(7) Nonlinear Functions and Related Topics

In this last video, I will cover basic points about

- The *inverse* of a function, *increasing* / *decreasing* functions,
- How a *multi-variate* (multiple variable) function looks through important examples,
- When and how to sum functions *horizontally* or *vertically*,
- How *compound interest* works, and how to calculate the *net present value* of a payoff stream,
- The summation sign that you might see in statistical or economic tables.

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(7) Inverse Function

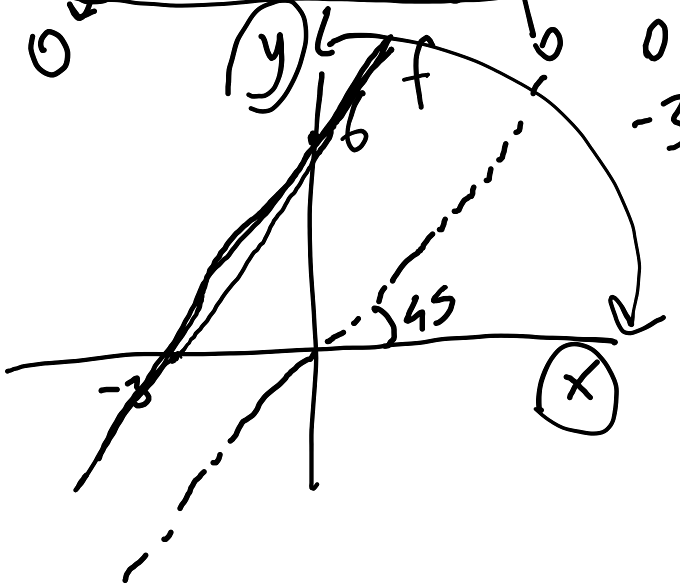
- ***Inverse of a function*** f, called f^{-1}, **<u>reverses</u>** the mapping from the output **y** to the input **x**. It is as if <u>*expressing* **x** *in terms of* **y** *values*</u>, rather than **y** in terms of **x** values:

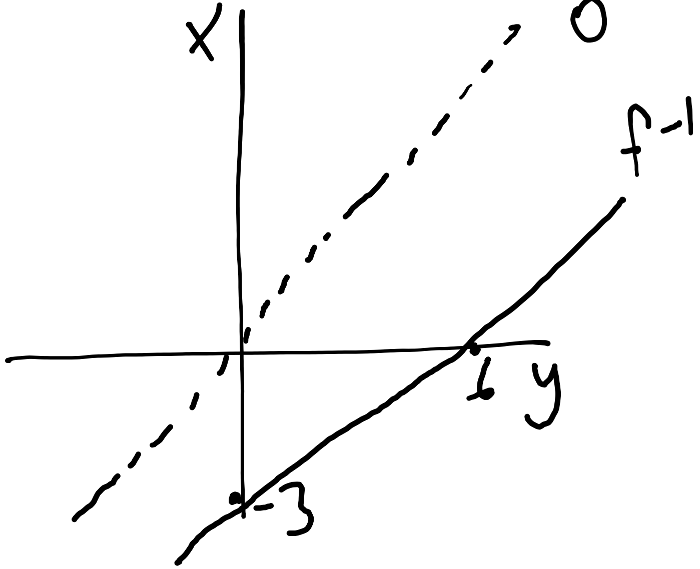
  $f(x) = y \longleftrightarrow f^{-1}(y) = x$

- $\mathbf{y = f(x) = 2x + 6} \rightarrow \underline{\mathbf{y - 6 = 2x}} \rightarrow \mathbf{x = \frac{y-6}{2}} = \boxed{\mathbf{\frac{1}{2}y - 3}} = \mathbf{f^{-1}(y)}$

  $0 = 2x + 6$

  $-3 = x$





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(7) Inverse Function

- For example, the *demand function* $\boxed{\mathbf{Q = 120 - 2p}}$ can be <u>reorganized</u> as $\mathbf{p = 60 - \frac{Q}{2}}$, which is called the ***inverse demand.***

$f(p) = Q = 120 - 2p$

$2p = 120 - Q$

$\boxed{p = 60 - \frac{Q}{2}}$

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(7) Increasing and decreasing Functions

