RELATIVE CHOW CORRESPONDENCES AND THE GRIFFITHS GROUP

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In the monograph [FM-1], the author and Barry Mazur introduce a filtration on algebraic cycles on a (complex) projective variety which we called the topological filtration. This filtration, defined using a fundamental operation on the homotopy groups of cycle spaces, has an interpretation in terms of "Chow correspondences." The purpose of this paper is to give examples in which specific levels of this filtration are non-trivial. Thus, we obtain examples of cycles which lie in different levels of a naturally defined filtration of the Griffiths group (of cycles homologically equivalent to 0 modulo cycles algebraically equal to 0). Our examples are cycles on general complete intersections analyzed by Madhav Nori by means of his (rational) Lefschetz hyperplane theorem [N]. The relevance of Nori's examples is suggested by a description given in [F3] of the topological filtration closely resembling the filtration on cycles that Nori considers.

Nori's theorem is a result about cohomology and Nori's application to his filtration on cycles involves working with cycle classes in cohomology; our topological filtration lends itself less easily to a cohomological analysis. One difficulty we face is that the topological filtration of a smooth variety involves cycles on singular varieties. This provides considerable awkwardness for cycles on singular varieties need not have cycle classes in cohomology. Another difficulty is that Nori's application of his Lefschetz theorem to cycles involves the consideration of families of varieties over a quasi-projective base variety, whereas the machinery for studying the topological filtration has been formulated in the context of projective varieties. Thus, we are led to consider "relative Chow correspondences."

We briefly sketch the organization of the paper. Section 1 summarizes the context and results of Nori's paper which we shall use. In section 2, we extend to quasi-projective varieties the construction of Chow correspondences and graph mappings. More importantly, we interpret the Chow correspondence homomorphism of [FM-1] in terms of slant product, a familiar operation in homology theory. Section 3 presents our results, most notably Theorem 3.4 which is our strengthened version of one aspect of Nori's theorems about the Griffiths group. A few corollaries are given enabling us to obtain examples of varieties with non-trivial layers in the topological filtration on algebraic cycles. In section 4, we develop relative Chow correspondences in order to work with families of varieties as we encounter in Nori's context. Finally, section 5 completes the proof of Theorem 3.4.

As we have observed previously (cf. [FM-1], [F2]), one of the intriguing aspects of geometric techniques involving cycle spaces is that these techniques rarely require

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hypotheses of smoothness. Indeed, the difference between the topological filtration and the cohomological filtration introduced by Nori in [N] is that one begins with cycles homologically equivalent to 0 on possibly singular (rather than smooth) varieties. These techniques for cycle spaces are elementary in nature, so that we expect them to lead to further geometric properties of varieties. The results of this paper enable one to sometimes convert these geometric techniques to more familiar cohomological ones.

Throughout this paper, all varieties considered will be quasi-projective complex algebraic varieties.

We are especially grateful to Madhav Nori who encouraged us to reinterpret our geometric techniques in such a way that his remarkable Lefschetz theorem (Theorem 1.4 below) could be applied. We also thank Dick Hain for several useful conversations.

§1. NORI'S FILTRATION

In this inititial section, we briefly summarize those aspects of [N] which we shall employ or modify. We begin with a filtration on algebraic cycles on a smooth variety introduced by Nori.

Definition 1.1. *Let Y be a smooth variety of dimension n and let $CH_r(Y)$ denote the Chow group of algebraic r-cycles on Y modulo rational equivalence for some $r \geq 0$. Then $A_jCH_r(Y) \subset CH_r(Y)$ is the subgroup generated by the rational equivalence classes of those cycles ξ for which there exists some smooth, projective variety E of some dimension m, an $n+r-j$-cycle γ on $E \times Y$, and a j-cycle δ on E homologically equivalent to 0 (denoted $\delta \sim_h 0$) such that ξ is represented by*

$$p_{Y*}(\gamma \bullet p_E^*(\delta)), \quad \delta \sim_h 0.$$

Here, $p_{Y*} : CH_r(E \times Y) \to CH_r(Y)$ is proper push-forward of cycles, $\bullet$ is the intersection product on $CH_*(E \times Y)$, and $p_E^* : CH_{r-j}(E) \to CH_{r+m-j}(E \times Y)$ is flat pull-back of cycles. The condition $\delta \sim_h 0$ is taken to mean that the cycle class of δ in integral singular homology , $[\delta] \in H_{2r-2j}(E)$, is 0.

For Y projective, $\{A_jCH_r(Y)\}$ is an increasing filtration on $CH_r(Y)$; $A_0CH_r(Y)$ consists of those classes algebraically equivalent to 0; $A_rCH_r(Y)$ consists of those classes homologically equivalent to 0 [N;5.2]. In particular,

$$A_rCH_r(Y)/A_0CH_r(Y) \ = \ Griff_r(Y),$$

the Griffiths group of algebraic r-cycles homologically equivalent to 0 modulo those cycles algebraically equivalent to 0.

The following spells out the notational conventions which establish the context for the Nori-Lefschetz theorem.

Conventions 1.2. *Assume that X is a projective, smooth variety of dimension $n+h$, let S denote $\prod_{i=1}^h \mathbf{P}(\Gamma(X, O_X(a_i)))$ with $min\{a_1, \ldots, a_h\} \geq N_X(n)$, and consider a smooth morphism $\mathcal{E} \to S$. The positive number $N_X(n)$, which depends upon n, X, and an ample line bundle $O_X(1)$ on X, is that of* [N;Thm4]. *We denote by $\mathcal{Y}_S \subset X \times S \equiv \mathcal{X}_S$ the incidence variety, with fibre $Y_s = \{x \in X : F_s(x) = 0\}$ over $s \in S$.*

The following theorem of M. Nori essentially doubles the range of the classical Lefschetz hyperplane theorem provided that one considers cohomology with rational coefficients and replaces a single complete intersection by a general family of complete intersections with sufficiently high degree.

Theorem 1.3. *(* [N;Thm 4]*) Adopt the conventions of 1.2 and let $\mathcal{E} \to S$ be smooth. Then*

$$H^k(\mathcal{E} \times_S \mathcal{X}_S, \mathcal{E} \times_S \mathcal{Y}_S; \mathbf{Q}) = 0 \quad \textit{for } k \leq 2n.$$

Nori applies Theorem 1.3 to obtain the following interesting result about his filtration. Our major goal is to prove the analogous result (Theorem 3.4 below) for the topological filtration. In Theorem 1.4 (and Theorem 3.4), one can simply take $j = r-1$; for $j < r-1$, Nori obtains a stronger condtion on ζ than the vanishing of $[\zeta]$.

Theorem 1.4. [N;6.1] *Adopt the notation and conventions of (1.2). If $\zeta \in Z_{r+h}(X)$ satisfies*

$$i_s^!(\zeta) \;\in\; A_j CH_r(Y_s), \quad j < r$$

for almost all $s \in S$ where $i_s : Y_s \to X$ is the restriction of $i : \mathcal{Y} \to \mathcal{X}$, then

$$[\zeta] = 0 \in H_{2(r+h)}(X, \mathbf{Q}).$$

In particular, if X is itself a complete intersection of dimension $2r+2$ whose algebraic homology in middle dimension has rank at least 2 and if $h = 1$, then for almost all $s \in S$

$$A_r CH_r(Y_s)/A_{r-1}CH_r(Y_s) \otimes \mathbf{Q} \neq 0.$$

§2. CHOW VARIETES AND CORRESPONDENCE HOMOMORPHISMS

After recalling the notation and terminology of Chow varieties, we extend to quasi-projective varieties the formulation of correspondence homomorphisms introduced by the author and B. Mazur for projective varieties. We then proceed to interpret these homomorphisms in cohomological terms.

Throughout this section, $U,\ V$ will denote quasi-projective varieties of pure dimension $m,\ n$. We shall let $X,\ Y$ denote projective varieties, typically projective closures of U, V. We recall that once a projective embedding of Y is chosen, then one has Chow Varieties $C_{j,d}(Y)$ of effective j-cycles on Y of degree d (for integers $j, d \geq 0$) and one considers the **Chow monoid**

$$\mathcal{C}_j(Y) = \coprod_{d=0}^{\infty} C_{j,d}(Y)$$

whose isomorphism type is independent of the choice of projective embedding of Y [B]. We provide $\mathcal{C}_j(Y)$ with the analytic topology and form its **näive group completion** $Z_j(Y)$ whose homotopy type is that of the homotopy theoretic group completion of the topological monoid $\mathcal{C}_j(Y)$ (cf. [LiF], [F-G]). The underlying discrete group $Z_j(Y)^{disc}$ of $Z_j(Y)$ is the group of algebraic j-cycles on Y.

Assume now that $V \subset Y$ is a projective closure with Zariski closed complement $Y_\infty \subset Y$. We consider the quotient topological monoid $C_j(Y)/C_j(Y_\infty)$ and its näive group completion $Z_j(V)$. The homotopy type of $Z_j(V)$ depends only upon V and not the choice of projective closure $V \subset Y$.

We shall frequently use the **s-operation** first introduced in [FM-1] for projective varieties, extended to quasi-projective varieties in [F2]. Recall that this operation takes the form

$$s : Z_j(V) \to \Omega^2 Z_{j-1}(V)$$

and can be viewed heuristically as taking a j-cycle ζ to a $\mathbf{P}^1 \simeq S^2$ paramterized family of $j-1$-cycles obtained by intersecting ζ with a Lefschetz pencil of hyperplane sections.

Proposition 2.1. *Let V be a quasi-projective variety, let $V \subset Y$ be a projective closure, and let $Y_\infty = Y - V$.*

(a.) $\pi_0 Z_r(V)$ is the group of algebraic equivalence classes of algebraic r-cycles on V.

(b.) $\pi_i Z_0(V)$ is naturally isomorphic to $H_i^{BM}(V) \simeq H_i(Y, Y_\infty)$, the Borel-Moore homology of V (provided with its classical topology as an analytic space).

