \equiv \mathbf{F}[c_1, \ldots , c_n],\ \deg(c_i) = 2i\,.$$

The reader should recall that $S$ is isomorphic to the mod $\ell$ cohomology ring of the classifying space $\mathrm{BGL}_n(\mathbf{C}_{\mathrm{top}})$ of the complex Lie group $GL_n(\mathbf{C}_{\mathrm{top}})$ or to the mod $\ell$ cohomology ring of $\mathrm{BU}_n$. Let $\tilde{S}$ denote the graded polynomial algebra

$$\tilde{S} \equiv \mathbf{F}[t_1, ..., t_n],\ \deg(t_i) = 2\,.$$

Then $\tilde{S}$ is isomorphic to the mod $\ell$ cohomology ring of the classifying space $BT_n(\mathbf{C}_{\mathrm{top}})$ of a maximal torus of $\mathrm{GL}_n(\mathbf{C}_{\mathrm{top}})$ or to the mod $\ell$ cohomology ring of the classifying space $BT_n$ of the group $T_n \subset \mathrm{U}_n$ of diagonal matrices. The restriction map

$$S \cong H^*(\mathrm{BU}_n, \mathbf{Z}/\ell) \ \to\ H^*(BT_n, \mathbf{Z}/\ell) \cong \tilde{S}$$

includes $S \subset \tilde{S}$ as the ring of symmetric polynomials in the variables $t_i$.

6.1 Remark: If $X^A$ is a good mod $\ell$ model for the $R$ algebra $A$, then for each $n > 0$ there is a fibration $\mathrm{BGL}_{n,X^A} \to X^A$ with homotopy fibre $\mathbf{F}_\omega(\mathrm{BU}_n)$ (see 1.8) and the technique of 1.8 produces a parallel fibration $BT_{n,X^A} \to X^A$ with homotopy fibre $\mathbf{F}_\omega(BT_n)$. There is a map $BT_{n,X^A} \to \mathrm{BGL}_{n,X^A}$ which on fibres is the completion of the natural map $BT_n \to \mathrm{BU}_n$. We will denote the spaces of sections of these fibrations by, respectively, $\mathrm{BGL}_n(X^A)$ and $BT_n(X^A)$. If $X^A$ is a space (as opposed to a pro-space), then $\mathrm{BGL}_n(X^A)$ is the space of sections of the ordinary fibration $\lim\{\mathrm{BGL}_{n,X_A}\} \to X_A$ with homotopy fibre $\mathbf{F}_\infty(\mathrm{BU}_n)$; a similar remark holds for $BT_n(X^A)$. According to the final statement in 1.8 (as extended to cover $T_{n,A}$ as well as $\mathrm{GL}_{n,A}$) there is up to homotopy a commutative diagram

$$\begin{array}{ccccc} BT_n(A) & \longrightarrow & BT_n(A_{\mathrm{et}}) & \xrightarrow{\simeq} & BT_n(X^A) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{BGL}_n(A) & \longrightarrow & \mathrm{BGL}_n(A_{\mathrm{et}}) & \xrightarrow{\simeq} & \mathrm{BGL}_n(X^A) \end{array}.$$

6.2 Lemma. *Let $A$ be a finitely generated $R$ algebra and let $f^A : X^A \to K(\Lambda, 1)$ be a good* mod $\ell$ *model for $A$. Assume that $X^A$ has the* mod $\ell$ *homology of a bouquet of $r > 0$ circles, and further assume that the composite $X^A \to K(\Lambda,1) \to K(\lambda,1)$ is homotopically trivial. Then there are gradings on $H^*(\mathrm{BGL}_n(X^A), \mathbf{Z}/\ell)$ and on $H^*(BT_n(X^A), \mathbf{Z}/\ell)$ compatible with the natural map (6.1) between these two cohomology rings, and a commutative diagram*

$$\begin{array}{ccc} \mathrm{gr}\ H^*(\mathrm{BGL}_n(X^A), \mathbf{Z}/\ell) & \longrightarrow & \mathrm{gr}\ H^*(BT_n(X^A), \mathbf{Z}/\ell) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \Omega^*_S \otimes_S \cdots \otimes_S \Omega^*_S & \longrightarrow & \Omega^*_{\tilde{S}} \otimes_{\tilde{S}} \cdots \otimes_{\tilde{S}} \Omega^*_{\tilde{S}} \end{array}$$

*in which the lower horizontal arrow is the $r$–fold tensor power of the map $\Omega^*_{S/\mathbf{F}} \to \Omega^*_{\tilde{S}/\mathbf{F}}$ of DeRham complexes induced by $S \to \tilde{S}$. In particular the map $H^*(\mathrm{BGL}_n(X^A), \mathbf{Z}/\ell) \to H^*(BT_n(X^A), \mathbf{Z}/\ell)$ is injective.*