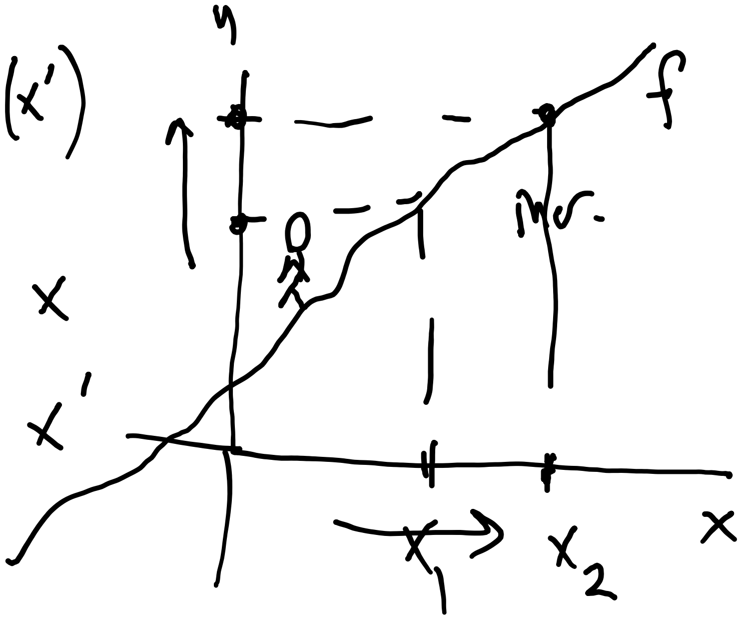
- For the *inverse function* $\mathbf{f}^{-1}$ to be well defined, the function **f** should be ***monotonic:***

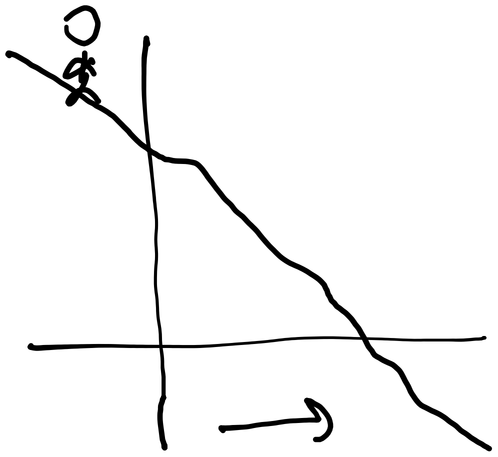
  either ***increasing:*** $\mathbf{x_2 > x_1}$ implies $\mathbf{f(x_2) > f(x_1)}$

  or ***decreasing***: $\mathbf{x_2 > x_1}$ implies $\mathbf{f(x_2) < f(x_1)}$

$f(x) = y = f(x')$

$f^{-1}(y) = ??$ x, x'





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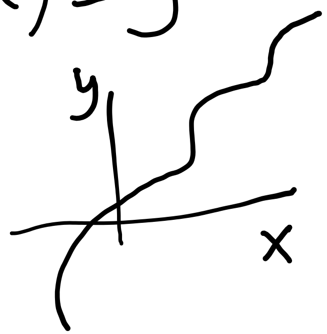
(7) Functions of many variables

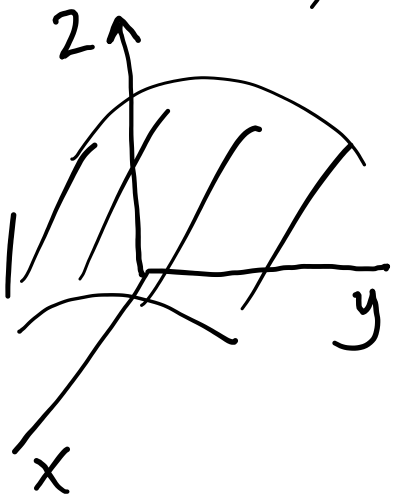
- A multi-variable function has <u>more than one</u> input (*independent*) variable mapped to an output (*dependent*) variable, for example: the *utility function*

$$u(x,y) = \boxed{\sqrt{x} + y^2}$$

$f(x) = y$

$f(x,y) = z$





x	y	u(x,y)
4	2	
4	3	
9	2	
9	3	

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(7) Functions of many variables

