Cycle Spaces and Intersection Theory

Eric M. Friedlander* and Ofer Gabber

to John Milnor on his 60th birthday

This paper constitutes a preliminary discussion of joint work in progress as presented by the first author at Stony Brook in June 1991 at the symposium in honor of John Milnor. Our results include a construction of an intersection pairing on spaces of algebraic cycles on a given smooth complex quasi-projective variety, thereby providing a ring structure in "Lawson homology." We verify that the Lawson homology for quasi-projective varieties satisfies sufficiently many of the "standard properties" of a good homology theory that it admits a theory of Chern classes from algebraic K-theory. The reader familiar with higher Chow groups of S. Bloch might find it useful to view Lawson homology as a topological analogue of that theory. In fact, we exhibit a tantalizing map from Bloch's higher Chow groups to Lawson homology.

Our subject is algebraic geometry. Until we discuss "algebraic bivariant cycle theory" in section 4, our ground field will always be the complex numbers $\mathbf{C}$. We shall consider *projective varieties*, reduced schemes over $\mathbf{C}$ which admit a closed embedding in some (complex) projective space P^N; as such, a projective variety is the zero locus of a family of homogeneous polynomials $\{F_\alpha(X_0, ..., X_N)\}$. The fact that we consider polynomial equations characterizes our study as algebraic geometry. More generally, we shall consider *quasi-projective varieties* which are complements of (algebraic) embeddings of one projective variety in another.

Our program is to study invariants of a given quasi-projective variety X using the group of all algebraic cycles on X of some fixed dimension r. The Lawson homology groups, $L_rH_*(X)$, are defined to be the homotopy groups of the "space of algebraic r-cycles on X." Recall that an algebraic $r-$cycle on X is a formal sum

$$Z \;=\; \Sigma m_i Y_i \quad , \; m_i \in \mathbf{Z}$$

where each Y_i is an irreducible closed subvariety of X of dimension r. If each m_i is positive, we say that Z is an *effective* $r-$ cycle. Typically, one considers equivalence classes of algebraic cycles. For example, the usual intersection theory

*Partially supported by the N.S.F. and NSA Grant # MDA904-90-H-4006

for smooth varieties involves the intersection of rational equivalence classes of cycles (elements of the “Chow ring”) whose “modern” formulation by W. Fulton and R. MacPherson [Fu] has much influenced the present work.

Rather than study equivalence classes of algebraic $r-$cycles, we study the topological group of all algebraic $r-$cycles on X. This point of view was initiated by Blaine Lawson in his path-breaking paper [L]. As motivation for this point of view, we recall the following fundamental theorem of A. Dold and R. Thom [D-T]: the homology groups of a finite simplicial complex T can be computed as the homotopy groups of the topological group A(T) defined as the free abelian group on the points of T. In [A], F.J. Almgren generalized this theorem by considering the topological group $I_r(T)$ of integral r-cycles (in the sense of currents) on T; in the special case $r = 0$, we have $I_0(T) = A(T)$. Almgren proved that the homotopy groups of $I_r(T)$ equal the homology of T (shifted in degree by r). Lawson had the daring and insight to "algebraicize" Almgren's investigations, replacing integral r-cycles by algebraic r-cycles. In the special case in which T equals the complex projective space P^N, Lawson proved that the space of effective $r-$cycles suitably “stabilized” also has homotopy groups which are the homology groups of P^N.

From the point of view of an algebraic geometer looking for new invariants, the story so far is not promising. However, Lawson achieved much more than his computations of the homotopy groups of cycle spaces for projective spaces. Namely, his “Lawson suspension theorem” [L] for the homotopy groups of the suitably stabilized space of effective $r-$cycles of a projective variety X provides the key property which leads to the development of “Lawson homology theory.” In particular, this theorem provides the key step permitting the author and Barry Mazur [F-M] to obtain qualitative results showing how these invariants interpolate between “algebraic cycles modulo algebraic equivalence” and (singular) homology of X. As discussed in [F-M], a naturally constructed filtration on singular homology (with $\mathbf{Q}-$coefficients) associated to Lawson homology is closely related to the Hodge filtration and Grothendieck's geometric filtration.

Although we consider Lawson homology in this paper, the reader might be interested in the “morphic cohomology theory” developed by the author and H.B. Lawson ([F-L]). This cohomology theory is constructed using the same formalism as Lawson homology but with a definition of an algebraic cocycle (i.e., an equidimensional family of algebraic cycles) which is not formally dual to that of an algebraic cycle. In many geometric contexts, “morphic cohomology” may well be a more natural invariant than Lawson homology and it is this theory that we algebraicize in section 4.

The algebro-geometric input for Lawson homology arises from consideration of *Chow varieties* of effective r-cycles on a projective variety X provided with a given (closed) embedding into some projective space P^N. Namely, a construction due in its final form to Chow and van der Waerden [C-W] associates in a 1-1 manner to an effective r-cycle Z on P^N of degree d a polynomial F_Z homogeneous of degree d in each of $(r+1)$ sets of $(N+1)-$ variables. As Z ranges over all effective r-cycles Z of degree d supported on X, the coefficients of these "Chow forms" constitute the points of a projective variety, denoted $C_{r,d}(X)$. Formal sum of cycles determines the *Chow monoid*

$$C_r(X) \equiv \coprod_{d \geq 0} C_{r,d}(X)$$

of effective r-cycles on X. The space of algebraic r-cycles, $Z_r(X)$, is defined as a "group completion" of this monoid provided with its analytic topology.

In brief, this paper is organized as follows. Section 1 discusses P. Lima-Filho's definition of the Lawson homology of quasi-projective varieties in terms of naive group completions and sketches alternate proofs we have found. In section 2, we sketch a proof of the "homotopy property" for Lawson homology (generalizing Lawson's suspension theorem) and use this to define the intersection operator associated to a (Cartier) divisor. Consequences include the "projective bundle theorem" and the independence of projective embedding of the s-operation of [F-M]. The extension of this intersection operator for an irreducible projective variety X to a pairing of cycle spaces

$$Z_r(X) \ \wedge \ \mathrm{Div}(X)^+ \ \to \ Z_{r-1}(X)$$

is discussed in section 3, where $\mathrm{Div}(X)^+$ is the homotopy-theoretic group completion of the topological monoid of effective Cartier divisors on X. For a variety smooth but not necessarily projective, we also present in section 3 our intersection pairing in the derived category

$$\tilde{Z}_r(X) \ \otimes \ \tilde{Z}_s(X) \ \to \ \tilde{Z}_{r+s-d}(X)$$

We then verify that this structure is sufficient to define Chern classes from algebraic K-theory to Lawson homology. The map from Bloch's higher Chow groups to Lawson homology is described in section 4 together with a tentatively defined refined version of Lawson homology which captures rational equivalence rather than simply algebraic equivalence of algebraic cycles and which is "bivariant" in the sense of [F-L].

We are indebted to P. Lima-Filho for sharing preprints of his recent work. C. Soulé made the valuable suggestion that our results should be formulated in terms of the derived category. The first author gratefully thanks I.H.E.S. for its warm hospitality during the formative stages of this work.

1. Lawson homology

In this section, we recall the foundational constructions of Lawson homology. We shall find it convenient to work in the category $D(\mathbf{Z})$, the additive category of chain complexes localized with respect to quasi-isomorphisms, maps of chain complexes inducing isomorphisms in homology (cf. [V] for a discussion of the derived category of cochain complexes). By a theorem of A. Dold and D.M. Kan [D], sending a simplicial abelian group C to its normalized chain complex $N(C)$ determines an equivalence of categories such that $\pi_*(C) = H_*(N(C))$. Since the singular complex functor Sing.(-) sends an abelian topological group to a simplicial abelian group, we conclude that the functor

$$A \ \mapsto\ \tilde{A} \ \equiv\ \mathrm{NSing.}(A)$$

induces an equivalence of categories between the category of abelian topological groups (in the sense of Kelley spaces) localized with respect to continuous homomorphisms which are weak equivalences and the full subcategory of $D(\mathbf{Z})$ of non-negatively graded objects. Under this correspondence, a product $A \times B$ of abelian topological groups is sent to the sum $\tilde{A} \oplus \tilde{B}$ of corresponding chain complexes; similarly, a bilinear map $A \times B \to C$ is sent to a pairing $\tilde{A} \otimes \tilde{B} \to \tilde{C}$ of corresponding chain complexes.

Let us begin with a re-formulation of Lawson homology. Recall that if X is a projective variety, then $Z_r(X)$ denotes the topological group of algebraic r-cycles, whose topology is the quotient topology associated to the surjective map $C_r(X)^{\times 2} \to Z_r(X)$ where $C_r(X)$ is the Chow monoid of effective algebraic r-cycles on X. The homeomorphism type of $Z_r(X)$ is independent of the choice of closed embedding of X in some projective space. For $r > \dim X$, $Z_r(X)$ consists of a single point (associated to the empty r-cycle on X).

Definition 1.1. Let X be a projective variety and let r be a non-negative integer. For any integer $n \geq 2r$, we define the Lawson homology group $L_rH_n(X)$ as

$$L_rH_n(X) \ \equiv\ \pi_{n-2r}(Z_r(X)) \ \cong\ H_{n-2r}(\tilde{Z}_r(X)).$$

The above definition implicitly employs an important theorem of P. Lima-Filho [Li-1]: the natural map of H-spaces $C_r(X) \to Z_r(X)$ is homotopy equivalent to $C_r(X) \to \Omega BC_r(X)$, the homotopy-theoretic group completion. We offer an alternative, and somewhat more general, formulation of Lima's theorem which may shed some light on the group completion process. We first introduce a definition of a "tractable" action.

Definition 1.2. Let M be a topological monoid and let T be a topological space on which M acts on the right. Then T is said to be a *tractable* M-space if it satisfies the following property: T admits an increasing filtration $\{T_n; n \geq -1)\}$ such that T_{-1} is empty, T is the topological union of the T_n, and for each $n > 0$ the inclusion $T_{n-1} \subset T_n$ fits in a pushout square (with R_0 also empty)

$$\begin{array}{ccc} R_n \times M & \to & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \to & T_n \end{array}$$

whose upper horizontal arrow is induced by some cofibration $R_n \subset S_n$ of Hausdorff spaces. The topological monoid M is said to be tractable if it is abelian and if the diagonal action of M on $T = M^{\times 2}$ is tractable. The *naive group completion* M^+ of an abelian topological monoid M is defined to be the quotient of $M^{\times 2}$ under this diagonal action with the quotient topology.

The following proposition verifies that our definition of a tractable M-space is sufficiently general to include the examples one encounters in the study of Chow monoids.

Proposition 1.3. Let X be a projective variety, $Y \subset X$ a closed subvariety, and $r \in \mathbf{N}$.
i.) $C_r(Y)$ is a tractable monoid.
ii.) The inclusion $C_r(Y) \subset C_r(X)$ provides $C_r(X)$ with the structure of a tractable $C_r(Y) -$ space.
iii.) The quotient $C_r(X)/C_r(Y)$ is a tractable monoid.

sketch of proof. To prove i.), let $M(d)$ denote $C_{r,d}(Y)$ (with the analytic topology), let M denote $C_r(Y)$, T the space $M^{\times 2}$ with the diagonal $M -$ action. We denote by T_n the M-subspace of T generated by $M(a) \times M(b)$ with $\nu(a,b) \leq n$ (for a suitable bijection $\nu : \mathbf{N}^{\times 2} \to \mathbf{N}$):

$$T_n \equiv [\cup_{\nu(a,b) \leq n} M(a) \times M(b)] \cdot M$$

Set

$$S_n \equiv M(a_n) \times M(b_n) \quad , \quad \nu(a_n, b_n) = n$$

and denote by R_n the following subspace of S_n :

$$R_n \equiv \text{image}\{\cup_{c>0} M(a_n - c) \times M(b_n - c) \times M(c) \to M(a_n) \times M(b_n)\}$$

Then we readily check that we have a pushout diagram of topological spaces

$$\begin{array}{ccc} R_n \times M & \to & S_n \times M \\ \downarrow & & \downarrow \\ T_{n-1} & \to & T_n \end{array}$$

Recall from [H] that any projective variety Z admits a triangulation by semi-algebraic simplices which can be chosen so that any specified finite collection of semi-algebraic closed subsets consists of subcomplexes. Since the image of a semi-algebraic set under a continuous algebraic map such as the multiplication maps

$$M(a-c) \times M(b-c) \times M(c) \ \rightarrow \ M(a) \times M(b)$$

is again semi-algebraic, we can inductively provide $S_n \ \equiv \ M(a_n) \times M(b_n)$ with a semi-algebraic triangulation so that $R_n \subset S_n$ is a subcomplex. This completes the proof of i.)

For ii.), we continue to let M denote $C_r(Y)$ but now let T denote $C_r(X)$. Define T_n , S_n, and R_n by

$$T_n \ \equiv \ [\cup_{d \leq n} C_{r,d}(X)] \cdot M$$

$$S_n \ \equiv \ C_{r,n}(X)$$

$$R_n \ \equiv \ \text{image}\{\cup_{c>0} C_{r,n-c}(X) \times M(c) \rightarrow \ C_{r,n}(X)\}.$$

For iii.), we let M denote $C_r(X)/C_r(Y)$, C_d denote $C_{r,d}(X)$, C'_d denote $C_{r,d}(Y)$, and T denote $M \times M$. Define T_n , S_n, and R_n by

$$T_n \ \equiv \ \text{image}\{\cup_{\nu(a,b) \leq n} C_a \times C_b \rightarrow M \times M\} \cdot M$$

$$S_n \ \equiv \ C_{a_n} \times C_{b_n}$$

$$\begin{aligned} R_n \ \equiv \ & \text{image}\{\cup_{c>0} C_{a_n-c} \times C_{b_n-c} \times C_c \rightarrow \ C_{a_n} \times C_{b_n}\} \\ & \cup \ \text{image}\{\cup_{c>0} C'_c \times C_{a_n-c} \times C_{b_n} \rightarrow \ C_{a_n} \times C_{b_n}\} \\ & \cup \ \text{image}\{\cup_{c>0} C_{a_n} \times C_{b_n-c} \times C'_c \rightarrow \ C_{a_n} \times C_{b_n}\}. \ \blacksquare \end{aligned}$$

The tractability of the abelian topological monoid $C_r(X)$ enables us to deduce Lima's theorem (asserting that $C_r(X) \ \rightarrow \ Z_r(X)$ is a homotopy equivalent to $C_r(X) \rightarrow \Omega BC_r(X)$) as a consequence of the following theorem and of the well-known results of D. Quillen (to appear as an appendix of [F-M]) giving that if A is an abelian simplicial monoid and if $A \rightarrow A^+$ is the homomomorphism of simplicial monoids which is level-by-level group completion which we can view as $A^{\times 2}/A$, then the induced map $H_*(A) \rightarrow H_*(A^+)$ is localization of the action of $\pi_0(A)$.