(c.) $s^r \circ \pi : Z_r(V) \xrightarrow{\pi} \pi_0 Z_r(V) \xrightarrow{s} \pi_2 Z_{r-1}(V) \xrightarrow{s} \cdots \xrightarrow{s} \pi_{2r} Z_0(V) \simeq H_{2r}^{BM}(V)$ is the cycle map.

(d.) The Griffiths group of algebraic r-cycles on V homologically equivalent to 0 modulo algebraic equivalence equals the quotient

$$ker\{Z_r(V) \xrightarrow{s^r \circ \pi} \pi_{2r} Z_0(V)\}/ker\{Z_r(V) \xrightarrow{\pi} \pi_0 Z_r(V)\}.$$

Proof. A cycle $\zeta = \sum m_i W_i$ on V is algebraically equivalent to 0 if and only its closure $\overline{\zeta} = \sum m_i \overline{W}_i$ on Y is algebraically equivalent to a cycle supported on Y_∞. Thus, (a.) follows from the special case in which $V = Y$ is projective [F1;1.8] and the following commutative square of surjective maps:

$$\begin{array}{ccc} Z_r(Y) & \longrightarrow & Z_r(V) \\ \downarrow & & \downarrow \\ \pi_0 Z_r(Y) & \longrightarrow & \pi_0 Z_r(V) \end{array}$$

We view the Dold-Thom theorem as providing a natural quasi-isomorphism between the chain complex associated to the simplicial abelian group of singular simplices on $Z_0(Y)$ and the chain complex of singular chains on Y (cf. [FM-1;appB]). Thus, the 5-Lemma enables us to extend the Dold-Thom theorem to prove (b.).

In the special case in which $V = Y$ is projective, (c.) is proved in [FM-1;6.4]. The general case follows from the surjectivity of $Z_r(Y) \to Z_r(V)$ and the commutativity of the following diagram:

$$\begin{array}{ccccc} \pi_0 Z_r(Y) & \xrightarrow{s^r} & \pi_{2r} Z_0(Y) & \xrightarrow{\simeq} & H_{2r}(Y) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0 Z_r(V) & \xrightarrow{s^r} & \pi_{2r} Z_0(V) & \xrightarrow{\simeq} & H_{2r}^{BM}(V) \end{array}$$

Finally, (d.) follows from parts (a.), (b.), (c.) and the definition of the Griffiths group of r-cycles as the group of algebraic equivalence classes of r-cycles homologically equivalent to 0. $\square$

We now begin the process of extending the constructions of [FM-1] and [F2] to quasi-projective varieties.

Definition 2.2. *A* **Chow correspondence**

$$f = (\overline{f}, f_\infty) : U \to \mathcal{C}_j(V)$$

is represented by the following data: choices of projective closures $U \subset X, V \subset Y$ with Zariski closed complements $Y_\infty \subset Y, X_\infty \subset X$ and a pair of morphisms $\overline{f} : X \to \mathcal{C}_j(Y),\ f_\infty : X_\infty \to \mathcal{C}_j(Y_\infty)$. The data $U \subset X', V \subset Y', \overline{g} : X' \to \mathcal{C}_j(Y'), g_\infty : X'_\infty \to \mathcal{C}_j(Y'_\infty)$ will be viewed as the same Chow correspondence as $(\overline{f}, f_\infty)$ if the maps $U \to \mathcal{C}_j(Y)/\mathcal{C}_j(Y_\infty),\ U \to \mathcal{C}_j(Y')/\mathcal{C}_j(Y'_\infty)$ become equal after making the evident identification $\mathcal{C}_j(Y)/\mathcal{C}_j(Y_\infty) \simeq \mathcal{C}_j(Y')/\mathcal{C}_j(Y'_\infty)$.

Proposition 2.3. *(cf.*[FM-1], [F2]*) A Chow correspondence $f : U \to \mathcal{C}_j(V)$ determines* **graph mappings**

$$\Gamma_f : Z_{r-j}(U) \to Z_r(V),\ r \geq j$$

induced by the construction which sends an irreducible closed subvariety $W \subset X$ of dimension $r - j$ to the "trace" of the cycle on $X \times Y$ associated to the composition $W \subset X \xrightarrow{\overline{f}} \mathcal{C}_j(Y)$.

Moreover, f determines a **Chow Correspondence homomorphism**

$$\Phi_f : H_*^{BM}(U) \to H_{*+2j}^{BM}(V).$$

Finally, if $[\delta] \in H_{2r}^{BM}(U)$ denotes the cycle class of $\delta \in Z_r(U)$, then

$$\Phi_f([\delta]) = [\Gamma_f(\delta)].$$

Proof. The construction mentioned in the statement of the proposition is the construction presented in [F2] for the graph mapping in the case in which $X = U$ and $Y = V$ are projective. The naturality of this construction provides the following commutativity diagram:

$$\begin{array}{ccccc} Z_{r-j}(X_\infty) & \longrightarrow & Z_{r-j}(X) & \longrightarrow & Z_{r-j}(U) \\ \Gamma_{f_\infty}\big\downarrow & & \Gamma_{\overline{f}}\big\downarrow & & \big\downarrow\Gamma_f \\ Z_r(Y_\infty) & \longrightarrow & Z_r(Y) & \longrightarrow & Z_r(V) \end{array} \tag{2.3.1}$$

where $U \subset X, V \subset Y$ are projective closures with Zariski closed complements $X_\infty \subset X, Y_\infty \subset Y$. The map f induces $s^j \circ f_* : Z_0(U) \to Z_j(V) \to \Omega^{2j} Z_0(V)$. The asserted map Φ_f is the map on homotopy groups induced by $s^j \circ f_*$ (using the isomorphism of (2.1.b)):

$$\Phi_f = (s^j \circ f_*)_\# : H_*^{BM}(U) \simeq \pi_* Z_0(U) \to \pi_{*+2j} Z_0(V) \simeq H_{*+2j}^{BM}(V). \tag{2.3.2}$$

The equality $\Phi_f([\delta]) = [\Gamma_f(\delta)]$ follows from the commutativity of (2.3.1) and the corresponding result for projective varieties [FM-1;6.4]. $\square$

Let $\mathcal{D}$ denote the derived category of bounded below chain complexes of abelian groups. If V is a smooth variety of (complex) dimension n with projective closure

Y, then Poincaré duality implies that cap product with the fundamental class $[Y]$ determines a quasi-isomorphism of chain complexes

$$\cap[Y] : C^*(V)[2n] \xrightarrow{\sim} C_*(Y, Y_\infty), \tag{2.4.0}$$

where

$$(C^*(V))_{-i} = Hom(C_i(V), \mathbf{Z})$$

is the group of simplicial cochains on V (with respect to some triangulation of V) of codegree i so that $(C^*(V)[2n])_k = Hom(C_{2n-k}(V), \mathbf{Z})$.

For any quasi-projective variety U, we define the hypercohomology of V (with respect to the classical topology on V) with coefficients in a bounded below chain complex C_* as

$$\mathbf{H}^i(U; C_*) \equiv Hom_{\mathcal{D}}(C_*(U), C_*[i]).$$

In the special case that C_* is the degenerate chain complex whose only non-zero term is the abelian group A in degree 0, then $\mathbf{H}^i(U, A)$ equals the singular cohomology group $H^i(U, A)$. More generally, the Künneth Theorem and (2.4.0) imply that

$$\mathbf{H}^i(U; C_*(Y, Y_\infty)[-2n]) = H^i(U \times V, \mathbf{Z}). \tag{2.4.1}$$

The following proposition generalizes to quasi-projective varieties and refines to integral cohomology the formulation of the total characteristic class in rational cohomology given in [FM-2;1.5] for Chow correspondences of projective varieties.

Proposition 2.4. *If V is smooth of dimension n, then a Chow correspondence $f : U \to \mathcal{C}_j(V)$ determines the* **characteristic class**

$$\langle f \rangle \in Hom_{\mathcal{D}}(C_*(X, X_\infty), C_*(Y, Y_\infty)[-2j]) \simeq H^{2(n-j)}((X, X_\infty) \times V; \mathbf{Z})$$

where $U \subset X$ is a projective closure with complement X_∞.

Proof. The Chow correspondence $f : U \to \mathcal{C}_j(V)$ induces the homomorphism of topological abelian groups $s^j \circ f : Z_0(U) \to Z_j(V) \to \Omega^{2j} Z_0(V)$. Applying the singular complex functor, we obtain a map of chain complexes $Sing.(Z_0(U)) \to Sing.(\Omega^{2j}(Z_0(V))$ which is quasi-isomorphic to $C_*(X, X_\infty) \to C_*(Y, Y_\infty)[-2j]$. Using (2.4.1), we reinterpret this as a class in $H^*((X, X_\infty) \times V, \mathbf{Z})$. □

We conclude this section by reformulating the Chow correspondence homomorphism in cohomological terms. We refer the reader to [Sp;6.1], [D;VII.13] for a discussion of the slant product pairing

$$-/- : Hom((C \otimes C')_n, R) \otimes (C_p \otimes R) \to Hom(C'_{n-p}, R) \tag{2.4.1}$$

for chain complexes C, C' of modules over a commutative ring k and a k-algebra R.

Proposition 2.5. *Adopt the hypotheses and notation of Proposition 2.4.*

(a.) For any $\delta \in H_i^{BM}(U, \mathbf{Z}) \simeq H_i(X, X_\infty; \mathbf{Z})$,

$$\Phi_f(\delta) = (\langle f \rangle / \delta)^\vee \in H_{i+2j}^{BM}(V, \mathbf{Z}),$$

the Poincaré dual of the class $\langle f \rangle / \delta \in H^{2(n-j)-i}(V, \mathbf{Z})$ (so that $(\langle f \rangle / \delta)^\vee$ is given by cap product of $\langle f \rangle / \delta$ with the fundamental class $[V]$ of V).

