A Mean-field Stochastic Control Problem with Partial Observations

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Abstract

In this paper we are interested in a new type of mean-field-type, non-Markovian stochastic control problems with partial observations. More precisely, we assume that the coefficients of the controlled dynamics depend not only on the paths of the state, but also on the conditional law of the state, given the observation to date. Our problem is strongly motivated by the recent study of the mean field games and the related McKean-Vlasov stochastic control problem, but with added aspects of path-dependence and partial observation. We shall first investigate the well-posedness of the state-observation dynamics, with combined reference probability measure arguments in nonlinear filtering theory and the Schauder fixed point theorem. We then study the stochastic control problem with a partially observable system in which the conditional law appears nonlinearly in both the coefficients of the system and cost function. As a consequence the control problem is intrinsically “time-inconsistent”, and we prove that the Pontryagin Stochastic Maximum Principle holds in this case and characterize the adjoint equations, which turn out to be a new form of mean-field type BSDEs.

Keywords. Conditional mean-field SDEs, non-Markovian stochastic control system, nonlinear filtering, stochastic maximum principle, mean-field backward SDEs.


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1 Introduction

In this paper we are interested in the following mean-field-type stochastic control problem, on a given filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0})\):
\[
\begin{cases}
  dX_t = \mathbb{E}\{b(t, \varphi_{\wedge t}, \mathbb{E}[X_t|\mathcal{G}_t], u)|_{\varphi=X,u=u_s}dt + \mathbb{E}\{\sigma(t, \varphi_{\wedge t}, \mathbb{E}[X_t|\mathcal{G}_t], u)|_{\varphi=X,u=u_s}dB_t, \\
  X_0 = x,
\end{cases}
\]
where \(B\) is an \(\mathbb{F}\)-Brownian motion, \(b\) and \(\sigma\) are measurable functions satisfying reasonable conditions, \(\varphi_{\wedge t}\) and \(X_{\wedge t}\) denotes the continuous function and process, respectively, “stopped” at \(t\); \(\mathbb{G} \triangleq \{\mathcal{G}_t\}_{t \geq 0}\) is a given filtration that could involve the information of \(X\) itself, and \(u = \{u_t : t \geq 0\}\) is the “control process”, assumed to be adapted to a filtration \(\mathbb{H} = \{\mathcal{H}_t\}_{t \geq 0}\), where \(\mathcal{H}_t \subseteq \mathcal{F}_t^X \lor \mathcal{G}_t, t \geq 0\). We note that if \(\mathcal{G}_t = \{\emptyset, \Omega\}\) for all \(t \geq 0\) (i.e., the conditional expectation in (1.1) becomes expectation), \(\mathcal{H}_t = \mathcal{F}_t^X\), and coefficients are “Markovian” (i.e., \(\varphi_{\wedge t} = \varphi_t\)), then the problem becomes a stochastic control problem with McKean-Vlasov dynamics and/or a Mean-field game (see, for example [7, 8, 9] in its “forward” form, and [2, 3, 4] in its “backward” form). On the other hand, when \(\mathbb{G}\) is a given filtration, this is the so-called conditional mean-field SDE (CMFSDE for short) studied in [11]. We note that in that case the conditioning is essentially “open-looped”.

