Algebraic Cocycles on Normal, Quasi-Projective Varieties

Eric M. Friedlander*

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Introduction

Blaine Lawson and the author introduced algebraic cocycles on complex algebraic varieties in [FL-1] and established a duality theorem relating spaces of algebraic cocycles and spaces of algebraic cycles in [FL-2]. This theorem has non-trivial (and perhaps surprising) applications in several contexts. In particular, duality enables computations of "algebraic mapping spaces" consisting of algebraic morphisms. Moreover, duality appears to be an important property in motivic cohomology/homology (cf. [F-V]).

In this paper, we extend the theory of [FL-1], [FL-2] to quasi-projective varieties. (Indeed, our duality theorem is an assertion of a natural homotopy equivalence from cocycle spaces to cycle spaces and thus is a refinement of the duality theorem of [FL-2] when specialized to projective varieties.) One can view this work as developing an algebraic bivariant theory for complex quasi-projective varieties which is closely based on algebraic cycles. On the other hand, one can also view the resulting spaces of algebraic cocycles as function complexes equipped with a natural topology. Thus, the theory of cycle spaces, cocycle spaces, and duality has both a formal role in providing invariants for algebraic varieties (closely related to classical invariants and problems as seen in [F-2]) and a more explicit role in the analysis of heretofore inaccessible function complexes.

Our consideration of quasi-projective varieties enables computations as exemplified in §7. Many local calculations, useful even for projective varieties, should now be accessible. Other applications of this theory in the quasi-projective context can be found in §6.

Duality for cocycle and cycle spaces should be viewed as a somewhat sophisticated generalization of the comparison of Cartier and Weil divisors on a (smooth) variety. From this point of view, one does indeed expect that the theory developed for projective varieties to extend to quasi-projective varieties. The essential difficulty in providing such an

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extension is the formulation of a suitable definition of the topological monoid $\mathfrak{C}_r(Y)(U)$ of effective cocycles on a normal, quasi-projective variety U with values in a projective variety Y. On the one hand, these cocycles should be related by a "duality map" to cycles on the product $U \times Y$; again, one wants the group completion of the space of effective cocycles to provide "sensible" homotopy groups; further, one requires that this space be contravariant with respect to U, covariant with respect to Y. Indeed, one would like that the definition be algebraic in nature. As the reader will see, $\mathfrak{C}_r(Y)(U)$ and its "naïve group completion" $\mathcal{Z}_r(Y)(U)$ do meet our criteria for a useful working definition.

The defining property of the topological monoid $\mathfrak{C}_r(Y)(U)$ of effective cocycles on a normal variety U is that this be the quotient of the monoid of effective cycles on $X \times Y$ equidimensional over U modulo cycles on $X_\infty \times Y$, where $U \subset X$ is a projective closure with complement X_∞. The formalism of tractable monoids (introduced by O. Gabber and the author in [F-G]) enables us to work with this monoid and its group completion. For quasi-projective range V, we provide a definition of cocycles with values in a pair (Y, Y_∞), where $Y_\infty \subset Y$ are projective and $V = Y - Y_\infty$. Of primary interest is the case $V = \mathbf{A}^n$ with evident compactification $\mathbf{P}^n$. As a consequence of our duality theorem (Theorem 5.4), we conclude that the space of cocycles on U with values in the pair (Y, Y_∞) has homotopy type depending only upon U and $V = Y - Y_\infty$ provided that U, Y, Y_∞ satisfy certain smoothness hypotheses. To complete the formalism of cycle spaces, we also introduce the space of cocycles on U with support in $U_0 \subset U$.

Basic properties proved in §3 include the fundamental ones of covariant functoriality with respect to Y and contravariant functoriality with respect to U. Homotopy invariance with respect to bundle projections and a projective bundle theorem are also proved. As recognized in [FL-1], the cocycle analogue of Lawson's "algebraic suspension theorem" of [L] is valid. Mayer-Vietoris sequences are established which are useful for calculations. Moreover, homotopy groups of spaces of algebraic cocycles on U naturally map to cohomology groups of U, one of the original motivating aspects of algebraic cocycles in [FL-1].

The duality map $\mathcal{D} : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_{r+m}(U \times Y)$, $m = dim U$, is the map on naïve group completions induced by the natural inclusion of effective cocycles into effective cycles. We show that this duality map enjoys all the good properties established in [FL-2] in the case of projective varieties. Our proof of duality (i.e., that $\mathcal{D}$ is a homotopy equivalence under appropriate hypotheses of smoothness) in §5 follows along the lines of [FL-2]; in particular, the essential ingredient of the proof of duality is the "Moving Lemma for Cycles of Bounded Degree" established by the author and Blaine Lawson in [FL-3].

We anticipate many applications of duality both for projective and quasi-projective varieties, a few of which were presented in [FL-2]. In this paper, we provide evident extensions of those results to quasi-projective varieties as well as obtain results not heretofore proven even for projective varieties. For example, we extend the construction of Chern classes given in [FL-1] to algebraic vector bundles not necessarily generated by their global sections (cf. Remark 6.4.). The families of examples presented in §7 are a first sampler of computations of non-trivial homotopy groups of the topological monoids $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$.

Throughout, X and Y will denote reduced schemes proper over the complex field $\mathbf{C}$ of pure dimension m and n respectively which admit a (Zariski) closed embedding in some projective space. We shall refer to such schemes X and Y as projective varieties of dimension m and n respectively. We shall consider consider arbitrary Zariski closed subvarieties $X_\infty \subset X$ and $Y_\infty \subset Y$; thus, X_∞ and Y_∞ are reduced but not necessarily irreducible closed subschemes of X and Y respectively. We denote by $U \subset X$ and $V \subset Y$ the Zariski open complements of X_∞ and Y_∞. We shall let r, t denote non-negative integers with $r \leq n = dimY$, $t \leq m = dimX$. We shall usually view locally closed algebraic subsets of projective spaces with their analytic topology and state explicitly when subsets are to be viewed as open or closed in the Zariski topology.

We recall that an (effective) algebraic r-cycle on a variety Y is a (non-negative) integral combination of irreducible subvarieties of Y each of dimension r. If $Z = \sum_i n_i V_i$ is such a cycle, its support $|Z|$ is the Zariski closed subset $\cup_i V_i \subset Y$. Our study involves the consideration of Chow varieties (cf. [S]). In particular, we shall consider various topological submonoids of the Chow monoid

$$\mathfrak{C}_{r+m}(X \times Y) \stackrel{\text{def}}{=} \coprod_d C_{r+m,d}(X \times Y)$$

where $C_{r+m,d}(X \times Y)$ is a (Zariski) closed algebraic subset of an appropriate complex projective space whose points correspond naturally to effective $r+m$-cycles on $X \times Y$ of degree d (with respect to some unspecified embedding of $X \times Y$ in a projective space).

The author is especially indebted to H. Blaine Lawson, for this paper is an extension of earlier collaborative work. On the other hand, the formal and algebraic nature of this work is such that the author alone should be held responsible for the specific results discussed below. We also thank the referee who suggested that we prove that the cycle and cocycle spaces we consider are C.W. complexes, thereby refining our results to be assertions about homotopy type rather than weak homotopy type.

1. Cocycles on Normal Varieties.

We begin by introducing the monoids which occur in our definition of the cocycle space $\mathfrak{C}_r(Y)(U)$. We then summarize the key properties of tractable monoids and observe their applicability in our context. The new property we verify is that the spaces we consider admit the structure of C. W. complexes. We conclude this section by defining $\mathfrak{C}_r(Y)(U)$ and identifying its topology.

Consider the incidence correspondence

$$\mathcal{I} \subset \mathcal{C}_{r+m}(X \times Y) \times (X \times Y)$$

consisting of triples (Z, x, y) such that $(x, y) \in X \times Y$ lies in the support $|Z|$ of Z. Consider the composition of this closed embedding and the projection $\mathfrak{C}_{r+m}(X \times Y) \times (X \times Y) \to \mathfrak{C}_{r+m}(X \times Y) \times X$,

$$p : \mathcal{I} \to \mathfrak{C}_{r+m}(X \times Y) \times X,$$

and let

$$p_U : \mathcal{I}_U \to \mathcal{C}_{r+m}(X \times Y) \times U$$

denote the restriction of p_U above U. We denote by $\mathcal{I}_{\mathcal{Z},u}$ the fibre of p_U above $(\mathcal{Z}, u)$. By upper semi-continuity of dimension of the fibres of p_U,

$$\mathcal{W} \stackrel{\text{def}}{=} \{(Z, u) : dim\mathcal{I}_u \geq r+1\} \subset \mathcal{C}_r(X \times Y) \times U$$

is a Zariski closed subset of $\mathfrak{C}_r(X \times Y) \times U$. Let $\pi : \mathcal{C}_{r+m}(X \times Y) \times U \to \mathcal{C}_{r+m}(X \times Y)$ denote the projection and let $(-)^c$ denote the operation of taking complements.

Recall that a subset S of an algebraic variety V is said to be constructible if it is a finite union of subsets each of which is locally closed in V with respect to the Zariski topology. If $S \subset V$ is a constructible subset of a variety V, then the inclusion $S' \subset S$ of subset of S is said to be a constructible embedding if S' is also a constructible subset of V.

Definition 1.1. With notation as above, we define

$$\mathcal{E}_r(Y)(U) \quad \stackrel{\text{def}}{=} \quad \pi(\mathcal{W})^c \quad \subset \quad \mathfrak{C}_{r+m}(X \times Y), \tag{1.1.1}$$

to be the topological submonoid consisting of those effective $r+m$-cycles on $X \times Y$ whose restrictions to $U \times Y$ are equidimensional over U of relative dimension r. The embedding of (1.1.1) is **constructible**, in the sense that it is a disjoint union of constructible embeddings.

Moreover, the embedding $\mathfrak{C}_{r+m}(X_\infty \times Y) \subset \mathfrak{C}_{r+m}(X \times Y)$ factors through an embedding

$$\mathfrak{C}_{r+m}(X_\infty \times Y) \subset \mathcal{E}_r(Y)(U) \tag{1.1.2}$$

which is Zariski closed (in the sense that it is a disjoint union of Zariski closed embeddings of algebraic varieties).

Warning. As used above (and throughout this paper), the terminology of an effective cycle Z on $U \times Y$ equidimensional of relative dimension r over U refers to a cycle whose fibres above points of U are either empty or of pure dimension r. In particular, if U is reducible, then such a cycle need not dominate U even if it does not lie in $\mathfrak{C}_{r+m}(X_\infty \times Y)$.

We recall that the **naïve group completion** M^+ of an abelian topological monoid M with the cancellation property is the quotient (with the quotient topology) of $M \times M$ by the equivalence relation consisting of pairs $(m_1, m_2), (n_1, n_2)$ with the property that $m_1 + n_2 = m_2 + n_1$. In general, the relationship between the algebraic invariants of M and M^+ is obscure at best. Moreover, even if M is algebro-geometric (e.g., the Chow monoid $\mathfrak{C}_{r+m}(X \times Y)$), M^+ appears to have no such geometric structure.

Nonetheless, in our context of Chow monoids this construction of naïve group completion turns out to be quite reasonable. As formalized by O. Gabber and the author in [F-G], a tractable monoid M has the property that M^+ is obtained by successive push-out diagrams which enables one to identify the homotopy type of M^+ and view it in some sense as algebro-geometric provided that M itself is algebro-geometric.

With the example of $\mathcal{E}_r(Y)(U)$ in mind, we now introduce the formalism of tractability.

Definition 1.2. The action of an abelian topological monoid on a topological space T is said to be **tractable** if T is the topological union of inclusions

$$\emptyset = T_{-1} \subset T_0 \subset T_1 \subset \dots$$

such that for each $n > 0$ $T_{n-1} \subset T_n$ fits into a push-out square of M-equivariant maps (with R_0 empty)

$$\begin{array}{ccc} R_n \times M & \rightarrow & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \rightarrow & T_n \end{array} \tag{1.2.1}$$

whose upper horizontal arrow is induced by a cofibration $R_n \subset S_n$ of Hausdorff spaces. The monoid M itself is said to be **tractable** if the diagonal action of M on $M \times M$ is tractable.

Lemma 1.3. *Let T be a tractable space for the abelian topological monoid M. If T has a presentation as in Definition 1.2 with each $R_n \subset S_n$ a relative C.W. complex, then T/M admits the structure of a C.W. complex.*

Proof. Since T_n/M fits in the push-out square

$$\begin{array}{ccc} R_n & \subset & S_n \\ \downarrow & & \downarrow \\ T_{n-1}/M & \rightarrow & T_n/M \end{array} \tag{1.3.1}$$

we conclude that $T_{n-1}/M \rightarrow T_n/M$ is a relative C.W. complex and thus an induction argument immediately implies that T_n/M is a C.W. complex. Consequently, $colim_n(T_n/M)$ is also a C.W. complex.

We conclude that it suffices to verify that the natural continuous bijection

$$colim_n(T_n/M) \rightarrow (colim_n T_n)/M = T/M$$

is a homeomorphism. This follows from the observation that $Y \subset T/M$ is closed iff $\pi^{-1}Y \subset T$ is closed iff $\pi^{-1}Y \cap T_n \subset T_n$ is closed for each n iff $\pi_n^{-1}Y_n \subset T_n$ is closed for each n iff $Y_n \subset T_n/M$ is closed for each n iff $Y \subset colim_n(T_n/M)$ is closed (where $\pi : Y \rightarrow Y/M, \pi_n : Y_n \rightarrow Y_n/M$ are the projections and where Y_n equals $Y \cap T_n/M$.) $\square$

The importance for us of the existence of the structure of a C.W. complex on a cycle space is the following well known fact (cf. [Sp;7.6.24]).