(b.) If $\alpha \in H^{2m-i}(U, \mathbf{Z})$ is the restriction of some $\overline{\alpha} \in H^{2m-i}(X, \mathbf{Z})$, then

$$\langle f \rangle / (\alpha \cap [X]) = pr_{V!}(\langle f \rangle \cdot pr_X^*(\overline{\alpha})),$$

where $pr_X^* : H^*(X, \mathbf{Z}) \to H^*(X \times V, \mathbf{Z})$ and $pr_{V!} : H^*(X \times V, \mathbf{Z}) \to H^{*-2m}(V, \mathbf{Z})$ is the Gysin map.

Proof. Essentially by definition of the slant product,

$$\langle f \rangle \in (C_*(X, X_\infty)^\# \otimes C^*(V)[2n-2j])_{2j-2n}$$

sends $c \in C_i(X, X_\infty)$ to its image under the map $C_*(X, X_\infty) \to C_*(Y, Y_\infty)$ defining $\langle f \rangle$. Granted how this map was constructed using Poincaré duality, we immediately conclude that this map sends the homology class $\delta \in H_i(X, X_\infty; \mathbf{Z})$ to $\Phi_f(\delta)^\vee$.

To prove (b.), we use the equalities

$$\langle f \rangle / (\alpha \cap [X]) = \langle f \rangle / (\overline{\alpha} \cap [X]) = (\langle f \rangle \cdot p_X^*(\overline{\alpha}))/[X],$$

the first evident by inspection and the second a special case of [D;6.1.4]. Thus, (b.) follows from the evident equality

$$(\langle f \rangle \cdot p_X^*(\overline{\alpha}))/[X] = pr_{V!}(\langle f \rangle \cdot pr_X^*(\overline{\alpha})).$$

□

§3. TOPOLOGICAL FILTRATION

Our objective is to exhibit classes in specific stages of the following topological filtration. Note that there is no hypothesis of smoothness in the definition.

Definition 3.1. *(cf.* [FM-1]*) Let V be a quasi-projective variety. The j-th stage of the* **topological filtration** *on $Z_r(V)$ is defined to be*

$$S_j Z_r(V) \equiv ker\{Z_r(V) \xrightarrow{\pi} \pi_0 Z_r(V) \xrightarrow{s^j} \pi_{2j} Z_{r-j}(V)\}.$$

Clearly, $\{S_j Z_r(V)\}$ is an increasing filtration on $Z_r(V)$. In the notation of (3.1), the Griffiths group of r-cycles equals $S_r(V)/S_0(V)$.

As defined in Definition 3.1, the topological filtration on algebraic cycles has no evident homological interpretation. However, such an interpretation is indeed available, as we recall in the following theorem.

Theorem 3.2. *(cf.* [F2;3.2]]*) Let Y be a projective variety. Then $S_j Z_r(Y) \subset Z_r(Y)$ is the subgroup generated by r-cycles of the form $\Gamma_f(\delta)$, where $f : W \to \mathcal{C}_{r-j}(Y)$ is a Chow correspondence from a projective variety W of dimension $2j+1$ and δ is an j-cycle on W homologically equivalent to 0.*

As observed in [F2;3.3], Theorem 3.2 implies the following

Corollary 3.3. *For any smooth projective variety Y,*

$$A_jCH_r(Y) \ \subset \ S_jZ_r(Y)/(\sim_{rat}).$$

We now state our main theorem, our analogue of Nori's result Theorem 1.4. To apply this to exhibit non-trivial filtrations, we use its contrapositive: we begin with some algebraic cycle ζ on X which is not homologically trivial and conclude the non-triviality in the penultimate level of the topological filtration of its restriction to $Y_t \subset X$.

The proof of Theorem 3.4 will be given in §5, after a discussion of relative characteristic classes in §4. We abuse notation by letting $i_T : \mathcal{Y} \to \mathcal{X} = X \times T$ denote the pull-back (more properly denoted $\mathcal{Y}_T \to \mathcal{X}_T$) of $\mathcal{Y}_S \subset \mathcal{X}_S$ via $T \to S$.

Theorem 3.4. *Adopt the notation and conventions of (1.2), and let $T \to S$ be an etale map. If $\zeta \in Z_{r+h}(X)$ satisfies*

$$i_t^!(\zeta) \ \in \ S_jZ_r(Y_t), \ j < r \quad \textit{almost all } t \in T.$$

where $i_t : Y_t \to X$ is the restriction of $i_T : \mathcal{Y} \to \mathcal{X} = X \times T$, then

$$[\zeta] = 0 \in H_{2(r+h)}(X, \mathbf{Q}).$$

In particular, if X is itself a complete intersection of dimension $2r+2$ whose algebraic homology in middle dimension has rank at least 2 and if $h = 1$, then there exists $t \in T$ with

$$S_rZ_r(Y_t)/S_{r-1}Z_r(Y_t) \ \otimes \mathbf{Q} \neq 0.$$

In view of Corollary 3.3, Theorem 3.4 is stronger than Theorem 1.4. As in that theorem, we could simply take $j = r-1$ in its statement.

The next proposition shows one easy way that Theorem 3.4 provides examples of cycles lying in levels of the topological filtration lower than the penultimate level.

Proposition 3.5. *Let Y be a smooth projective variety and consider an algebraic cycle $\gamma \in Z_k(Y)$ satisfying $\gamma \neq 0 \in S_jZ_k(Y)/S_{j-1}Z_k(Y) \otimes \mathbf{Q}$ for some $j, 0 < j \leq k$. Let P be a projective smooth variety of dimension m and consider $\gamma \times P \in Z_{k+m}(Y \times P)$. Then for any $p \in P$*

$$\gamma \times \{p\} \neq 0 \in S_jZ_k(Y \times P)/S_{j-1}Z_k(Y \times P) \otimes \ \mathbf{Q}$$

and

$$\gamma \times P \neq 0 \in S_jZ_{k+m}(Y \times P)/S_{j-1}Z_{k+m}(Y \times P) \otimes \ \mathbf{Q}.$$

Proof. We give a proof of the second assertion concerning $\gamma \times P$. The proof of the first assertion is similar (and even easier): to prove the first assertion we would replace in the proof below intersection with $Y \times \{p\}$ by the projection of cycles $\mathcal{C}_{k-j+1}(Y \times P) \to \mathcal{C}_{k-j+1}(Y)$.

Since $\gamma \in S_jZ_k(Y)$, there exists some Chow correspondence $f : W \to \mathcal{C}_{k-j}(Y)$ with $dim(W) = 2j+1$ and some $\delta \in \mathcal{C}_j(W)$ such that $\gamma = \Gamma_f(\delta)$ and $[\delta] = 0$ in $H_{2j}(W)$. Define $g : W \to \mathcal{C}_{k+m-j}(Y \times P)$ by sending $w \in W$ to $f(w) \times P$. Then $\Gamma_g(\delta) = \gamma \times P$, so that $\gamma \times P \in S_j(Y \times P)$.

Suppose that there exists some $h : W' \to \mathcal{C}_{k+m-j+1}(Y \times P)$ with $dim(W') = 2j-1$ and some $\xi \in \mathcal{C}_{j-1}(W')$ such that some multiple of $\gamma \times P$ equals $\Gamma_h(\xi)$ and $[\xi] = 0$ in $H_{2j-2}(W')$. As argued in [FL-2], for N sufficiently large and for some Zariski neighborhood $\mathcal{O}$ of $0 \in \mathbf{A}^1$, we may find an algebraic homotopy

$$\Theta_N : \mathcal{C}_{k+m-j+1}(Y \times P) \times \mathcal{O} \to \mathcal{C}_{k+m-j+1}(Y \times P)$$

such that Θ_N restricted to $\mathcal{C}_{k+m-j+1}(Y \times P) \times \{0\}$ is multiplication by N and for any $0 \neq t \in \mathcal{O}$ the restriction θ_t of Θ_N to $\mathcal{C}_{k+m-j+1}(Y \times P) \times \{t\}$ has image consisting of $(k+m-j+1)$-cycles on $Y \times P$ meeting $Y \times \{p\}$ properly for all $p \in P$. Thus, sending $w' \in W'$ to $\theta_t(h(w')) \bullet (Y \times \{p\})$ determines a Chow correspondence $g : W' \to \mathcal{C}_{k-j+1}(Y)$ with $\Gamma(\xi)$ rationally equivalent to some multiple of γ. This contradicts the assumption that $\gamma \neq 0 \in S_j Z_k(Y)/S_{j-1}Z_k(Y) \otimes \mathbf{Q}$ (since $S_{j-1}Z_k(Y)$ is closed under rational equivalence). $\square$

As an immediate corollary of Theorem 3.4 and Proposition 3.5, we obtain examples in which the topological filtration has several non-trivial associated graded pieces of specified level. (In view of Proposition 3.5, one may find examples of Y satisfying the hypothesis of Corollary 3.6 by taking products of examples given in Theorem 3.4)

Corollary 3.6. *Assume that Y is a projective smooth variety with the property that there exist algebraic cycles $\gamma \neq 0 \in S_k Z_k(Y)/S_{k-1}Z_k(Y) \otimes \mathbf{Q}$, and $\gamma' \neq 0 \in S_{k'} Z_{k'}(Y)/S_{k'-1}Z_{k'}(Y) \otimes \mathbf{Q}$ with $k < k'$. Let P, P' be projective smooth varieties of dimensions m, m' satisfying $k + m' = k' + m$. Then*

$$S_k Z_{k+m'}(Y \times P \times P')/S_{k-1}Z_{k+m'}(Y \times P \times P') \otimes \mathbf{Q} \neq 0,$$

$$0 \neq S_{k'} Z_{k+m'}(Y \times P \times P')/S_{k'-1}Z_{k+m'}(Y \times P \times P') \otimes \mathbf{Q}.$$

§4. RELATIVE CHARACTERISTIC CLASSES

We fix a connected projective variety $\overline{T}$ of pure (complex) dimension τ and a non-empty Zariski open subset $T \subset \overline{T}$ with closed complement $T_\infty \subset \overline{T}$. We consider projective maps $p_{\overline{\mathcal{E}}} : \overline{\mathcal{E}} \to \overline{T}, p_{\overline{\mathcal{Y}}} : \overline{\mathcal{Y}} \to \overline{T}$ and denote by $p_{\mathcal{E}} : \mathcal{E} \to T, p_{\mathcal{Y}} : \mathcal{Y} \to T$ the restrictions of these maps to $T \subset \overline{T}$. We let $\mathcal{E}_\infty$ denote $\overline{\mathcal{E}} - \mathcal{E}$ and $\mathcal{Y}_\infty$ denote $\overline{\mathcal{Y}} - \mathcal{Y}$.