The problem that this paper is particularly focusing on is when \(\mathcal{G}_t = \mathcal{F}_t^Y, t \geq 0\), where \(Y\) is an “observation process” of the dynamics of \(X\), i.e., the case when the pair \((X,Y)\) forms a “close-looped” or “coupled” CMFSDE. More precisely, we shall consider the following partially observed controlled dynamics (assuming \(b = 0\) for notational simplicity):
\[
\begin{cases}
  dX_t = \mathbb{E}\{\sigma(t, \varphi_{\wedge t}, \mathbb{E}[X_t|\mathcal{F}_t^Y], u)|_{\varphi=X,u=u_s}dB_t; \\
  dY_t = h(t, X_t)dt + \hat{\sigma}dB_t^2; \\
  X_0 = x, \quad Y_0 = 0.
\end{cases}
\]
Here \(X\) is the “signal” process that can only be observed through \(Y\), \((B^1, B^2)\) is a standard Brownian motion, and \(\hat{\sigma}\) is a constant. We should note that in SDEs (1.2) the conditioning filtration \(\mathbb{F}^Y\) now depends on \(X\) itself, therefore it is much more convoluted than the CMFSDE we have seen in the literature. Furthermore, the path-dependent nature of the coefficients makes the SDE essentially non-Markovian. Such form of CMFSDEs, to the best of our knowledge, has not been explored fully in the literature.

Our study of the CMFSDE (1.2) is strongly motivated by the following variation of the mean field game in a finance context, which would result in a McKean-Vlasov stochastic control problem with partial observation as we are proposing (see, [8], for a more detailed background). Consider a firm whose fundamental value, under the risk neutral measure
\( \mathbb{P}^0 \) with zero interest, evolves as following SDE with “stochastic volatility” \( \sigma = \sigma(t, \omega) \), 
\((t, \omega) \in [0, \infty) \times \Omega; \)
\[ X_t = x + \int_0^t \sigma(s, \cdot) dB^1_s, \quad t \geq 0, \tag{1.3} \]
where \( B^1 \) is the intrinsic noise from inside the firm. We assume that such fundamental value process cannot be observed directly, but can be observed through a stochastic dynamics (e.g., its stock value) via an SDE:
\[ Y_t = \int_0^t h(s, X_s) ds + B^2_t, \quad t \geq 0, \tag{1.4} \]
where \( B^2 \) is the noise from the market, which we assume is independent of \( B^1 \) (this is by no means necessary, we can certainly consider the filtering problem with correlated noises).

Now let us assume that the volatility \( \sigma \) in (1.3) is affected by the actions of a large number of investors, and all can only make decisions based on the information from the process \( Y \). Therefore, in light of the argument of [8] we could start from a Mean-field game in which each individual investor has a private state dynamics of the form:
\[ dU^i_t = \sigma^i(t, U^i_{\lambda t}, \tilde{\nu}^N_t, \alpha^i_t) dB^2^i_t, \quad t \geq 0, \quad i = 1, 2, \ldots \tag{1.5} \]
where \( B^{1,i} \)'s are independent Brownian motions, and \( \tilde{\nu}^N \) denotes the empirical conditional distribution of \( U = (U^1, \ldots, U^N) \), given the (common) observation \( Y = \{ Y_t : t \geq 0 \} \), that is, \( \tilde{\nu}^N \triangleq \frac{1}{N} \sum_{j=1}^N \delta_{\mathbb{E}[U^j_t | F^Y_t]} \), where \( \delta_x \) denotes the Dirac measure at \( x \). More precisely, the notation in (1.5) means (see, e.g., [8]),
\[ \sigma^i(t, U^i_{\lambda t}, \tilde{\nu}^N_t, \alpha^i_t) \triangleq \int_{\mathbb{R}} \tilde{\sigma}^i(t, U^i_{\lambda t}, y, \alpha^i_t) \tilde{\nu}^N_t(dy) \]
\[ = \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}} \tilde{\sigma}^i(t, U^j_{\lambda t}, y, \alpha^i_t) \delta_{\mathbb{E}[U^j_t | F^Y_t]}(dy) \tag{1.6} \]
\[ = \frac{1}{N} \sum_{j=1}^N \tilde{\sigma}^i(t, U^j_{\lambda t}, \mathbb{E}[U^j_t | F^Y_t], \alpha^i_t). \]
Here, \( \tilde{\sigma}^i \)'s are the functions defined on appropriate (Euclidean) spaces.