Theorem 1.4. A tractable monoid has the "cancellation property" (i.e., mn = mp implies n=p). If M is any abelian topological monoid with the "cancellation property" and if T a tractable M-space, then the natural map

$$\text{Sing.}T/\text{Sing.}M \ \rightarrow \ \text{Sing.}(T/M)$$

is a weak equivalence of simplicial sets.

sketch of proof. If M is a tractable monoid, then $M \times M$ as an M-set (with the diagonal action) is equivalent to a disjoint union of copies of M with M-action given by addition. This easily implies that a tractable monoid M has the cancellation property.

If M is an abelian topological monoid with the cancellation property and if T is a tractable M-space, then the action of Sing.M on Sing.T is torsion free. To prove the asserted homotopy equivalence, one first verifies that $B(S,P) \to S/P$ is a weak equivalence, where B(S,P) is the diagonal of the simplicial bar construction associated to a torsion free action of an abelian simplicial monoid P on a simplicial set S (so that $B(S,P)_n = S_n \times P_n^{\times n}$). Viewing this map as the diagonal of a map of bisimplicial sets, one is reduced to proving that $p : B(T,A) \to T/A$ is a weak equivalence where T is a set on which the discrete abelian monoid A acts torsion freely. Indeed, using the hypothesis of a torsion free action, each fiber $p^{-1}(\bar{t})$ "is" the simplicial compex associated to a partially ordered set; this complex is contractible since any finite subcomplex is contained in a subcomplex associated to a partially ordered subset with a largest element.

It then suffices to prove that the natural projection

$$B(\mathrm{Sing}.T, \mathrm{Sing}.M) \to \mathrm{Sing}.(T/M)$$

is a weak equivalence. (This does not require the monoid M to be abelian.) We inductively assume that

$$B(\mathrm{Sing}.T_{n-1}, \mathrm{Sing}.M) \to \mathrm{Sing}.(T_{n-1}/M)$$

is a weak equivalence. Let Π_n denote the pushout square of Definition 1.2. We readily verify that $B(\mathrm{Sing}.\Pi_n, \mathrm{Sing}.M)$ is a homotopy-theoretic pushout square of simplicial sets by comparing this commutative square with the intermediate square $B(\Gamma_n, \mathrm{Sing}.M)$, where Γ_n is the simplicial pushout square associated to $\mathrm{Sing}.(R_n \times M) \to \mathrm{Sing}.T_n$ and $\mathrm{Sing}.(R_n \times M) \to \mathrm{Sing}.(S_n \times M)$. Moreover, $B(\mathrm{Sing}.\Pi_n, \mathrm{Sing}.M)$ maps to the commutative square

$$\begin{array}{ccc} \mathrm{Sing}.R_n & \to & \mathrm{Sing}.S_n \\ \downarrow & & \downarrow \\ \mathrm{Sing}.(T_{n-1}/M) & \to & \mathrm{Sing}.(T_n/M) \end{array}$$

which is also a homotopy-theoretic pushout square, since it is obtained by applying Sing.(-) to a pushout square. The theorem now follows by comparing homotopy pushout squares. ■

We recall from [F-M] a heuristic interpretation of the Lawson homology groups. We begin with the observation that $L_rH_{2r}(X)$ is the group of algebraic r-cycles modulo algebraic equivalence [F], whereas the Dold-Thom theorem can be interpreted as asserting that $L_0H_n(X) = H_n(X)$, the singular homology of X. In [F-M], an operation

$$s: \; L_rH_n(X) \;\to\; L_{r-1}H_n(X)$$

is introduced with the property that $s^{\circ r}: L_rH_{2r}(X) \to H_{2r}(X)$ is the "cycle map" sending an algebraic r-cycle Z to its fundamental homology class $[Z] \in H_{2r}(X)$. The constructions of [F-M] justify the interpretation of the image in $H_n(X)$ of $L_rH_n(X)$ under $s^{\circ r}$ as consisting of homology classes "with at least r algebraic parameters." Namely, a homotopy class of maps $\alpha : S^{n-2r} \to Z_r(X)$ corresponds to a family of algebraic r-cycles on X parametrized by the sphere S^{n-2r}. The "fundamental homology class" of the total space of this family is the image of α in $H_n(X)$.

The definition of the Lawson homology of quasi-projective varieties is due to P. Lima-Filho [Li-2]. Although this is a natural extension of Definition 1.1, the reader should take note of the fact that in the special case of 0-cycles this extension yields the homology of locally finite chains on X and not the usual singular homology of X.

Definition 1.5. (cf. Lima-Filho [Li-2]) Let X be a quasi-projective variety with projective closure $\bar{X}$. We define $Z_r(X)$ by

$$Z_r(X) \;\equiv\; Z_r(\bar{X})/Z_r(\bar{X} - X)$$

and we define the Lawson homology groups of X to be

$$L_rH_n(X) \;\equiv\; \pi_{n-2r}(Z_r(X)) \simeq H_{n-2r}(\tilde{Z}_r(X))$$

which by Theorem 1.6 below is independent of the choice $X \subset \bar{X}$ of projective closure.

Covariant functorality

$$f_*: \; \tilde{Z}_r(X) \to \tilde{Z}_r(Y)$$

for a proper map $f : X \to Y$ follows as in [F]. Contravariant functorality

$$g^* : \tilde{Z}_r(X) \;\to\; \tilde{Z}_{r+e}(X')$$

for a flat morphism $g : X' \to X$ of pure relative dimension e is also valid. For X' and X projective, this is verified in [F]; similar techniques apply in the somewhat more complicated context of algebraic varieties which are only locally closed in projective space. Namely, one verifies that for projective closures $\bar{X}, \bar{X}'$, the pull-back by g determines a closed correspondence from $C_r(\bar{X})$ to $C_{r+e}(\bar{X}')$ which induces a well defined set-theoretic map $g^* : Z_r(X) \to Z_{r+e}(X')$ which is necessarily continuous.

When considering certain functoriality properties, we shall occasionally admit non-reduced quasi-projective schemes, but the invariants that we consider invariably depend only upon the associated reduced schemes.

The following theorem, originally due to P. Lima-Filho, justifies the preceding definition.

Theorem 1.6. ([Li-1]) Retain the notation of Definition 1.5. Then $\tilde{Z}_r(X)$ is independent (in the derived category) of the choice of projective embedding $X \subset \bar{X}$. Moreover, if $Y \subset X$ is a (Zariski) closed subset with complement $U \subset X$, then

$$\tilde{Z}_r(Y) \ \rightarrow \ \tilde{Z}_r(X) \ \rightarrow \ \tilde{Z}_r(U)$$

gives a distinguished triangle of chain complexes, thereby yielding a natural "localization" long exact sequence in Lawson homology

$$... \rightarrow L_rH_n(Y) \rightarrow L_rH_n(X) \rightarrow L_rH_n(U) \rightarrow L_rH_{n-1}(X) \rightarrow ...$$

sketch of proof. To prove that $\tilde{Z}_r(X)$ is independent of projective embedding $X \subset \bar{X}$, we identify $Z_r(X)$ with the naive group completion of the quotient monoid $C_r(\bar{X}, \bar{X} - X) \equiv C_r(\bar{X})/C_r(\bar{X} - X)$. In comparing the result for two different compactifications, one easily reduces to the case in which one compactification $X \subset \bar{X}$ is dominated by another $X \subset \hat{X}$ so that one has a continuous bijective map $\Psi : C_r(\hat{X}, \hat{X} - X) \ \rightarrow \ C_r(\bar{X}, \bar{X} - X)$; one proves that Ψ is also closed by showing for any $d \geq 0$ that cycles of $C_r(\bar{X}, \bar{X} - X)$ which are images of cycles of $C_r(\bar{X})$ of degree bounded by d lie in the image under Ψ of cycles of some bounded degree in $C_r(\hat{X})$. This is verified by a straight-forward argument using noetherian induction and generic flatness: namely, any algebraic family of subschemes on $\bar{X}$ parametrized by some variety C is generically flat over C, and the closure in $\hat{X} \times C$ of the intersection of the family with $X \times C$ is generically flat over C.

To prove the second assertion, one easily reduces to the case in which X is projective. Thanks to Proposition 1.3, it then suffices to prove the following: Let $M \subset N$ be a closed immersion of tractable monoids such that N is tractable as an $M-$space and such that the quotient monoid N/M is also tractable. Then the sequence of topological groups

$$M^+ \ \rightarrow \ N^+ \ \rightarrow \ (N/M)^+$$

yields a distinguished triangle of chain complexes. This is proved by applying Theorem 1.4 to show that the vertical maps of the following commutative diagram of simplicial abelian groups

$$\begin{array}{ccccccccc}
0 & \rightarrow & (\mathrm{Sing.}(M))^+ & \rightarrow & (\mathrm{Sing.}(N))^+ & \rightarrow & (\mathrm{Sing.}(N)/\mathrm{Sing.}(M))^+ & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Sing.}(M^+) & \rightarrow & \mathrm{Sing.}(N^+) & \rightarrow & \mathrm{Sing.}((N/M)^+) & &
\end{array}$$

are weak equivalences. ■

As a first consequence of Theorem 1.6, we construct the "Lawson analogue" of Quillen's local-to-global spectral sequence in algebraic K-theory [Q]. Denote by $S_p(X)$ (or, S_p) the set of all closed subsets $Y \subset X$ with $\dim(Y) \leq p$ and by S_p/S_{p-1} the partially ordered set of pairs

$$S_p/S_{p-1} \;=\; \{(Y,Y') : Y' \subset Y \subset X \;\; \text{closed}; \dim(Y') < p; \dim(Y) \leq p\}.$$

We define

$$L_rH_n(X;p) \;\equiv\; \mathrm{colim}_{S_p} L_rH_n(Y)$$

$$L_rH_n(X;p,p-1) \;\equiv\; \mathrm{colim}_{S_p/S_{p-1}} L_rH_n(Y-Y').$$

Finally, if $x \in X$ is a point of dimension p, we abuse notation by denoting the closure of $\{x\}$ by $\{\bar{x}\}$ and we define

$$L_rH_n(x) \;\equiv\; \mathrm{colim}_{U \subset \{\bar{x}\}} L_rH_n(U)$$

where the (direct) limit is taken over non-empty Zariski open subsets of $\{\bar{x}\} \in S_p$.

Proposition 1.7. For any quasi-projective variety X and any $r \geq 0$, there is a spectral sequence of homological type of the following form:

$$E^1_{p,q} \;=\; \oplus_{\{\bar{x}\} \in S_p/S_{p-1}} L_rH_{p+q}(x) \;\Rightarrow\; L_rH_{p+q}(X)$$

sketch of proof . Theorem 1.6 implies that

$$L_rH_n(X;p,p-1) \;\cong\; \oplus_{\{\bar{x}\} \in S_p/S_{p-1}} L_rH_n(x)$$

as well as the long exact sequence

$$\ldots \to L_rH_n(X;p-1) \to L_rH_n(X;p) \to L_rH_n(X;p,p-1)$$

$$\to L_rH_{n-1}(X;p-1) \to \ldots$$

The construction of an exact couple and resulting spectral sequence follow in the familiar way. ■

2. Intersection with divisors

We begin by recalling the "Lawson suspension theorem", a result fundamental to all that follows. If X is a projective variety provided with a specified embedding $X \subset P^N$ in some projective space, we consider the "algebraic suspension" $\Sigma X \subset$

P^{N+1} given as the cone of X with vertex some point $x_\infty \in\ P^{N+1} - P^N$. Algebraic suspension induces a continuous homomorphism

$$\Sigma :\ Z_r(X)\ \rightarrow\ Z_{r+1}(\Sigma X).$$

Theorem 2.1. (Lawson [L]) Let $X \subset P^N$ be a projective variety provided with an embedding into projective space. Then

$$\Sigma :\ Z_r(X)\ \rightarrow\ Z_{r+1}(\Sigma X)$$

is a weak equivalence. Equivalently,

$$\Sigma :\ \tilde{Z}_r(X)\ \rightarrow\ \tilde{Z}_{r+1}(\Sigma X)$$

is an isomorphism in the derived category $D(\mathbf{Z})$.

The beginnings of intersection theory on cycle spaces were apparent in [F-M]. The first author and B. Mazur considered the following question: can one provide a definition of intersection of all algebraic r-cycles on $X \subset P^N$ with a given hyperplane section H? An affirmative answer is given by the following construction, which reveals in simple form several aspects of our intersection theory.

Construction 2.2. ([F-M]). Let $X \subset P^N$ be a projective variety provided with an embedding into projective space. Then for any $r \geq 0$ the composition

$$\Sigma^{-1} \circ i :\ \tilde{Z}_{r+1}(X)\ \rightarrow\ \tilde{Z}_{r+1}(\Sigma X)\ \rightarrow\ \tilde{Z}_r(X)$$

represents intersection with a hyperplane section H, where $i : X \rightarrow \Sigma X$ is the natural inclusion and $\Sigma : \tilde{Z}_r(X)\ \rightarrow\ \tilde{Z}_{r+1}(\Sigma X)$ is the quasi-isomorphism of Theorem 2.1.

Observe that this gives a "value" to intersection of H with an r-cycle contained in H. Rather than move such a cycle off H and then intersect, we use the flexibility given by the derived category to invert the algebraic suspension map. The key ingredient in this construction is, of course, Theorem 2.1. In fact, we view Theorem 2.1 as a new form of more classical "Moving Lemmas." The assertion that $\Sigma^{-1} \circ i$ represents intersection with H is justified by several criteria: it is compatible with intersection with the homology class of H when one uses the $s-$operation to pass to homology; on cycles which intersect H properly, it is given by the intersection-theoretic construction of intersecting with H.

We proceed to generalize Construction 2.2 to provide an intersection theory for (Cartier) divisors more general than a hyperplane section. At the same time, we shall eliminate the condition that X be projective. The key to this generalization is Proposition 2.3, the "Homotopy Property" for Lawson homology generalizing

Theorem 2.1 (which is essentially the special case in which X is projective and $E = O_X(1)$).

Let X be a quasi-projective variety and E a locally free, coherent O_X −module of rank e. Denote by $\pi : V(E) \to X$ the associated vector bundle over X ($V(E)$ is the relative spectrum of the symmetric algebra over O_X of the O_X-dual E^* of E).