The aim of this section is to develop some aspects of Chow correspondences and correspondence homomorphisms relative to our fixed base T. In particular, Proposition 4.5 refines the characteristics class $\langle f \rangle$ of Proposition 2.4 by formulating a relative characteristic class $\langle f/T \rangle$ in the cohomology the fibre product of $\mathcal{E}$ and $\mathcal{Y}$ over T. This refinement is required in order to be able to apply the Nori-Lefschetz Theorem.

We begin our relativiation of aspects of §2 with the following simple but useful definition.

Definition 4.1. *For each $j, d \geq 0$, we define the* **relative Chow variety** *(of effective j-cycles of degree d in some fibre of $\overline{\mathcal{Y}}/\overline{T}$) to be the fibre product*

$$C_{j,d}(\overline{\mathcal{Y}}/\overline{T}) \equiv \mathcal{C}_j(\overline{\mathcal{Y}}) \times_{\mathcal{C}_j(\mathbf{P}^N \times \overline{T})} [C_{j,d}(\mathbf{P}^N) \times \overline{T}],$$

where $\overline{\mathcal{Y}} \subset \mathbf{P}^N \times \overline{T}$ is a closed embedding whose composition with the projection is the structure map $\overline{\mathcal{Y}} \to \overline{T}$. We further define

$$\mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T}) \equiv \coprod_{d=0}^{\infty} C_{j,d}(\overline{\mathcal{Y}}/\overline{T}), \quad \mathcal{Z}_j(\overline{\mathcal{Y}}/\overline{T}) \equiv [\mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})]^{+\overline{T}}$$

where $[-]^{+\overline{T}}$ denotes the naïve fibre-wise group completion over $\overline{T}$.

We define

$$\mathcal{C}_j(\mathcal{Y}/T) \equiv \mathcal{C}_j(\overline{Y}/\overline{T}) \times_{\overline{T}} T, \quad \mathcal{Z}_j(\mathcal{Y}/T) \equiv \mathcal{Z}_j(\overline{Y}/\overline{T}) \times_{\overline{T}} T$$

The naïve fibrewise group completion $[\mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})]^{+\overline{T}}$ is defined as a quotient space of $\mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T}) \times_{\overline{T}} \mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})$. This can be realized as the colimit of a sequence of pushout squares exactly as naïve group completions constructed in [F-G]. As argued in [F3], this construction yields C.W. complexes.

As established in the next proposition, our relative Chow varieties provide a naïve version of the relative cycles functor restricted to normal varieties. The interested reader should consult [S-V] for a more sophisticated and complete investigation of relative cycles.

Proposition 4.2. *If U is a quasi-projective variety over T, then a morphism $f : U \to \mathcal{C}_j(\mathcal{Y}/T)$ over T naturally determines an effective cycle $Z_{f/T}$ on $U \times_T \mathcal{Y}$ equidimensional over U of relative dimension j. If U is normal, then sending such a morphism f to $Z_{f/T}$ is a 1-1 correspondence.*

Proof. As seen in [F1;1.4], the composition $f : U \to \mathcal{C}_j(\mathcal{Y}/T) \to \mathcal{C}_j(\overline{\mathcal{Y}})$ determines the cycle $Z_f \subset U \times \overline{\mathcal{Y}}$. To verify that this cycle lies in $U \times_T \mathcal{Y} \subset U \times \overline{\mathcal{Y}}$, it suffices to prove this for the pre-composition of f with an arbitrary point $\nu : Spec\mathbf{C} \to U$. In this case, the support of $Z_{f\circ\nu}$ equals that of the cycle parametrized by the Chow point $f \circ \nu \in \mathcal{C}_j(\overline{\mathcal{Y}})$ (cf. [F1;1.3]) which is clearly contained in $\mathcal{Y}_\nu \subset U \times_T \mathcal{Y}$.

If U is normal, then the 1-1 correspondence proved in [FL-1;1.5] in the absolute case (i.e., $T = Spec\mathbf{C}$) restricts to the asserted 1-1 correspondence by the argument given immediately above. □

In order to relativize our discussion of §2, we shall consider presheaves of chain complexes on $\overline{T}$. If $T' \subset \overline{T}$ is an analytic open subset, then we shall consider the topological abelian monoid $Hom_{Lif}(T', \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}}$ of Lifschitz maps from T' to $\mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T})$ over $\overline{T}$ provided with the compact-open topology. We associate to this monoid the chain complex

$$Norm\{[Sing.(Hom_{Lif}(T', \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}})]^+\}, \tag{4.3.0}$$

the normalized chain complex of the simplicial abelian group obtained by group completing the simplicial monoid obtained by applying the singular complex functor to $Hom_{Lif}(T', \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}}$.

In order to formalize our discussion, we shall consider the abelian category $\mathcal{S}_B$ of presheaves of bounded below chain complexes of abelian groups on a topological space B. $\mathcal{S}_B$ has the structure of a triangulated category whose distinguished triangles are triple $P \to Q \to R$ with the property that

$$0 \to P(U) \to Q(U) \to R(U) \to 0$$

is a short exact sequence for every open subset $U \subset B$. We say that $P \to Q$ is a quasi-isomorphism if the induced map on fibres at each point of B is an isomorphism; alternatively, if the kernel and cokernel of this map have acyclic fibres. Finally, we denote by $\mathcal{D}_B$ the localization of $\mathcal{S}_B$ with respect to the thick subcategory of those $P \in \mathcal{S}_B$ with the property that each fibre of P is acyclic (cf. [F4]).

Theorem 4.3. *If $p_{\mathcal{Y}} : \mathcal{Y} \to T$ is smooth as well as projective of relative dimension n, then $Z_0(\mathcal{Y}/T) \to T$ is locally (on T for the analytic topology) a product projection with fibres $Z_0(Y_t)$, where Y_t is the fibre of $\mathcal{Y} \to T$ above $t \in T$.*

Moreover, let $\underset{\sim}{Z_0}(\overline{\mathcal{Y}}/\overline{T})$ denote the sheaf of chain complexes on $\overline{T}$ sending an analytic open subset $T' \subset \overline{T}$ to the chain complex of (4.3.0). Then the restriction of $\underset{\sim}{Z_0}(\overline{\mathcal{Y}}/\overline{T})$ to $T \subset \overline{T}$, $\underset{\sim}{Z_0}(\mathcal{Y}/T)$, is quasi-isomorphic to $\mathbf{R}p_{\mathcal{Y}*}\mathbf{Z}[2n]$ (where the cochain complex $\mathbf{R}p_{\mathcal{Y}*}\mathbf{Z}$ is indexed as a chain complex vanishing in positive degrees).

Proof. A sufficiently small tubular neighborhood of $Y_t \subset \mathcal{Y}$ has projection in T containing an ϵ-neighborhood N_t of $t \in T$ whose preimage in $\mathcal{Y}$ admits the structure of a product with the property that the restriction of $\mathcal{Y} \to T$ to N_t is a product projection. Then the restriction of $Z_0(\mathcal{Y}/T) \to T$ above N_t is also a product projection.

As discussed in [FL-3], the graph of a Lifschitz map $\overline{T} \to \mathcal{C}_0(\overline{\mathcal{Y}})$ is a well defined integral cycle of (real) dimension 2τ on $\overline{T} \times \overline{\mathcal{Y}}$ which we project to $\overline{\mathcal{Y}}$. This graphing construction determines a continuous map

$$Hom_{Lif}(\overline{T}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}} \to \mathcal{Z}_{2\tau}(\overline{\mathcal{Y}})$$

where $\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}})$ denotes the topological abelian group of integral cycles on $\overline{\mathcal{Y}}$ of (real) dimension 2τ and where $Hom_{Lif}(-,-)_{\overline{T}}$ is given the compact open topology.

We consider a basis of open sets $\mathcal{O}$ of T with the property that $\overline{\mathcal{O}} \subset \overline{T}$ (the closure of $\mathcal{O}$ in $\overline{T}$) is contained in T, and both $\overline{\mathcal{O}}$ and $\mathcal{O}^c \equiv \overline{T} - \mathcal{O}$ are compact Lifschitz neighborhood retracts. We define

$$Hom_{Lif}(\mathcal{O}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}} \equiv im\{Hom_{Lif}(\overline{\mathcal{O}}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}} \to Hom_{cont}(\mathcal{O}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}}\},$$

so that as above we have a well defined continuous graph mapping

$$Hom_{Lif}(\mathcal{O}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}} \to \mathcal{Z}_{2\tau}(\overline{\mathcal{Y}})/\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}} \times_{\overline{T}} \mathcal{O}^c)$$

sending $f : \mathcal{O} \to \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))$ to the projection of the closure of its graph in $\overline{\mathcal{O}} \times \overline{\mathcal{Y}}$. This map in turn induces a map of simplicial abelian groups

$$[Sing.(Hom_{Lif}(\mathcal{O}, \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}}]^+ \to Sing.(\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}})/\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}} \times_{\overline{T}} \mathcal{O}^c)). \tag{4.3.1}$$

We interpret F. Almgren's theorem [A] as the assertion of a quasi-isomorphism

$$Norm\{Sing.(\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}})/\mathcal{Z}_{2\tau}(\overline{\mathcal{Y}} \times_{\overline{T}} \mathcal{O}^c)\} \simeq Norm\{Sing.(\mathcal{Z}_0(\overline{\mathcal{Y}})/\mathcal{Z}_0(\overline{\mathcal{Y}} \times_{\overline{T}} \mathcal{O}^c)\}[-2\tau].$$

For small polydisks $T' \subset T$, we recall [FL-3] that the natural inclusion

$$Hom_{Lif}(T', \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}} \to Hom_{cont}(T', \mathcal{C}_0(\overline{\mathcal{Y}}/\overline{T}))_{\overline{T}}$$

of function spaces with the compact open topology is a deformation retract. Thus, the first assertion of this theorem implies that the homology groups of the left hand side of (4.3.1) are the cohomology groups of Y_t whenever $p_{\mathcal{Y}}$ is proper and smooth and $\mathcal{O} = T'$ is a small polydisk around $t \in T'$, whereas the Dold-Thom theorem implies that the homology of the right hand side is Borel-Moore homology of the pre-image in $\mathcal{Y}$ of the polydisk. As argued in [FL-3], the graph mapping of (4.3.1) induces an isomorphism on these homotopy groups since $\mathcal{Y} \to T$ is smooth.