Recollection 1.4. *Let $f : A \rightarrow B$ be a weak homotopy equivalence between spaces A, B having the homotopy type of C.W. complexes. Then f is a homotopy equivalence.*

If M is a topological monoid, then we denote by $B[M]$ its classifying space and by $\Omega B[M]$ the loop space on this classifying space. We recall that $\Omega B[M]$, which we call the homotopy-theoretic group completion, is the usual group completion considered by topologists (cf. [F-M;appQ]).

The following theorem summarizes the topological consequences of tractability that we shall require.

Theorem 1.5. *Assume that $\mathcal{E}_r \subset \mathfrak{C}_{r+m}(X \times Y) \stackrel{\text{def}}{=} \mathfrak{C}_r$ is a topological submonoid of $\mathfrak{C}_r$ whose embedding is constructible.*
(a.) $\mathcal{E}_r$ is a tractable monoid which admits the structure of a C.W. complex.
(b.) The natural homotopy class of maps of H-spaces

$$\Omega \circ B[\mathcal{E}_r] \quad \to \quad [\mathcal{E}_r]^+$$

is a weak homotopy equivalence.
(c.) If $\mathcal{F}_r \subset \mathcal{E}_r$ is a Zariski closed submonoid, then $\mathcal{E}_r$ is tractable as a $\mathcal{F}_r$-space and the quotient monoid (with the quotient topology) $\mathcal{E}_r/\mathcal{F}_r$ is also a tractable monoid admitting the structure of a C.W. complex.
(d.) For $\mathcal{F}_r \subset \mathcal{E}_r$ as in (c.), the following is a fibration sequence (i.e., induces a long exact sequence in homotopy groups) of spaces each of which admits the structure of a C.W. complex

$$[\mathcal{F}_r]^+ \to [\mathcal{E}_r]^+ \to [\mathcal{E}_r/\mathcal{F}_r]^+.$$

Proof. The tractability of $\mathcal{E}_r$ in (a.) is verified in [FL-2;T.3] (which is itself merely a modification of [FG;1.3]). The fact that $\mathcal{E}_r$ admits the structure of a C.W. complex is an immediate consequence of the triangulation of semi-algebraic sets as proved in [H-2]. The weak homotopy equivalence of (b.) is established in [FL-2;T.4]. The tractability properties asserted in part (c.) are also proved in [FL-2;T.3]; the fact that $\mathcal{E}_r/\mathcal{F}_r$ admits the structure of a C.W. complex follows from Lemma 1.3 and the observation that each $R_n \subset S_n$ in the presentation of $\mathcal{E}_r$ as a $\mathcal{F}_r$ space is a Zariski closed embedding of constructible spaces and thereby admits the structure of a relative C.W. complex. This latter fact is a consequence of the following result proved in the appendix of [Fl]: if $\overline{R}_n \subset \overline{S}_n$, $\overline{S}_n - S_n \subset \overline{S}_n$ are simplicial embeddings of finite polyhedra and if $R_n = \overline{R}_n \cap S_n$, then $R_n \subset S_n$ is a polyhedral pair. The fact that the sequence in part (d.) is a fibration sequence is established in the proof of [F-G;1.6]. To verify that each of the spaces occurring in this sequence admits the structure of a C.W. complex, we let M denote any of the tractable monoids $\mathcal{F}_r, \mathcal{E}_r, \mathcal{E}_r/\mathcal{F}_r$ and apply Lemma 1.3 to the tractable M space $M^{\times 2}$ as in the proof of part (c.). □

In the following definition of effective cocycles, we assume that the quasi-projective variety U is normal. Indeed, the same definition could be given for any quasi-projective variety U and Corollary 1.7 and Proposition 1.8 would remain valid without the hypothesis of normality on U. On the other hand, normality is needed for Proposition 1.9 and (more importantly) to establish functoriality in Proposition 3.3.

Definition 1.6. Let U be a normal, quasi-projective variety. We define the monoid of **effective cocycles** on U equidimensional of relative dimension r in Y to be the following quotient monoid (with the quotient topology)

$$\mathfrak{C}_r(Y)(U) \quad \stackrel{\text{def}}{=} \quad \mathcal{E}_r(Y)(U)/\mathfrak{C}_{r+m}(X_\infty \times Y). \tag{1.6.1}$$

We define the topological abelian group of cocycles on U equidimensional of relative dimension r in Y to be the naïve group completion of $\mathfrak{C}_r(Y)(U)$,

$$\mathcal{Z}_r(Y)(U) \quad \stackrel{\text{def}}{=} \quad [\mathfrak{C}_r(Y)(U)]^+. \tag{1.6.2}$$

Theorem 1.5 immediately provides the following corollary which shows that Definition 1.6 agrees with the definitions of [FL-1] and [FL-2] for the special case in which U equals the projective variety X.

Corollary 1.7. *As in (1.4), let U be a normal, quasi-projective variety. The natural map $\mathfrak{C}_r(Y)(U) \to \Omega \circ B\mathfrak{C}_r(Y)(U)$ determines a weak homotopy equivalence*

$$\Omega \circ B[\mathfrak{C}_r(Y)(U)] \quad \to \quad \mathcal{Z}_r(Y)(U). \qquad (1.7.1).$$

Moreover, the following sequence of topological abelian groups is a fibration sequence:

$$\mathcal{Z}_{r+m}(X_\infty \times Y) \quad \to \quad [\mathcal{E}_r(Y)(U)]^+ \quad \to \quad \mathcal{Z}_r(Y)(U). \tag{1.7.2}$$

We recall (cf. [LF-2], [F-G]) that the monoid of effective cocycles on $U \times Y$ is defined as the quotient (with the quotient topology)

$$\mathfrak{C}_{r+m}(U \times Y) \quad \stackrel{\text{def}}{=} \quad \mathfrak{C}_{r+m}(X \times Y)/\mathfrak{C}_r(X_\infty \times Y).$$

Proposition 1.8. *Let U be a normal, quasi-projective variety. The topological embedding (1.1.1) induces a topological embedding*

$$\mathfrak{C}_r(Y)(U) \quad \subset \quad \mathfrak{C}_{r+m}(U \times Y). \tag{1.8.1}$$

In particular, the homeomorphism type of $\mathfrak{C}_r(Y)(U)$ is independent of the choice of projective closure $U \subset X$.

Proof. Since $\mathcal{E}_r(Y)(X) \subset \mathfrak{C}_{r+m}(X \times Y)$ is saturated for $\mathfrak{C}_{r+m}(X_\infty \times Y)$ and since $\mathfrak{C}_{r+m}(X_\infty \times Y)$ is closed in $\mathfrak{C}_{r+m}(X \times Y)$ and thus also in $\mathcal{E}_r(Y)(X)$, we conclude easily that (1.8.1) is a topological embedding.

The second assertion follows from the fact that the homeomorphism type of $\mathfrak{C}_{r+m}(U \times Y)$ is independent of projective closure $U \subset X$ as shown in [LF] and [F-G]. $\square$

In [FL-1], Lawson and the author considered "continuous algebraic maps" (i.e., morphisms from the weak normalization of the domain) from a quasi-projective variety U to the Chow monoid $\mathfrak{C}_r(Y)$. Since we restrict our attention to normal varieties U, such maps are always morphisms. We recall from [F-1] that any morphism $f : U \to \mathfrak{C}_r(Y)$ admits a "graph" $Z_f \subset U \times Y$ and that $Z_f \neq Z_g$ whenever $f \neq g$. (The graph Z_f can be defined as the Zariski closure in $U \times Y$ of the effective cycle on Y_{SpecK} associated to the restriction of f to the generic point $SpecK \in U$.) Moreover, sending a morphism f to its graph is a bijection whenever U is normal: this is shown in [FL-1;1.5], where proof of bijectivity is local and thus applies to quasi-projective U.

The following characterization of the topology on $\mathfrak{C}_r(Y)(U)$ is verified in Appendix C of [FL-2].

Proposition 1.9. [FL-2;C.3] *Let $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ denote the abelian monoid of morphisms from a normal, quasi-projective variety U to $\mathfrak{C}_r(Y)$. Thus, the graphing construction*

$$\mathcal{G} : \mathfrak{Mor}(U, \mathfrak{C}_r(Y)) \stackrel{\cong}{\to} \mathfrak{C}_r(Y)(U)$$

is an isomorphism. Then the topology on $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ inherited from that on $\mathfrak{C}_r(Y)(U)$ via $\mathcal{G}$ is characterized by the following property: a sequence $\{f_n; n \in \mathbf{N}\} \subset \mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ converges for this topology if and only if

(i.) $\{f_n; n \in \mathbf{N}\}$ converges when viewed in $Hom_{cont}(U, \mathfrak{C}_r(Y))$ provided with the compact-open topology.

(ii.) The associated sequence $\{Z_n; n \in \mathbf{N}\} \subset \mathfrak{C}_r(Y)(U)$ of graphs has the property that for some locally closed Zariski embedding $U \times Y \subset \mathbf{P}^N$, there is a positive integer E such that each Z_n has closure $\bar{Z}_n \subset \mathbf{P}^N$ of degree $\leq E$.

We call this topology on $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ inherited from that of $\mathfrak{C}_r(Y)(U)$ the **topology of convergence with bounded degree**.

Remark 1.10. There are several other definitions of the space of cocycles on U with values in Y which come readily to mind but which appear to us to be less useful.

(a.) If one considered the monoid of effective cocycles to be $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ for U not normal, then it would be difficult to identify this as an accessible submonoid of $\mathfrak{C}_{r+m}(U \times Y)$. Even for U normal, if one were to define the monoid of effective cocycles to be $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ with the compact-open topology (as suggested in [FL-1]), Proposition 1.9 tells us that we would fail to have a continuous map to $\mathfrak{C}_{r_m}(U \times Y)$ which we require for duality.

(b.) Another possible approach is to define effective cocycles on U with values in Y as a quotient of effective cocycles on X with values on Y, for the latter is well understood thanks to [FL-2]. This definition has the strong disadvantage that it depends (even as a discrete monoid) upon the choice of projective closure $U \subset X$.

(c.) A third alternative is to retain our definition of effective cocycle on U with values in Y but to define the topological abelian group of all cocycles on U with values in Y as the subgroup of $\mathcal{Z}_{r+m}(U \times Y)$ generated by $\mathfrak{C}_r(Y)(U)$. One has a continuous bijection from $\mathcal{Z}_r(Y)(U)$ to this subgroup, but this bijection is not a homeomorphism.

This formulation suffers from the fact that its algebraic invariants have no evident relationship to those of $\mathfrak{C}_r(Y)(U)$.

2. Relative Cocycles and Cocycles with Support.

In this section, we define the topological abelian group of algebraic cocycles on U with values in a pair (Y, Y_∞). This is to be viewed as our approximation of a suitable definition of cocycles on U with values in the quasi-projective variety $V = Y - Y_\infty$. A special case of particular interest is the pair $(\mathbf{P}^t, \mathbf{P}^{t-1})$. To complete the formalism, we also define cocycles with support.

We remind the reader that Y is assumed to be projective (i.e., to admit a Zariski closed embedding in some projective space) and $Y_\infty \subset Y$ is a Zariski closed embedding. Throughout this section, the quasi-projective variety U is assumed to be normal.

Definition 2.1. We define the topological submonoid $\mathcal{F}_r(Y_\infty)(U)$ by

$$\mathcal{F}_r(Y_\infty)(U) \overset{\text{def}}{=} \mathcal{E}_r(Y_\infty)(U) + \mathfrak{C}_{r+m}(X_\infty \times Y) \subset \mathcal{E}_r(Y)(U).$$

We define the topological monoid $\mathfrak{C}_r(Y \backslash Y_\infty)(U)$ of effective algebraic cocycles on U equidimensional of relative dimension r in (Y, Y_∞) by

$$\mathfrak{C}_r(Y \backslash Y_\infty)(U) \overset{\text{def}}{=} \mathcal{E}_r(Y)(U)/\mathcal{F}_r(Y_\infty)(U). \tag{2.1.1}$$

We define the topological abelian group $\mathcal{Z}_r(Y \backslash Y_\infty)(U)$ to be the naïve group completion of $\mathfrak{C}_r(Y \backslash Y_\infty)(U)$,

$$\mathcal{Z}_r(Y \backslash Y_\infty)(U) \overset{\text{def}}{=} [\mathfrak{C}_r(Y \backslash Y_\infty)(U)]^+. \tag{2.1.2}$$

Theorem 1.5 easily implies the following properties of our definition of relative cocycles.

Proposition 2.2. *The topological monoid $\mathfrak{C}_r(Y \backslash Y_\infty)(U)$ of (2.1.1) is also given as the following quotient*

$$\mathfrak{C}_r(Y \backslash Y_\infty)(U) = \mathfrak{C}_r(Y)(U)/\mathfrak{C}_r(Y_\infty)(U).$$

The natural map

$$\Omega \circ B[\mathfrak{C}_r(Y \backslash Y_\infty)(U)] \rightarrow \mathcal{Z}_r(Y \backslash Y_\infty)(U)$$

is a weak homotopy equivalence.