Proposition 2.3. Let X be a quasi-projective variety and E a locally free, coherent O_X − module of rank e. Then sending a cycle Z on X to its pull-back via π on $V(E)$ determines a quasi-isomorphism

$$\pi^* : \tilde{Z}_r(X) \ \to \ \tilde{Z}_{r+e}(V(E)).$$

sketch of proof. Since $\pi : V(E) \to X$ is flat, π^* is well defined. One uses the localization sequence to reduce the proof of the theorem to the case in which $V(E)$ is a trivial rank e bundle. Arguing by induction on e, one easily reduces to the case in which $V(E)$ is a trivial, rank 1 bundle. Using localization once again, one reduces to the case that X admits a projective closure $\bar{X}$ such that E is the restriction of $O_{\bar{X}}(1)$. Finally, localization reduces us to the special case in which X is a projective variety and $E = O_X(1)$. ■

A first generalization of Construction 2.2 is to define for $r \geq 0$

$$c_1(L) \equiv (\pi^*)^{-1} \circ o_* : \ \tilde{Z}_{r+1}(X) \ \to \ \tilde{Z}_{r+1}(V(L)) \ \to \ \tilde{Z}_r(X)$$

for any quasi-projective variety X and any locally free, rank 1 O_X − module L, where o_* is the map induced by the inclusion of the 0-section $X \subset V(L)$.

We proceed to sharpen this definition of $c_1(L)$ so that intersection with the (effective) divisor D takes values in $\tilde{Z}_r(|D|)$, where $i_D : |D| \subset X$ denotes the (closed) embedding of the support of D in X with complement $U \subset X$. Consider the composition

$$\mathrm{res} \circ s_{D*} : \ Z_{r+1}(X) \ \to \ Z_{r+1}(V(L)) \ \to \ Z_{r+1}(V(L)_{|U})$$

where $s_D : X \to V(L)$ is the map associated to the canonical section of $L = O(D)$. Then $\mathrm{res} \circ s_{D*}$ admits a natural homotopy to the 0-map; namely, for varying t with $0 \leq t < \infty$, composition of $\mathrm{res} \circ s_{D*}$ with fibre-wise multiplication by $1+t$ together with the 0-map for $t = \infty$ provides such a homotopy. This homotopy determines a map to the homotopy fibre of $Z_{r+1}(V(L)) \to Z_{r+1}(V(L)_{|U})$

$$Z_{r+1}(X) \ \to \ \mathrm{htyfib}(Z_{r+1}(V(L)) \to Z_{r+1}(V(L)_{|U}))$$

which by the localization theorem provides a map

$$Z_{r+1}(X) \ \to \ Z_{r+1}(V(i_D^* L))$$

in the derived category of topological abelian groups. Said a bit differently, since the homotopy consists of a family of continuous group homomorphisms, it provides a chain homotopy between $\mathrm{res} \circ s_{D*} : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(L)_{|U})$ and the 0-map and thus a map in the derived category

$$\sigma_D : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(i_D^* L)).$$

Observe that σ_D is determined (as a map in $D(\mathbf{Z})$) by our choice of homotopy relating $\mathrm{res} \circ s_{D*}$ to the 0-map.

Theorem 2.4. Let X be a quasi-projective variety and let D, D' be effective Cartier divisors on X. For any $r \geq 0$, define

$$i_D^! \equiv (\pi^*)^{-1} \circ \sigma_D : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(i_D^* L)) \to \tilde{Z}_r(|D|)$$

where $\sigma_D : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(i_D^* L))$ is the map in $D(\mathbf{Z})$ constructed above using the canonical section of V(L) for $L = O(D)$ and the natural homotopy of $\mathrm{res} \circ s_{D*}$ to the 0-map.
a.) If $Z_{r+1}(X, D)$ (mapping to $Z_{r+1}(X)$) denotes the naive group completion of the submonoid of $C_{r+1}(X) \equiv C_{r+1}(\bar{X})/C_{r+1}(\bar{X} - X)$ of effective cycles which meet D properly with the k-topology, then $i_D^!$ restricted to $\tilde{Z}_{r+1}(X, D)$ is induced by the usual intersection with D.
b.) The composition

$$c_1(L) \equiv i_{D*} \circ i_D^! : \tilde{Z}_{r+1}(X) \to \tilde{Z}_r(|D|) \to \tilde{Z}_r(X)$$

depends only on the isomorphism class of $L = O(D)$.
c.) Additivity:

$$i_D^! + i_{D'}^! = i_{D+D'}^! : \tilde{Z}_{r+1}(X) \to \tilde{Z}_r(|D| \cup |D'|).$$

d.) Commutativity for any $r \geq 1$:

$$i_D^! \circ i_{D'}^! = i_{D'}^! \circ i_D^! : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r-1}(|D| \cap |D'|)$$

(where the composition is defined using the evident generalization of the construction of $i_D^!$ to pseudo-divisors as in Proposition 3.2 if $|D|$ and $|D'|$ do not meet properly).

sketch of proof: Let D^ω denote the Weil divisor associated to D. Then the usual definition (in the case of proper intersection) of the intersection of a subscheme $f : V \subset X$ with D is $(f^*D)^\omega$. The pull-back of this cycle to $V(L)$ is the cycle of the scheme-theoretic normal cone $C_W(V)$ (cf. [Fu]) of the scheme-theoretic intersection $W = V \cap D$ in V. On the other hand (see the discussion prior to Proposition 3.3), the deformation space $Q_D(X)$ of the deformation to the normal cone which maps to $V(L)$ gives an explicit (continuous) lifting of the homotopy for σ_D to a homotopy

$$Z_{r+1}(X,D) \times [1,\infty] \to Z_{r+1}(V(L)),$$

thereby specifying σ_D on $Z_{r+1}(X,D)$ as sending a variety V which meets D properly to $C_W(V)$.

Composing s_D with scalar multiplication by 1-t for $0 \leq t \leq 1$ gives a homotopy through continuous group homomorphisms relating $s_{D*} : Z_{r+1}(X) \to Z_{r+1}(V(L))$ to the map $o_{L*} : Z_{r+1}(X) \to Z_{r+1}(V(L))$ induced by inclusion of the 0-section $X \subset V(L)$. Thus, one may in the definition of $i_D^!$ replace s_{D*} by the map o_* provided that one uses as homotopy $(-) + ts_D$ for $0 \leq t \leq \infty$. In particular,

$$c_1(L) \ = \ (\pi^*)^{-1} \circ s_{D*} \ = \ (\pi^*)^{-1} \circ o_*$$

is independent (as a map in $D(\mathbf{Z})$) of the choice of D as asserted in b.).

To relate $i_D^! + \ i_{D'}^!$ to $i_{D+D'}^!$, we employ the natural flat map

$$\tau : \ V(L \oplus L') \ \to \ V(L \otimes L')$$

given by sending the local section $(s,s') \in (L \oplus L')(U)$ to $ss' \in (L \otimes L')(U)$. Then,

$$\tau^* \circ o_{L \otimes L' *} \ = \ pr_1^* \circ o_{L*} + pr_2^* \circ o_{L'*}.$$

We demonstrate the existence of a homotopy between the 0-homotopies

$$\tau^* \circ (o_{L \otimes L' *} + ts_D s_{D'}) \ , \ pr_1^* \circ (o_{L*} + ts_D) + pr_2^* \circ (o_{L'*} + ts_{D'})$$

(over the complement of $|D| \cup |D'|$). This reduces to exhibiting a homotopy in P^2 (the completion of a single fibre) between the closures of two families of degree 2 1-cycles in A^2

$$\{xy = t\} \quad , \quad \{(x-t)(y-t) = 0\}$$

whose existence follows from the fact that $C_{1,2}(P^2)$ is simply connected.

Finally, to prove commutativity, we employ the following commutative "3×3 diagram" whose rows and columns give distinguished triangles

$$\begin{array}{ccccc}
\tilde{Z}_{r+1}(V(L'')_{|D| \cap |D'|}) & \to & \tilde{Z}_{r+1}(V(L'')_{|D'|}) & \to & \tilde{Z}_{r+1}(V(L'')_{V'}) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{Z}_{r+1}(V(L'')_{|D|}) & \to & \tilde{Z}_{r+1}(V(L'')) & \to & \tilde{Z}_{r+1}(V(L'')_U) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{Z}_{r+1}(V(L'')_V) & \to & \tilde{Z}_{r+1}(V(L'')_{U'}) & \to & \tilde{Z}_{r+1}(V(L'')_{U \cap U'})
\end{array}$$

where $L'' = L \oplus L'$, $U = X - |D|$, $U' = X - |D'|$, V $= |D| - |D| \cap |D'|$, $V' = |D'| - |D| \cap |D'|$, . The composition $i_{D'}^! \circ i_D^!$ is defined by first lifting s_{D*} : $\tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(L))$ to $\tilde{Z}_{r+1}(V(L)_{|D|})$, then lifting $s_{D'*} : \tilde{Z}_{r+1}(V(L)_{|D|}) \to$

$\tilde{Z}_{r+1}(V(L'')_{|D|})$ to $\tilde{Z}_{r+1}(V(L'')_{|D|\cap|D'|})$ with the liftings given by the homotopies given by fibrewise multiplication by scalars, then applying the inverse of π^* : $\tilde{Z}_{r-1}(|D| \cap |D'|) \to \tilde{Z}_{r+1}(V(L'')_{|D|\cap|D'|})$. Similar homotopies not only determine the composition $i_D^! \circ i_{D'}^!$, but also determine a lifting of $(s_D, s_{D'})_* : \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(V(L''))$ to $\tilde{Z}_{r+1}(V(L'')_{|D|\cap|D'|})$ as the double "homotopy fibre" of the lower right hand square of the above diagram, which realizes both compositions of the assertion. In particular, we conclude commutativity as asserted in d.). ■

Observe that the alternate construction of $c_1(L)$ given in the proof of part b.) of Theorem 2.4 yields the map $\Sigma^{-1} \circ i$ of Proposition 2.2 in the special case in which X is provided with a closed embedding $X \subset P^N$ and $L = O_X(1)$. Thus, part c.) of Theorem 2.4 provides an extension of the h-operation of [F-M] to an action of Pic(X).

Once one has a good definition of intersection with a divisor, one obtains a "projective bundle theorem" for Lawson homology.

Proposition 2.5. Let E be a rank $k+1$ vector bundle over a quasi-projective variety X, let $p : P(E) \to X$ denote $\mathrm{Proj}(\mathrm{Sym}_{O_X} E^*)$ over X, and let $O_{P(E)}(1)$ denote the canonical line bundle on P(E) defined as a quotient of p^*E^*. Then

$$\Phi \equiv \sum_{0\leq j\leq k} c_1(O_{P(E)}(1))^j \circ p^* : \bigoplus_{0\leq j\leq k} \tilde{Z}_{r+j}(X) \to \tilde{Z}_{r+k}(P(E))$$

is a quasi-isomorphism for any $r \geq 0$.

sketch of proof: Using the localization sequence and the naturality of Φ, we reduce ourselves to the case in which E is trivial. For $t \leq k$, let $p_t : X \times P^t \to X$ denote the projection and let $i_t : X \times P^t \to X \times P^k$ denote the map induced by a standard linear embedding $P^t \subset P^k$. The equality (in D($\mathbf{Z}$))

$$c_1(L)^{\circ k-j} \circ p_k^* = i_{j*} \circ p_j^* : \tilde{Z}_{r+k-j}(X) \to \tilde{Z}_{r+k}(X \times P^k)$$

implies that Φ (for $P(E) = X \times P^k$ and $E = pr_2^* O_{P^k}(1)$) is equivalent to

$$\Phi_k \equiv \sum_{0\leq j\leq k} i_{j*} \circ p_j^*.$$

On the other hand, Φ_k fits in a commutative square of simplicial abelian groups

$$\begin{array}{ccc} \bigoplus_{1\leq j\leq k} \tilde{Z}_{r+j}(X) & \to & \bigoplus_{0\leq j\leq k} \tilde{Z}_{r+j}(X) \\ \Phi_{k-1}\downarrow & & \downarrow\Phi_k \\ \tilde{Z}_{r+k}(X \times P^{k-1}) & \to & \tilde{Z}_{r+k}(X \times P^k) \end{array}$$

We therefore obtain a map of distinguished triangles

$$\begin{array}{ccccc}
\bigoplus_{1\leq j\leq k}\tilde{Z}_{r+j}(X) & \to & \bigoplus_{0\leq j\leq k}\tilde{Z}_{r+j}(X) & \to & \tilde{Z}_r(X) \\
\Phi_{k-1}\downarrow & & \downarrow\Phi_k & & \downarrow\pi^* \\
\tilde{Z}_{r+k}(X\times P^{k-1}) & \to & \tilde{Z}_{r+k}(X\times P^k) & \to & \tilde{Z}_{r+k}(X\times A^k)
\end{array}$$

The proof is completed using induction, the 5-Lemma, and the "homotopy property" for the trivial bundle projection $X\times A^k\to X$. ■

Using Proposition 2.5, we define as in [Fu] the *Segre class operators* associated to a locally free, coherent O_X- module E of rank $k+1$:

$$s_i(E)\equiv p_*\circ c_1(O_{P(E)}(1))^{k+i}\circ p^*:\tilde{Z}_{r+i}(X)\to$$

$$\tilde{Z}_{r+k+i}(P(E))\to\tilde{Z}_r(P(E))\to\tilde{Z}_r(X)$$

for any $r\geq 0$. The commutativity of $c_1(pr_1^*(O_{P(E)}(1)))$ and $c_1(pr_2^*(O_{P(E')}(1)))$ on $\tilde{Z}_*(P(E)\times_X P(E'))$ enables one to verify that

$$s_i(E)\circ s_j(E')\ =\ s_j(E')\circ s_i(E)$$

for all locally free, coherent O_X- modules E, E' and all $i,j\geq 0$. As in [Fu], this suffices to define the *Chern class operators*

$$c_i(E):\ \tilde{Z}_{r+i}(X)\ \to\tilde{Z}_r(X)$$

using the formalism

$$\sum_{i\geq 0}c_i(E)t^i\ =\ (\sum_{j\geq 0}s_j(E)t^j)^{-1}.$$

When E is a direct sum of line bundles $L_1\oplus...\oplus L_r$, then $P(E)$ is equipped with r hyperplanes with empty intersection. Interpreting the fact that the composition of the corresponding intersection operators is 0, one concludes that $c_i(E)$ is the i-th elementary symmetric polynomial in the $c_1(L_j)$'s. From this and "splitting principle" arguments, one verifies the expected formulas for the Chern classes of vector bundles obtained as extensions, as tensor products, and in terms of other standard operations.