We now assume that each investor chooses a strategy, and tries to minimize the cost:
\[ J^i(\alpha^i) \triangleq \mathbb{E}\left\{ \Phi^i(U^i_T) + \int_0^T L^i(t, U^i_{\lambda t}, \tilde{\nu}^N_t, \alpha^i_t) \right\}. \tag{1.7} \]
Following the argument of Lasry and Lions [16] (see also [8, 9]), if we assume that the game is symmetric (i.e., \( \sigma^i = \sigma \), for all \( i \)) and that the number of investors becomes large,
then under suitable technical conditions, one can find (approximate) Nash equilibriums through a limiting dynamics, and assign each investor the unified strategy $\alpha$, determined by a McKean-Vlasov type dynamics:

$$dX_t = \sigma(t, X_{\land t}, \mu_t, \alpha_t)dB^1_t, \quad t \geq 0,$$

where $\mu$ is the conditional distribution of $X_t$ given $\mathcal{F}^Y_t$, and

$$\sigma(t, X_{\land t}, \mu_t, u_t) \triangleq \int \sigma(t, X, y, u_t)\mu_t(dy) = \mathbb{E}\{\sigma(t, \varphi_{\land t}, \mathbb{E}[X_t]|\mathcal{F}^Y_t], u)\}|_{\varphi=X,u=u_s}.$$

Furthermore, the value function becomes, with similar notations,

$$V(x) = \inf_{\alpha} J(\alpha) \triangleq \mathbb{E}\{\Phi(X_T) + \int_0^T L(t, X_{\land t}, \mu_t, \alpha_t)\}.$$

We note that (1.8) and (1.9), together with (1.4), form a stochastic control problem with McKean-Vlasov dynamics and partial observations, as we are proposing.

The main objective of this paper is twofold: We shall first study the exact meaning as well as the well-posedness of the dynamics, and then investigate the Stochastic Maximum Principle for the corresponding stochastic control problem. For the wellposedness of (1.2) we shall use a scheme that combines the idea of [7] and the techniques of nonlinear filtering, and prove the existence and uniqueness of the solution to SDE (1.8) via Schauder’s fixed point theorem on $\mathcal{P}_2(\Omega)$, the space of probability measures with finite second moment, endowed with the $2$-Wasserstein metric. We note that the important elements in this argument include the so-called reference probability space that is often seen in the nonlinear filtering theory and the Kallianpur-Striebel formula (cf. e.g., [1, 19]), which enable us to define the solution mapping.

Our next task is to prove Pontryagin’s Maximum Principle for our stochastic control problem. The main idea is similar to earlier works of the first two authors ([4, 17]), with some significant modifications. In particular, since in the present case the control problem can only be carried out in a weak form, due to the lack of strong solution of CMFSDE, the existence of the common reference probability space is essential. Consequently, extra effort is needed to overcome the complexity caused by the change of probability measures, which, together with the path-dependent nature of the underlying dynamic system, makes even the first order adjoint equation more complicated than the traditional ones. To the best of our knowledge, the resulting mean-field backward SDE is new.

The rest of the paper is organized as follows. In section 2 we provide all the necessary preparations, including some known facts of nonlinear filtering. In sections 3 and 4 we
prove the well-posedness of the partially observable dynamics. In section 5 we introduce the stochastic control problem, and in section 6 we study the variational equations and give some important estimates. Finally, in section 7 we prove the Pontryagin maximum principle.

2 Preliminaries

Throughout this paper we consider a complete filtered probability space \((\Omega, \mathcal{F}, P; \mathbb{F})\), where \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is a filtration satisfying the usual hypotheses. We often consider the canonical probability space in which \(\Omega \triangleq C_0([0, \infty); \mathbb{R}^{2d}) = \{\omega \in C([0, \infty); \mathbb{R}^{2d}) : \omega_0 = 0\}\) and \(\mathbb{F}\) is the natural filtration on \(\Omega\) (generated by the canonical process \(B_t(\omega) = \omega_t, \omega \in \Omega\)). For simplicity, throughout this paper we assume \(d = 1\), and all the processes are 1-dimensional, although the higher dimensional cases can be argued similarly without substantial difficulties. We let \(\mathcal{P}(\Omega)\) denote the space of all probability measures on \((\Omega, \mathcal{F})\), and \(\mathbb{Q}_0\) be the (2-dimensional) Wiener measure.