- A multi-variable function has <u>more than one</u> input (*independent*) variable mapped to an output (*dependent*) variable, for example: the ***utility function***

$$\mathbf{u(x,y) = \sqrt{x} + y^2}$$

x	y	u(x,y)
4	2	$\sqrt{4} + 2^2 = 6$
4	3	$\sqrt{4} + 3^2 = 11$
9	2	$\sqrt{9} + 2^2 \neq 7$
9	3	$\sqrt{9} + 3^2 = 12$

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(7) Functions of many variables

output capital labor

Consider the ***production function*** $\mathbf{Y = F(K, L) = 3 \cdot K^{0.4} \cdot L^{0.6}}$

- We can plug in different values for K and L to find the values of the function:

$$\mathbf{F(3, 2) = 7.056} \qquad \mathbf{F(2, 5) = 8.453} \qquad \ldots\text{etc.}$$

$(ab)^k = a^k \cdot b^k$

- Notice that $\mathbf{F(2K, 2L) = 3 \cdot (2K)^{0.4} \cdot (2L)^{0.6} = 3 \cdot 2^{0.4} \cdot K^{0.4} \cdot 2^{0.6} \cdot L^{0.6}}$

$$\mathbf{= 2^{0.4} \cdot 2^{0.6} \cdot (3 \cdot K^{0.6} \cdot L^{0.6}) = 2^1 \cdot F(K, L)} \rightarrow \textbf{constant returns to scale}$$

$a^k a^t = a^{k+t}$

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(7) Functions of many variables

$\boxed{\mathbf{F(K,L) = A \cdot K^{a} \cdot L^{b}}}$ has

$0.4 + 0.6 = 1$

- *Constant* returns to scale $\leftrightarrow$ $\mathbf{a + b = 1}$
- *Increasing* returns to scale $\leftrightarrow$ $\mathbf{a + b > 1}$
- *Decreasing* returns to scale $\leftrightarrow$ $\mathbf{a + b < 1}$
- A function of this form is called a ***Cobb-Douglas function***.

$$F(cK, cL) = A \cdot (cK)^a \cdot (cL)^b = A\, c^a K^a c^b L^b$$

$c > 1$

$$= c^a c^b \boxed{A K^a L^b} = c^{a+b} \cdot F(K, L)$$

$$c^{a+b} > c \iff a+b > 1$$

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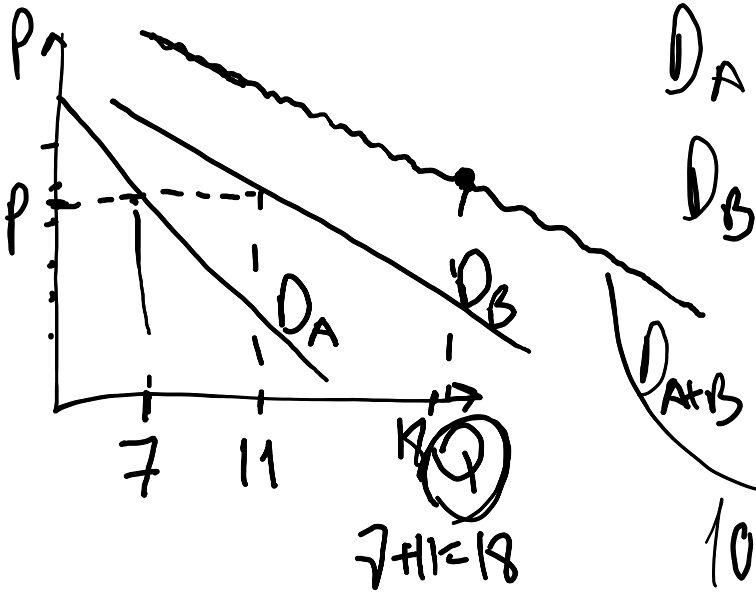
(7) Summing functions horizontally vs. vertically

- When summing functions <u>horizontally</u>, express the variable on <u>horizontal</u> axis (**Q**) in terms of the other variable (**p**) and sum the <u>right-hand sides</u>.