Furthermore, the following **localization sequence** *of topological abelian groups*

$$\mathcal{Z}_r(Y_\infty)(U) \rightarrow \mathcal{Z}_r(Y)(U) \rightarrow \mathcal{Z}_r(Y \backslash Y_\infty)(U)$$

is a fibration sequence.

Proof. The equality

$$\mathcal{E}_r(Y)(U)/\mathcal{F}_r(Y_\infty)(U) = \mathfrak{C}_r(Y)(U)/\mathfrak{C}_r(Y_\infty)(U)$$

is verified by inspection.

Since addition in $\mathcal{E}_r(Y)(U)$ is a proper map (the restriction of addition in $\mathfrak{C}_{r+m}(X \times Y)$) and since both $\mathcal{E}_r(Y_\infty)(U) \subset \mathcal{E}_r(Y)(U)$ and $\mathfrak{C}_{r+m}(X_\infty \times Y)$ are Zariski closed, we conclude that $\mathcal{F}_r(Y_\infty)(U) \subset \mathcal{E}_r(Y)(U)$ is also Zariski closed. Thus, the fact that

$$\mathcal{Z}_r(Y\backslash Y_\infty)(U) \to \Omega \circ B[\mathfrak{C}_r(Y\backslash Y_\infty)(U)]$$

is a weak homotopy equivalence follows from Theorem 1.5.

Consider the following diagram

$$\begin{array}{ccccc}
[\mathfrak{C}_{r+m}(X_\infty \times Y)]^+ & \to & [\mathcal{F}_r(Y_\infty)(U)]^+ & \to & \mathcal{Z}_r(Y_\infty)(U) \\
= \downarrow & & \downarrow & & \downarrow \\
[\mathfrak{C}_{r+m}(X_\infty \times Y)]^+ & \to & [\mathcal{E}_r(Y)(U)]^+ & \to & \mathcal{Z}_r(Y)(U) \\
\downarrow & & \downarrow & & \downarrow \\
* & \to & \mathcal{Z}_r(Y\backslash Y_\infty)(U) & \to & \mathcal{Q}
\end{array}$$

where $\mathcal{Q}$ is the homotopy fibre of $B[Z_r(Y_\infty)(U)] \to B[Z_r(Y)(U)]$, the delooping of the upper right vertical map. Each of the columns and the upper two rows of this diagram are fibration sequences. We conclude by the "3×3 Lemma" that $Z_r(Y\backslash Y_\infty)(U) \to \mathcal{Q}$ is a homotopy equivalence, thereby implying the asserted fibration sequence. $\square$

Remark 2.3. One could define $\mathfrak{C}_r(V)(U)$ as the topological submonoid of $\mathfrak{C}_{r+m}(U \times V)$ consisting of equidimensional cycles, thereby giving a definition which depends only upon U, V, but not upon the projective closure Y. However, such a definition would not appear to have good properties (e.g., functoriality, comparison with the naïve group completion and homotopy-theoretic group completion, fibration sequences), in view of the fact that the natural map

$$\mathfrak{C}_r(Y\backslash Y_\infty)(U) \ \to \ \mathfrak{C}_{r+m}(U \times V) \quad , V = Y - Y_\infty$$

is a continuous monomorphism but not necessarily a topological embedding.

For example, we can take $X = \mathbf{P}^1 = U$ and Y_∞ to be some point $\infty \in \mathbf{P}^1$. Consider the cycles Z_n in $\mathbf{P}^1 \times \mathbf{P}^1$ given by the equations $y = x/n$ in the affine chart $\mathbf{A}^1 \times \mathbf{A}^1$. Then the sequence $\{W_n = Z_{n+1} - Z_n\}$ converges in $\mathcal{Z}_0(\mathbf{A}^1)(\mathbf{P}^1)$ to the graph of the function which is everywhere 0 on $\mathbf{P}^1$. On the other hand, the sequence $\{W_n\}$ does not converge in

$$\mathcal{Z}_0(\mathbf{P}^1\backslash\{0\})(\mathbf{P}^1) = [\mathfrak{C}_0(\mathbf{P}^1)(\mathbf{P}^1)/\mathfrak{C}_0(\mathbf{P}^1)(\{0\})]^+$$

since each of the sequences $\{Z_n\}, \{Z_{n+1}\}$ converges to $\mathbf{P}^1 \times \{0\} + \{\infty\} \times \mathbf{P}^1$ which does not lie in $\mathfrak{C}_0(\mathbf{P}^1)(\mathbf{P}^1)$.

Proposition 2.2 implies that the following definition of $\mathcal{Z}^t(U)$ generalizes to quasi-projective U the definition of algebraic cocycle spaces given in [FL1], [FL-2]. As we shall see, these spaces for U smooth are homotopy equivalent to corresponding cycle spaces. This is one justification of our view of $\mathcal{Z}^t(U)$ as the contravariant aspect of the bivariant $\mathcal{Z}_r(Y)(U)$.

Recall that t is a non-negative integer $\leq dimX = dimU$.

Definition 2.4. We define the topological abelian group of t-cocycles on X to be

$$\mathcal{Z}^t(U) \quad \stackrel{\text{def}}{=} \quad \mathcal{Z}_0(\mathbf{P}^t \backslash \mathbf{P}^{t-1})(U).$$

For the sake of completeness, we introduce the following definition of the space of cocycles on U with supports in a closed subvariety $U_0 \subset U$. Taking homotopy groups gives a definition of cohomology groups with support, leading to a theory satisfying most of the properties of a "Bloch-Ogus" theory (cf. [B-O]). Indeed, the one property that we lack is a version of Poincaré duality which involves a pairing of cycle spaces (rather than our formulation of duality given in §5).

Definition 2.5. Let $U_0 \subset U$ be a closed subvariety of the quasi-projective variety U. Then we define the topological abelian group of codimension t cocycles on U supported on U_0 by

$$Z^t_{U_0}(U) \quad \equiv \quad htyfib\{Z^t(U) \to Z^t(U - U_0)\}.$$

Remark 2.6. Although we shall rarely explicitly discuss $Z^t_{U_0}(U)$, our duality theorems provide some understanding of these spaces. For example, Proposition 6.1 in conjunction with Remark 4.4 implies the existence of a homotopy equivalence

$$Z^t_{U_0}(U) \quad \cong \quad Z^t(U_0)$$

provided that $U_0 \subset U$ is a closed immersion of *smooth* varieties. More generally, if we know only that U is smooth, then Theorem 5.2 implies that the duality map induces a homotopy equivalence

$$Z^t_{U_0}(U) \quad \stackrel{\cong}{\to} \quad Z_{m-t}(U_0).$$

One can view this homotopy equivalence as a form of Alexander Duality.

3. Basic Properties.

In this section, we prove some basic properties of our topological abelian groups of algebraic cocycles. The primary one is functoriality. Another is the existence of a natural map to singular cohomology. We also show that the algebraic suspension theorem, first proved for cycle spaces by Lawson in [L] and subsequently for cocycle spaces in the context of projective varieties in [FL-1], remains valid for cocycles on a quasi-projective variety U. We conclude this section with a Mayer-Vietoris fibration sequence. Once again, throughout this section U will denote a normal quasi-projective variety of dimension m embedded as a Zariski open subset in the projective variety X.

Proposition 3.1. *Let $f : Y \to Y'$ be a morphism of projective varieties. Then proper push-forward of cycles via $1 \times f : U \times Y \to U \times Y'$ determines continuous homomorphisms*

$$f_* : \mathfrak{C}_r(Y)(U) \to \mathfrak{C}_r(Y')(U) \ , \ f_* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y')(U).$$

Moreover, if f restricts to $Y_\infty \to Y'_\infty$, then push forward of cycles determines continuous homomorphisms on relative cocycle spaces

$$f_* : \mathfrak{C}_r(Y \backslash Y_\infty)(U) \to \mathfrak{C}_r(Y' \backslash Y'_\infty)(U) \ , \ f_* : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \to \mathcal{Z}_r(Y' \backslash Y'_\infty)(U).$$

Proof. We recall that $(1 \times f)_* : \mathfrak{C}_{r+m}(X \times Y) \to \mathfrak{C}_{r+m}(X \times Y')$ is a continuous algebraic map (cf. [F]). Since this clearly restricts to $(1 \times f)_* : \mathfrak{C}_{r+m}(X_\infty \times Y) \to \mathfrak{C}_{r+m}(X_\infty \times Y')$, we obtain

$$(1 \times f)_* : \mathfrak{C}_{r+m}(U \times Y) \to \mathfrak{C}_{r+m}(U \times Y'). \tag{3.1.1}$$

Moreover, if $Z \subset U \times Y$ is an irreducible cycle equidimensional over U of relative dimension r, then $(1 \times f)_*(Z)$ restricted to some irreducible component of U is either 0 (because the dimension of $(1 \times f)(U)$ is $< r + m$) or has fibres which are generically of dimension r over U. By the upper semi-continuity of dimension of the fibres of $(1 \times f)(Z) \to U$, we conclude that $(1 \times f)_*(Z)$ in this latter case is equidimensional of relative dimension r over U. Thus, (3.1.1) restricts to $f_* : \mathfrak{C}_r(Y)(U) \to \mathfrak{C}_r(Y')(U)$ which determines $f_* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y')(U)$ via naïve group completion.

The map on spaces of effective relative cocycles is merely the quotient of the map f_* constructed above; this being well defined by naturality. Finally, the map on topological abelian groups of relative cocycles is the group completion of this map. $\square$

The following lemma shows that even though Chow varieties do not represent families of cycles (and, in particular, there is no universal family of cycles on Chow varieties), they do provide a sort of universality for the coarser context of supports of cycles.

Lemma 3.2. *Let $\mathcal{I}_{r,d} \subset C_{r,d}(Y) \times Y$ denote the incidence correspondence consisting of pairs (Z, y) with $y \in |Z|$. If $f : U \to C_{r,d}(Y)$ is any morphism, then the support of the graph Z_f of f is given by*

$$|Z_f| \ = \ U \times_{C_{r,d}(Y)} \mathcal{I}_{r,d}.$$

Proof. We recall that Z_f is defined as follows: for each generic point $\eta : SpecK \to U$, let $Z_\eta = \sum n_j V_{\eta,j}$ denote the cycle on Y_K with Chow point $f(\eta)$; let $\bar{V}_{\eta,j} \subset U \times Y$ denote the closure of $V_{\eta,j}$; then we define

$$Z_f \stackrel{\text{def}}{=} \sum_\eta \sum_j n_j \bar{V}_{\eta,j}.$$

Since $\mathcal{I}_{r,d} \subset \mathfrak{C}_{r,d}(Y) \times Y$ is Zariski closed, the equality $|\bar{V}_{\eta,j}| = \overline{|V_{\eta,j}|}$ immediately implies the inclusion

$$|Z_f| \subset U \times_{\mathfrak{C}_{r,d}(Y)} \mathcal{I}_{r,d}.$$

To prove the reverse inclusion, it suffices to show for any $u \in U$ that $|Z_{f(u)}| \subset |Z_f|$. Since u is the specialization of some generic point η, there exists a smooth curve C and a map $g : C \to U$ sending the generic point γ to η and sending some closed point $c \in C$ to u. Then the graph of $f \circ g$, $Z_{f \circ g}$, is flat over C and has the property that cycle associated to the scheme theoretic fibre of $Z_{f \circ g}$ above any point of C is the cycle $Z_{f \circ g(c)}$ (i.e., the cycle with Chow point $f \circ g(c)$) [F;1.3]. In particular, we conclude that $|Z_{f \circ g(c)}|$ lies in the closure of $|Z_{f \circ g(\gamma)}|$ (which equals all of $|Z_{f \circ g}|$). This implies that $|Z_{f \circ g(c)}| = |Z_{f(u)}|$ lies in the closure of $|Z_{f \circ g(\eta)}| = |Z_{f(\eta)}|$ as required. □

Using Lemma 3.2, we now establish the contravariant functoriality of $\mathfrak{C}_r(Y)(U)$ with respect to U. The reader is referred to [S-V] for a proof of functoriality for normal varieties over general fields (requiring the mastery of technicalities arising from purely inseparable extensions).

Proposition 3.3. *Let $g : U' \to U$ be a morphism (of normal quasi-projective varieties). Then composition with g determines continuous homomorphisms*

$$g^* : \mathfrak{C}_r(Y)(U) \to \mathfrak{C}_r(Y)(U') \ , \ g^* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y)(U').$$

If g is a regular closed immersion of codimension c, then g^* is the restriction of the intersection-theoretic pull-back $g^! : \mathcal{Z}_{r+m}(U \times Y) \to \mathcal{Z}_{r+m-c}(U' \times Y)$ (cf. [Fu]).

Moreover, for $Y_\infty \subset Y$ a closed subvariety, g^* induces

$$g^* : \mathfrak{C}_r(Y \backslash Y_\infty)(U) \to \mathfrak{C}_r(Y \backslash Y_\infty)(U') \ , \ g^* : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \to \mathcal{Z}_r(Y \backslash Y_\infty)(U').$$

Proof. Composition with g determines a continuous homomorphism of mapping spaces with the compact-open topology

$$Hom_{cont}(U, \mathfrak{C}_r(Y)) \to Hom_{cont}(U', \mathfrak{C}_r(Y)).$$

Using Proposition 1.9, we conclude that to prove the continuity of g^* it suffices to prove the following:

if $\{f_n : U \to \mathfrak{C}_r(Y)\}$ is a sequence of maps whose graphs $\{Z_n\}$ have closures in some $\mathbf{P}^N$ of bounded degree, then the graphs $\{Z'_n\}$ associated to $\{f_n \circ g\}$ likewise have closures of bounded degree.