We observe that there is a natural quasi-isomorphism

$$C_*(A,B) \simeq Norm\{Sing.(\mathcal{Z}_0(A)/Z_0(B))\}$$

for any polyhedral pair $B \subset A$, where $C_*(A,B)$ denotes the singular chain complex of the pair. Thus,

$$Norm\{Sing.(\mathcal{Z}_0(\overline{\mathcal{Y}})/\mathcal{Z}_0(\overline{\mathcal{Y}}\times_{\overline{T}}T^{'c})\}[-2\tau] \simeq C_*(\overline{\mathcal{Y}},\overline{\mathcal{Y}}\times_{\overline{T}}T^{'c})[-2\tau] \simeq C^*(p_{\mathcal{Y}}^{-1}(T'))[2n],$$

where the second quasi-isomorphism is given by Poincaré duality (cf. (2.4.0)). The observation that sending an open subset $V \subset \mathcal{Y}$ to $C^*(V)$ is a flasque presheaf of chain complexes on $\mathcal{Y}$ implies the quasi-isomorphism

$$C^*(p_{\mathcal{Y}}^{-1}(T')) \simeq \mathbf{R}p_{\mathcal{Y}*}\mathbf{Z},$$

thereby completing the proof. □

Sending an irreducible subvariety of $\mathbf{P}^n \times \mathbf{P}^m$ defined by bi-homogeneous equations $\{F_1(x,t),\dots,F_k(x,t)\}$ to the irreducible subvariety of $\mathbf{P}^{n+1} \times \mathbf{P}^m$ given by the same equations determines a natural morphism, the **relative algebraic suspension**,

$$\Sigma_{\mathbf{P}^m} : \mathcal{C}_r(\mathbf{P}^n \times \mathbf{P}^m) \to \mathcal{C}_{r+1}(\mathbf{P}^{n+1} \times \mathbf{P}^m).$$

We denote by $\Sigma_{\overline{T}}\overline{\mathcal{Y}} \subset \mathbf{P}^{N+1} \times \overline{T}$ the image of $\overline{\mathcal{Y}} \subset \mathbf{P}^N \times \overline{T}$ under such a relative algebraic suspension map. This construction determines a morphism over $\overline{T}$

$$\Sigma_{\overline{T}} : \mathcal{C}_r(\overline{\mathcal{Y}}/\overline{T}) \to \mathcal{C}_{r+1}(\Sigma_{\overline{T}}\overline{\mathcal{Y}}/\overline{T}).$$

If $X \subset \mathbf{P}^M, Y \subset \mathbf{P}^N$ are projective varieties, then the algebraic join $X\#Y \subset \mathbf{P}^M\#\mathbf{P}^N = \mathbf{P}^{M+N+1}$ is the subvariety defined by the union of the homogeneous equations defining X and Y. This can be viewed as the subvariety of $\mathbf{P}^{M+N+1}$ consisting of points lying on some line from a point of X to a point of Y. If $\overline{\mathcal{X}}/\overline{T}, \overline{\mathcal{Y}}/\overline{T}$ are projective families over a projective variety $\overline{T}$, then the **relative algebraic join** $\overline{\mathcal{X}}\#_{\overline{T}}\overline{\mathcal{Y}}$ is the subvariety of $\overline{\mathcal{X}}\#\overline{\mathcal{Y}}$ consisting of points lying on some line from a point of $\overline{\mathcal{X}}$ to a point of $\overline{\mathcal{Y}}$ all of which project to the same point of $\overline{T}$.

Applying the relative algebraic join determines a continuous algebraic map over T

$$\#_{\overline{T}} : \mathcal{C}_r(\overline{\mathcal{Y}}/\overline{T}) \times \mathcal{C}_0(\mathbf{P}^1 \times \overline{T}/\overline{T}) \to \mathcal{C}_{r+1}(\Sigma^2_{\overline{T}}\overline{\mathcal{Y}}/\overline{T}).$$

We may interpret the relative algebraic suspension $\Sigma_{\overline{T}}$ as the special case of the following relative algebraic join construction in which $\overline{\mathcal{Y}} \to \overline{T}$ is taken to be the identity map.

Proposition 4.4. *Relative algebraic suspension admits a homotopy inverse*

$$\Sigma_T^{-1} : Z_{r+1}(\Sigma_T \mathcal{Y}/T) \to Z_r(\mathcal{Y}/T).$$

Consequently, we may define a relative s-map over T

$$s_T = \Sigma_T^{-2} \circ \#_T : Z_r(\mathcal{Y}/T) \wedge S^2 \to Z_{r+1}(\Sigma_T^2 \mathcal{Y}/T) \to Z_{r-1}(\mathcal{Y}/T)$$

with adjoint denoted also by s_T:

$$s_T : Z_r(\mathcal{Y}/T) \to \Omega_T^2 Z_{r-1}(\mathcal{Y}/T),$$

where $\Omega_T W \subset \Omega W$ denotes the subspace of the free loop space of a topological space W over T equipped with a section $\omega : T \to W$ consisting of loops each lying above some $t \in T$ and based at $\omega(t)$.

Proof. For each $d > 0$ and all e sufficiently large with respect to d, there exists an algebraic homotopy

$$C_{r+1,\leq d}(\mathbf{P}^{N+1}) \times \mathcal{O} \to C_{r+1,\leq de}(\mathbf{P}^{N+1})$$

relating multiplication by e and a map with image contained in the image of $\Sigma : C_{r,\leq de}(\mathbf{P}^N) \to C_{r+1,\leq de}(\mathbf{P}^{N+1})$, where $\mathcal{O}$ is a Zariski open subset of $0 \in \mathbf{A}^1$ (cf. [F1]). This homotopy extends to

$$C_{r+1,\leq d}(\mathbf{P}^{N+1} \times \overline{T}/\overline{T}) \times \mathcal{O} \to C_{r+1,\leq de}(\mathbf{P}^{N+1} \times \overline{T}/\overline{T})$$

by taking the constructions of the original homotopy and formally extending them so as to be independent of $t \in \overline{T}$, where degree now refers to the first component of multi-degree in $\mathbf{P}^{N+1} \times \overline{T}$. Embedding $\overline{\mathcal{Y}}$ in $\mathbf{P}^N \times \overline{T}$ and thereby $\Sigma_{\overline{T}} \overline{\mathcal{Y}}$ in $\mathbf{P}^{N+1} \times \overline{T}$, we easily see this extended homotopy restricts to

$$C_{r+1,\leq d}(\Sigma_{\overline{T}} \overline{\mathcal{Y}}/\overline{T}) \times \mathcal{O} \to C_{r+1,\leq de}(\Sigma_{\overline{T}} \overline{\mathcal{Y}}/\overline{T})$$

relating fibre-wise multiplication by e to a map with image contained in the image of

$$\Sigma_{\overline{T}} : C_{r,de}(\overline{\mathcal{Y}}/\overline{T}) \to C_{r+1,\leq de}(\Sigma_{\overline{T}} \overline{\mathcal{Y}}/\overline{T}).$$

This homotopy clearly restricts to

$$C_{r+1,\leq d}(\Sigma_T \mathcal{Y}/T) \times \mathcal{O} \to C_{r+1,\leq de}(\Sigma_T \mathcal{Y}/T).$$

The arguments of [L], [F1] now apply to establish the fact that $\Sigma_T : Z_r(\mathcal{Y}/T) \to Z_{r+1}(\Sigma_T \mathcal{Y}/T)$ is a weak homotopy equivalence (over T). The fact that this is a homotopy equivalence follows from [F3;1.3]. $\square$

Using the relative s-map s_T of Proposition 4.4, we now exhibit relative characteristic classes for relative Chow correspondences.

Proposition 4.5. *A* **relative Chow correspondence**

$$\overline{f}/\overline{T} : \overline{\mathcal{E}} \to \mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})$$

is a morphism over $\overline{T}$. Let f/T denote the restriction of this morphism above T. If $p_{\mathcal{Y}} : \mathcal{Y} \to T$ is smooth (as well as proper) of relative dimension n, then such a relative Chow correspondence naturally determines a **relative characteristic class**

$$\langle f/T \rangle \in H^{2n-2j}(\mathcal{E} \times_T \mathcal{Y}, \mathbf{Z}).$$

Proof. The relative s-map of Proposition 4.4 determines a map in $\mathcal{D}_T$:

$$s_{T*} : \underset{\sim}{Z}_j(\mathcal{Y}/T) \to \underset{\sim}{Z}_0(\mathcal{Y}/T)[-2j],$$

whereas $\overline{f}/\overline{T}$ induces by naturality of (4.3.0) the map

$$(f/T)_* : \underset{\sim}{Z}_0(\mathcal{E}/T) \to \underset{\sim}{Z}_0(\mathcal{Y}/T).$$

Thus, assuming $p_{\mathcal{Y}}$ is smooth, $\overline{f}/\overline{T}$ determines

$$\langle f \rangle \in Hom_{\mathcal{D}_T}(\underset{\sim}{Z}_0(\mathcal{E}/T), \underset{\sim}{Z}_0(\mathcal{Y}/T)[-2j]) \simeq Hom_{\mathcal{D}_T}(\underset{\sim}{Z}_0(\mathcal{E}/T), \mathbf{R}p_{\mathcal{Y}*}\mathbf{Z}[2n-2j]).$$