- **<u>EXAMPLE:</u>** Deriving the <u>market demand</u> from <u>individual demands</u>



$$
\begin{aligned}
D_A \quad & Q_A = 120 - 2p \\
D_B \quad + \; & Q_B = 240 - 4p \\
\hline
& Q_{A+B} = Q_A + Q_B = 360 - 6p
\end{aligned}
$$

100 indi. each $Q_i = 120 - 2p$

$$Q_V = 100 \times Q_i = 100 \times (120 - 2p) = 12{,}000 - 200p$$

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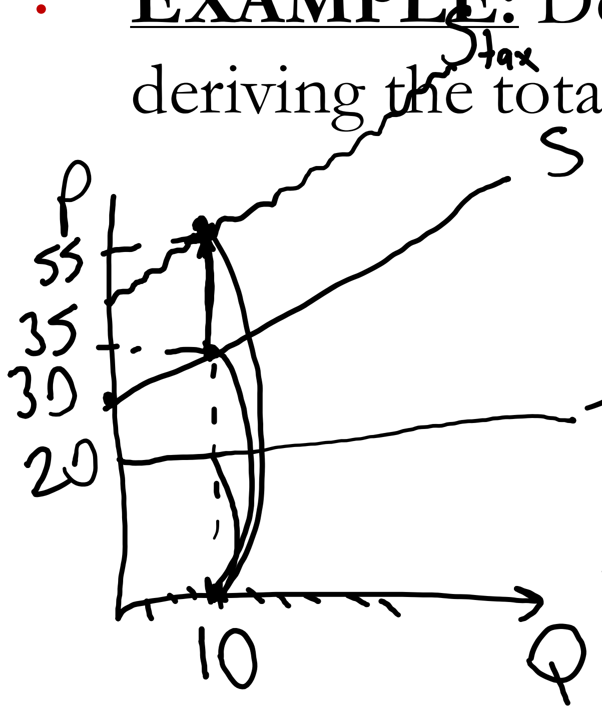
(7) Summing functions horizontally vs. vertically

- When summing functions <u>vertically</u>, express the variable on <u>vertical</u> axis (**p**) in terms of the other variable (**Q**) and sum the <u>right hand sides.</u>
- **<u>EXAMPLE:</u>** Deriving the supply after a per-unit tax being imposed… or deriving the total cost TC(Q) function from fixed and variable costs for a firm…



S

$$\boxed{Q = 2p - 60}$$

10

$Q + 60 = 2p$

tax = 20

$$p_s = \frac{Q}{2} + 30$$

$$+ \quad p_t = 20$$

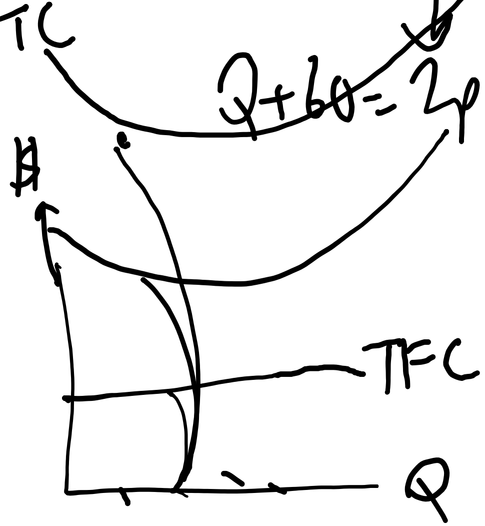
$$p_{s\,after\,tax} = \frac{Q}{2} + 50$$

10

$TFC(Q) = 200$

$+ \quad TVC(Q) = Q^2 - 5Q$

$TC(Q) = Q^2 - 5Q + 200$



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(7) Compound Interest rates, Net Present Value