PROOF: By replacing $X^A$ if necessary (see 1.8) we can assume that $X^A$ is a bouquet $\vee S^1$ of $r$ circles and that $\mathrm{BGL}_n(X^A)$ (resp. $BT_n(X^A)$) is the space of sections of a fibration over $\vee S^1$ with fibre $\mathbf{F}_\infty(\mathrm{BU}_n)$ (resp. $\mathbf{F}_\infty(BT_n)$) (see 6.1). The triviality of $\vee S^1 \to K(\Lambda, 1) \to K(\lambda, 1)$ implies that $\pi_1(\vee S^1)$ acts trivially on the mod $\ell$ cohomology groups of the fibres in these fibrations. The existence of the given commutative diagram is a direct consequence of 4.3; the gradings on the cohomology rings in question are the gradings associated to appropriate collapsing Eilenberg-Moore spectral sequences (4.1). Let $S(r+1)$ and $\tilde{S}(r+1)$ denote the $(r+1)$–fold tensor powers of $S$ and $\tilde{S}$ respectively. The indicated map on DeRham complexes is the natural map

$$\mathrm{Tor}^*_{S(r+1)}(S, S) \to \mathrm{Tor}^*_{\tilde{S}(r+1)}(\tilde{S}, \tilde{S})$$

induced by $S \to \tilde{S}$. One can view this map explicitly by first mapping the Koszul resolution of $S$ over $S(r+1)$ to the Koszul resolution of $\tilde{S}$ over $\tilde{S}(r+1)$, then tensoring the corresponding resolutions by $S$ or $\tilde{S}$. Inspection verifies that this map is injective. For another proof of this injectivity see [**14**, Lemma 9]. ∎

Lemma 6.2 leads quickly to the result we want.

6.3 PROPOSITION. *Assume that $\ell$ is an odd regular prime and let $A$ denote the ring $R[\zeta] = \mathbf{Z}[1/\ell, \zeta]$. Let $r = (\ell+1)/2$. Then the natural map (1.4) $\mathrm{BGL}_n(A) \to \mathrm{BGL}_n(A_{\mathrm{et}})$ induces a monomorphism*

$$H^*(\mathrm{BGL}_n(A_{\mathrm{et}}), \mathbf{Z}/\ell) \to H^*(\mathrm{BGL}_n(A), \mathbf{Z}/\ell).$$

*Moreover, $H^*(\mathrm{BGL}_n(A_{\mathrm{et}}), \mathbf{Z}/\ell)$ admits a filtration for which the associated graded module is isomorphic to $\Omega^*_S \otimes_S \cdots \otimes_S \Omega^*_S$, the $r$–fold tensor power of the DeRham complex of $S$.*

REMARK: The proof of 6.3 goes through without change if $A$ is the coordinate algebra of a multiply punctured plane over the algebraic closure of a field of characteristic different from $\ell$, i.e., in the genus 0 case of 3.1. The proof fails for affine curves of higher genus, since $\mathrm{Pic}(A)$ then contains divisible subgroups (cf. 5.3).

PROOF OF 6.3: It follows from 1.9 that $A$ has a good mod $\ell$ model $X^A$ which is a bouquet of circles; this is also a consequence of 2.1, since the fact that $\mathbf{R}P^\infty \vee S^1$ is a model for $R$ implies easily that the cover of this space described in 2.5 is a model for $R[\zeta]$. Since $A$ contains $\mu_\ell$, the composite $X^A \to K(\Lambda, 1) \to K(\lambda, 1)$ is trivial. By 6.2 and the discussion in 6.1, then, the map $BT_n(A_{\mathrm{et}}) \to \mathrm{BGL}_n(A_{\mathrm{et}})$ induces a monomorphism on mod $\ell$ cohomology. By 5.3, the map $BT_n(A) \to BT_n(A_{\mathrm{et}})$

induces an isomorphism on mod $\ell$ cohomology. The commutative diagram in 6.1 gives the desired injectivity statement. The formula for $\mathrm{gr}\{H^*(\mathrm{BGL}_n(A_{\mathrm{et}}, \mathbf{Z}/\ell)\}$ is from 6.2. ∎

There is a stable version of Proposition 6.3 that holds for $R$ as well as for $R[\zeta]$.