Observe that the canonical map $\overline{\mathcal{E}} \to \mathcal{C}_0(\overline{\mathcal{E}}/\overline{T})$ over $\overline{T}$ determines a canonical map in $\mathcal{S}_{\mathcal{E}}$:

$$\kappa_{\mathcal{E}} : \mathbf{Z} \to p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{E}/T)). \tag{4.5.1}$$

(For any open subset $\mathcal{W} \subset \mathcal{E}$, the evident map $\mathcal{W} \to \mathcal{C}_0(\mathcal{E}/T)$ determines an element in degree 0 of the chain complex $p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{E}/T))(\mathcal{W})$.) Together with the natural isomorphism

$$p_{\mathcal{E}}^*\mathbf{R}p_{\mathcal{Y}*}\mathbf{Z} \simeq \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z},$$

where $p_{\mathcal{E}\times_T\mathcal{Y}} : \mathcal{E} \times_T \mathcal{Y} \to \mathcal{E}$ is the pull-back via $p_{\mathcal{E}}$ of $p_{\mathcal{Y}}$, this gives us a natural map

$$Hom_{\mathcal{D}_T}(\underset{\sim}{Z}_0(\mathcal{E}/T), \mathbf{R}p_{\mathcal{Y}*}\mathbf{Z}[2n-2j]) \to Hom_{\mathcal{D}_{\mathcal{E}}}(\mathbf{Z}, \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z}[2n-2j]) \quad (4.5.2).$$

Finally, the right hand side of (4.5.2) is identified using the following isomorphisms

$$Hom_{\mathcal{D}_{\mathcal{E}}}(\mathbf{Z}, \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z}[2n-2j])) = \mathbf{H}^{2n-2j}(\mathcal{E}, \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z}) = H^{2n-2j}(\mathcal{E} \times_T \mathcal{Y}, \mathbf{Z}), \tag{4.5.3}$$

where the first isomorphism can be taken to be the definition of the hypercohomology of $\mathcal{E}$ with coefficients in the complex of sheaves $\mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z}$ and the second equality is a form of the Serre spectral sequence for $p_{\mathcal{E}\times_T\mathcal{Y}}$. □

We continue our study of relative Chow correspondences by relating the relative characteristic class $\langle f/T \rangle$ of Proposition 4.5 to the characteristic class of f as formulated in Proposition 2.4.

Proposition 4.6. *Assume that T is a smooth variety and that $p_{\mathcal{Y}} : \mathcal{Y} \to T$ is smooth as well as proper of relative dimension n. Then the embedding $\eta_{\mathcal{Y}} : \mathcal{E} \times_T \mathcal{Y} \subset \mathcal{E} \times \mathcal{Y}$ determines a map (in the derived category $\mathcal{D}_T$) of chain complexes on $\mathcal{E}$*

$$\eta_{\mathcal{Y}!} : \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z} \simeq p_{\mathcal{Y}}^*\mathbf{R}p_{\mathcal{Y}*}\mathbf{Z} \to \pi_{\mathcal{E}}^*\mathbf{R}\pi_{\mathcal{Y}*}\mathbf{Z}[2\tau] \simeq \mathbf{R}\pi_{\mathcal{E}\times\mathcal{Y}*}\mathbf{Z}[2\tau]$$

where $p_{\mathcal{E}\times_T\mathcal{Y}} : \mathcal{E} \times_T \mathcal{Y} \to \mathcal{E}$, $\pi_{\mathcal{E}\times\mathcal{Y}} : \mathcal{E} \times \mathcal{Y} \to \mathcal{E}$ are the projections.

Moreover, consider a relative Chow correspondence $\overline{f}/\overline{T} : \overline{\mathcal{E}} \to \mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})$ and let $f : \mathcal{E} \to \mathcal{C}_j(\mathcal{Y})$ denote the Chow correspondence obtained by restricting $\overline{f}$ above $T \subset \overline{T}$ and composing with $C_j(\mathcal{Y}/T) \subset \mathcal{C}_j(\mathcal{Y})$. Then, assuming that $p_{\mathcal{Y}}$ is smooth, the map

$$\eta_{\mathcal{Y}!} : H^{2n-2j}(\mathcal{E} \times_T \mathcal{Y}, \mathbf{Z}) \to H^{2n+2\tau-2j}(\mathcal{E} \times \mathcal{Y}, \mathbf{Z})$$

sends $\langle f/T\rangle$ to the restriction of $\langle f\rangle \in H^{2n-2j}((\overline{\mathcal{E}}, \mathcal{E}_\infty) \times \mathcal{Y}; \mathbf{Z})$.

Proof. By Theorem 4.3 and the smoothness of $p_{\mathcal{E}}$ and $\mathcal{Y}$, the embedding $Z_0(\mathcal{Y}/T) \subset Z_0(\mathcal{Y} \times T/T) = Z_0(\mathcal{Y}) \times T$ induces a map of complexes of presheaves on T

$$\mathbf{R}p_{\mathcal{Y}*}\mathbf{Z}[2n] \simeq \underset{\sim}{Z}_0(\mathcal{Y}/T) \to \underset{\sim}{Z}_0(\mathcal{Y} \times T/T) \simeq \mathbf{R}\pi_{\mathcal{Y}*}\mathbf{Z}[2n + 2\tau]$$

where $\pi_{\mathcal{Y}} : \mathcal{Y} \times T \to T$. Applying the exact functor $p_{\mathcal{E}}^*$, where $\pi_{\mathcal{E}} : \mathcal{E} \to T$, we obtain

$$\eta_! : \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z} \to \mathbf{R}\pi_{\mathcal{E}\times\mathcal{Y}*}\mathbf{Z}[2\tau] \qquad (4.6.1).$$

By (4.5.3), $\eta_!$ induces a map on cohomology

$$\eta_{\mathcal{Y}!} : H^{2n-2j}(\mathcal{E} \times_T \mathcal{Y}, \mathbf{Z}) \to H^{2n+2\tau-2j}(\mathcal{E} \times \mathcal{Y}, \mathbf{Z}).$$

We consider the commutative square in $\mathcal{S}_{\mathcal{E}}$ (i.e., of complexes of presheaves on $\mathcal{E}$):

$$\begin{array}{ccc}
p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{E}/T)) & \longrightarrow & p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{Y}/T)[-2j]) \simeq \mathbf{R}p_{\mathcal{E}\times_T\mathcal{Y}*}\mathbf{Z}[2n-2j] \\
\big\downarrow & & \big\downarrow \\
p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{E} \times T/T)) & \longrightarrow & p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{Y} \times T/T)[-2j]) \simeq \mathbf{R}\pi_{\mathcal{E}\times\mathcal{Y}*}\mathbf{Z}[2n+2\tau-2j].
\end{array} \qquad (4.6.2)$$

By definition, $\langle f/T\rangle$ is obtained from the top row of (4.6.2) by composing with the canonical map $\kappa_{\mathcal{E}}$ of (4.5.1). Observe that $\langle f\rangle$ when represented as a class in $Hom_{\mathcal{D}}(\underset{\sim}{Z}_0(\mathcal{E}), \underset{\sim}{Z}_0(\mathcal{Y})[-2j])$, where $\underset{\sim}{\mathcal{Z}}_0(\mathcal{E})$ denotes $Norm\{[Sing.(\mathcal{C}_0(\overline{\mathcal{E}})/\mathcal{C}_0(\mathcal{E}_\infty))]^+\}$, determines a map of constant sheaves on $\mathcal{E}$ quasi-isomorphic to the lower horizontal map of (4.6.2). Moreover, $p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{Y}\times T/T))$ is a chain complex of flasque sheaves on $\mathcal{E}$, so that we may identify the homology in degree $2j$ of $\Gamma(\mathcal{E}, p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{Y}\times T/T))$ with $H^{2n+2\tau-2j}(E\times\mathcal{Y}, \mathbf{Z})$. On the other hand, the composition of $\kappa_{\mathcal{E}} : \mathbf{Z} \to p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{E}/T))$ with the left vertical and lower horizontal maps of (4.6.2) is identified in this way with the global section in degree $2j$ of $p_{\mathcal{E}}^*(\underset{\sim}{Z}_0(\mathcal{Y} \times T/T))$ corresponding to the restriction of $\langle f\rangle$. $\square$

We conclude this section with the following refinement of Proposition 2.5.

Proposition 4.7. *Let $\overline{f}/\overline{T} : \overline{\mathcal{E}} \to \mathcal{C}_j(\overline{\mathcal{Y}}/\overline{T})$ be a relative Chow correspondence and assume that $p_y : \mathcal{Y} \to T$ is smooth (as well as proper). Then for any $\overline{\alpha} \in H^k(\overline{\mathcal{E}}, \mathbf{Z})$,*

$$\Phi_f(\alpha \cap [\overline{\mathcal{E}}])^\wedge = \langle f \rangle/(\alpha \cap [\overline{\mathcal{E}}]) = pr_{\mathcal{Y}*}(\langle f/T \rangle \cdot p_{\mathcal{E}}^*(\alpha)),$$

where $\alpha \in H^k(\mathcal{E}, \mathbf{Z})$ denotes the restriction of $\overline{\alpha}$.