Another consequence of Proposition 2.5 is an alternate interpretation of the s-operation of [F-M] , an interpretation independent of a choice of projective embedding. Recall that this operation is defined as the map on homotopy groups given by pairing with the canonical class in $\pi_2(Z_0(P^1))$ associated to the composition

$$\Sigma^{-2}\circ\gamma:\ Z_{r+1}(X)\ \times\ Z_0(P^1)\ \to\ Z_{r+2}(\Sigma^2X)\ \to\ Z_r(X)$$

where $\gamma(Z,p) \equiv Z\#p$, the join of $Z \subset P^N$ and $p \in P^1$ inside

$$\Sigma^2 X \ \cong \ X\#P^1 \subset \ P^N\#P^1 = P^{N+2}$$

defined as the union of all lines from points of Z to p.

Corollary 2.6. For any quasi-projective variety X and any $r \geq 0$, consider the composition

$$i_X^! \circ \tilde{\omega} : \ \tilde{Z}_{r+1}(X) \ \otimes \ \tilde{Z}_0(P^1) \ \to \ \tilde{Z}_{r+1}(X \times P^1) \ \to \tilde{Z}_r(X)$$

where $\omega(Z,p) \equiv Z \times \{p\}$ and where i_X embeds X as the divisor $X \times \infty$ on $X \times P^1$. We define

$$s' : \ L_{r+1}H_n(X) \ \to \ L_rH_n(X)$$

as the associated map in homology given by pairing with the canonical class in $H_2(\tilde{Z}_0(P^1))$. In the special case in which X is a projective variety, provided with a closed embedding into some P^N, s' equals the s-operation of [F-M].

sketch of proof. Consider the closed subset W of $X \times P^1 \times P^{N+2}$ consisting of triples (x,y,z) with z lying on the line from x to y. One readily checks that W is obtained by blowing up $X\#P^1$ and that the projection map $\pi : W \to X \times P^1$ can be identified with

$$P(L \oplus 1) \ \to \ X \times P^1 \quad , \quad L \ = \ pr_1^* O_X(1) \otimes pr_2^* O_{P^1}(-1).$$

Moreover, the map

$$\omega : Z_{r+1}(X) \ \times \ Z_0(P^1) \to Z_{r+2}(\Sigma^2 X)$$

factors naturally through $Z_{r+2}(W)$. Thus, we obtain the following commutative diagram

$$\begin{tikzcd}
\tilde{Z}_{r+1}(X) \otimes \tilde{Z}_0(P^1) \arrow[r] \arrow[d] & \tilde{Z}_{r+2}(\Sigma^2(X)) \arrow[r] & \tilde{Z}_r(X) \\
\tilde{Z}_{r+1}(X \times P^1) \arrow[r, "\pi^*"] & \tilde{Z}_{r+2}(W) \arrow[u, "p_*"] &
\end{tikzcd}$$

with the s-operation determined by the upper row.

We verify that the quasi-isomorphism

$$(pr_1^*, i_{X*}) : \tilde{Z}_r(X) \oplus \tilde{Z}_{r+1}(X) \to \tilde{Z}_{r+1}(X \times P^1)$$

of the projective bundle theorem is inverse to $(i_X^!, pr_{1*})$. On the other hand, the composition

$$pr_{1*} \circ \omega : \ Z_{r+1}(X) \times Z_0(P^1) \to Z_{r+1}(X \times P^1) \to Z_{r+1}(X)$$

restricts to the 0-map on $Z_{r+1}(X)\times(Z_0(P^1))^o$, where $(Z_0(P^1))^o \subset Z_0(P^1)$ denotes the connected component of 0. This implies that

$$H_*(\tilde{Z}_{r+1}(X)) \otimes H_2(\tilde{Z}_0(P^1)) \ \to\ H_{*+2}(\tilde{Z}_{r+1}(X \times P^1)) \ \to\ H_{*+2}(\tilde{Z}_{r+1}(X))$$

is the 0-map, so that the s-map factors through projection on the summand $pr_1^*\tilde{Z}_r(X)$ of $\tilde{Z}_{r+1}(X \times P^1)$. We complete the proof by observing that the following composition

$$\Sigma^{-2} \circ p_* \circ \pi^* \circ pr_1^* :\ \tilde{Z}_r(X) \to$$

$$\tilde{Z}_{r+1}(X \times P^1) \to \tilde{Z}_{r+2}(W) \to \tilde{Z}_{r+2}(\Sigma^2(X)) \to \tilde{Z}_r(X)$$

is the identity. ■

We conclude this section with the "Lawson analogue of Gersten's Conjecture" which provides a resolution of sheaves associated to Lawson homology and a consequent determination of the E^2 – term of the spectral sequence of Proposition 1.7. Our proof follows that of D. Quillen's original proof of "Gersten's Conjecture" for algebraic K-theory [Q] and the subsequent analysis of S. Bloch and A. Ogus [B-O].

Proposition 2.7. Let X be a smooth quasi-projective variety of pure dimension m and let $\mathcal{LH}_{r,n}$ denote the sheaf on X associated to the presheaf sending U to $L_rH_n(U)$. Then for any $r, n \geq 0$ the sheafification of the E^1 – term plus differential d^1 of the spectral sequence of Proposition 1.7 determines the following exact sequence of sheaves on X

$$0 \ \to\ \ \mathcal{LH}_{r,n} \ \to\ \oplus_{\{\bar{x}\}\in S_m}\ i_x(L_rH_n(x)) \to$$

$$\oplus_{\{\bar{x}\}\in S_{m-1}}\ i_x(L_rH_{n-1}(x)) \to \ldots \to$$

$$\oplus_{\{\bar{x}\}\in S_0}\ i_x(L_rH_{n-m}(x)) \ \to\ 0$$

where $i_x(L_rH_j(x))$ denotes the constant sheaf $L_rH_j(x)$ on $\{\bar{x}\}$ extended by 0 to all of X. Consequently, the spectral sequence of Proposition 1.7 is of the following form

$$E^2_{p,q} \ =\ H^{m-p}(X,\ \mathcal{LH}_{r,m+q}) \ \Rightarrow\ L_rH_{p+q}(X).$$

sketch of proof. The long exact seqeunces occurring in the proof of Proposition 1.7

$$\ldots \to L_rH_n(U;p{-}1) \to L_rH_n(U;p) \to L_rH_n(U;p,p{-}1) \to L_rH_{n-1}(U;p{-}1) \to \ldots$$

determine long exact sequences of associated sheaves on X

$$\ldots \to\ \ \mathcal{LH}_{r,n}(p-1) \to\ \mathcal{LH}_{r,n}(p) \to\ \mathcal{LH}_{r,n}(p,p-1) \to\ \mathcal{LH}_{r,n-1}(p-1) \to \ldots$$

As argued in the proof of Proposition 1.7,

$$\mathcal{LH}_{r,n}(p,p-1) \;=\; \oplus_{\{\bar{x}\}\in S_p/S_{p-1}}\, i_x(L_rH_n(x)).$$

Since the sheaves $i_x(L_rH_n(x))$ are flasque, the theorem follows easily once one proves that

$$\mathcal{LH}_{r,n}(p-1) \to \quad \mathcal{LH}_{r,n}(p) \quad ,\ p \leq m$$

is the 0-map.

This vanishing is equivalent to the following local effaceability: given any closed point $x \in X$ and given some class $\alpha \in L_rH_n(W)$ for some locally closed $W \subset X$ of dimension $\leq p-1$ containing x, then there exists some locally closed $Y \subset X$ of dimension $\leq p$ containing W as a closed subset and an open $U \subset X$ containing x such that the restriction of α is sent to 0 via

$$L_rH_n(W\cap U) \;\to\; L_rH_n(Y\cap U).$$

Following Quillen, after possibly shrinking X to some affine neighborhood of $x \in X$ we find a divisor W' in X containing W which fits in the following diagram whose squares are cartesian

$$\begin{array}{ccccc} V & \subset & V' & \to & X \\ \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle g'} \\ W & \subset & W' & \to & A^{m-1} \end{array}$$

where g' is smooth of relative dimension 1 at $x \in X$, W is closed in W', and the map $W' \to A^{m-1}$ defined as the restriction of g' to $W' \subset X$ is a finite map (cf. [B-O]).

An easy diagram chase reduces the proof of local effaceability to that of the effaceability of the natural section $i : W \subset V$ restricted to some open neighborhood in V of the (finite) inverse image of $x \in X$. Since g restricted to this inverse image is smooth, upon shrinking V we may assume that g is smooth and thus that $i : W \subset V$ is a Cartier divisor; further shrinking of V permits us to assume that $i : W \subset V$ is principal. The effaceability of $i_* : L_rH_n(W) \to L_rH_n(V)$ now follows from the equalities

$$i_W^! \circ g^* = 1 :\ L_rH_n(W) \to L_{r+1}H_{n+2}(V) \to L_rH_n(W)$$

$$i_* \circ i_W^! = 0 :\ L_{r+1}H_{n+2}(V) \to L_rH_n(W) \to L_rH_n(V)$$

the first of which holds because $i(W)$ meets g^*Z transversally for any algebraic r-cycle Z on W, the second follows from the fact that $W \subset V$ is a principal divisor. ■

3. Intersection of cycle spaces and Chern classes

In this section, we discuss two intersection pairings. The first, which requires that X be an irreducible projective variety, is of the form

$$Z_{r+1}(X) \ \wedge \ \mathrm{Div}(X)^{+} \ \rightarrow \ Z_r(X)$$

where $\mathrm{Div}(X)^{+}$ is the homotopy-theoretic group completion of the topological monoid of effective Cartier divisors on X. Using this pairing, we offer further insight into the operations of [F-M]. Our second pairing is much influenced by the constructions of MacPherson and Fulton in [F] for the Chow ring of algebraic cycles modulo rational equivalence. For a quasi-projective variety X, we first define a Gysin homomorphism

$$i_W^! : \tilde{Z}_{r+c}(X) \rightarrow \tilde{Z}_r(W)$$

for $i : W \subset X$ a regular closed embedding of codimension c; this generalizes the construction of $i_D^!$ for a Cartier divisor D. In a now-familiar manner, this leads to our intersection pairing

$$\tilde{Z}_r(X) \ \otimes \ \tilde{Z}_s(X) \ \rightarrow \ \tilde{Z}_{r+s-n}(X)$$

for X smooth of dimension n and $r+s \leq n$. As a consequence of the resulting ring structure on Lawson homology , we verify for smooth varieties the existence of Chern classes from algebraic K-theory to Lawson homology.

Recall from [G] that if X is a (complex) projective variety, then the group of isomorphism classes of invertible sheaves on X is representable by a commutative group scheme

$$\mathrm{Pic}(X) \ \equiv \coprod_{\alpha \in NS(X)} \mathrm{Pic}_\alpha(X)$$

each component of which is a quasi-projective variety. By construction, $\mathrm{Pic}_\alpha(X) \times X$ has a universal invertible sheaf L_α. Then $\mathrm{Div}(X)$ the disjoint union of schemes over Pic(X)

$$\mathrm{Div}(X) \ \equiv \coprod_{\alpha \in NS(X)} \mathrm{Div}_\alpha(X),$$

where $\mathrm{Div}_\alpha(X)$ equals Proj(-) of the symmetric algebra of some coherent sheaf F_α with dual $pr_{1*}L_\alpha = E_\alpha$. If F_α is locally free, then $\mathrm{Div}_\alpha(X) = P(E_\alpha)$.

The following theorem provides yet another construction of the s-map of [F-M] as well as verifies that the operators $c_1(L)$ for $L \in \mathrm{Pic}(X)$ depend only upon the class of L in NS(X).

Theorem 3.1. Let X be an irreducible projective variety and let D_0 be a Cartier divisor on X such that $O(D_0)$ is very ample. Denote by $\mathrm{Div}(X)^{+}$ the (infinite) mapping telescope of successive additions by D_0,

$$\mathrm{Div}(X)^+ \;\equiv\; \mathrm{Tel}\{\mathrm{Div}(X),\ (- + D_0)\}.$$

Then $\mathrm{Div}(X) \to \mathrm{Div}(X)^+$ is a homotopy-theoretic group completion. Moreover, for any $r \geq 0$ there is a pairing

$$\rho : \ Z_{r+1}(X) \ \wedge \ \mathrm{Div}(X)^+ \ \to \ Z_r(X)$$

which induces a natural pairing on homotopy groups

$$L_{r+1}H_n(X) \ \otimes \ \pi_i(\mathrm{Div}(X)^+) \ \to \ L_rH_{n-2+i}(X)$$

This pairing restricted to $\pi_0(\mathrm{Div}(X)^+)$ gives the operators

$$c_1(L) : L_{r+1}H_n(X) \to \ L_rH_{n-2}(X)$$

and restricted to the distinguished generator of $\pi_2(\mathrm{Div}(X)^+) \cong \mathbf{Z}$ is the s-map of [F-M].

sketch of proof . For notational convenience, let $\mathcal{D}_\alpha$ denote $\mathrm{Div}_\alpha(X)$, let $\alpha_0 \in NS(X)$ denote the class of D_0, and let $\mathcal{D}_{\alpha+n}$ denote $\mathcal{D}_{\alpha+n\alpha_0}$. For any $\alpha \in NS(X)$, there exists some $n(\alpha) \geq 0$ such that whenever $n \geq n(\alpha)$ then $E_{\alpha+n} \equiv pr_{1*}L_{\alpha+n}$ is a locally free rank sheaf on $\mathcal{D}_{\alpha+n}$ of rank ≥ 2 and $(E_{\alpha+n})_p = H^0(X, (L_{\alpha+n})_p)$ for all $p \in \mathrm{Pic}_{\alpha+n}(X)$. In particular, whenever $n \geq n(\alpha)$, $\mathcal{D}_{\alpha+n} \ = P(E_{\alpha+n})$; let $M_{\alpha+n}$ denote the invertible sheaf on $\mathcal{D}_{\alpha+n} \times X$ obtained by tensoring $pr_1^* O_{\mathcal{D}_{\alpha+n}}(1)$ with the pull-back of $L_{\alpha+n}$; this is canonically the invertible sheaf associated to the "universal divisor."