6.4 Proposition. *Assume that $\ell$ is an odd regular prime and let $A$ denote either the ring $R = \mathbf{Z}[1/\ell]$ or the ring $R[\zeta]$. Then the natural map (1.4) $\mathrm{BGL}(A) \to \mathrm{BGL}(A_{\mathrm{et}})$ induces a monomorphism*

$$H^*(\mathrm{BGL}(A_{\mathrm{et}}), \mathbf{Z}/\ell) \to H^*(\mathrm{BGL}(A), \mathbf{Z}/\ell) \ .$$

6.5 Remark: The mod $\ell$ cohomology of $\mathrm{BGL}(A_{\mathrm{et}})$ can be determined easily from 2.3 if $A = R$ and by an analogous calculation or from [**5**, 4.6] if $A = R[\zeta]$. With almost no extra effort (cf. [**5**, p. 143]) results similar to 6.4 can be obtained for any number ring $R'$ between $R$ and $R[\zeta]$, or for any normal etale extension $R''$ of such an $R'$ of degree a power of $\ell$. (Observe for instance that by 1.9 or 2.1 there are many normal etale extensions of $R[\zeta]$ of degree a power of $\ell$: these correspond to normal subgroups of finite index in a free pro–$\ell$ group on $(\ell+1)/2$ generators.)

Proof of 6.4: For $A = R[\zeta]$ the result follows from passing to an inverse limit over $n$ with 6.3; on both sides the cohomology groups in any given dimension stablilize with $n$ to a fixed value. Recall from [**4**, 6.4] that the map $\mathrm{BGL}(A)^+ \to \mathrm{BGL}(A_{\mathrm{et}})$ is a map of infinite loop spaces commuting with transfer. Consider the following commutative diagram whose horizontal arrows are induced on the one hand by the transfer and on the other hand by the ring homomorphism $i : R \to R[\zeta]$ (in this diagram $H^*$ denotes $H^*(-, \mathbf{Z}/\ell)$):

$$\begin{array}{ccccc}
H^*\,\mathrm{BGL}(R) & \xrightarrow{\mathrm{tr}^*} & H^*\,\mathrm{BGL}(R[\zeta]) & \xrightarrow{i^*} & H^*\,\mathrm{BGL}(R) \\
\uparrow & & \uparrow & & \uparrow \\
H^*\,\mathrm{BGL}(R_{\mathrm{et}}) & \xrightarrow{\mathrm{tr}^*} & H^*\,\mathrm{BGL}(R[\zeta]_{\mathrm{et}}) & \xrightarrow{i^*} & H^*\,\mathrm{BGL}(R_{\mathrm{et}})
\end{array} .$$

Recall that the composite $\mathrm{BGL}(R)^+ \xrightarrow{i} \mathrm{BGL}(R[\zeta])^+ \xrightarrow{\mathrm{tr}} \mathrm{BGL}(R)^+$ is multiplication by $(\ell - 1)$, in view of the fact that $R[\zeta]$ is a free $R$ module of rank $(\ell - 1)$; for the same reason the composite $\mathrm{BGL}(R_{\mathrm{et}}) \to \mathrm{BGL}(R[\zeta]_{\mathrm{et}}) \to \mathrm{BGL}(R_{\mathrm{et}})$ is multiplication by $(\ell - 1)$. Since $(\ell - 1)$ is prime to $\ell$, this implies that both horizontal composites in the above diagram are isomorphisms. The middle vertical arrow is a monomorphism by the considerations above. The fact that the left vertical arrow is a monomorphism follows at once. ∎

We will state our final proposition only for the ring $R$, although there is a version for $R[\zeta]$ as well as for the other rings mentioned in 6.5. Let $D_n \subset \mathrm{GL}_{n(\ell-1)}(R)$ be the image of the diagonal subgroup $(R[\zeta]^*)^n \subset \mathrm{GL}_n(R[\zeta])$ under the transfer map $\mathrm{GL}_n(R[\zeta]) \to \mathrm{GL}_{n(\ell-1)}(R)$ obtained by choosing a basis for $R[\zeta]$ as an $R$ module.

6.6 PROPOSITION. *Suppose that $\ell$ is an odd regular prime and that the Lichtenbaum-Quillen conjecture holds at $\ell$ for $\mathbf{Z}$; equivalently (2.4), assume that the map $\mathrm{BGL}(R) \to \mathrm{BGL}(R_{\mathrm{et}})$ induces an isomorphism on* mod $\ell$ *cohomology. Then $H^*(\mathrm{BGL}(R), \mathbf{Z}/\ell)$ is detected on the groups $D_n$, in the sense that if $x \in H^*(\mathrm{BGL}(R), \mathbf{Z}/\ell)$ is non-zero, then there exists $n$ such that $x$ has nonzero image under the composite*

$$H^*(\mathrm{BGL}(R), \mathbf{Z}/\ell) \to H^*(\mathrm{BGL}_{n(\ell-1)}(R), \mathbf{Z}/\ell) \to H^*(BD_n, \mathbf{Z}/\ell) \ .$$