- Suppose you deposit **P** dollars at *annual interest rate* **i** (Ex: 7% interest, $\mathbf{i = 0.07}$)
- After **t** years you will have $\boxed{\boxed{\mathbf{P}} \cdot (\mathbf{1} + \boldsymbol{i})^{t}}$
- So, **P** dollars now is worth $\boldsymbol{P} \cdot (\mathbf{1} + \boldsymbol{i})^{\boldsymbol{t}}$ dollars **t** years later. Or alternatively;
- **X** dollars received **t** years from now has ***present value*** of $\dfrac{\mathbf{X}}{(\mathbf{1+i})^{\mathbf{t}}}$

$100(1+i)$

$\$100$; $100 + 100i$; $\boxed{100(1+i)}(1+i)$

$t=0$; $t=1$; $t=2$

$?$; X

$t=0$; $t=1$

$? \cdot (1+i) = X$

$? = \dfrac{X}{1+i}$

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(7) Compound Interest rates, Net Present Value

- Assume you are given **C** dollars each year starting from now, <u>for eternity</u>.
- *Now next year the following year….*

t=0	t=1	t=2	t=3	t=4….		time
C	C	C	C	C	C …	

- ***Net Present Value (NPV)*** of this stream of payoffs will be:

$$\mathbf{NPV} = \mathbf{C} + \boxed{\frac{\mathbf{C}}{(\mathbf{1}+\mathbf{i})^{\mathbf{1}}}} + \frac{\mathbf{C}}{(\mathbf{1}+\mathbf{i})^{\mathbf{2}}} + \frac{\mathbf{C}}{(\mathbf{1}+\mathbf{i})^{\mathbf{3}}} + \cdots$$

- $\frac{\mathbf{1}}{\mathbf{1+i}} = \boldsymbol{\delta}$ is sometimes called the ***discount factor***.

$\in (0,1)$

$$\mathbf{NPV} = \mathbf{C} + \boldsymbol{C\delta} + \boldsymbol{C\delta^2} + \boldsymbol{C\delta^3} + \cdots$$

$100\delta \leftarrow 100$

$t=1$

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(7) Compound Interest rates, Net Present Value

- ***Net Present Value (NPV)*** of this stream of payoffs will be:

$$\mathbf{NPV} = C + C\delta + C\delta^2 + C\delta^3 + \cdots = C(1 + \cancel{\delta} + \cancel{\delta^2} + \cancel{\delta^3} + \cancel{\cdots})$$

$$-\quad \delta \cdot \mathbf{NPV} = C(\cancel{\delta} + \cancel{\delta^2} + \cancel{\delta^3} + \cancel{\delta^4} + \cdots)$$

$$(1 - \delta) \cdot \mathbf{NPV} = C$$

$i = 5\%$

$$\mathbf{NPV} = \boxed{\frac{C}{1-\delta}} = \frac{C}{1 - \frac{1}{1+i}} = \frac{C}{\frac{1+i-1}{1+i}} = C \cdot \frac{1+i}{i} = \boxed{C \cdot \left(1 + \frac{1}{i}\right)}$$

$$\frac{1+i}{1+i} - \frac{1}{1+i} \qquad \frac{i}{1+i}$$

$$100 \times \left(1 + \frac{1}{0.05}\right) = 2100$$

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(7) Summation Sign, Price Indices

- $\sum_{i=2}^{i=5}(2i^2 - 1) = (2 \cdot 2^2 - 1) + (2 \cdot 3^2 - 1) + (2 \cdot 4^2 - 1) + (2 \cdot 5^2 - 1)$

  $= 7 + 17 + 31 + 49 = \boxed{104}$

  $\sum_{a}^{b} f(i)$

- In general, $\sum_{i=a}^{i=b} f(i) = f(a) + f(a+1) + \cdots + f(b-1) + f(b)$
- In the previous example on a stream of **C** dollars each year, we would have:

  $2(a+b) = 2a + 2b$

- **NPV** $= \boxed{\sum_{t=0}^{t=\infty} \frac{C}{(1+i)^t}} = C \cdot \sum_{t=0}^{t=\infty} \frac{1}{(1+i)^t}$

  $= C + \frac{C}{1+i} + \frac{C}{(1+i)^2} + \cdots$

  $\sum_{i=2}^{i=5}(2i^2 - 1) = 2\sum_{i=2}^{i=5} i^2 - \sum_{i=2}^{i=5} 1$


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