For each $n \geq n(\alpha)$, we consider the composition maps

$$\rho_{\alpha+n} = pr_{1*} \circ c_1(M_{\alpha+n}) \circ \omega : \ Z_{r+1}(X) \times \mathcal{D}_{\alpha+n} \to Z_{r+1}(X \times \mathcal{D}_{\alpha+n})$$

$$\to \ Z_r(X \times \mathcal{D}_{\alpha+n}) \to Z_r(X)$$

where $\omega(Z, D) = Z \times \{D\}$. Observe that the pull-back of $M_{\alpha+n}$ via

$$\mathcal{D}_{\alpha+n,\alpha_0} \ \equiv \mathcal{D}_{\alpha+n} \times \mathcal{D}_{\alpha_0} \to \mathcal{D}_{\alpha+n+1}$$

is $pr_1^* M_{\alpha+n} \otimes pr_2^* M_{\alpha_0}$. This implies the homotopy commutativity of the following diagram relating $\rho_{\alpha+n} + \rho_{\alpha_0}$ to $\rho_{\alpha+n+1}$

$$\begin{array}{ccccc}
Z_{r+1}(X) \times \mathcal{D}_{\alpha+n,\alpha_0} & \to & Z_{r+1}(X \times \mathcal{D}_{\alpha+n,\alpha_0}) & \to & Z_r(X \times \mathcal{D}_{\alpha+n,\alpha_0}) \\
\downarrow & & \downarrow & & \downarrow \\
Z_{r+1}(X) \times \mathcal{D}_{\alpha+n+1} & \to & Z_{r+1}(X \times \mathcal{D}_{\alpha+n+1}) & \to & Z_r(X \times \mathcal{D}_{\alpha+n+1})
\end{array}$$

where the top row is ω followed by

$$c_1(pr_1^*M_{\alpha+n}) + c_1(pr_2^*M_{\alpha_0}) \; = \; c_1(pr_1^*M_{\alpha+n} \otimes pr_2^*M_{\alpha_0})$$

the bottom row is ω followed by $c_1(M_{\alpha+n+1})$, and the vertical maps are induced by the monoid pairing $\mathcal{D}_{\alpha+n,\alpha_0} \to \mathcal{D}_{\alpha+n+1}$.

We conclude that

$$\rho_{\alpha+n} + c_1(O(D_0)) \circ \mathrm{pr}_1, \rho_{\alpha+n+1} \circ (id \times (- + D_0)) : Z_{r+1}(X) \times \mathcal{D}_{\alpha+n} \to Z_r(X)$$

are homotopic. This enables us to define ρ as follows: on the copy (which we denote by $\mathcal{D}_{\alpha,k}$) of $\mathcal{D}_\alpha$ in the telescope indexed by k (i.e., after k additions of D_0), ρ is defined to be

$$\rho_{|\mathcal{D}_{\alpha,k}} = \rho_{\alpha+n(\alpha)} \circ (- + n(\alpha)D_0) - (n(\alpha) + k)c_1(D_0).$$

So defined, ρ determines a map (well defined up to homotopy when restricted to finite subcomplexes of a C.W. realization) on $Z_{r+1}(X) \wedge \mathrm{Div}(X)^+$. Additivity (up to homotopy on finite complexes of a C.W. realization) in the first factor follows from additivity of each ρ_α; additivity (up to homotopy on finite complexes of a C.W. realization) in the second factor follows from a homotopy commutative diagram of the above form (with α_0 replaced by a general α').

As argued in [F] (which is made explicit only in the smooth case), $\mathrm{Div}(X)^+$ fits in a fibration sequence

$$P^\infty \;\to\; \mathrm{Div}(X)^+ \;\to\; \mathrm{Pic}(X)$$

thereby determining the homotopy groups of $\mathrm{Div}(X)^+$. One readily verifies that $\pi_1(\mathrm{Pic}(X))$ acts trivially on $\pi_2(P^\infty)$, so that $\mathrm{Div}(X)^+$ is a simple space. Since $\mathrm{Div}(X) \to \mathrm{Div}(X)^+$ has the effect on homology of localizing the action of $\pi_0(\mathrm{Div}(X)) = NS(X)$ and since $\mathrm{Div}(X)^+$ maps to the homotopy theoretic group completion in view of its definition as a mapping telescope, we conclude that $\mathrm{Div}(X) \to \mathrm{Div}(X)^+$ is a homotopy theoretic group completion.

By construction, the restriction of ρ to $Z_{r+1}(X) \times \{D\}$ is

$$c_1(L \otimes L_0^{n(\alpha)}) - n(\alpha)c_1(L_0) \;,\; L_0 \equiv O(D_0),\; L \equiv O(D),\; D \in \mathcal{D}_\alpha.$$

Arguing as in the proof of Corollary 2.6, we verify that the s-map is obtained from

$$\rho_{\alpha+n(\alpha)} - c_1(L \otimes L_0^{n(\alpha)}) : \; Z_{r+1}(X) \wedge \mathcal{D}_{\alpha+n(\alpha)} \;\to\; Z_r(X)$$

by pairing with the image of the distinguished generator of $\pi_2(P^{N_\alpha}) \cong \mathbf{Z}$, where P^{N_α} is the fibre of

$$\tau_\alpha : \; \mathcal{D}_{\alpha+n(\alpha)} \to \mathrm{Pic}_{\alpha+n(\alpha)}(X)$$

and $L \equiv O(D)$ with $D \in \mathcal{D}_\alpha$. ■

With additional effort, one can realize $\mathrm{Div}(X)^+$ as the group completion of the monoid Div(X) with a suitable topology. For X smooth, this is a special case of Lima-Filho's result used in Definition 1.1. More generally, one can construct a model for $\mathrm{Div}(X)^+$ using a desingularization of X.

The following construction presents useful maps closely related to the Gysin map $i^!_D$ of Theorem 2.4.

Proposition 3.2. Let L be an invertible sheaf on the quasi-projective variety X, and let $i : Y \subset X$ be a closed subvariety. Then the data of a trivialization $L_{|U} \simeq O_U$, where $U = X - Y$, determines a natural map

$$\tilde{Z}_r(U) \to \tilde{Z}_r(V(i^*L) - o(Y)).$$

Moreover, the additional data of a trivialization of i^*L determines a lifting of this map to a natural map

$$\tilde{Z}_r(U) \to \tilde{Z}_r(V(i^*L))$$

which fits in the following square

$$\begin{array}{ccc} \tilde{Z}_r(X) & \stackrel{o_*}{\to} & \tilde{Z}_r(V(L)) \\ \downarrow & & \uparrow \\ \tilde{Z}_r(U) & \to & \tilde{Z}_r(V(i^*L)) \end{array}$$

commutative in the derived category $D(\mathbf{Z})$. The diagonal map of this square, $\tilde{Z}_r(X) \to \tilde{Z}_r(V(i^*L))$, coincides with the map σ_D defined as in section 2 for D equal to the pseudo-divisor given by the trivialization on U.

sketch. The inclusion $o : U \to V(L) - o(Y)$ (or alternatively, that given by the trivializing section) induces $\tilde{Z}_r(U) \to \tilde{Z}_r(V(L) - o(Y))$ whereas the trivialization $L_{|U} \simeq O_U$ determines a null homotopy for the composition

$$\tilde{Z}_r(U) \to \tilde{Z}_r(V(L) - o(Y)) \to \tilde{Z}_r(V(L)_{|U}).$$

Hence, the first map is obtained using the localization distinguished triangle

$$\tilde{Z}_r(V(i^*L) - o(Y)) \to \tilde{Z}_r(V(L) - o(Y)) \to \tilde{Z}_r(V(L)_{|U}).$$

To obtain the second map, observe that the trivialization of i^*L determines a null-homotopy for the inclusion $\tilde{Z}_r(Y) \to \tilde{Z}_r(V(i^*L))$ which determines a splitting of the distinguished triangle

$$\tilde{Z}_r(V(i^*L)) \to \tilde{Z}_r(V(i^*L) - o(Y)) \to \tilde{Z}_r(Y)[1]$$

given by localization. We then define the asserted second map as the composition of the first map and the map splitting the preceding distinguished triangle. ■

We shall employ the *deformation to the normal cone* construction for a regular embedding $i_W : W \to X$, the construction represented by the following diagram with pull-back squares

$$\begin{array}{ccccc} N_W X & \to & Q_W X & \leftarrow & X \times A^1 \\ \downarrow & & \downarrow f & & \downarrow \\ \{\infty\} & \to & P^1 & \leftarrow & A^1 \end{array}$$

In the above diagram, $N_W X$ is the normal bundle of i_W and the deformation space $Q_W X$ is defined by

$$Q_W X \;=\; Bl(X \times P^1 / W \times \{\infty\}) - Bl(X/W),$$

where $Bl(X \times P^1 / W \times \{\infty\})$ is the blow-up of $X \times P^1$ along $W \times \{\infty\}$ and Bl(X/W) is the blow-up of X along W. We denote by j: $X \times A^1 \subset Q_W X$ the inclusion of the complement of $N_W X \subset Q_W X$ and by $\epsilon : X \to Q_W X$ the inclusion $X \times \{1\} \subset X \times A^1 \subset Q_W X$. Observe that the line bundle $O_{P^1}(\infty)$ admits a natural trivialization when restricted to A^1 as well as when restricted to the point $\{\infty\}$. Consequently, $O_Q(\infty) \equiv f^* O_{P^1}(\infty)$ is provided with trivializations when restricted to $X \times A^1$ and to $Q_W X - (X \times A^1) = N_W X$, so that Proposition 3.2 provides a specialization map

$$s_{X/W} : \; \tilde{Z}_{r+1}(X \times A^1) \;\to\; \tilde{Z}_r(N_W X).$$

As we see below, this specialization map has an alternate description which will prove convenient for the proof of Theorem 3.4.

Proposition 3.3. Let $i_W : W \to X$ be a regular embedding of codimension c and $r \geq 0$. Then the composition

$$s_{X/W} \circ pr_1^* : \; \tilde{Z}_r(X) \to \tilde{Z}_{r+1}(X \times A^1) \to \tilde{Z}_r(N_W X)$$

is equivalent (in $D(\mathbf{Z})$) to the map

$$\tilde{\epsilon}_* : \tilde{Z}_r(X) \to \tilde{Z}_r(N_W X)$$

defined as the lifting of $\epsilon_* : \tilde{Z}_r(X) \to \tilde{Z}_r(Q_W X)$ determined by the natural null-homotopy of the composition of $j^* \circ i_* \; : \tilde{Z}_r(X) \to \tilde{Z}_r(Q_W X) \to \tilde{Z}_r(X \times A^1)$.

Furthermore, if W has codimension 1 so that W may be viewed as a Cartier divisor and if $r \geq 1$, then the composition of this map and the inverse of π^* : $\tilde{Z}_{r-1}(W) \to \tilde{Z}_r(N_W X)$ is equivalent (in $D(\mathbf{Z})$) to the Gysin map

$$i_W^! : \; \tilde{Z}_r(X) \;\to\; \tilde{Z}_{r-1}(W)$$

of Theorem 2.4.

sketch of proof. For notational simplicity, let Q denote $Q_W X$, N denote $N_W X \equiv Q - (X \times A^1)$, Q' denote $Q - (X \times \{0\})$, and $X \times G_m$ be identified with $Q' - N$. We provide Q with a G_m action as follows: on $Q - N = X \times A^1$, the action is via multiplication on the second factor and on N the action is by fibre-wise multiplication with respect to the structure of N as a vector bundle over W. We provide $V(O_{Q'}(\infty)) - o(N)$ with the structure of a G_m torsor over Q by defining

$$p : V(O_{Q'}(\infty)) - o(N) \to Q$$

as the map which sends a "point" (x,s) with $x \epsilon Q'$ and $s \epsilon O_Q(\infty)_x$ to the point $(s/s_0).x$, where s_0 is the global section of $O_Q(\infty)$ defined by the coordinate function on P^1 (and which restricts to give the trivialization of $O_Q(\infty)_{|N}$ used in the construction of the specialization map).

Consider the following diagram constructed as in Proposition 3.2:

$$\begin{tikzcd}
\tilde{Z}_r(X) \arrow[r, "\tilde{\epsilon}_*"] \arrow[d, "="'] & \tilde{Z}_r(N) \arrow[d] & \\
\tilde{Z}_r(X) \arrow[r, "\epsilon_*"] \arrow[d, "pr_1^*"'] & \tilde{Z}_r(Q) \arrow[r] \arrow[d, "p^*"'] & \tilde{Z}_r(X \times A^1) \arrow[d, "p_|^*"] \\
\tilde{Z}_{r+1}(X \times G_m) \arrow[r, "s_{\infty *}"] & \tilde{Z}_{r+1}(V(O_{Q'}(\infty)) - o(N)) \arrow[r] & \tilde{Z}_{r+1}(V(O_{Q'}(\infty))_{|X \times G_m}) \\
\tilde{Z}_{r+1}(X \times G_m) \arrow[r, "\tilde{s}_{\infty *}"] \arrow[u, "="] & \tilde{Z}_{r+1}(V(O_N(\infty)) - o(N)) \arrow[r, "\rho"] \arrow[u] & \tilde{Z}_{r+1}(V(O_N(\infty)))
\end{tikzcd}$$

where s_∞ is the global section of $O_Q(\infty)$ obtained by pulling-back the global section of $O_{P^1}(\infty)$ associated to the function 1 on P^1, $\tilde{s}_{\infty *}$ is the lifting determined by the trivialization of $O_{Q'}(\infty)_{|X \times G_m}$ determined by s_∞, and ρ is the "splitting" determined by the trivializing section $s_{0|N}$. The definition of p implies the (strict) equality $p^* \circ \epsilon_* = s_{\infty *} \circ pr_1^*$. We conclude that $\tilde{s}_{\infty *}$ is related to $\tilde{\epsilon}_*$ by pull-back via the "twisted projection" $p_{|N} : V(O_N(\infty)) - o(N) \to N$, since one sees that the homotopies defining them are related.

By construction, $s_{X/W} = \pi^{*-1} \circ \rho \circ \tilde{s}_{\infty *}$. Consequently, to prove that $s_{X/W} \circ pr_1^* = \tilde{\epsilon}_*$, it suffices to prove that

$$\pi^{*-1} \circ \rho \circ p_{|N}^* : Z_r(N) \to Z_{r+1}(V(O_N(\infty)) - o(N)) \to Z_{r+1}(V(O_N(\infty)) \to Z_r(N)$$

is the identity in the derived category. This would be clear if the twisted projection were replaced by the bundle projection π (on the complement of $o(N)$). But $p^* = \pi^*$ on G_m-invariant cycles, and in particular on cycles on N which come from $Z_{r-c}(W)$. Thus, if $r \geq c$, our re-interpretation of $s_{X/W} \circ pr_1^*$ follows by applying the "homotopy property" (i.e., Proposition 2.3). To deal with lower dimensional cycles, one may "stabilize" by crossing with an affine space.