Proof. The proof consists in the straight-forward verification of the commutativity of the following diagram:

$$\begin{array}{ccccc}
H^{BM}_{2m+2\tau-k}(\mathcal{E},\mathbf{Z}) & \otimes H^{2n+2\tau-2j}((\overline{\mathcal{E}},\mathcal{E}_\infty)\times\mathcal{Y},\mathbf{Z}) & \xrightarrow{/} & H^{2n+k-2m-2j}(\mathcal{Y},\mathbf{Z}) \\
\uparrow & \uparrow & & \uparrow = \\
H_{2m+2\tau-k}(\overline{\mathcal{E}},\mathbf{Z}) & \otimes H^{2n+2\tau-2j}(\overline{\mathcal{E}}\times\mathcal{Y},\mathbf{Z}) & \xrightarrow{/} & H^{2n+k-2m-2j}(\mathcal{Y},\mathbf{Z}) \\
\cap[\overline{\mathcal{E}}]\uparrow & \uparrow = & & \uparrow \pi_{\mathcal{Y}!} \\
H^k(\overline{\mathcal{E}},\mathbf{Z}) & \otimes H^{2n+2\tau-2j}(\overline{\mathcal{E}}\times\mathcal{Y},\mathbf{Z}) & \xrightarrow{\bullet} & H^{2n+2\tau+k-2j}(\overline{\mathcal{E}}\times\mathcal{Y},\mathbf{Z}) \\
\downarrow & \downarrow & & \downarrow \\
H^k(\mathcal{E},\mathbf{Z}) & \otimes H^{2n+2\tau-2j}(\mathcal{E}\times\mathcal{Y},\mathbf{Z}) & \xrightarrow{\bullet} & H^{2n+2\tau+k-2j}(\mathcal{E}\times\mathcal{Y},\mathbf{Z}) \\
=\uparrow & \eta_{\mathcal{Y}!}\uparrow & & \eta_{\mathcal{Y}!}\uparrow \\
H^k(\mathcal{E},\mathbf{Z}) & \otimes H^{2n-2j}(\mathcal{E}\times_T\mathcal{Y},\mathbf{Z}) & \xrightarrow{\bullet} & H^{2n+k-2m-2j}(\mathcal{E}\times_T\mathcal{Y},\mathbf{Z})
\end{array}$$

and the observation that for any $\overline{\beta} \in H^{t+2\tau}(\overline{\mathcal{E}} \times \mathcal{Y}, \mathbf{Z})$ and $\beta' \in H^t(\mathcal{E} \times_T \mathcal{Y}, \mathbf{Z})$ with equal images in $H^{t+2\tau}(\mathcal{E} \times \mathcal{Y}, \mathbf{Z})$, $\pi_{\mathcal{Y}!}(\overline{\beta}) = p_{\mathcal{Y}!}(\beta') \in H^{t-2m}(\mathcal{Y}, \mathbf{Z})$. □

§5 PROOF OF THEOREM 3.4

The following proposition enables us to interpret in terms of relative Chow correspondences the condition that each member of a family of cycles belongs to a given level of the topological filtration.

Proposition 5.1. *Let X be a projective variety and consider $\zeta \in Z_{r+h}(X)$. Let T be a smooth quasi-projective variety of dimension τ with projective closure $\overline{T}$, $p_{\overline{\mathcal{Y}}} : \overline{\mathcal{Y}} \to \overline{T}$ a projective map with smooth restriction $p_{\mathcal{Y}} : \mathcal{Y} \to T$ above T, and $i_T : \mathcal{Y} \to \mathcal{X} = X \times T$ a closed immersion with the property that the restriction $\zeta_t = \epsilon_t^!(\zeta) \in Z_r(Y_t)$ of ζ lies in $S_j Z_r(Y_t)$ for all $\epsilon_t : \{t\} \to T$.*

After possibly replacing T by some etale open, there exists some projective, flat map $\mathcal{E} \to T$ of relative dimension $2j+1$, some relative Chow correspondence $\overline{f}/\overline{T} : \overline{\mathcal{E}} \to \mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})$, and relative Chow correspondences

$$\overline{\sigma}^+/\overline{T} : \overline{T}^+ \to C_j(\overline{\mathcal{E}}/\overline{T}), \quad \overline{\sigma}^-/\overline{T} : \overline{T}^- \to C_j(\overline{\mathcal{E}}/\overline{T})$$

such that

$$\Gamma_f(\delta_T) = i_T^!(\zeta \times T) \in Z_{r+\tau}(\mathcal{Y}), \quad [\delta_t] = 0 \in H_{2j}(E_t, \mathbf{Z}),\ \forall t \in T.$$

where

$$\delta_T \equiv \Gamma_{\sigma^+}(T) - \Gamma_{\sigma^-}(T) \in Z_{j+\tau}(\mathcal{E}), \quad \delta_t \equiv \delta^+(t) - \delta^-(t) = \epsilon_t^!(\delta_T)$$

and $\overline{T}^+, \overline{T}^-$ *map projectively onto* $\overline{T}$ *via morphisms which are isomorphisms above* $T \subset \overline{T}$.

Proof. The condition that an r-cycle ξ on Y_t lies in $S_j Z_r(Y_t)$ is equivalent to the condition that there exists a j-cycle δ_t on $\mathcal{C}_{r-j}(Y_t)$ homologically equivalent to 0 with the property that $\xi_t = tr(\delta_t)$, where tr is the trace map $Z_j(\mathcal{C}_{r-j}(Y_t)) \to Z_r(Y_t)$ of [FL-1]. Let

$$[\mathcal{C}_j(\mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})/\overline{T})^{\times 2}_{\overline{T}}]_{hom}$$

denote the kernel of the map

$$\mathcal{C}_j(\mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})/\overline{T}) \times_{\overline{T}} \mathcal{C}_j(\mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})/\overline{T}) \to \bigoplus_{t \in \overline{T}} H_{2r}(Y_t)$$

sending (δ, δ') to $[tr(\delta) - tr(\delta')]$.

Let $\zeta = \zeta^+ - \zeta^-$ be a minimal representation of $\zeta \in Z_{r+h}(X)$ as a difference of effective cycles. Consider the projection

$$[\mathcal{C}_j(\mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})/\overline{T})^{\times 2}_{\overline{T}}]_{hom} \times_{\mathcal{C}_r(\overline{Y}/\overline{T}) \times_{\overline{T}} \mathcal{C}_r(\overline{\mathcal{Y}}/\overline{T})} T \to T \tag{5.1.1}$$

where the two maps determining the fibre product are the trace map (two times) and the map $T \to \mathcal{C}_r(\overline{Y}/\overline{T}) \times_{\overline{T}} \mathcal{C}_r(\overline{Y}/\overline{T})$ sending $t \in T$ to (ζ_t^+, ζ_t^-) (where $\zeta_t^\pm = \epsilon_t^!(\zeta^\pm)$). Our hypothesis on ζ implies that (5.1.1) has image containing the open set of all those $t \in T$ for which $\zeta_t = \zeta_t^+ - \zeta_t^-$ is a minimal decomposition.

Replacing T by an etale open if necessary, we may assume that this map admits a section

$$\tilde{\sigma} : T \to [\mathcal{C}_j(\mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})/\overline{T})^{\times 2}_{\overline{T}}]_{hom}$$

sending $t \in T$ to a pair of j-cycles $\delta_t^\pm$ on $\mathcal{C}_{r-j}(Y_t)$ whose difference is homologically trivial.

As argued in [FM2;4.3], the Lefschetz theorem for singular varieties of [A-F] implies the existence for a given $t \in T$ of a $(2j+1)$-dimensional closed subvariety $E_t \subset \mathcal{C}_{r-j}(Y_t)$ such that $\delta_t = \delta_t^+ - \delta_t^-$ is supported on E_t and $[\delta_t] = 0 \in H_{2j}(E_t, \mathbf{Z})$. (We construct E_t by successively taking a hyperplane section of $\mathcal{C}_{r-j}(Y_t)$ which contains the singular locus of the previous hyperplane section as well as the support of δ_t.) We extend this to our relative context as follows. We apply the theorem on generic flatness to appropriate components of $\mathcal{C}_{r-j}(\mathcal{Y}/T)$ over T in order to successively choose a hyperplane section flat over T containing the singularities of the fibres over T of the previously defined hyperplane section as well as the support of $\tilde{\sigma}(T)$. We thus obtain (after replacing T by a possibly smaller Zariski open subset) a closed subvariety $\mathcal{E} \subset \mathcal{C}_{r-j}(\mathcal{Y}/T)$ which is flat of relative dimension $2j+1$ over T, whose fibres E_t support δ_t, and on which δ_t is homologically trivial.

We define the relative Chow correspondence

$$\overline{f}/\overline{T} : \overline{\mathcal{E}} \to \mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})$$

to be the closure of the embedding $\mathcal{E} \subset \mathcal{C}_{r-j}(\mathcal{Y}/T) \subset \mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})$. Moreover, we define

$$\sigma = (\sigma^+, \sigma^-, 1_T) : T \to \mathcal{C}_j(\mathcal{E}/T) \times_T \mathcal{C}_j(\mathcal{E}/T) \times_{\mathcal{C}_r(\overline{\mathcal{Y}}/\overline{T}) \times_T \mathcal{C}_r(\overline{\mathcal{Y}}/\overline{T})} T$$

to be the section induced by $\tilde{\sigma}$, so that $\delta_t = \sigma^+(t) - \sigma^-(t)$ is a homologically trivial j-cycle on for all $t \in T$. (We thus obtain the relative Chow correspondences $\overline{\sigma}^{\pm} : \overline{T}^{\pm} \to \mathcal{C}_j(\overline{\mathcal{E}}/\overline{T})$ by setting $\overline{T}^{\pm}$ to be the closures of the graphs of $\sigma^{\pm}$ in $\overline{T} \times \mathcal{C}_j(\mathcal{E}/T)$.)

By construction, $\Gamma_f(\delta_T)$ is an equidimensional $r+\tau$-cycle on $\mathcal{Y}$ with the property that its specialization to any $t \in T$ equals ζ_t. Thus, $\Gamma_f(\delta_T) = (\zeta \times T)_{|\mathcal{Y}}$. Moreover, in our construction we arranged that $[\delta_t] = 0 \in H_{2j}(E_t, \mathbf{Z})$. $\square$

Proof of Theorem 3.4

We consider $\zeta \in Z_{r+h}(X)$ such that $i_t^!(\zeta) \in S_{r-j}Z_r(Y_t)$ for almost all $s \in S$ (where $j < r$). Apply Proposition 5.1 to obtain $\delta_{\overline{T}} = \Gamma_{\overline{\sigma}^+}(\overline{T}) - \Gamma_{\overline{\sigma}^-}(\overline{T})$ in $Z_j(\overline{\mathcal{E}})$. By replacing $\delta_{\overline{T}}$ by a multiple if necessary, we may assume that $\delta_{\overline{T}} = p_*(\delta'_{\overline{T}})$ for some $\delta'_{\overline{T}} \in Z_j(\overline{\mathcal{E}}')$, where $p : \overline{\mathcal{E}}' \to \overline{\mathcal{E}}$ is a proper birational map with $\overline{\mathcal{E}}'$ smooth (i.e., a resolution of singularities of $\overline{\mathcal{E}}$). Let $\overline{\mathcal{E}}'' \to \overline{\mathcal{E}}' \times_{\overline{\mathcal{E}}} \overline{\mathcal{E}}'$ be a resolution of singularities and let $p_1, p_2 : \overline{\mathcal{E}}'' \to \overline{\mathcal{E}}'$ be the two projections. Denote by $\overline{f}'/\overline{T} : \overline{\mathcal{E}}' \to \mathcal{C}_{r-j}(\overline{\mathcal{Y}}/\overline{T})$ the Chow correspondence given by the composition $\overline{f}/\overline{T} \circ p'$.