Choose projective closures $U \subset X, U' \subset X'$ together with a map $\tilde{g} : X' \to X$ extending g. Let $\bar{Z}_n$ denote the closure in $X \times Y$ of Z_n. By Lemma 3.2, $X' \times_X |\bar{Z}_n|$ is the closure in $X' \times Y$ of the support $|Z'_n|$ of Z'_n. Granted that the degrees of $\bar{Z}_n$ are bounded, we conclude that the sum of the multiplicities of the components of Z_n and thus also of Z'_n are bounded. Hence, it suffices to prove that the degrees of irreducible components of $X' \times_X |\bar{Z}_n| \subset X' \times Y$ are bounded. We conclude that it suffices to prove that whenever $\{V_n\}$ is a sequence of irreducible subvarieties of $X \times Y$ of bounded degree then $\{V_n \times_X X'\}$ is also a sequence of bounded degree. We can further reduce the problem to the assertion that for any $d > 0$ there exists some n_d such that every irreducible cycle V on $X \times Y$ of degree d has the property that the degree of $V \times_X X'$ has degree bounded by n_d.

We consider the family (parametrized by an open subset U of $C_{m+r,d}(X \times Y)$) of all irreducible cycles on $X \times Y$ of degree d. Pull-back via $\tilde{g} \times 1$ gives us a family $\{V'_u; u \in U\}$ of subvarieties of $X' \times Y$ parametrized by U. This family is flat when restricted to some open dense subset $U_1 \subset U$ and hence $\{V'_u; u \in U_1\}$ is a family of constant degree. Similarly, there is an open dense subset U_2 of $U - U_1$ such that this family is flat when restricted to U_2 and hence $\{V'_u; u \in U_2\}$ is a family of constant degree. Since U is finite dimensional, this process (" noetherian induction") eventually stops so that we conclude that the degrees of $\{V'_u; u \in U\}$ are bounded.

If $g : U' \to U$ is a regular closed immersion, then we may apply [F-M;3.2] which asserts that the intersection-theoretic pull-back $g^!$ constructed in [Fu] is given on effective cocycles by composition with g. Thus, $g^* = g^!$.

It is evident that g^* so defined as composition with g restricts to $g^* : \mathfrak{C}_r(Y_\infty)(U) \to \mathfrak{C}_r(Y_\infty)(U')$ and therefore induces a map on quotient monoids $g^* : \mathfrak{C}_r(Y \backslash Y_\infty)(U) \to \mathfrak{C}_r(Y \backslash Y_\infty)(U')$ and their naïve group completions. □

As demonstrated in [FL-1], the homotopy groups of the cocycle spaces naturally map to singular cohomology. We verify that this map remains well defined with the definition of cocycle spaces $Z_r(Y \backslash Y_\infty)(U)$ given in §2.

Proposition 3.4 *There is a natural map*

$$\Phi : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \quad \to \quad Hom_{cont}(U, \mathcal{Z}_r(V)), \quad V = Y - Y_\infty.$$

In the special case $(Y, Y_\infty) = (\mathbf{P}^t, \mathbf{P}^{t-1})$ *and* $r = 0$, *this map*

$$\Phi : \mathcal{Z}^t(U) \quad \to \quad Hom_{cont}(U, \mathcal{Z}_0(\mathbf{A}^t))$$

induces on j*-th homotopy groups* π_j *a map of the form*

$$\Phi_* : L^t H^{2t-j}(U) \stackrel{\text{def}}{=} \pi_j(\mathcal{Z}^t(U)) \quad \to \quad H^{2t-j}(U). \tag{3.4.1}$$

Proof. Proposition 1.9 implies that the natural inclusion

$$\mathfrak{C}_r(Y)(U) \quad\rightarrow\quad Hom_{cont}(U, \mathfrak{C}_r(Y))$$

is continuous, thereby inducing

$$\mathfrak{C}_r(Y)(U) \quad\rightarrow\quad Hom_{cont}(U, \mathcal{Z}_r(Y)).$$

The naturality of this map when applied to $Y_\infty \to Y$ implies that this induces

$$\mathfrak{C}_r(Y\backslash Y_\infty)(U) \quad\rightarrow\quad Hom_{cont}(U, \mathcal{Z}_r(V))$$

since $\mathcal{Z}_r(V) = \mathcal{Z}_r(Y)/\mathcal{Z}_r(Y_\infty)$. The naïve group completion of this last map provides the asserted map Φ.

The second assertion follows from the observation that $\mathcal{Z}_0(\mathbf{A}^t)$ is a model for the Eilenberg-MacLane space $K(\mathbf{Z}, 2t)$. □

We next verify the **homotopy invariance** of $\mathcal{Z}_r(Y)(U)$ with respect to U. Unfortunately, the proof does not apply to prove the more general assertion that $p^* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y)(E)$ is a homotopy equivalence for any affine torsor $p : E \to U$. Using the duality theorem, this more general assertion is proved for U smooth in Proposition 6.3.

Proposition 3.5. *As usual, asssume that U', U are normal and consider an algebraic homotopy $G : U' \times \mathbf{A}^1 \to U$ relating two morphisms $g, g' : U' \to U$. Then G induces a continuous homotopy*

$$G_{\mathcal{Z}} : \mathcal{Z}_r(Y)(U) \times \mathbf{A}^1 \quad\rightarrow\quad \mathcal{Z}_r(Y)(U')$$

relating $g^*, g'^* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y)(U')$.

Consequently, if $p : E \to U$ is the projection of an algebraic vector bundle, then $p^* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y)(E)$ is a homotopy equivalence.

Proof. We define $G_{\mathcal{Z}}$ as the composition

$$\mathcal{Z}_r(Y)(U) \times \mathbf{A}^1 \stackrel{G^*\times 1}{\rightarrow} \mathcal{Z}_r(Y)(U' \times \mathbf{A}^1) \times \mathbf{A}^1 \stackrel{ev}{\rightarrow} \mathcal{Z}_r(Y)(U').$$

To verify that $G_{\mathcal{Z}}$ is continuous, it suffices to prove that the evaluation map ev is continuous. Since $U' \times \{t\} \subset U' \times \mathbf{A}^1$ is a regular immersion, evaluation at t has the effect on a cycle $Z \in \mathcal{Z}_r(Y)(U' \times \mathbf{A}^1)$ of sending Z to its intersection theoretic fibre above t; in particular, effective cocycles of some bounded degree on $U' \times \mathbf{A}^1 \times Y$ are sent via evaluation at t to effective cocycles of bounded degree on $U' \times Y$. Consequently, Proposition 1.9 and the well-behaved nature of the compact-open topology with respect to evaluation imply that ev is continuous.

If $p : E \to U$ is the bundle projection of an algebraic vector bundle, then clearly p admits an algebraic homotopy $E \times \mathbf{A}^1 \to E$ relating the identity to $o \circ p$, where o denotes the 0-section $o : U \to E$. □

Remark 3.6. A suitably general homotopy invariance property would follow from the verification that whenever U is written as a union of Zariski open subsets U_1, U_2 the following triple

$$\mathfrak{Mor}(U, \mathfrak{C}_r(Y)) \to \mathfrak{Mor}(U_1, \mathfrak{C}_r(Y)) \oplus \mathfrak{Mor}(U_2, \mathfrak{C}_r(Y)) \to \mathfrak{Mor}(U_1 \cap U_2, \mathfrak{C}_r(Y))$$

determines upon naïve group completion a distinguished triangle in the derived category (and hence a Mayer-Vietoris exact sequence). Our present techniques fail to prove such a result for several reasons. First, this triple is not a short exact sequence of topological monoids. Second, it is not clear that the required tractability condition for this triple is valid.

We recall that one of fundamental properties of cycles spaces is the algebraic suspension theorem proved by Lawson in [L]. This theorem asserts that sending a cycle Z on a projective variety Y to its "algebraic suspension" $\not{\Sigma}(Z)$ on the algebraic suspension $\not{\Sigma}(Y)$ of Y induces a weak homotopy equivalence

$$\not{\Sigma} : \mathcal{Z}_r(Y) \to \mathcal{Z}_{r+1}(\not{\Sigma}(Y)). \tag{3.6.1}$$

This theorem was extended to cocycle spaces in [FL-1;3.3] (see also [FL-2;1.7]: relative algebraic suspension $\not{\Sigma}_X$ induces a weak homotopy equivalence

$$\not{\Sigma}_X : \mathcal{Z}_r(Y)(X) \to \mathcal{Z}_{r+1}(\not{\Sigma}Y)(X). \tag{3.6.2}$$

By Recollection 1.4 and Theorem 1.5, (3.6.1) and (3.6.2) are in fact homotopy equivalences.

We now verify that this algebraic suspension theorem extends to the quasi-projective context.

Proposition 3.7. *Relative algebraic suspension $\not{\Sigma}_U : \mathfrak{C}_r(Y)(U) \to \mathfrak{C}_{r+1}(\not{\Sigma}Y)(U)$ induces a homotopy equivalence*

$$\not{\Sigma}_U : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r+1}(\not{\Sigma}Y)(U). \tag{3.7.1}$$

Moreover, if $Y_\infty \subset Y$ is a closed subvariety, then $\not{\Sigma}_U$ induces a homotopy equivalence

$$\not{\Sigma}_U : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r+1}(\not{\Sigma}Y \backslash \not{\Sigma}Y_\infty)(U). \tag{3.7.2}$$

Proof. The proof of [FL-1;3.3] applies to prove that

$$\not{\Sigma}_U : \mathcal{E}_r(Y)(U) \quad \to \quad \mathcal{E}_{r+1}(\not{\Sigma}Y)(U)$$

induces a weak homotopy equivalence on homotopy-theoretic group completions

$$\not{\Sigma}_U : \Omega \circ B[\mathcal{E}_r(Y)(U)] \quad \to \quad \Omega \circ B[\mathcal{E}_{r+1}(\not{\Sigma}Y)(U)],$$

for the argument only involves algebraic suspension in the second factor of cycles on $U \times Y$ and explicitly permits the first factor to be quasi-projective. Moreover, this weak homotopy equivalence restricts to the weak homotopy equivalence

$$\not\Sigma_{X_\infty} : \Omega \circ B[\mathfrak{C}_{r+m}(X_\infty \times Y)] \quad \to \quad \Omega \circ B[\mathfrak{C}_{r+m}(X_\infty \times \not\Sigma Y)].$$

Thus, Corollary 1.7 enables us to conclude that

$$\not\Sigma_U : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r+1}(\not\Sigma Y)(U)$$

is also a weak homotopy equivalence; Recollection 1.4 and Theorem 1.4 now imply that this map is a homotopy equivalence.

Using the first assertion in conjunction with the fibration sequence of Corollary 1.7, the second assertion follows by the 5-Lemma. □

The following Mayer-Vietoris fibration sequence with respect to the covariant argument Y is an elementary consequence of our definitions and Theorem 1.5.

Proposition 3.8. *Assume that Y can be written as a union of closed subvarieties, $Y = Y_1 \cup Y_2$. Let $i_1 : Y_1 \to Y, j_1 : Y_1 \cap Y_2 \to Y_1$ denote the closed immersions and let i_2, j_2 denote the corresponding closed immersions for Y_2. Then the short exact sequences of abelian topological monoids*

$$0 \to \mathfrak{C}_r(Y_1 \cap Y_2)(U) \stackrel{j_{1*} \oplus j_{2*}}{\to} \mathfrak{C}_r(Y_1)(U) \oplus \mathfrak{C}_r(Y_2)(U) \stackrel{i_{1*} - i_{2*}}{\to} \mathfrak{C}_r(Y)(U) \to 0$$

$$0 \to \mathfrak{C}_{r+m}(U \times (Y_1 \cap Y_2)) \stackrel{j_{1*} \oplus j_{2*}}{\to} \mathfrak{C}_{r+m}(U \times Y_1) \oplus \mathfrak{C}_{r+m}(U \times Y_2) \stackrel{i_{1*} - i_{2*}}{\to} \mathfrak{C}_{r+m}(U \times Y) \to 0$$

determine by naïve group completion the following fibration sequences

$$\mathcal{Z}_r(Y_1 \cap Y_2)(U) \to \mathcal{Z}_r(Y_1)(U) \oplus \mathcal{Z}_r(Y_2)(U) \to \mathcal{Z}_r(Y)(U) \qquad (3.8.1)$$

$$\mathcal{Z}_{r+m}(U \times (Y_1 \cap Y_2)) \to \mathcal{Z}_{r+m}(U \times Y_1) \oplus \mathcal{Z}_{r+m}(U \times Y_2) \to \mathcal{Z}_{r+m}(U \times Y) \qquad (3.8.2).$$

Proof. The short exact sequences follow from the evident observation that an irreducible cocycle in $\mathfrak{C}_r(Y)(U)$ (respectively, an irreducible cycle in $\mathfrak{C}_{r+m}(U \times Y)$) lies in the image of either $\mathfrak{C}_r(Y_1)(U)$ or $\mathfrak{C}_r(Y_2)(U)$ (resp., $\mathfrak{C}_{r+m}(U \times Y_1)$ or $\mathfrak{C}_{r+m}(U \times Y_2)$). We observe that $j_{1*} \oplus j_{2*}$ is a closed immersion, for it is the restriction to $\mathfrak{C}_r(Y_1)(U) \oplus \mathfrak{C}_r(Y_2)(U)$ (resp., $\mathfrak{C}_{r+m}(U \times Y_1) \oplus \mathfrak{C}_{r+m}(U \times Y_2)$) of the closed immersion

$$\mathfrak{C}_{r+m}(X \times (Y_1 \cap Y_2)) \to \mathfrak{C}_{r+m}(X \times Y_1) \oplus \mathfrak{C}_{r+m}(X \times Y_2).$$

Thus, the fact that (3.6.1) and (3.6.2) are fibration sequences follows from Theorem 1.5. □

4. Duality Map.

This section defines our duality map from spaces of cocycles to spaces of cycles and verifies that this map is compatible with various constructions. Many of these verifications are little different than those of [FL-2], so that we can refer to proofs given there; on the other hand, the more delicate nature of functoriality in the quasi-projective case requires alternate proofs of various compatibility properties.