Assume now that W is a Cartier divisor with associated line bundle L and let $U = X - W$. We recall that $Bl(X \times P^1/W \times \{\infty\})$ can be described as the closure of the image of the embedding $X \times A^1 \subset P(L \oplus O) \times P^1$ sending (x,t) to $(ts(x),t)$ where s is a global section "1" of $L = O(W)$ with 0-locus W. The projection

$$f : Bl(X \times P^1/W \times \{\infty\}) \subset P(L \oplus O) \times P^1 \to P(L \oplus O)$$

restricts to a proper map $f : Q \to V(L)$. Moreover, this map further restricts to $N \tilde{\to} V(L)_{|W}$, $f^{-1}(V(L)_{|U}) \subset Q - N$, $f \circ \epsilon : X \to V(L)$ is the canonical section, and the map f transforms the homotopy used in the construction of $\tilde{\epsilon}_*$ to the homotopy used in the construction of σ_W (as discussed prior to Theorem 2.4).

Therefore, we obtain the following diagram of complexes

$$\begin{array}{ccccc}
\tilde{Z}_r(X) & \stackrel{\tilde{\epsilon}_*}{\to} & \text{htyfib}\{\tilde{Z}_r(Q) \to \tilde{Z}_r(Q-N)\} & \leftarrow & \tilde{Z}_r(N) \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \text{htyfib}\{\tilde{Z}_r(Q) \to \tilde{Z}_r(f^{-1}(V(L)_{|U}))\} & & \downarrow \\
\downarrow & & f_* \downarrow & & \downarrow \\
\tilde{Z}_r(X) & \to & \text{htyfib}\{\tilde{Z}_r(V(L)) \to \tilde{Z}_r(V(L)_{|U})\} & \leftarrow & \tilde{Z}_r(N)
\end{array}$$

In the above diagram, the lower left horizontal map is that employed in the construction of the Gysin map of Theorem 2.4, the upper central vertical arrow is the natural restriction map, the left and right vertical maps are the identity, and the right horizontal maps are the equivalences given by the localization theorem. The commutativity of this diagram implies that $\pi^{*-1} \circ s_{X/W} \circ pr_1^*$ equals $i_W^!$ in D($\mathbf{Z}$) as asserted. ■

Theorem 3.4 below introduces our generalization to regular embeddings of the Gysin map for Cartier divisors of Theorem 2.4.

Theorem 3.4. Let X be a quasi-projective scheme and $i_W : W \to X$ be a regular (closed) embedding of codimension c. Then for any $r \geq 0$ there is a naturally defined Gysin map in $D(\mathbf{Z})$

$$i_W^! : \tilde{Z}_{r+c}(X) \to \tilde{Z}_r(W)$$

such that

a.) If $Z_{r+c}(X,W)$ (mapping to $Z_{r+c}(X)$) denotes the naive group completion of the monoid of effective $r{+}c$ cycles on X which meet W properly, then $i_W^!$ restricted

to $\tilde{Z}_{r+c}(X,W)$ is induced by the map $Z_{r+c}(X,W) \to Z_r(W)$ sending a cycle Z to the usual intersection-theoretic intersection $Z \cdot W \in Z_r(W)$.
b.) If $c=1$, then $i_W^!$ agrees with the construction of Theorem 2.4.
c.) If $i_V : V \to W$ is a regular (closed) embedding of codimension d and if $r \geq d$, then

$$(i_W \circ i_V)^! \ = \ i_V^! \circ i_W^! : \quad \tilde{Z}_{r+c}(X) \ \to \ \tilde{Z}_{r-d}(V).$$

d.) If $g : X' \to X$ is flat of relative dimension e and if $g' : W' \equiv X' \times_X W \to W$ denotes the pull-back of g via i_W, then

$$i_{W'}^! \circ g^* = {g'}^* \circ i_W^! : Z_{r+c}(X) \to Z_{r+e}(W').$$

Similarly, if $g : X' \to X$ is proper, then

$$i_W^! \circ g_* = {g'}_* \circ i_{W'}^! : Z_{r+c}(X') \to Z_r(W).$$

sketch of proof. We define $i_W^!$ as the composition

$$\tilde{Z}_{r+c}(X) \ \to \tilde{Z}_{r+c+1}(X \times A^1) \to \ \tilde{Z}_{r+c}(N_W X) \ \cong \ \tilde{Z}_r(X)$$

defined as composition of pr_1^*, specialization associated to deformation to the normal cone, and the inverse of π^* given by the "homotopy property".

As argued for (2.4.a), scheme-theoretic deformation to the normal cone when restricted to $Z_{r+c}(X,W)$ provides a lifting $Z_{r+c}(X,W) \times I \to Z_{r+c}(Q_W X)$ of the null-homotopy $Z_{r+c}(X) \times I \to Z_{r+c}(X \times A^1)$ and thus explicitly determines the lifting to $Z_{r+c}(N_W X)$. This construction sends an effective cycle $Z \in Z_{r+c}(X,W)$ given as a subscheme to the cycle associated to its scheme theoretic normal cone $C_{Z \cap W}(Z)$ which is shown in [Fu] to equal $\pi^*(Z \cdot W)$.

Part b.) is a re-statement of the second assertion of Proposition 3.3.

To prove c.), we consider the following diagram

$$\begin{array}{ccccc}
\tilde{Z}_{r+c}(X) & \stackrel{\epsilon_*}{\to} & \tilde{Z}_{r+c}(Q_V X) & \to & \tilde{Z}_{r+c}(X \times A^1) \\
i_W^! \downarrow & & \downarrow q_W^! & & \downarrow (i_W \times 1)^! \\
\tilde{Z}_r(W) & \stackrel{\epsilon_*}{\to} & \tilde{Z}_r(Q_V W) & \to & \tilde{Z}_r(W \times A^1)
\end{array}$$

where the vertical maps are Gysin maps for the regular embeddings $i_W : W \to X$, $q_W : Q_V W \to Q_V X$, $i_W \times 1 : W \times A^1 \to X \times A^1$. Using the alternate description of the Gysin map provided in Proposition 3.3, we readily verify the commutativity in $D(\mathbf{Z})$ of this diagram. Let $\pi_X : N_V X \to V$ and $\pi_W : N_V W \to V$ denote the bundle projections. By Proposition 3.3, $\pi_X^* \circ (i_W \circ i_V)^!$ (respectively, $\pi_W^* \circ i_V^!$) is realized by lifting the natural homotopy for the upper (resp., lower) horizontal composition. We identify the Gysin map $i_W^!$ as the composition of the map $\tilde{Z}_{r+c}(X) \to htyfib(\tilde{Z}_{r+c}(Q_W X) \to \tilde{Z}_{r+c}(X \times A^1))$ determined by the natural null-homotopy for $\tilde{Z}_{r+c}(X) \to \tilde{Z}_{r+c}(X \times A^1)$ and the derived category inverse of the natural map $\pi^* : \tilde{Z}_r(W) \to htyfib(\tilde{Z}_{r+c}(Q_W X) \to \tilde{Z}_{r+c}(X \times A^1))$, with parallel descriptions for the other vertical maps. The commutativity in D$\mathbf{Z}$ of the above diagram does not suffice to prove c.); we must rigidify this diagram by working with maps of chain complexes.

Using the preceding formulation of the Gysin map, we expand the above diagram to be a commutative diagram of chain complexes, inserting a middle row (for notational convenience, denoted $A \to B \to C$) between the upper row $A' \to B' \to C'$ and the lower row $A'' \to B'' \to C''$ of the above diagram. We may construct a null-homotopy for the middle row which is strictly compatible with that of the upper and lower rows, thereby obtaining a commutative diagram of chain complexes

$$\begin{array}{ccccc}
A' & \to & htyfib(B' \to C') & \tilde{\leftarrow} & ker(B' \to C') \\
\downarrow & & \downarrow & & \downarrow \\
A & \to & htyfib(B \to C) & \tilde{\leftarrow} & ker(B \to C) \\
\wr\uparrow & & \wr\uparrow & & \wr\uparrow \\
A'' & \to & htyfib(B'' \to C'') & \tilde{\leftarrow} & ker(B'' \to C'')
\end{array}$$

where $\tilde{\to}$ denotes a quasi-isomorphism. The perimeter of this diagram becomes the following commutative square in D$\mathbf{Z}$

$$\begin{array}{ccc}
\tilde{Z}_{r+c}(X) & \to & \tilde{Z}_{r+c}(N_V X) \\
{\scriptstyle i_W^!}\downarrow & & \downarrow{\scriptstyle n_W^!} \\
\tilde{Z}_r(W) & \to & \tilde{Z}_r(N_V W)
\end{array}$$

where $n_W^!$ is the Gysin map for the bundle inclusion $n_W : N_V W \to N_V X$. The verification of c.) is completed by observing that

$$\pi_W^* = n_W^! \circ \pi_X^* : \ \tilde{Z}_{r-d}(V) \ \to \ \tilde{Z}_{r+c}(N_V X) \ \to \ \tilde{Z}_r(N_V W).$$

The proof of d.) follows easily from the alternate description of the specializatin map given in Proposition 3.3 together with the identification

$$Q_{W'} X' \ = \ X' \times_X Q_W X. \qquad \blacksquare$$

The following intersection pairing for a smooth variety X follows easily from the existence of a Gysin map for the diagonal embedding

$$\Delta : \ X \ \to \ X \times X.$$

Theorem 3.5. If $f : X \to Y$ is a map of quasi-projective varieties with Y smooth of pure dimension c, then f determines a natural pairing

$$\tilde{Z}_r(X) \ \otimes \ \tilde{Z}_s(Y) \ \to \ \tilde{Z}_{r+s-c}(X)$$

whenever $r + s \geq c$.

In particular, if X is a smooth variety of dimension n and if $r + s \geq n$, then the resulting *intersection pairing*

$$\tilde{Z}_r(X) \ \otimes \ \tilde{Z}_s(X) \ \to \ \tilde{Z}_{r+s-n}(X)$$

has the following properties:
a.) If $Z_r(X, W)$ (mapping to $Z_r(X)$) (respectively, $Z_s(X, V)$ to $Z_s(X)$) is the naive group completion of those effective cycles which meet $W \in Z_s(X)$ (resp, $V \in Z_r(X)$) properly, then the restriction of this pairing to $\tilde{Z}_r(X, W)$ (resp., $\tilde{Z}_s(X, V)$) is induced by the usual intersection pairing.
b.) The pairing is commutative and associative (in $D(\mathbf{Z})$).
c.) Applying $H_0(-)$ to this pairing yields the usual intersection pairing on algebraic equivalence classes.

sketch of proof. If Y is smooth, then the graph Γ_f of f is regularly embedded in $X \times Y$. Moreover, sending $(Z, V) \in Z_r(X) \times Z_s(Y)$ to $Z \times V \in Z_{r+s}(X \times Y)$ is a continuous bilinear map. Thus, the Gysin map for the embedding $\Gamma_f \subset X \times Y$ determines the pairing

$$i^!_{\Gamma_f} \circ \times : \ \tilde{Z}_r(X) \otimes \tilde{Z}_s(Y) \to \tilde{Z}_{r+s}(X \times Y) \to \tilde{Z}_{r+s-c}(\Gamma_f) \cong \tilde{Z}_{r+s-c}(X)$$

where $c = \dim(Y)$.

If (V,W) intersect properly on X, then $V \times W$ meets $\Delta(X) \subset X \times X$ properly. Thus, a.) follows from (3.4.a). Associativity follows from (3.4.c) applied to

$$(\Delta \times 1) \circ \Delta \ = \ (1 \times \Delta) \circ \Delta : \ X \ \to \ X \times X \times X$$

whereas commutativity follows immediately by transport of structure.

To prove c.), we recall that the ring structure on the Chow groups (of cycles modulo rational equivalence) is given by the same construction as we have used: one uses deformation to the normal cone to define a Gysin map for the diagonal embedding and then applies this Gysin map to the product of cycles. Thus, applying $\pi_0(-)$ to our construction yields the image of the Chow ring structure on cycles modulo algebraic equivalence. ■

We conclude this section by exhibiting Chern classes from the (higher) algebraic K-theory of a smooth variety to Lawson homology.

Construction 3.6. Let X be a smooth quasi-projective variety of pure dimension d. Then for any $p > 0$ and any i with $0 \leq i \leq d$, we define Chern classes

$$c_{i,p} : K_p(X) \ \to \ L_{d-i}H_{2d-2i+p}(X) \ .$$

sketch . If C_*, C'_* are chain complexes, then we denote by $\mathrm{Hom}(C_*, C'_*)$ the chain complex of homomorphisms given in degree m by the following (infinite) product

$$\mathrm{Hom}(C_*, C'_*)_m \ = \ \prod_{i-j=m} \mathrm{Hom}(C_j, C'_i).$$

If Y_* is a simplicial object in the category of disjoint unions of equidimensional varieties and if the face maps of Y_* are flat, then we define $\mathrm{Hom}(Y_*, \tilde{Z}^r)$ analogously to be the chain complex whose term of degree m is given by

$$\mathrm{Hom}(Y_*, \tilde{Z}^r)_m \ = \ \prod_{i-j=m} \tilde{Z}^r(Y_j)_i$$

where $\tilde{Z}^r(Y_j)_i$ is the group in degree i of the complex $\tilde{Z}^r(Y_j)$ defined as

$$\tilde{Z}^r(Y_j) \ \equiv \ \prod_{\alpha \in \pi_0(Y_j)} \tilde{Z}^r(Y_{j,\alpha})$$

where $Y_{j,\alpha}$ is the connected component of Y_j indexed by $\alpha \in \pi_0(Y_j)$ and Z^r denotes cycles of *codimension* r. (There are different conventions for the signs in the differential.)

A map between filtered chain complexes which induces a quasi-isomorphism of associated graded complexes is itself a quasi-isomorphism provided that the chain complexes are complete for the filtration. The useful property of these Hom-complexes is that they are complete with respect to the decreasing filtration given at level p by taking products with $j \geq p$.

We recall "Jouanolou's device" which exhibits an affine variety $\mathrm{Spec}A$ and a morphism $\mathrm{Spec}A \to X$ which is an affine bundle map of some relative dimension c. The proof of the "homotopy property" proves that $\tilde{Z}_r(X) \to \tilde{Z}_{r+c}(\mathrm{Spec}A)$ is a quasi-isomorphism; algebraic K-theory satisfies a similar property (without a shift in degree) [Q] which implies that $K_*(X) \to K_*(\mathrm{Spec}A)$ is an isomorphism. Thus, to exhibit Chern classes for X, it suffices to assume that $X = \mathrm{Spec}A$. In what follows, we shall stabilize the cycle spaces by replacing $Z^s(Y)$ by $\mathrm{colim}_n Z^s(Y \times A^n)$.