Let $\mathcal{E}'', \mathcal{E}'$ denote the restrictions of $\overline{\mathcal{E}}'', \overline{\mathcal{E}}'$ above T. Since $p_{\mathcal{E}''} : \mathcal{E}'' \to T, p_{\mathcal{E}'} : \mathcal{E}' \to T$ are dominant morphisms of smooth varieties, we may replace T by a possibly smaller non-empty Zariski open with the additional property that $p_{\mathcal{E}''}, p_{\mathcal{E}'}$ are smooth (as well as $(\Gamma_f(\delta_T) = i_{T!}(\zeta \times T)$, and $[\delta_t] = 0$ for $t \in T$).

Observe that

$$\Phi_{f'}([\delta'_T]) = i_T^!([\zeta \times T]).$$

Let

$$\alpha' = [\delta'_T]^\wedge \in H^{2m-2j}(\overline{\mathcal{E}}', \mathbf{Q})$$

denote the Poincaré dual of $[\delta'_{\overline{T}}] \in H_{2j+2\tau}(\overline{\mathcal{E}}', \mathbf{Q})$. By Proposition 2.5,

$$\Phi_{f'}([\delta'_T])^\wedge = pr_{\mathcal{Y}!}(\langle f \rangle \cdot pr^*_{\overline{\mathcal{E}}'}(\alpha')) \in H^{2n-2r}(\mathcal{Y}, \mathbf{Q}).$$

Thus, by Proposition 4.7,

$$i_T^!([\zeta \times T])^\wedge = p_{\mathcal{Y}!}(\langle f/T \rangle \cdot p^*_{\mathcal{E}'}(\alpha')) \in H^{2n-2r}(\mathcal{Y}, \mathbf{Q})$$

where we have abused notation with α' also denoting the image in $H^{2j+2}(\mathcal{E}', \mathbf{Q})$ of $\alpha' \in H^{2j+2}(\overline{\mathcal{E}}', \mathbf{Q})$ and where $p_{\mathcal{E}} : \mathcal{E} \times_T \mathcal{Y} \to \mathcal{E}$.

We consider the following diagram

$$\begin{tikzcd}
H^s(\mathcal{E}',\mathbf{Q}) \otimes & H^u(\mathcal{E}' \times_T \mathcal{Y},\mathbf{Q}) \arrow[r, "\bullet"] & H^{s+u}(\mathcal{E}' \times_T \mathcal{Y},\mathbf{Q}) \arrow[r, "p_{\mathcal{Y}!}"] & H^{s+u-2m}(\mathcal{Y},\mathbf{Q}) \\
H^s(\mathcal{E}',\mathbf{Q}) \otimes \arrow[u, "="] \arrow[d, "\epsilon_t^*"'] & H^u(\mathcal{E}' \times_T \mathcal{X},\mathbf{Q}) \arrow[u, "i_T^*"'] \arrow[d, "(\epsilon_t\times\epsilon_t)^*"'] \arrow[r, "\bullet"] & H^{s+u}(\mathcal{E}' \times_T \mathcal{X},\mathbf{Q}) \arrow[u, "i_T^*"] \arrow[d, "(\epsilon_t\times\epsilon_t)^*"] \arrow[r, "p_{\mathcal{X}!}"] & H^{s+u-2m}(\mathcal{X},\mathbf{Q}) \arrow[u, "i_T^*"] \arrow[d, "\epsilon_t^*"] \\
H^s(E_t',\mathbf{Q}) \otimes & H^u(E_t' \times X,\mathbf{Q}) \arrow[d, "(p_i\times 1)^*"'] \arrow[r, "\bullet"] & H^{s+u}(E_t' \times X,\mathbf{Q}) \arrow[r, "p_{X!}"] & H^{s+u-2m}(X,\mathbf{Q}) \\
H^s(E_t'',\mathbf{Q}) \otimes \arrow[u, "p_{i!}"] & H^u(E_t'' \times X,\mathbf{Q}) \arrow[r, "\bullet"] & H^{s+u}(E_t'' \times X,\mathbf{Q}) \arrow[u, "p_{i\times 1!}"] \arrow[r, "p_{X!}"] & H^{s+u-2m}(X,\mathbf{Q}) \arrow[u, "="]
\end{tikzcd} \tag{5.2.1}$$

where $\epsilon_t : \{t\} \to T$, $i_T : \mathcal{E}' \times_T \mathcal{Y} \to \mathcal{E}' \times_T \mathcal{X}$, m equal to the relative dimension of $\mathcal{E} \to \mathcal{Y}$. The commutativity of the upper and middle squares of (5.2.1) are evident, whereas the "commutativity" of the lower squares for $i = 1, 2$ is a consequence of the formula $f_!(f^*\alpha \cdot \beta) = \alpha \cdot f_!(\beta)$ for the cohomology of smooth manifolds M, N related by a continuous map $f : M \to N$ (dual to the more familiar equality in homology $f_*(f^*(\alpha) \cap \beta^\wedge) = \alpha \cap f_*(\beta^\wedge)$).

We shall trace through this diagram with

$$\alpha' \in im\{H^{2m-2j}(\overline{\mathcal{E}}',\mathbf{Q}) \to H^{2m-2j}(\mathcal{E}',\mathbf{Q})\}, \quad \langle f'/T\rangle \in H^{2n-r+2j}(\mathcal{E}' \times_T \mathcal{Y},\mathbf{Q}),$$

so that $s = 2m-2j, u = 2n-2r+2j$. By Proposition 5.1, we may take $m = 2j+1$. Then we have the following values:

$$s = 2j+2,\ u = 2n-2r+2j,\ s+u = 2n-2r+4j+2,\ s+u-2m = 2n-2r.$$

By Theorem 1.3, the second and right-most upper vertical arrow of (5.2.1) are isomorphisms (assuming $j < r$). Thus, there exists (a unique) $\gamma \in H^u(\mathcal{E}' \times_T \mathcal{X},\mathbf{Q})$ restricting to $\langle f'/T\rangle \in H^u(\mathcal{E}' \times_T \mathcal{Y},\mathbf{Q})$. Moreover,

$$p_{\mathcal{X}!}(\gamma \cdot p_{\mathcal{E}}^*(\alpha')) = pr^*([\zeta]^\wedge) \in H^{s+u-2m}(\mathcal{X},\mathbf{Q}), \tag{5.2.2}$$

since i_T^* (the right-most upper vertical arrow of (5.3.1)) is an isomorphism. Since $\langle f'\rangle$ is the restriction of $\langle f\rangle \in H^u(\mathcal{E}\times\mathcal{Y},\mathbf{Q})$ and since $H^u(\mathcal{E}'' \times_T \mathcal{Y},\mathbf{Q}) \simeq H^u(\mathcal{E}'' \times_T \mathcal{X},\mathbf{Q})$ by another application of Theorem 1.3, we conclude that

$$(p_1 \times 1)^*(\gamma) = (p_2 \times 1)^*(\gamma) \in H^u(\mathcal{E}'' \times_T \mathcal{Y},\mathbf{Q}). \tag{5.2.3}$$

Let $\alpha_t' = \epsilon_t^*(\alpha') \in H^{2j+2}(E_t',\mathbf{Q}), \gamma_t = (\epsilon_t \times \epsilon_t)^*(\gamma) \in H^u(E_t' \times X,\mathbf{Q})$. By (5.2.2) and the commutativity of the middle squares of (5.2.1), we have the equality

$$pr_{X!}(\gamma_t \cdot pr_{E_t'}^*(\alpha_t')) = [\zeta]^\wedge \in H^{s+u-2m}(X,\mathbf{Q}). \tag{5.2.4}$$

Recall that a theorem of P. Deligne [De] asserts the exactness of

$$H_*(E_t,\mathbf{Q}) \xleftarrow{p_*} H_*(\mathcal{E}_t',\mathbf{Q}) \xleftarrow{p_{1*}-p_{2*}} H_*(E_t'',\mathbf{Q}).$$

Since $p_*([\delta'_t]) = [\delta_t] = 0$, we may find $\beta_t \in H_{2j}(E''_t, \mathbf{Q})$ with the property that

$$p_{1*}(\beta_t) - p_{2*}(\beta_t) = [\delta'_t] \in H_{2j}(E''_t, \mathbf{Q}).$$

Stated in terms of cohomology, we may find $\alpha''_t \in H^{2j+2}(E''_t, \mathbf{Q})$ such that

$$p_{1!}(\alpha''_t) - p_{2!}(\alpha''_t) = \alpha'_t \in H^{2j+2}(E'_t, \mathbf{Q}).$$

The "commutativity" of the the bottom squares of (5.2.1) together with (5.2.3) and (5.2.4) now implies the required vanishing:

$$[\zeta]^\wedge = p_{X!}(\gamma_t \bullet p^*_{E'_t}(p_{1!}\alpha''_t - p_{2!}\alpha''_t)) = p_{X!}((p_1 \times 1)^* \gamma'_t \bullet p^*_{E''_t}\alpha''_t - (p_2 \times 1)^* \gamma'_t \bullet p^*_{E''_t}\alpha''_t) = 0.$$

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