We retain our notational conventions, including the consideration of a Zariski open subvariety $V \subset Y$ with not necessarily irreducible complement $Y_\infty \subset Y$. As in previous sections, U will denote a normal quasi-projective variety with projective closure $U \subset X$.

Definition 4.1. We define the duality map

$$\mathcal{D} : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times Y) \tag{4.1.1}$$

to be the map on naïve group completions induced by (1.8.1).

For any closed subvariety $Y_\infty \subset Y$, we define

$$\mathcal{D} : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times V) \ \ , \quad V = Y - Y_\infty \tag{4.1.2}$$

to be the map on naïve group completions induced by the following map defined as a quotient (cf. Proposition 2.2)

$$\mathfrak{C}_r(Y \backslash Y_\infty)(U) \equiv \mathfrak{C}_r(Y)(U)/\mathfrak{C}_r(Y_\infty)(U) \to \mathfrak{C}_r(U \times V) \equiv \mathfrak{C}_r(U \times Y)/\mathfrak{C}_r(U \times Y_\infty)$$

of maps of the form (1.8.1).

For any $t \leq m$, we define the duality map

$$\mathcal{D} : \mathcal{Z}^t(U) \quad \to \quad \mathcal{Z}_{m-t}(U) \tag{4.1.3}$$

as the homotopy class of maps given by the composition of $\mathcal{D} : \mathcal{Z}^t(U) \equiv \mathcal{Z}_0(\mathbf{P}^t/\mathbf{P}^{t-1})(U) \to \mathcal{Z}_m(U \times \mathbf{A}^t)$ and a homotopy inverse of the natural homotopy equivalence $\mathcal{Z}_{m-t}(U) \to \mathcal{Z}_m(U \times \mathbf{A}^t)$.

In the following proposition, we verify that the duality map $\mathcal{D}$ of (4.1.1) is natural with respect to functorial constructions on cycles and cocycles.

Proposition 4.2. *Let $f : Y \to Y'$ be a morphism of projective algebraic varieties. Then the continuous homomorphism $f_* : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_r(Y')(U)$ of Proposition 3.1 fits in the following commutative square*

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{r+m}(U \times Y) \\ f_* \downarrow & & \downarrow (1 \times f)_* \\ \mathcal{Z}_r(Y')(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{r+m}(U \times Y') \end{array} \quad . \tag{4.2.1}$$

Let $g : \tilde{Y} \to Y$ be a flat map of projective varieties of relative dimension k. Then g induces a continuous homomorphism

$$g^* : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_r(\tilde{Y})(U)$$

which fits in the following commutative square

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m}(U \times Y) \\ g^* \Big\downarrow & & \Big\downarrow (g \times 1)^* \\ \mathcal{Z}_{r+k}(\tilde{Y})(U) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m+k}(U \times \tilde{Y}) \end{array} . \tag{4.2.2}$$

Let $h : \tilde{U} \to U$ be a flat morphism of relative dimension e between (quasi-projective, normal) varieties. Then

$$h^* : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_r(Y)(\tilde{U})$$

fits in the following commutative square

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m}(U \times Y) \\ h^* \Big\downarrow & & \Big\downarrow (h \times 1)^* \\ \mathcal{Z}_r(Y)(\tilde{U}) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m+e}(\tilde{U} \times Y) \end{array} . \tag{4.2.3}$$

Let $i : U_0 \to U$ be a regular closed immersion of codimension c of normal quasi-projective varieties. Then

$$i^* : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_r(Y)(U_0)$$

fits in the following homotopy commutative square

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m}(U \times Y) \\ i^* \Big\downarrow & & \Big\downarrow (i \times 1)^! \\ \mathcal{Z}_r(Y)(U_0) & \overset{\mathcal{D}}{\to} & \mathcal{Z}_{r+m-c}(U_0 \times Y) \end{array} , \tag{4.2.4}$$

where $(1 \times i)^!$ is the Gysin map of [F-G] (well defined up to homotopy).

Proof. The commutativity of (4.2.1) follows immediately from the fact that f_* is induced by $(1 \times f_*)$.

To exhibit g^* fitting in the commutative diagram (4.2.2), it suffices to observe that

$$(g \times 1)^* : \mathfrak{C}_{r+m}(U \times Y) \ \to \ \mathfrak{C}_{r+m+k}(U \times \tilde{Y})$$

is a continuous algebraic map and restricts to $\mathfrak{C}_r(Y)(U) \to \mathfrak{C}_r(\tilde{Y})(U)$. The continuity is proved in [F-G;1.5ff], whereas the property of restriction is evident.

To verify the commutativity of (4.2.3), we must show the following: if $Z \subset U \times Y$ is an irreducible cycle equidimensional over U and corresponding to a map $j : U \to \mathfrak{C}_r(Y)$, then $(h \times 1)^*(Z)$ equals the cycle $Z_{j \circ h}$ corresponding to the map $j \circ h : \tilde{U} \to \mathfrak{C}_r(Y')$. This is verified by observing that the restrictions of $(h \times 1)^*(Z)$ and $Z_{j \circ h}$ to $Spec(K') \times Y'$ are equal, where $Spec(K) \to U$ is a generic point.

As verified in [F-G;3.4], the Gysin map

$$(i \times 1)^! : \mathfrak{C}_{r+m}(U \times Y) \quad \to \mathfrak{C}_{r+m-c}(U_0 \times Y)$$

can be represented (in the derived category, thus up to homotopy equivalence between spaces having the homotopy type of C.W. complexes) by intersection with $U_0 \times Y$ on the submonoid $\mathfrak{C}_{r+m}(U \times Y; U_0 \times Y)$ of those cycles which meet $U \times Y_0$ properly. Clearly, $\mathfrak{C}_r(Y)(U) \subset \mathfrak{C}_{r+m}(U \times Y; U_0 \times Y)$. On the other hand, by [F-M;3.2] the homomorphism $i^* : \mathfrak{C}_r(Y)(U) \to \mathfrak{C}_r(Y)(U_0)$ given by intersection with $U_0 \times Y$ equals that given sending a cycle represented by $j : U \to \mathfrak{C}_r(Y)$ to the cycle represented by $j \circ i : U_0 \to \mathfrak{C}_r(Y)$. $\square$

We state without proof the following relative version of Proposition 4.2 for the duality map $\mathcal{D}$ of (4.1.2) and its special case given in (4.1.3).

Proposition 4.3. *The relative versions of the commutative squares (4.2.1), (4.2.2), (4.2.3), and the homotopy commutative square (4.2.4) remain valid provided that one replaces* Y *by* $Y \backslash Y_\infty$ *in the left hand sides of these squares and by* $V = Y - Y_\infty$ *in the right hand sides. In particular, we have the following special cases of these relative versions of (4.2.3) and (4.2.4).*

If $h : \tilde{U} \to U$ *is a flat morphism of relative dimension* e*, then the following square commutes up to homotopy*

$$\begin{array}{ccc} \mathcal{Z}^t(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m-t}(U) \\ h^* \downarrow & & \downarrow (h \times 1)^* \\ \mathcal{Z}^t(\tilde{U}) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m+e-t}(\tilde{U}) \end{array} \quad . \qquad (4.3.1)$$

If $i : U_0 \to U$ *is a regular closed immersion of codimension* c*, then the following square commutes up to homotopy*

$$\begin{array}{ccc} \mathcal{Z}^t(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m-t}(U) \\ i^* \downarrow & & \downarrow (i \times 1)^! \\ \mathcal{Z}^t(U_0) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m-t-c}(U_0) \end{array} \quad . \qquad (4.3.2)$$

Remark 4.4. For any $t \leq m$ and any closed subvariety $U_0 \subset U$, one can define a duality map

$$\mathcal{D} : \mathcal{Z}^t_{U_0}(U) \quad \to \quad \mathcal{Z}_{m-t}(U_0) \qquad (4.4.1)$$

as follows. We consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}^t_{U_0}(U) & \to & \mathcal{Z}^t(U) & \xrightarrow{j^*} & \mathcal{Z}^t(U-U_0) \\ & & \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ \mathcal{Z}_{m-t}(U_0) & \to & \mathcal{Z}_{m-t}(U) & \xrightarrow{j^*} & \mathcal{Z}_{m-t}(U-U_0) \end{array} \tag{4.4.2}$$

where $j : U - U_0 \to U$ is the Zariski open complement of $U_0 \subset U$. In view of the facts that $\mathcal{Z}^t_{U_0}(U) = htyfib\{\mathcal{Z}^t(U) \to \mathcal{Z}^t(U - U_0)\}$, that $\mathcal{Z}_{m-t}(U_0)$ admits a natural equivalence to the homotopy fibre of $\mathcal{Z}_{m-t}(U) \to \mathcal{Z}_{m-t}(U - U_0)$, and that the square of (4.4.2) is commutative, we conclude that there is a natural homotopy class of maps as in (4.4.1) induced by (4.4.2). This map extends (4.4.2) to a map of fibration sequences.

We next proceed to exhibit a Gysin morphism on cocycles with respect to a regular embedding $\epsilon : Y_0 \to Y$. Essentially, we show that the Gysin map constructed in [F-G] on cycle spaces for a regular immersion $\epsilon : T_0 \to T$ restricts to a map on cocycle spaces. To carry out this argument, we require the fundamental ingredient of our duality theorem (namely, Proposition 5.1 of the next section), so that smoothness conditions are required.

Theorem 4.5. (cf. [FL-2;2.4]) *Let $\epsilon : Y_0 \to Y$ be a closed immersion of codimension e of smooth, projective varieties and assume that U is also smooth. Then there is a natural homotopy class of maps*

$$\epsilon^! : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r-e}(Y_0)(U)$$

which fits in the following homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(U \times Y) \\ \epsilon^!\downarrow & & \downarrow(1\times\epsilon)^! \\ \mathcal{Z}_{r-e}(Y_0)(U) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m-e}(U \times Y_0) \end{array} . \tag{4.5.1}$$

Moreover, if Y_∞ meets Y_0 properly and if V_0 denotes the Zariski open complement of $Y_1 = Y_0 \cap Y_\infty$, then $\epsilon^!$ admits a relative version

$$\epsilon^! : \mathcal{Z}_r(Y\backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r-e}(Y_0\backslash Y_1)(U)$$

which fits in the following homotopy commutative diagram

$$\begin{array}{ccc} \mathcal{Z}_r(Y\backslash Y_\infty)(U) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m}(U \times V) \\ \epsilon^!\downarrow & & \downarrow(1\times\epsilon)^! \\ \mathcal{Z}_{r-e}(Y_0\backslash Y_1)(U) & \xrightarrow{\mathcal{D}} & \mathcal{Z}_{r+m-e}(U \times V_0) \end{array} . \tag{4.5.2}$$

Proof. We define

$$\mathcal{E}_r(Y;Y_0)(U) \quad \subset \quad \mathcal{E}_r(Y)(U) \tag{4.5.3}$$

to be the submonoid of those effective $r+m$-cycles on $X \times Y$ which intersect $X \times Y_0$ properly and whose restrictions to $U \times Y, U \times Y_0$ are equidimensional over U. To verify that (4.5.3) is a constructible embedding, we proceed as follows. We first argue as for Definition 1.1, using the upper semi-continuity of the fibres of the projection

$$\mathcal{I} \cap [\mathcal{E}_r(Y)(U) \times (X \times Y_0)] \quad \to \quad \mathcal{E}_r(Y)(U)$$

where $\mathcal{I} \subset \mathfrak{C}_{r+m}(X \times Y) \times (X \times Y)$ is the incidence correspondence, thereby obtaining ${}'\mathcal{E}_r(Y)(U) \subset \mathcal{E}_r(Y)(U)$ consisting of cocycles on U intersecting $X \times Y_0$ properly. We then apply the same argument to the projection

$$\mathcal{J} \cap [{}'\mathcal{E}_r(Y)(U) \times (U \times Y_0)] \quad \to \quad {}'\mathcal{E}_r(Y)(U) \times U$$

where $\mathcal{J} \subset \mathfrak{C}_{r+m}(X \times Y) \times (X \times Y_0)$ consists of those (Z,x,y) such that $x,y \in |Z| \cap (X \times Y_0)$.