Fix a positive integer n. Let B_* denote the simplicial variety $\mathrm{BGL}(n, A) \otimes X$ given at level m as the disjoint union of copies of X indexed by $GL(n, A)^{\times m}$ and let

$$\pi : \ P_* \ = \ B(P_A^{n-1}, GL(n, A)) \ \to \ \mathrm{BGL}(n, A) \otimes X \ = \ B_*$$

denote the "universal bundle projection" with

$$P_m \;=\; P_A^{n-1} \times_X B_m \;=\; P^{n-1} \times B_m.$$

Denote by $V(O_{P_*}(1))$ the simplicial variety over P_* given at level m by

$$V(O_{P_*}(1))_m \;=\; V(O_{P_m}(1)).$$

We consider the following chain in $D(\mathbf{Z})$

$$c_1(O_{P_*}(1)):\ \mathrm{Hom}(P_*, \tilde{Z}^s) \to \mathrm{Hom}(V(O_{P_*}(1)), \tilde{Z}^{s+1}) \leftarrow \mathrm{Hom}(P^*, \tilde{Z}^{s+1})$$

where the second map is a quasi-isomorphism since each

$$\tilde{Z}^{s+1}(P_m) \;\to\; \tilde{Z}^{s+1}(V(O_{P_m}(1)))$$

is a quasi-isomorphism. We easily extend the proof of Proposition 2.5 to prove that

$$\sum_{0 \le j < n} c_1(O_{P_*}(1))^j \circ p^* :\ \bigoplus_{0 \le j < n} \mathrm{Hom}(B_*, \tilde{Z}^{r-j}) \to \mathrm{Hom}(P_*, \tilde{Z}^r)$$

is a quasi-isomorphism.

The multiplicative pairing of Theorem 3.5 can be extended to apply to the map $\pi : P_* \to B_*$ to give pairings (in the derived category)

$$\mathrm{Hom}(P_*, \tilde{Z}^r) \otimes \mathrm{Hom}(B_*, \tilde{Z}^s) \;\to\; \mathrm{Hom}(P_*, \tilde{Z}^{r+s})$$

which determines a $H_*(\mathrm{Hom}(B_*, \tilde{Z}^*))$ – module structure on $H_*(\mathrm{Hom}(P_*, \tilde{Z}^*))$. The quasi-isomorphism implies that $H_*(\mathrm{Hom}(P_*, \tilde{Z}^*))$ is a free $H_*(\mathrm{Hom}(B_*, \tilde{Z}^*))$-module of rank n. Following Grothendieck, we define Chern classes

$$c_{i,n} \in H_0(\mathrm{Hom}(B_*, \tilde{Z}^i))$$

by the equation

$$c_1(O_{P_*}(1))^n + c_1(O_{P_*}(1))^{n-1} \cdot \pi^*(c_{1,n}) + \ldots + \pi^*(c_{n,n}) = 0.$$

($c_1(O_{P_*}(1))^n$ is applied to the fundamental cycle.) We interpret $H_0(\mathrm{Hom}(B_*, \tilde{Z}^i))$ as the group of homotopy classes of maps from the chain complex of BGL(n,A) to $\tilde{Z}^i(X)$, so that our Chern classes $c_{i,n}$ determine maps

$$C_{i,n}:\ H_*(\mathrm{BGL}(n, A)) \;\to\; H_*(\tilde{Z}^i(X)).$$

One verifies that these maps stabilize to determine

$$C_i:\ H_*(\mathrm{BGL}(A)) \;\to\; H_*(\tilde{Z}^i(X))$$

We compose these stable maps with the Hurewicz homomorphism

$$K_*(A) \;=\; \pi_*(\mathrm{BGL}(A)^+) \;\to\; H_*(\mathrm{BGL}(A))$$

to obtain the asserted Chern classes. ■

We anticipate that the above construction can be extended to the Quillen K-theory of coherent sheaves on a possibly singular variety using the sheaf-theoretic techniques of [Gi].

4. Relationships to other theories

In this section, we exhibit a natural map from S. Bloch's higher Chow groups [B] to Lawson homology for (complex) quasi-projective varieties. We then introduce the *algebraic bivariant cycle complex* whose homology is an analogue of bivariant morphic cohomology recently introduced in [F-L] (a bivariant generalization of Lawson homology). Our tentative new theory is defined for arbitrary pairs of quasi-projective varieties over an arbitrary field; when the contravariant variable is a point, our theory maps to Lawson homology and in some respects resembles Bloch's theory.

We recall the definition of S. Bloch's higher Chow groups [B]. Let X be an equi-dimensional quasi-projective variety over a field k. For each $s \geq 0$, one considers the following simplicial abelian group

$$z^s(X,*):\;\Delta^{op}\;\to\;(Ab)$$

defined by sending the m-simplex $[m]=\{0,...,m\}$ to the free abelian group $z^s(X,m)$ generated by irreducible subvarieties of $X\times\Delta[m]$ of codimension s which meet each face $1\times\partial : X\times\Delta[k]\to X\times\Delta[m]$ properly. Here, $\Delta[m]$ denotes the "algebraic m-simplex" over k; namely,

$$\Delta[m]\;\equiv\;\mathrm{Spec}(\mathrm{k}[x_0,...,x_m]/\sum x_i-1).$$

Face and degeneracy maps of $z^s(X,*)$ are defined by intersection and pull-back of cycles induced by the (standard linear) face and degeneracy maps of $\Delta[m]$. Bloch defines the higher Chow groups by

$$CH^s(X,m)\;\equiv\;\pi_m(z^s(X,*))\;\cong\;H_m(\tilde{z}^s(X,*))$$

where $\tilde{z}^s(X,*)$ denotes the normalized chain complex associated to the simplicial abelian group $z^s(X,*)$.

Construction 4.1. Let X be a quasi-projective variety of pure dimension n over $\mathbf{C}$. Then there is a natural map in $D(\mathbf{Z})$

$$\tilde{z}^s(X,*) \ \to \ \tilde{Z}_r(X) \ \ , \ \ r = n-s \geq 0.$$

Consequently, there are natural maps from Bloch's higher Chow groups to Lawson homology groups indexed as follows:

$$CH^s(X,m) \ \to \ L_rH_{2r+m}(X) \ \ , \ \ r = n-s \geq 0.$$

sketch. If A is a simplicial abelian group, we define $\oplus_* A$ to be the naturally constructed simplicial abelian group with m-simplices

$$(\oplus_* A)_m \ = \ \bigoplus_{|\alpha|=m} A_{n_{\alpha(m)}}$$

where the direct sum is indexed by the m-tuples of composable maps of Δ

$$\alpha \ = \ (\alpha_1 : [n_{\alpha(0)}] \to [n_{\alpha(1)}], ..., \alpha_m : [n_{\alpha(m-1)}] \to \ [n_{\alpha(m)}]).$$

Using the natural isomorphism of functors

$$H_0(-) \ \cong \ \mathrm{colim}_{\Delta^{op}}(-)$$

from simplicial abelian groups to abelian groups (cf. [G-Z]), we obtain a homotopy class of natural homotopy equivalences of normalized chain complexes

$$\tilde{A} \ \to \ \tilde{\oplus}_* A$$

for any simplicial abelian group A.

For any composable m-tuple $\alpha = (\alpha_1, ..., \alpha_m)$ of Δ, we can associate in a natural way a composable m-tuple $\alpha' = (\alpha'_1, ..., \alpha'_m)$ of injective maps of Δ with $\alpha'(m) = \alpha(m)$. We consider the naive group completion $T^s(X,\alpha)$ of those effective $(r + n_{\alpha(m)})$ cycles on $X \times \Delta[n_{\alpha(m)}]$ which meet each face $X \times \Delta[n_{\alpha'(i)}]$ properly. We verify that

$$pr_1^* : \ Z_r(X) \ \to \ T^s(X,\alpha) \ \ , \ r = n-s$$

is a homotopy equivalence for each composable m-tuple α by applying the "homotopy property" to

$$Z_r(X) \to Z_{r+n_{\alpha'(0)}}(X \times \Delta[n_{\alpha'(0)}])$$

and then verifying that the map

$$Z_{r+n_{\alpha'(0)}}(X \times \Delta[n_{\alpha'(0)}]) \to T^s(X,\alpha)$$

(defined using a linear retraction of $\Delta[n_{\alpha(m)}]$ to $\Delta[n_{\alpha'(0)}]$) is the inclusion of a deformation retract. This latter fact follows from the proof of the following statement, proved using a flow to infinity as in [L]: for any Y and any non-negative n, r, d the pull-back map

$$pr_1^* : C_r(Y) \to C_{r+n,d}(Y \times \Delta[n], Y)$$

from cycles on Y to cycles on $Y \times \Delta[n]$ which meet Y properly is a deformation retract.

We denote by $\oplus_* T^s(X)$ the simplicial abelian topological group whose topological group of m-simplices is given by

$$\oplus_* T^s(X)_m \ \equiv \ \bigoplus_{|\alpha|=m} T^s(X, \alpha)$$

Then the asserted map is given by the following chain of maps

$$\tilde{z}^s(X,*) \to \tilde{\oplus}_* z^s(X,*) \to N \circ \mathrm{diag} \circ \mathrm{Sing.} \oplus_* T^s(X)$$
$$\leftarrow N \circ \mathrm{diag} \circ \mathrm{Sing.} \oplus_* Z_r(X) \leftarrow \tilde{Z}_r(X)$$

where the first map is homotopy equivalence described above in a general setting, the second is given by the natural inclusion of the discrete abelian group $z^s(X, n_{\alpha(m)})$ in the topological group $T^s(X, \alpha)$ for each composable m-tuple α, the third is the quasi-isomorphism induced by the equivalence $pr_1^* : Z_r(X) \to T^s(X, \alpha)$ for each α, and the last quasi-isomorphism arises by viewing $Z_r(X)$ as a constant simplicial topological abelian group and observing that the equivalence in the derived category $\tilde{A} \ \to \ \tilde{\oplus}_* A$ implies a similar equivalence for (the singular functor applied to) simplicial abelian topological groups. ■

The following observation suggests that this map might be highly non-trivial.

Proposition 4.2. Let X be a smooth, projective complex variety of pure dimension $m > 0$ and let n be a positive integer. The homology groups of the chain complexes

$$z^1(X,*) \otimes \mathbf{Z}/n \ , \quad \tilde{Z}_{m-1}(X) \otimes \mathbf{Z}/n$$

are abstractly isomorphic.

sketch of proof. This is merely a comparison of Bloch's computation of his higher Chow groups in codimension 1 [B] and the computation of Lawson homology for codimension 1 cycles [F]. ■

In fact, the map of Construction 4.1 induces an isomorphism in $\mathbf{Z}/n-$homology in degrees 0 and 1. Roughly speaking, this map in homology is associated to the identity map from Pic(X) viewed as a discrete group (which we denote $\mathrm{Pic}(X)^\delta$)

to Pic(X) viewed as a topological group. Viewed in terms of the "Isomorphism Conjecture" of [M], this isomorphism can be interpreted in terms of the map of classifying spaces

$$K(\mathrm{Pic}(X)^{\delta}, 1) \ \rightarrow \ \mathrm{BPic}(X)$$

Since Pic(X) is an extension of a finitely generated abelian group NS(X) by a complex torus, this last map induces an isomorphism in $\mathbf{Z}/n-$homotopy.

We now briefly consider an algebraic analogue of "bivariant morphic cohomology" (cf. [F-L]). Our construction has been inspired by A. Suslin's definition of "algebraic homology" [Su], S. Bloch's definition of higher Chow groups [B], and the bivariant point of view introduced by the author and H. B. Lawson [F-L]. Our analogue, which we denote $A_r(W, X)$, has various appealing features as indicated in Theorem 4.6 below. In short, it is a bivariant version of Lawson homology defined over an arbitrary field k with the role of algebraic equivalence being replaced by rational equivalence.

For the remainder of this section, k denotes an arbitrary (but fixed) field and $k \subset \bar{k}$ is a fixed algebraic closure. Without mention to the contrary, all varieties discussed below will be varieties over k.

Definition 4.3. Let Y be a projective variety with closed equivalence relation $R \subset Y \times Y$, let W be a quasi-projective variety with proper equivalence relation $S \subset W \times W$. A *continuous algebraic map* $f : W/S \rightarrow Y/R$ is a set theoretic function on geometric points

$$f: \ W(\bar{k})/S(\bar{k}) \ \rightarrow \ Y(\bar{k})/R(\bar{k})$$

which is induced by a closed subset $C_f \subset W \times Y$. If f is bijective and if C_f is proper over Y, then f is said to be an algebraic homeomorphism.

When $S = diag(W)$, we employ the notation $\mathrm{Hom}_{alg.cont}(W, Y/R)$. Definition 4.3 may be appropriately extended to k-schemes W possibly not of finite type. With this extended definition, one has for L a field extension of k with an algebraic closure $\bar{L}$

$$\mathrm{Hom}_{alg.cont}(Spec(L), Y/R) \cong (Y(\bar{L})/R(\bar{L}))^{Gal(\bar{L}/L)}$$

and for V a valuation ring over k

$$\mathrm{Hom}_{alg.cont}(Spec(V), Y/R) \cong \mathrm{Hom}_{alg.cont}(Spec(fract(V)), Y/R).$$

Composition of continuous algebraic maps is easily seen to be well defined. In the special case that $S = \mathrm{diag}(W)$ and $R = \mathrm{diag}(Y)$, this definition agrees with that of [F], for in this case $pr_1 : C_f \rightarrow W$ is a finite, surjective, radicial map.

As for Bloch's complex, we use in the definition below the algebraic simplices

$$\Delta[m] \ \equiv \ \mathrm{Spec}(\mathrm{k}[x_0, ..., x_m]/\sum x_i - 1)$$

together with their (algebraic) face and degeneracy maps.

Definition 4.4. Let W be a quasi-projective variety and Y a projective variety with closed equivalence relation $R \subset Y \times Y$. We define the simplicial set mor(W,Y/R) by setting

$$\mathrm{mor}(W, Y/R)_m \ \equiv \ \mathrm{Hom}_{\mathrm{alg.cont}}(W \times \Delta[m], Y/R)$$

for each $m \geq 0$ and by defining face and degeneracy maps as those induced by the face and degeneracy maps of $\Delta[m]$.