PROOF: There are commutative diagrams

$$\begin{array}{ccccc}
BT_n(R[\zeta]) & \xrightarrow{\alpha} & \mathrm{BGL}_n(R[\zeta]_{\mathrm{et}}) & \xrightarrow{\beta} & \mathrm{BGL}(R[\zeta]_{\mathrm{et}}) \\
\cong\downarrow & & & & \downarrow\mathrm{tr} \\
BD_n & \longrightarrow & \mathrm{BGL}_{n(\ell-1)}(R_{\mathrm{et}}) & \longrightarrow & \mathrm{BGL}(R_{\mathrm{et}})
\end{array}$$

in which the right hand vertical map is the transfer and induces a monomorphism on mod $\ell$ cohomology (cf. proof of 6.4). The map $\beta$ is an equivalence in a stable range [**4**, 4.5] which tends to infinity as $n$ gets large. By 6.2 (cf. proof of 6.3) the cohomology map $H^*(\alpha, \mathbf{Z}/\ell)$ is a monomorphism. It follows that $H^*(\mathrm{BGL}(R_{\mathrm{et}}), \mathbf{Z}/\ell)$ is detected on the groups $D_n$; if the Lichtenbaum-Quillen conjecture holds the same is true of $H^*(\mathrm{BGL}(R), \mathbf{Z}/\ell)$. ∎

REMARK: The groups $D_n$ which appear in 6.6 are closely analogous to the subgroups used by Quillen [**14**, Lemmas 12 and 13] to detect the cohomology of general linear groups over a finite field. Propositions 6.3, 6.4 and 6.6 are also true for $\ell = 2$ (in this case $R = R[\zeta] = \mathbf{Z}[1/2]$). This can be proved along the above lines using [**5**, 4.2]; it also follows directly from the recent work of Mitchell [**13**].

## §7. FURTHER QUESTIONS AND PROBLEMS

It seems reasonable to conclude this paper by mentioning a few of the many open questions related to what is discussed here. In some cases, the authors have obtained partial answers whereas in other cases the answers remain completely elusive. Probably the most compelling problem is to settle the Lichtenbaum–Quillen conjecture:

7.1 PROBLEM: Determine whether or not the map

$$\mathrm{BGL}(\mathbf{Z}[1/\ell]) \to \mathrm{BGL}(\mathbf{Z}[1/\ell]_{\mathrm{et}})$$

induces an isomorphism on mod $\ell$ cohomology.

Somewhat less ambitious is

7.2 QUESTION: Are there some general circumstances under which etale $K$-theory splits off of algebraic $K$-theory? More specifically, if $A$ is a regular noetherian ring of etale cohomological dimension $\leq 2$ with $1/\ell \in A$, does the map (1.4)

$$\mathbf{F}_\infty(\mathrm{BGL}(A)^+) \longrightarrow \mathbf{F}_\infty(\mathrm{BGL}(A_{\mathrm{et}})) \simeq \mathrm{BGL}(A_{\mathrm{et}})$$

have a right inverse? This would give a strong geometric explanation for the cohomological injectivity results in §6 and the homotopical surjectivity results in [**4**].

7.3 PROBLEM: Develop a tractable good mod $\ell$ model for $R = \mathbf{Z}[1/\ell]$ if $\ell$ is a prime which is not regular. It seems possible to do this if $\ell$ is an irregular prime which is well-behaved [**20**, p. 201] in a certain number theoretic sense.

7.4 QUESTION: To what extent can one modify the construction of $\mathrm{BGL}_n(A_{\mathrm{et}})$ to take into consideration the unstable phenomena Quillen describes in [**15**, p. 591]? Namely, Quillen for various $A$ produces classes in $H_*(\mathrm{BGL}_n(A), \mathbf{Z}/\ell)$ which vanish in $H_*(\mathrm{BGL}(A), \mathbf{Z}/\ell)$; his key assumption is that $\mathrm{Pic}(A)$ has nontrivial elements of finite order prime to $\ell$. In particular, if $\ell$ is regular his method produces such classes for $A = R[\zeta]$ whenever $R[\zeta]$ is not a principal ideal domain. These classes are not accounted for in $H_*(\mathrm{BGL}_n(R[\zeta]_{\mathrm{et}}), \mathbf{Z}/\ell)$.

7.5 QUESTION: How can one incorporate transfer more systematically into the modelling process? For example, transfer plays an important role in known surjectivity results (see, for example, [**4**]). Can one incorporate transfer into the model itself?

7.6 QUESTION: To what extent can the natural map

$$BG(A) \to BG(A_{\mathrm{et}})$$

of 1.4 be used to study the homology of $G(A)$ for group schemes $G_R$ over $R$ other than $GL_{n,R}$ and $T_{n,R}$?

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