Since $\mathfrak{C}_{r+m}(X_\infty \times Y) \subset \mathcal{E}_r(Y)(U)$ is a Zariski closed immersion, so is $\mathfrak{C}_{r+m}(X_\infty \times Y) \subset \mathcal{E}_r(Y;Y_0)(U)$. Set

$$\mathcal{Z}_r(Y;Y_0)(U) \quad \stackrel{\text{def}}{=} \quad [\mathfrak{C}_r(Y;Y_0)(U)]^+ \ , \ \ \mathfrak{C}_r(Y;Y_0)(U) \quad \stackrel{\text{def}}{=} \quad \mathcal{E}_r(Y;Y_0)(U)/\mathfrak{C}_{r+m}(X_\infty \times Y).$$

By Proposition 5.1, (4.5.3) induces a equivalence

$$[\mathcal{E}_r(Y;Y_0)(U)]^+ \to [\mathcal{E}_r(Y)(U)]^+. \tag{4.5.4}$$

Hence we may apply the 5-Lemma (in conjunction with Corollary 1.7) to the map of fibration sequences

$$\begin{array}{ccccc} \mathcal{Z}_{r+m}(X_\infty \times Y) & \to & [\mathcal{E}_r(Y;Y_0)(U)]^+ & \to & \mathcal{Z}_r(Y;Y_0)(U) \\ = \downarrow & & \downarrow & & \downarrow \\ \mathcal{Z}_{r+m}(X_\infty \times Y) & \to & [\mathcal{E}_r(Y)(U)]^+ & \to & \mathcal{Z}_r(Y)(U) \end{array}$$

to conclude that

$$\mathcal{Z}_r(Y;Y_0)(U) \quad \to \quad \mathcal{Z}_r(Y)(U)$$

is a homotopy equivalence.

We define the Gysin map $\epsilon^!$ as the composition

$$\epsilon^! : \mathcal{Z}_r(Y)(U) \ \simeq \mathcal{Z}_r(Y;Y_0)(U) \ \to \mathcal{Z}_{r-e}(Y_0)(U)$$

where the first map is the homotopy inverse of the homotopy equivalence established above and the second is the naïve group completion of the map $\mathfrak{C}_r(Y;Y_0)(U) \to \mathfrak{C}_{r-e}(Y_0)(U)$ given by intersection with $U \times Y_0$. So defined $\epsilon^!$ fits in the homotopy commutative square (4.5.1) by [F-G;3.4].

To exhibit $\epsilon^!$ in the relative case and prove that it fits in the homotopy commutative square (4.5.2), we first observe that the condition on $Z \in \mathcal{E}_r(Y)(U)$ supported on $U \times Y_\infty$ to meet $X \times Y_0$ properly in $U \times Y$ and hence have intersection with $U \times Y_0$ equidimensional over U is the same condition as the condition that Z when viewed in $\mathcal{E}_r(Y_\infty)(U)$ meet $X \times Y_1$ properly in $U \times Y_\infty$ and have intersection with $U \times Y_1$ equidimensional over U. Consequently, intersection with $U \times Y_0$ induces a well defined continuous map

$$\mathcal{E}_r(Y;Y_0)(U)/'\mathcal{F}_r(Y_\infty)(U) \quad \rightarrow \quad \mathcal{E}_{r-e}(Y_0)(U)/\mathcal{F}_{r-e}(Y_1)(U), \tag{4.5.5}$$

where $'\mathcal{F}_r(Y_\infty)(U) \equiv \mathcal{F}_r(Y_\infty)(U) \cap \mathcal{E}_r(Y;Y_0)(U)$. Since $\mathcal{F}_r(Y_\infty)(U) \subset \mathcal{E}_r(Y)(U)$ (cf. Definition 2.1) is a Zariski closed submonoid, so is $'\mathcal{F}_r(Y_\infty)(U) \subset \mathcal{E}_r(Y;Y_0)(U)$. A now familiar argument comparing fibration sequences and appealing to the 5-Lemma shows that the homotopy equivalence (4.5.4) implies that

$$[\mathcal{E}_r(Y;Y_0)(U)/'\mathcal{F}_r(Y_\infty)(U)]^+ \quad \rightarrow \quad \mathcal{Z}_r(Y\backslash Y_\infty)(U)$$

is a equivalence. Thus, we may define the relative Gysin map $\epsilon^!$ fitting in the homotopy commutative square (4.5.2) (by [F-G;3.4.2]) as the composition

$$\epsilon^! : \mathcal{Z}_r(Y\backslash Y_\infty)(U) \;\simeq\; [\mathcal{E}_r(Y;Y_0)(U)/'\mathcal{F}_r(Y_\infty)(U)]^+ \;\rightarrow\; \mathcal{Z}_{r-e}(Y_0;Y_1)(U)$$

where the first map is the homotopy inverse of the homotopy equivalence established above and the second is the naïve group completion of (4.5.4). $\square$

In [F-M], the operation $s : \Omega \circ B[\mathfrak{C}_r(X)] \wedge S^2 \rightarrow \Omega \circ B[\mathfrak{C}_r(\not\Sigma X)]$ was introduced and studied. In [F-G], this operation was extended to an operation $s : \mathcal{Z}_r(U) \wedge S^2 \rightarrow \mathcal{Z}_{r-1}(U)$ for cycles on a quasi-projective variety and was shown to be independent of the projective embedding. In [FL-2;2.5,2.6], this operation was refined for cocycles spaces on a smooth projective variety X with values in a smooth projective variety Y. (The smoothness hypotheses were required in order to employ the Gysin map of Proposition 4.5 to $Y \times \{0\} \subset Y \times \mathbf{P}^1$.) Indeed, the proofs given there apply verbatim with X replaced by a smooth, quasi-projective variety U.

Proposition 4.6. (cf. [FL-2;2.5,2.6]) *Let Y be a projective, smooth variety and let U be a smooth quasi-projective variety. The s-map determines a homotopy class of maps*

$$s : \mathcal{Z}_r(Y)(U) \quad \wedge S^2 \rightarrow \quad \mathcal{Z}_{r-1}(Y)(U)$$

which fits in the following homotopy commutative square:

$$\begin{array}{ccc} \mathcal{Z}_r(Y)(U) \wedge S^2 & \stackrel{\mathcal{D}\wedge 1}{\rightarrow} & \mathcal{Z}_{r+m}(U \times Y) \wedge S^2 \\ s\big\downarrow & & \big\downarrow s \\ \mathcal{Z}_{r-1}(Y)(U) & \stackrel{\mathcal{D}}{\rightarrow} & \mathcal{Z}_{r+m-1}(U \times Y) \end{array} . \tag{4.6.1}$$

Furthermore, the s-map determines a homotopy class of maps

$$s : \mathcal{Z}_r(Y\backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r-1}(Y\backslash Y_\infty)(U)$$

which fits in the following homotopy commutative square:

$$\begin{array}{ccc} \mathcal{Z}_r(Y\backslash Y_\infty)(U) \wedge S^2 & \stackrel{\mathcal{D}\wedge 1}{\to} & \mathcal{Z}_{r+m}(U \times V) \wedge S^2 \\ s\downarrow & & \downarrow s \\ \mathcal{Z}_{r-1}(Y\backslash Y_\infty)(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{r+m-1}(U \times V) \end{array} . \tag{4.6.2}$$

We single out the following special case of (4.6.2).

Corollary 4.7. *Let U be a smooth quasi-projective variety (of dimension m, as usual). Then there is a natural homotopy class of maps*

$$s : \mathcal{Z}^t(U) \quad \to \quad \mathcal{Z}^{t+1}(U)$$

which fits in the following homotopy commutative square

$$\begin{array}{ccc} \mathcal{Z}^t(U) \wedge S^2 & \stackrel{\mathcal{D}\wedge 1}{\to} & \mathcal{Z}_{m-t}(U) \wedge S^2 \\ s\downarrow & & \downarrow s \\ \mathcal{Z}^{t+1}(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m-t-1}(U) \end{array} . \tag{4.7.1}$$

Proof. By Proposition 3.5, there is a natural algebraic suspension homotopy equivalence

$$\not{\Sigma}_U : \mathcal{Z}^t(U) \equiv \mathcal{Z}_0(\mathbf{P}^t\backslash\mathbf{P}^{t-1})(U) \quad \to \quad \mathcal{Z}_1(\mathbf{P}^{t+1}\backslash\mathbf{P}^t)(U).$$

We define $\mathcal{Z}^t(U) \quad \to \quad \mathcal{Z}^{t+1}(U)$ to be the composition of this map and the relative s-map of Proposition 4.6:

$$s : \mathcal{Z}_1(\mathbf{P}^{t+1}\backslash\mathbf{P}^t)(U) \wedge S^2 \quad \to \mathcal{Z}_0(\mathbf{P}^{t+1}\backslash\mathbf{P}^t)(U) \equiv \mathcal{Z}^{t+1}(U).$$

The homotopy commutativity of (4.7.1) follows consideration of the following diagram

$$\begin{array}{ccccc} \mathcal{Z}_0(\mathbf{P}^t/\mathbf{P}^{t-1})(U) \wedge S^2 & \stackrel{\mathcal{D}\wedge 1}{\to} & \mathcal{Z}_m(U \times \mathbf{A}^t) \wedge S^2 & \stackrel{\pi^*\wedge 1}{\leftarrow} & \mathcal{Z}_{m-t}(U) \wedge S^2 \\ \not{\Sigma}_U\downarrow & & 1\times\pi^*\downarrow & & \downarrow = \\ \mathcal{Z}_1(\mathbf{P}^{t+1}/\mathbf{P}^t)(U) \wedge S^2 & \stackrel{\mathcal{D}\wedge 1}{\to} & \mathcal{Z}_{m+1}(U \times \mathbf{A}^{t+1}) \wedge S^2 & \stackrel{\pi^*\wedge 1}{\leftarrow} & \mathcal{Z}_{m-t}(U) \wedge S^2 \\ s\downarrow & & s\downarrow & & \downarrow s \\ \mathcal{Z}_0(\mathbf{P}^{t+1}/\mathbf{P}^t)(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_m(U \times \mathbf{A}^{t+1}) & \stackrel{\pi^*}{\leftarrow} & \mathcal{Z}_{m-t-1}(U) \end{array} .$$

The commutativity of the upper squares of this diagram is easily seen by inspection, the homotopy commutativity of the lower left square follows from (4.6.2), and the homotopy commutativity of the right lower square follows from the naturality of s. □

In [FL-2;2.7], the intersection product defined in [F-G] for cycle spaces of smooth varieties is shown to correspond to the fibrewise join product introduced in [FL-1]. The proof given there applies to quasi-projective U, so that we content ourselves with merely stating this result in our context of cocycles on a quasi-projective variety U.

Proposition 4.8. *If U is smooth and if $t+u$ are non-negative integers with $t+u \leq m$, then the fibre-wise join pairing $\#_U$ fits in a homotopy commutative diagram*

$$\begin{array}{ccccc} \mathcal{Z}_0(\mathbf{P}^t)(U) \times \mathcal{Z}_0(\mathbf{P}^u)(U) & \to & \mathcal{Z}^t(U) \times \mathcal{Z}^u(U) & \stackrel{\mathcal{D}\times\mathcal{D}}{\to} & \mathcal{Z}_{m-t}(U) \times \mathcal{Z}_{m-u}(U) \\ \#_U \downarrow & & & & \downarrow \bullet \\ \mathcal{Z}_0(\mathbf{P}^{t+u})(U) & \to & \mathcal{Z}^{t+u}(U) & \stackrel{\mathcal{D}}{\to} & \mathcal{Z}_{m-t-u}(U) \end{array} \tag{4.8.1}$$

where the left horizontal arrows are the defining projections and where $(-) \bullet (-)$ denotes the intersection product on cycle spaces.

Proof. See [FL-2;2.7]. □

5. Duality Theorems

In this section, we present various forms of duality relating spaces of algebraic cocycles and spaces of algebraic cycles. We retain our notational conventions on $X, Y, U, X_\infty, V, Y_\infty$, r and t of previous sections.

The following fundamental technical result is a consequence of the "Moving Lemma for Cycles of Bounded Degree" [FL-3].

Proposition 5.1. *Let $\mathcal{E}_r(Y)(U) \subset \mathfrak{C}_{r+m}(X \times Y)$ be the embedding of monoids of (1.6.1). If Y and U are both smooth, then this embedding induces a homotopy equivalence of naïve group completions*

$$\mathcal{D} : [\mathcal{E}_r(Y)(U)]^+ \quad \to \quad \mathcal{Z}_{r+m}(X \times Y).$$

Moreover, if $Y_0 \subset Y$ is a closed subvariety of some dimension $\geq n-r$ and if $\mathcal{E}_r(Y;Y_0)(U) \subset \mathcal{E}_r(Y)(U)$ is the submonoid of (4.5.3) consisting of cocycles which meet $X \times Y_0$ properly and whose intersections with $U \times Y_0$ are equidimensional over U, then the induced map of naïve group completions

$$[\mathcal{E}_r(Y;Y_0)(U)]^+ \quad \to \quad [\mathcal{E}_r(Y)(U)]^+$$

is a homotopy equivalence.