If X is a quasi-projective variety with projective closure $\bar{X}$, then for any $r, n \geq 0$ we write

$$Z_r(\bar{X}, \bar{X} - X)^{(n)} \ = \ Y_n/R_n \quad , \quad Y_n = \cup_{i,j \leq n} C_{r,i}(\bar{X}) \times C_{r,j}(\bar{X}).$$

where R_n is the equivalence relation on Y_n given by identifying pairs of cycles on $\bar{X}$ if their differences are equal when restricted to X. We define

$$\mathrm{mor}(W, Z_r(\bar{X}, \bar{X} - X)) \ \equiv \ \mathrm{colim}_n\{\mathrm{mor}(W, Z_r(\bar{X}, \bar{X} - X)^{(n)})\}.$$

The preceding definition differs from the construction of [F-L] in that we consider algebraic continuous maps into $Z_r(\bar{X}, \bar{X} - X)$ rather than a homotopy-theoretic group completion of maps into the Chow monoid $C_r(\bar{X})$. This difference permits us to obtain a theory for quasi-projective X. A second difference is that, following Bloch and Suslin, we consider the simplicial set of algebraic singular simplices rather than the topological space of maps, a difference which is clearly reflected in Theorem 4.6.b).

Proposition 4.5. Let W, X be quasi-projective varieties and let $X \subset \bar{X}$, $X \subset \hat{X}$ be projective closures. Then

$$\mathrm{mor}(W, Z_r(\hat{X}, \hat{X} - X)) \ , \ \mathrm{mor}(W, Z_r(\bar{X}, \bar{X} - X))$$

are naturally isomorphic simplicial abelian groups. We let $\mathrm{mor}(W, Z_r(X))$ denote $\mathrm{mor}(W, Z_r(\bar{X}, \bar{X} - X))$ for some projective closure $X \subset \bar{X}$ and define the *algebraic bivariant cycle complex*, $A_r(W, X)$, to be the normalized chain complex associated to $\mathrm{mor}(W, Z_r(X))$.

a.) Any map $h : V \to W$ induces a map

$$h^* : A_r(W, X) \ \to \ A_r(V, X).$$

b.) Any proper map $f : X \to Y$ induces a map

$$f_* : \ A_r(W,X) \ \to \ A_r(W,Y).$$

c.) Any flat map $g : X' \to X$ of constant relative dimension e induces a map

$$g^* : \ A_r(W,X) \ \to \ A_{r+e}(W,X').$$

sketch of proof. As discussed in the first part of the sketch of the proof of Theorem 1.6, to prove the independence of projective closure we may consider the situation in which the projective closure $\hat{X}$ dominates $\bar{X}$. Then the map of Chow monoids $C_r(\hat{X}) \to C_r(\bar{X})$ induces a map

$$\mathrm{mor}(W, Z_r(\hat{X}, \hat{X} - X)) \ \to \ \mathrm{mor}(W, Z_r(\bar{X}, \bar{X} - X)).$$

This map is seen by inspection to be injective; surjectivity is proved using the observation made in discussing Theorem 1.6 that effective r-cycles of X whose closures in $\bar{X}$ have degree $\leq d$ have closures in $\hat{X}$ of some bounded degree.

Contravariant functoriality with respect to maps $h : V \to W$ follows easily from the observation that composition of continuous algebraic maps is well defined. Since a proper map $f : X \to Y$ induces a homomorphism of Chow monoids $C_r(\bar{X}) \to C_r(\bar{Y})$ of suitably chosen projective closures, covariant functorality with respect to maps $f : X \to Y$ follows similarly. To prove contravariant functorality for flat maps $g : X' \to X$ of relative dimension e, we employ projective closures $\bar{X}', \bar{X}$. Let Y_n denote $\cup_{i,j \leq n} C_{r,i}(\bar{X}) \times C_{r,j}(\bar{X})$ and Y'_p denote $\cup_{i,j \leq p} C_{r+e,i}(\bar{X}') \times C_{r+e,j}(\bar{X}')$. Recall that $g^* : Z_r(X) \to Z_{r+e}(X')$ is induced by correspondences $\Gamma_{n,p} \subset Y_n \times Y'_p$ for $p >> n$. Given a continuous algebraic map $\phi : W \times \Delta[m] \to Z_r(X)$ represented by some correspondence $C_\phi \subset W \times \Delta[m] \times Y_n$, we consider the correspondence $C_{g^*(\phi)}$ defined as the composition of C_ϕ and the correspondence $\Gamma_{n,p}$. One easily verifies that $C_{g^*(\phi)} \subset W \times \Delta[m] \times Y'_p$ determines a set-theoretic function $(W \times \Delta[m])(\bar{k}) \to Z_{r+e}(X')(\bar{k})$. ■

We now verify a few properties of our algebraic bivariant cycle complex which suggest that this complex may prove to worthy of further study.

Theorem 4.6. Let W and X be quasi-projective varieties and $r \geq 0$.
a.) For k perfect, $H_0(A_r(*,X))$ is the Chow group of rational equivalence classes of r-cycles on X.
b.) If $k = \mathbf{C}$, there is a natural inclusion

$$A_r(*,X) \ \subset \ \tilde{Z}_r(X)$$

applying $H_0(-)$ to which yields the quotient map from r-cycles modulo rational equivalence to r-cycles modulo algebraic equivalence.

c.) If X is projective, provided with a given closed embedding in some projective space, then the suspension map induces a quasi-isomorphism

$$A_r(W,X) \ \rightarrow \ A_{r+1}(W,\Sigma X).$$

d.) Assume $k = \mathbf{C}$ and $N \geq r$. Then $Z_r(A^N)$ is an abelian topological group which has the homotopy type of $K(\mathbf{Z}, 2N-2r)$. There is a natural map of simplicial abelian groups

$$\Phi_{N,r}: \ \mathrm{mor}(W, Z_r(A^N)) \ \rightarrow \ \mathrm{hom}(W, Z_r(A^N))$$

where $\mathrm{hom}(W, Z_r(A^N))$ denotes the simplicial abelian group of continuous maps from W to $Z_r(A^N)$. Moreover, these maps $\Phi_{N,r}$ for increasing N determine a natural map

$$\mathrm{colim}_j H_*(A_{r+j}(W, A^{N+j})) \ \rightarrow \ H^{2N-2r-*}(W, \mathbf{Z}).$$

sketch of proof. To prove a.), one sees that for k perfect, $A_r(*,X)_0$ identifies with the discrete group of r-cycles on X. Also, the elements of $\mathrm{mor}(*, Z_r(X))_1$ correspond to continuous algebraic maps from the generic point of $\Delta[1]$ to $Z_r(\bar{X}, \bar{X}-X)$, which correspond to r-cycles on X defined over the perfect closure of the function field $k(t)$ of $\Delta[1]$, and this perfect closure is (for k perfect with $char(k) = p > 0$) the limit over n of rational function fields $k(t^{1/p^n})$. Furthermore, the simplicial face maps correspond to specialization of cycles, and the assertion follows by comparing the definition of $\pi_0(\mathrm{mor}(*, Z_r(X)))$ with one of the definitions of rational equivalence.

If $k = \mathbf{C}$, then a continuous algebraic map $\Delta[m] \rightarrow Z_r(X)$ is continuous for the analytic topology. Thus, $\mathrm{mor}(*, Z_r(X))$ is a sub-object of the simplicial abelian group $\mathrm{Sing}.Z_r(X)$. Since the inclusion map is the identity on 0-simplices, the induced map on $H_0(-)$ is necessarily the quotient map as asserted.

The proof of the Lawson suspension theorem given in [F] for a projective variety X provides for any $n > 0$ a continuous algebraic map

$$\theta_n: \ \cup_{d,e \leq n} C_{r+1,d}(\Sigma X) \times C_{r+1,e}(\Sigma X) \times J \ \rightarrow \ Z_{r+1}(\Sigma X)$$

where $J \subset A^1 \times A^1$ equals Spec(k[x,y]/(x-1)y). This map satisfies the following properties: at $(0,0) \in J$, θ_n is the natural projection; for all $(s,t) \in J$, θ_n restricted to $\cup_{d,e \leq n} C_{r,d}(X) \times C_{r,e}(X)$ is the natural projection; at $(1,1) \in J$, θ_n factors through the image of $\Sigma : Z_r(X) \rightarrow Z_{r+1}(\Sigma X)$. The map θ_n is constructed from the corresponding map on effective cycles of degree $\leq n$ which in turn is defined as the difference $T_{j+1} - T_j$, for some sufficiently large j. The map T_j is the composition of an algebraic homotopy relating multiplication by j on effective cycles of degree $\leq n$ to a map with image meeting $X \subset \Sigma X$ properly and an algebraic homotopy deforming effective cycles of arbitrary degree on ΣX which meet X properly to suspensions of cycles on X. One verifies that θ_n factors through a continuous algebraic map

$$\bar{\theta}_n : \; Z_{r+1}(\Sigma X)^{(n)} \times J \; \rightarrow \; Z_{r+1}(\Sigma X).$$

We obtain a commutative square of simplicial sets

$$\begin{array}{ccc} \mathrm{mor}(W, Z_r(X)^{(n)}) \times (\Delta[1] \vee \Delta[1]) & \rightarrow & \mathrm{mor}(W, Z_r(X)) \\ \Sigma \big\downarrow & & \big\downarrow \Sigma \\ \mathrm{mor}(W, Z_{r+1}(\Sigma X)^{(n)}) \times (\Delta[1] \vee \Delta[1]) & \rightarrow & \mathrm{mor}(W, Z_{r+1}(\Sigma X)) \end{array}$$

whose upper horizontal arrow is the constant homotopy for the inclusion and whose lower horizontal arrow is a homotopy between the inclusion and a map factoring through $\mathrm{mor}(W, Z_r(X))$. Since

$$H_*(A_r(W,X)) \; \equiv \; \pi_*(\mathrm{mor}(W, Z_r(X))) \; = \; \mathrm{colim}_n \pi_*(\mathrm{mor}(W, Z_r(X)^{(n)})),$$

the suspension isomorphism of part c) follows by applying $\pi_*(-)$ to the above commutative square.

The determination of the homotopy type of $Z_r(A^N)$ follows from Lawson's determination [L] of the Lawson homology of P^N together with the localization sequence. Since a continuous algebraic map

$$W \times \Delta[m] \; \rightarrow \; Z_r(X)$$

is continuous (for the analytic topology), there is a natural inclusion

$$\Phi_{N,r} : \; \mathrm{mor}(W, Z_r(A^N)) \; \subset \; \mathrm{hom}(W, Z_r(A^N)).$$

The final assertion of d.) follows from the obvious compatibility of $\Phi_{N,r}$ with suspension and the observation that the "homotopy property" asserts that

$$\Sigma : \; Z_r(A^N) \; \rightarrow \; Z_{r+1}(A^{N+1})$$

is a homotopy equivalence. ■

The reader might find it useful to contrast $mor(*, Z_r(X))$ with Bloch's complex $z^s(X,*)$ where $s = \mathrm{dimX} - r$. A major difference is that an m-simplex of Bloch's complex is a codimension s cycle on $\Delta[m] \times X$ which need not dominate $\Delta[m]$, whereas an $r+m$-effective cycle on $\Delta[m] \times X$ determines m-simplex of $mor(*, Z_r(X))$ only if it is equi-dimensional over $\Delta[m]$. A second, more subtle difference is that not every m-simplex of $mor(*, Z_r(X))$ arises as an m-simplex of Bloch's complex: the condition on an m-simplex of Bloch's complex of proper intersection of the cycle with faces of $\Delta[m] \times X$ is weakened for an m-simplex of $mor(*, Z_r(X))$ to the condition of existence of a well-defined specialization at every face (in fact, at every point of $\Delta[m]$) of the generically defined cycle. Namely, for an m-simplex given as a family of non-effective cycles over $\Delta[m]$, the condition of proper intersection with faces of $\Delta[m]$ is a condition on both positive and negative parts, whereas the condition of well-defined specialization permits cancellation of improper intersections of positive and negative parts.

What our newly defined "algebraic bivariant cycle theory" currently lacks is a localization theorem. Should such a theorem be valid for this theory, then many of our results for Lawson homology would apply to this theory as well.

References

[A] F.J. Almgren, Homotopy groups of the integral cycle groups. Topology 1 (1962), 257-299.

[B] S. Bloch, Algebraic cycles and higher K-theory, Adv. in Math. 61 (1986), 267-304.

[B-O] S. Bloch and A. Ogus, Gerten's conjecture and the homoogy of schemes, Ann. Scient. Ec Norm. Sup. 4^e serie, t. 7 (1974), 181-202.

[B-G] K.S. Brown and S.M. Gersten, Algebraic K-theory as generalized sheaf cohomology, Lecture Notes in Math 341, Springer-Verlag, (1973), 266-292

[C-W] W.L. Chow and B.L. van der Waerden, Math Ann 113 (1937), 692-704.

[D] A. Dold, Homology of symmetric products and other functors of complexes, Annals of Math 68 (1958), 54-80.

[D-T] A. Dold and R. Thom, Quasifaserungen und unendliche symmetrische produckte, Annals of Math 67 (1958), 239-281.

[F] E. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology, Compositio Mathematica 77 (1991), 55-93.

[F-L] E. Friedlander and H. B. Lawson, A theory of algebraic cocycles. To appear in Annals of Math.

[F-M] E. Friedlander and B. Mazur, Filtration on the homology of algebraic varieties. To appear.

[Fu] W. Fulton, *Intersection Theory*, Ergebnisse der Math, Springer-Verladg 1984.

[G-Z] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergebnisse der Math, Springer Verlag, 1967.

[Gi] H. Gillet, Riemann-Roch theorems for higher algebraic K-theory, Advan. in Math. 40 (1981), 203-289.

[G] A. Grothendieck, Technique de descente et theoremes d'existence en geometrie algebrique V. Les schemas de Picard: Theoremes d'existence. Seminaire Bourbaki, no. 232 (1962).

[H] H. Hironaka, Triangulations of algebraic sets, Proc. of Symposia in Pure Math 29 (1975), 165-185.
[L] H. B. Lawson, Algebraic cycles and homotopy theory, Annals of Math. 129 (1989), 253-291.
[Li-1] P. Lima-Filho, Completion and fibrations for topological monoids and excision for Lawson homology. Preprint.
[Li-2] P. Lima-Filho, Lawson homology for quasiprojective varieties. Preprint.
[M] J. Milnor, On the homology of Lie groups made discrete, Comment. Math. Helvetici 58 (1983), 72-85.
[Q] D. Quillen, Higher algebraic K-theory I, Lecture Notes in Math 341, Springer-Verlag, (1973), 85-147.
[Su] A. Suslin, Singular homology of abstract algebraic varieties, Luminy conference on regulators, 1987.
[V] J-L Verdier, Catégories dérivées, état 0, in SGA4$\frac{1}{2}$, Lecture Notes in Math 569, Springer-Verlag, 1977.

Department of Mathematics
Northwestern University
Evanston, IL. 60208
U.S.A.

Institut des Hautes Etudes Scientifiques
35 Route des Chartres
91440 Bures – sur – Yvette
France