Proof. Let

$$\begin{aligned} \pi : \mathfrak{C}_{r+m}(X \times Y) \times \mathfrak{C}_{r+m}(X \times Y) &\rightarrow \mathcal{Z}_{r+m}(X \times Y) \\ \pi' : \mathcal{E}_r(Y)(U) \times \mathcal{E}_r(Y)(U) &\rightarrow [\mathcal{E}_r(Y)(U)]^+ \end{aligned}$$

denote the canonical projection maps, and let

$$j : \mathcal{E}_r(Y)(U) \rightarrow \mathfrak{C}_{r+m}(X \times Y)$$

denote the embedding of (1.6.1). Then the filtration $\{K_e\}_{e=0}^{\infty}$ of $\mathcal{Z}_{r+m}(X \times Y)$ given by setting

$$K_e \equiv \pi \left\{ \coprod_{d+d' \leq e} \mathfrak{C}_{r+m,d}(X \times Y) \times \mathfrak{C}_{r+m,d'}(X \times Y) \right\}$$

is a good filtration in the sense of [FL-2;4.1]; namely, any map from a compact space K to $\mathcal{Z}_{r+m}(X \times Y)$ factors through some K_e. Consider the associated filtration $\{K'_e\}_{e=0}^{\infty}$ of $[\mathcal{E}_r(Y)(U)]^+$:

$$K'_e \equiv \pi' \left\{ \coprod_{d+d' \leq e} \mathcal{E}_{r,d}(Y)(U) \times \mathcal{E}_{r,d'}(Y)(U) \right\}$$

where $\mathcal{E}_{r,d}(Y)(U) = \mathfrak{C}_{r+m,d}(X \times Y) \cap \mathcal{E}_r(Y)(U)$. If K is compact and $f : K \rightarrow [\mathcal{E}_r(Y)(U)]^+$ is continuous, then $(j \circ f)(K)$ lies in some K_e, so that $f(K)$ lies in some K'_e. We conclude that $\{K'_e\}_{e=0}^{\infty}$ is also a good filtration.

Let e be any positive integer $\geq$ the degrees of $\{u\} \times Y \subset X \times Y$ for all $u \in U$. Then [FL-2;3.1], the "Moving Lemma for Cycles of Bounded Degree", implies the existence of

$$\phi_e : K_e \times I \rightarrow \mathcal{Z}_{r+m}(X \times Y) \quad , \quad \phi'_e : K'_e \times I \rightarrow \mathcal{Z}_r(Y)(U)$$

satisfying the conditions of a very weak deformation retract in the sense of [FL-2;4.1]. Namely, ϕ'_e covers ϕ_e with respect to j; $(\phi_e)\big|_{K_e \times \{0\}}, (\phi'_e)\big|_{K'_e \times \{0\}}$ are the natural inclusions; and $(\phi_e)\big|_{K_e \times \{t\}}$ lifts to $[\mathcal{E}_r(Y)(U)]^+$ for any $t \neq 0$. Thus, $\mathcal{D} : [\mathcal{E}_r(Y)(U)]^+ \rightarrow \mathcal{Z}_{r+m}(X \times Y)$ is easily seen to be a weak homotopy equivalence using the easy technical lemma [FL-2;4.2]. Since these spaces have the homotopy type of C.W. complexes, $\mathcal{D}$ is in fact a homotopy equivalence.

To prove the second assertion, recall that the Moving Lemma enables one to move s-cycles of degree $\leq e$ on $\mathfrak{C}_{r+m}(X \times Y)$ so that the resulting cycles intersect properly (off the singular locus of $X \times Y$) all effective cycles of degree $\leq e$ and of dimension $\geq m - s$ [FL-3;3.2]. We apply this result to move effective cycles in $\mathfrak{C}_{r+m}(X \times Y)$ with respect to the cycles $u \times Y, u \times Y_0; u \in U$ and the cycle $X \times Y_0$. Thus, the preceding argument applies to prove that

$$\mathcal{E}_r(Y; Y_0)(U) \rightarrow \mathcal{Z}_{r+m}(X \times Y)$$

is also a homotopy equivalence. The second assertion now follows. $\square$

The following duality theorem follows easily from Proposition 5.1.

Theorem 5.2. *As usual, let U be a quasi-projective variety of dimension m and let Y be a projective variety of dimension n. If both U and Y are smooth, then the duality map of (4.1.1)*

$$\mathcal{D} : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times Y)$$

is a homotopy equivalence for any $r \leq m$.

Furthermore, if $Y_\infty \subset Y$ is a smooth, closed subvariety with Zariski open complement $V \subset Y$, then the relative duality map of (4.1.2)

$$\mathcal{D} : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times V)$$

is also a homotopy equivalence for any $r \leq m$.

Specializing to $Y = \mathbf{P}^t, Y_\infty = \mathbf{P}^{t-1}$, we conclude that the duality map of (4.1.3)

$$\mathcal{D} : \mathcal{Z}^t(U) \quad \to \quad \mathcal{Z}_{m-t}(U)$$

is also a homotopy equivalence for any $t \geq 0$ (where

$$\mathcal{Z}_{m-t} \equiv \mathcal{Z}_0(\not\Sigma^{t-m} U) \quad , \quad m < t$$

as in [F-M]).

Proof. We consider the following diagram

$$\begin{array}{ccccc} \mathcal{Z}_{r+m}(X_\infty \times Y) & \to & [\mathcal{E}_r(Y)(U)]^+ & \to & \mathcal{Z}_r(Y)(U) \\ = \big\downarrow & & \mathcal{D}\big\downarrow & & \big\downarrow \mathcal{D} \\ \mathcal{Z}_{r+m}(X_\infty \times Y) & \to & \mathcal{Z}_r(X \times Y) & \to & Z_r(U \times Y) \end{array} . \qquad (5.2.1)$$

Both the rows of (5.2.1) are fibration sequences: the top by Corollary 1.5, the bottom by [F-G;1.6]. Consequently, the fact that the duality map $\mathcal{D}$ is a homotopy equivalence follows from Proposition 5.1 and an application of the 5-Lemma.

In the relative case, we consider the following diagram

$$\begin{array}{ccccc} \mathcal{Z}_r(Y_\infty)(U) & \to & \mathcal{Z}_r(Y)(U) & \to & \mathcal{Z}_r(Y \backslash Y_\infty)(U) \\ \mathcal{D}\big\downarrow & & \mathcal{D}\big\downarrow & & \big\downarrow \mathcal{D} \\ \mathcal{Z}_{r+m}(U \times Y_\infty) & \to & \mathcal{Z}_{r+m}(U \times Y) & \to & \mathcal{Z}_{r+m}(U \times V) \end{array} . \qquad (5.2.2)$$

The upper row of (5.2.2) is a fibration sequence by Proposition 2.2, whereas the lower row is a fibration sequence by [F-G;1.6] once again. Since the left and middle maps are homotopy equivalences by the first part of our theorem, the 5-Lemma implies that the relative duality map $\mathcal{D}$ is also a homotopy equivalence. $\square$

We recall that the homotopy groups of $\mathcal{Z}^t(U)$ and $\mathcal{Z}_r(U)$ are called "morphic cohomology groups" and "Lawson homology groups" respectively. These are indexed as follows:

$$L^t H^k(U) \stackrel{\text{def}}{=} \pi_{2t-k}(\mathcal{Z}^t(U)) \quad , \quad L_r H_k(U) \stackrel{\text{def}}{=} \pi_{k-2r}(\mathcal{Z}_r(U)).$$

Using this notation, we re-state the relative case of Theorem 5.2 (with $Y = \mathbf{P}^t, Y_\infty = \mathbf{P}^{t-1}$).

Corollary 5.3. *Let U be a smooth variety of dimension m and let $0 \leq k \leq 2t, 2m$. Then the duality map $\mathcal{D} : Z^t(U) \to Z_{m-t}(U)$ of (4.1.3) induces isomorphisms*

$$\boxed{L^t H^k(U) \simeq L_{m-t} H_{2m-k}(U).}$$

Applying the Mayer-Vietoris sequence of Proposition 3.8, we obtain the following mild generalization of Theorem 5.2.

Corollary 5.4. *Let $Y_1, \ldots, Y_k$ be smooth projective varieties and assume that each multiple intersection $Y_{i_1} \cap \cdots \cap Y_{i_j}$ is also smooth. Let Y denote the union of the Y_i's, $Y = Y_1 \cup \cdots \cup Y_k$. If U is a smooth quasi-projective variety, then*

$$\mathcal{D} : \mathcal{Z}_r(Y)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times Y)$$

is a homotopy equivalence.

Proof. We proceed by induction on k, the case $k = 1$ provided by Theorem 5.2. Let $Y' = Y_1 \cup \cdots \cup Y_{k-1}$ and let $Y'' = Y' \cap Y_k$. We consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}_r(Y'')(U) & \to & \mathcal{Z}_r(Y')(U) \oplus \mathcal{Z}_r(Y_k)(U) & \to & \mathcal{Z}_r(Y)(U) \\ \mathcal{D}\downarrow & & \mathcal{D} \oplus \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ \mathcal{Z}_{r+m}(U \times Y'') & \to & \mathcal{Z}_{r+m}(U \times Y')(U) \oplus \mathcal{Z}_{r+m}(U \times Y_k) & \to & \mathcal{Z}_{r+m}(U \times Y) \end{array}.$$

By Proposition 3.8, both rows are fibration sequences. Thus, induction and the 5-Lemma imply that $\mathcal{D} : \mathcal{Z}_r(Y)(U) \to \mathcal{Z}_{r+m}(U \times Y)$ is a homotopy equivalence. $\square$

We recall that Hironaka's resolution of singularities asserts that any smooth quasi-projective variety V admits a smooth projective closure Y with the property that $Y - V = Y_\infty$ is a divisor with normal crossings [H-1]. In particular, such a "complement at infinity" satisfies the conditions on Y of Theorem 5.4.

Corollary 5.5. *Let $Y_\infty \subset Y$ be a closed immersion of projective varieties both of which can be written as a union of smooth closed subvarieties whose multiple intersections are also smooth (e.g., Y_∞ might be a divisor with normal crossings in a smooth projective variety Y). Then the relative duality map (4.1.2)*

$$\mathcal{D} : \mathcal{Z}_r(Y \backslash Y_\infty)(U) \quad \to \quad \mathcal{Z}_{r+m}(U \times V)$$

is a homotopy equivalence.

In particular, the homotopy type of $\mathcal{Z}_r(Y\backslash Y_\infty)(U)$ depends only upon U and V and not their projective closures $U \subset X$ and $V \subset Y$.

Proof. We consider the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Z}_r(Y_\infty)(U) & \rightarrow & \mathcal{Z}_r(Y)(U) & \rightarrow & \mathcal{Z}_r(Y\backslash Y_\infty)(U) \\ \mathcal{D}\downarrow & & \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ \mathcal{Z}_{r+m}(U \times Y_\infty) & \rightarrow & \mathcal{Z}_{r+m}(U \times Y) & \rightarrow & \mathcal{Z}_{r+m}(U \times V) \end{array}.$$

By Proposition 2.2, the upper row is a fibration sequence; by [LF-2] or [F-G], the lower row is also a fibration sequence. Thus, the corollary follows from Corollary 5.4 and the 5-Lemma. □

6. First Consequences.

Using Theorem 5.4, we define a Gysin map for cocycle spaces compatible with the duality map. We can view this next proposition as a supplement to Propositions 4.2 and 4.3.

Proposition 6.1. *As in Theorem 5.4, let Y be a union of smooth projective varieties whose multiple intersections are also smooth. Consider a closed embedding $i : U_0 \subset U$ of smooth quasi-projective varieties of codimension c with Zariski open complement $U' \subset U$. Then there exists a homotopy class of maps (i.e., a* **Gysin map***)*

$$i_! : \mathcal{Z}_{r+c}(Y)(U_0) \rightarrow \mathcal{Z}_r(Y)(U) \tag{6.1.1}$$

which fits in the following map of fibration sequences

$$\begin{array}{ccccc} \mathcal{Z}_{r+c}(Y)(U_0) & \xrightarrow{i_!} & \mathcal{Z}_r(Y)(U) & \xrightarrow{j^*} & \mathcal{Z}_r(Y)(U') \\ \mathcal{D}\downarrow & & \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ \mathcal{Z}_{r+m}(U_0 \times Y) & \xrightarrow{i_*} & \mathcal{Z}_{r+m}(U \times Y) & \xrightarrow{j^*} & \mathcal{Z}_{r+m}(U' \times Y) \end{array}. \tag{6.1.2}$$

Moreover, if $Y_\infty \subset Y$ is also a union of smooth projective varieties whose multiple intersections are also smooth, then there exists a homotopy class of maps

$$i_! : \mathcal{Z}_{r+c}(Y\backslash Y_\infty)(U_0) \rightarrow \mathcal{Z}_r(Y\backslash Y_\infty)(U) \tag{6.1.3}$$

which fits in the following map of fibration sequences

$$\begin{array}{ccccc} \mathcal{Z}_{r+c}(Y\backslash Y_\infty)(U_0) & \xrightarrow{i_!} & \mathcal{Z}_r(Y\backslash Y_\infty)(U) & \xrightarrow{j^*} & \mathcal{Z}_r(Y\backslash Y_\infty)(U') \\ \mathcal{D}\downarrow & & \mathcal{D}\downarrow & & \downarrow\mathcal{D} \\ \mathcal{Z}_{r+m}(U_0 \times V) & \xrightarrow{i_*} & \mathcal{Z}_{r+m}(U \times V) & \xrightarrow{j^*} & \mathcal{Z}_{r+m}(U' \times V) \end{array}. \tag{6.1.4}$$

Proof. Using Theorem 5.2, we define $i_!$ for (6.1.1) and (6.1.3) by

$$i_! \stackrel{\mathrm{def}}{=} \mathcal{D}^{-1} \circ i_* \circ \mathcal{D}. \tag{6.1.5}$$

So defined, $i_!$ fits in homotopy commutative diagrams (6.1.2) and (6.1.4). Since the vertical maps are homotopy equivalences, the fact that the bottom rows of these diagrams constitute fibration sequences implies that the top rows are as well. $\square$

We restate Proposition 6.1 in terms of the notation used in Corollary 5.3.

Corollary 6.2. *Let $i : U_0 \subset U$ be a Zariski closed immersion of smooth subvarieties of pure codimension c and let $j : U' \subset U$ denote the Zariski open complement. Then the duality map determines an isomorphism of long exact sequences*

$$\begin{array}{ccccccccc} \cdots \to & L^{s-c}H^{k-2c}(U_0) & \xrightarrow{i_!} & L^sH^k(U) & \xrightarrow{j^*} & L^sH^k(U') & \to \cdots \\ & \mathcal{D}\big\downarrow & & \mathcal{D}\big\downarrow & & \big\downarrow\mathcal{D} & \\ \cdots \to & L_{m-s}H_{2m-k}(U_0) & \xrightarrow{i_*} & L_{m-s}H_{2m-k}(U) & \xrightarrow{j^*} & L_{m-s}H_{2m-k}(U') & \to \cdots \end{array}.$$

Observe that Theorem 5.2 implies that

$$\pi_j(\mathcal{Z}^m(U)) \xrightarrow{\mathcal{D}} \pi_j(\mathcal{Z}_0(U)) \simeq H_j^{BM}(U)$$

is an isomorphism, where H_*^{BM} denotes Borel-Moore homology. This suggests, but does not imply, that the map $\Phi_* : \pi_j(\mathcal{Z}^m(U)) \to H^{2m-j}(U)$ of (3.4.1) is an isomorphism. For $U = X$ projective, the map Φ_* is an isomorphism thanks to the compatibility of the duality map with the Poincaré duality map demonstrated in [FL-2;4.4].

The duality isomorphism permits us to extend the homotopy invariance property proved in Proposition 3.5 to arbitrary affine torsors over a smooth base U.

Proposition 6.3. *Let $\pi : E \to U$ be an affine torsor for some smooth quasi-projective variety U. Then*

$$\pi^* : \mathcal{Z}^t(U) \to \mathcal{Z}^t(E)$$

is a homotopy equivalence.

Proof. Since $\pi : E \to U$ is locally for the Zariski topology on U a product projection $U \times \mathbf{A}^e \to U$, we conclude as in [F-G;2.3] that $\pi^* : \mathcal{Z}_r(U) \to \mathcal{Z}_{r+e}(E)$ is a homotopy equivalence, where e denotes the fibre dimension of π. The assertion now follows from Theorem 5.2 and the commutative diagram (4.3.1). $\square$

Remark 6.4. As realized by Blaine Lawson and the author, Proposition 6.3 permits one to extend the Chern classes defined in [FL-1] for vector bundles generated by their global sections to all vector bundles on a smooth, quasi-projective variety. Namely, given any such U, "Jouanolou's device" (cf. [Q])provides an affine torsor

$$p_J : J_U \quad \to \quad U$$

with U_J an affine variety. Then, given any algebraic vector bundle $E \to U$, we consider $p_J^*(E) \to J_U$ which is a vector bundle generated by its global sections. Thus, $p_J^*(E)$ is associated to some morphism $f_E : J_U \to Grass^e(\mathbf{P}^N)$. Embedding $Grass^e(\mathbf{P}^N)$ in $\mathfrak{C}_{N-e}(\mathbf{P}^N)$, we conclude that f_E determines a element in $\mathfrak{C}_{N-e}(\mathbf{P}^N)(J_U)$. The algebraic suspension theorem enables us to associate to this map an element

$$\langle f_E \rangle \quad \in \pi_0(\mathcal{Z}_0(\mathbf{P}^e)(J_U)) = \pi_0(\mathcal{Z}_0(\mathbf{P}^e)(U)).$$

Finally, the splitting construction of [FL-1] enables one to obtain from $\langle f_E \rangle$ elements

$$\langle f_E \rangle_t \; \in \; L^t H^{2t}(U) \quad , \quad 0 \leq t \leq e.$$

□

In [F-G;2.5], a **projective bundle theorem** was proved in the following form. Let E be a rank $e+1$ algebraic vector bundle over a smooth quasi-projective variety U and let $p : \mathbf{P}(E) \to U$ denote $Proj(Sym_{\mathcal{O}_X} E^*)$ over U. Let $c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ denote the "first Chern class operator for the canonical line bundle" $\mathcal{O}_{\mathbf{P}(E)}(1)$ on $\mathbf{P}(E)$ defined in terms of intersection with a global section. Then the following is a homotopy equivalence:

$$\mathcal{P} \stackrel{\text{def}}{=} \sum_{0 \leq j \leq e} c_1(\mathcal{O}_{\mathbf{P}(E)}(1))^j \circ p : \prod_{0 \leq j \leq e} \mathcal{Z}_{t+j}(U) \quad \to \quad \mathcal{Z}_{t+e}(\mathbf{P}(E)) \qquad (6.5.0).$$

This result is the key to the construction of further Chern classes in Lawson homology introduced by O. Gabber and the author in [F-G].

Theorem 5.2 immediately gives us the following cocycle version of (6.5.0).

Proposition 6.5. *Assume that U is smooth. With notation as above,*

$$\mathcal{D}^{-1} \circ \mathcal{P} \circ \mathcal{D} : \prod_{0 \leq j \leq e} \mathcal{Z}^{m-t-j}(U) \quad \to \quad \mathcal{Z}^{m-t}(\mathbf{P}(E))$$

is a homotopy equivalence, where $\mathcal{D}^{-1}$ is a homotopy inverse of $\mathcal{D}$.

Remark 6.6. As presented in [F-G], the construction of Chern classes requires the smoothness of U. If one could find a direct proof of Proposition 6.5 which did not use duality and therefore did not require the smoothness of U, then one should be able to extend that construction to algebraic vector bundles on normal varieties which are not smooth.

7. Examples

In this final section, we show how known computations of Lawson homology (i.e., homotopy groups of cycle spaces) permit computations of homotopy groups of cocycle spaces. Such computations appear highly non-trivial if one views (using Proposition 1.9) these cocycle spaces as the naïve group completions of $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$, the topological monoid of morphisms from U to the Chow monoid $\mathfrak{C}_r(Y)$.

We introduce the following alternate notation for cocycle spaces

$$\mathfrak{Mor}(U, \mathcal{Z}_r(Y)) \equiv [\mathfrak{Mor}(U, \mathfrak{C}_r(Y)]^+ = \mathcal{Z}_r(Y)(U)$$

in order to emphasize this mapping complex point of view and to compare more easily with the computations of [FL-2].

We begin by recalling that the homotopy groups of $\mathfrak{Mor}(U, \mathcal{Z}_r(Y))$ are merely the stabilized homotopy groups of $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$.

Proposition 7.1. Assume that U is normal and let $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$ denote the topological abelian monoid of morphisms from U to $\mathfrak{C}_r(Y)$ with the topology that of convergence of bounded degree (as in Proposition 1.9). Then

$$\pi_0 \mathfrak{Mor}(U, \mathcal{Z}_r(Y)) \ = \ [\Pi]^+ \ \ , \ \ \Pi \ \stackrel{\text{def}}{=} \ \pi_0 \mathfrak{Mor}(U, \mathfrak{C}_r(Y)).$$

For each connected component $\alpha \in \Pi$, let $\mathfrak{C}_{r,\alpha}(Y)(U)$ denote the corresponding connected component of $\mathfrak{C}_r(Y)(U)$ and choose some $Z_\alpha \in \mathfrak{C}_{r,\alpha}(Y)(U)$. Let $\{\alpha_n\}$ denote a sequence in which each element of a generating set of Π occurs infinitely often among the α_n's and set α_0 equal to the 0-component. Then for any $i > 0$, $\pi_i \mathfrak{Mor}(U, \mathcal{Z}_r(Y))$ equals the direct limit of the sequence given by translation by Z_{α_n}:

$$\rightarrow \pi_i \mathfrak{Mor}(U, \mathfrak{C}_{r, \sum_{j<n} \alpha_j}(Y)) \stackrel{(Z_{\alpha_n}+)_*}{\rightarrow} \pi_i \mathfrak{Mor}(U, \mathfrak{C}_{r, \sum_{j\leq n} \alpha_j}(Y)) \stackrel{(Z_{\alpha_{n+1}}+)_*}{\rightarrow} \dots$$

Proof. By Proposition 1.2, it suffices to identify the homotopy groups of the homotopy-theoretic group completion $\Omega B[\mathfrak{C}_r(Y)(U)]$. The computation of $\pi_*(\Omega B[\mathfrak{C}_r(Y)(U)])$ as the indicated direct limit is given in [F;2.6]. $\square$

Example 7.2. As a first, relatively trivial example, we consider $\mathfrak{Mor}(\mathbf{A}^m, \mathcal{Z}_r(Y))$. Then the homotopy invariance of Proposition 3.4 enables us to conclude that evaluation at $0 \in \mathbf{A}^m$ determines a homotopy equivalence

$$\mathfrak{Mor}(\mathbf{A}^m, \mathcal{Z}_r(Y)) \rightarrow \mathcal{Z}_r(Y).$$

In particular, setting $r = 0$, we conclude the homotopy equivalence

$$\mathfrak{Mor}(\mathbf{A}^m, \mathcal{Z}_0(Y)) \ \ \simeq \ \ \prod K(H_i(Y), i)$$

by the Dold-Thom theorem. $\square$

Example 7.3. We next consider the example of $\mathfrak{Mor}(\mathbf{A}^m - \{0\}, \mathcal{Z}_r(Y))$. By Proposition 6.1, we have the following map of fibration sequences:

$$\begin{array}{ccccc}
\mathfrak{Mor}(\{0\}, \mathcal{Z}_{r+m}(Y)) & \to & \mathfrak{Mor}(\mathbf{A}^m, \mathcal{Z}_r(Y)) & \to & \mathfrak{Mor}(\mathbf{A}^m - \{0\}, \mathcal{Z}_r(Y)) \\
= \downarrow & & \mathcal{D} \downarrow & & \downarrow \mathcal{D} \\
\mathcal{Z}_{r+m}(Y) & \to & \mathcal{Z}_{r+m}(Y \times \mathbf{A}^m) & \stackrel{j^*}{\to} & \mathcal{Z}_{r+m}(Y \times (\mathbf{A}^m - \{0\}))
\end{array} .$$

We observe that the map on homotopy groups induced by j^* admits a section, for the inverse of algebraic suspension $\Sigma\!\!\!/^m : \mathcal{Z}_r(Y) \to \mathcal{Z}_{r+m}(Y \times \mathbf{A}^m)$ is given by the Gysin map associated to the regular immersion $\{1\} \subset \mathbf{A}^m$ which factors through $\mathbf{A}^m - \{0\}$. Thus, we conclude that

$$\pi_i \mathfrak{Mor}(\mathbf{A}^m - \{0\}, \mathcal{Z}_r(Y)) \simeq \pi_i \mathcal{Z}_r(Y) \oplus \pi_{i-1} \mathcal{Z}_{r+m}(Y).$$

In particular, we conclude using Proposition 7.1 the existence of interesting elements in the homotopy of $\mathfrak{C}_r(Y)(\mathbf{A}^m - \{0\}) = \mathfrak{Mor}(\mathbf{A}^m - \{0\}, \mathfrak{C}_r(Y))$ which reflect the structure of $\mathcal{Z}_{r+m}(Y)$. $\square$

Example 7.4. By work of Lima-Filho [LF-1], any generalized flag manifold Y (or more generally, any projective variety Y with a "cell decomposition") has the property that $\pi \mathcal{Z}_r(Y)$ is naturally isomorphic to $H_{i+2r}(Y)$. Since the product of varieties with a cell decomposition again has such a decomposition, we can make explicit computations of homotopy groups of cocycle spaces as follows. Let $X_\infty \subset X$ be a closed immersion of projective varieties with a cell decomposition (e.g., a projective embedding of a generalized flag manifold X_∞ in a projective space $X = \mathbf{P}^m$) and let Y also be a projective variety with a cell decomposition. Set $U = X - X_\infty$. Then we conclude that $\mathfrak{Mor}(U, \mathcal{Z}_r(Y))$ has the homotopy type of $\mathcal{Z}_{r+m}(U \times Y)$. Thus,

$$\pi_* \mathfrak{Mor}(U, \mathcal{Z}_r(Y)) = H_{2r+2m+i}(X \times Y, X_\infty \times Y).$$

Once again, by applying Proposition 7.1, we conclude the existence of interesting elements in the homotopy of $\mathfrak{Mor}(U, \mathfrak{C}_r(Y))$. $\square$

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