Next, for a given \(T > 0\) we denote \(C_T = C([0, T])\), and let \(\mathcal{B}(C_T)\) be the natural \(\sigma\)-algebra on \(C_T\). Consider the space of all probability measures on \((C_T, \mathcal{B}(C_T))\), denoted by \(\mathcal{P}(C_T)\), and for \(p \geq 1\) we let \(\mathcal{P}_p(C_T) \subseteq \mathcal{P}(C_T)\) be those that have finite \(p\)-th moment. We recall that the \(p\)-Wasserstein metric on \(\mathcal{P}(C_T)\) is defined as a mapping \(W_p: \mathcal{P}(C_T) \times \mathcal{P}(C_T) \mapsto \mathbb{R}_+\) such that for all \(\mu, \nu \in \mathcal{P}(C_T)\),

\[
W_p(\mu, \nu) \triangleq \inf \left\{ \int_{C_T^2} \|x - y\|_p^p \pi(dx, dy) \right\}^{1/p}; \; \pi \in \mathcal{P}_p(C_T) \text{ with marginals } \mu \text{ and } \nu. \tag{2.1}
\]

In this paper we shall use the 2-Wasserstein metric \(W_2\), and we shall abbreviate \((\mathcal{P}_2(C_T), W_2)\) by \(\mathcal{P}_2(C_T)\). Since \(C_T\) is a separable Banach space, it is known that \(\mathcal{P}_2(C_T)\) is a separable and complete metric space. Furthermore, it is known that (cf. e.g., [18]), for \(\mu_n, \mu \in \mathcal{P}_2(C_T)\),

\[
\lim_{n \to \infty} W_2(\mu_n, \mu) = 0 \iff \mu_n \wto \mu \text{ in } \mathcal{P}_2(C_T) \text{ and } \sup_n \int_{\Omega} \|\varphi\|_{C_T}^2 \mu_n(d\varphi) < \infty. \tag{2.2}
\]

Next, for any \(\mathbb{P} \in \mathcal{P}(\Omega)\), \(p, q \geq 1, \mathbb{G} \subseteq \mathbb{F}\), and any Banach space \(\mathbb{X}\), we denote \(L^p(\mathbb{P}; \mathbb{X})\) to be all \(\mathbb{X}\)-valued \(L^p\)-random variables under \(\mathbb{P}\). In particular, we denote \(L^p(\mathbb{P}; \Omega)\) to be all real valued \(L^p\)-random variables under \(\mathbb{P}\). Further, we denote by \(L^p_{\mathbb{G}}(\mathbb{P}; [0, T])\) the \(L^p\)-space of all \(\mathbb{G}\)-adapted processes \(\eta\), such that

\[
\|\eta\|_{p,q,\mathbb{P}} \triangleq \mathbb{E}^\mathbb{P}\left\{ \left[ \int_0^T |\eta|^q dt \right]^{p/q} \right\}^{1/p} < \infty. \tag{2.3}
\]
If \( p = q \), we simply write \( L^p_G(\mathbb{P}; [0, T]) \overset{\Delta}{=} L^p(|\mathbb{P}; L^p([0, T])) \). Finally, we define \( L^\infty_G(\mathbb{P}; [0, T]) \overset{\Delta}{=} \bigcap_{p>1} L^p_G(\mathbb{P}; [0, T]) \) and \( L^\infty_G(\mathbb{P}; C_T) \overset{\Delta}{=} \bigcap_{p>1} L^p_G(\mathbb{P}; C_T) \), where \( L^p_G(\mathbb{P}; C_T) \) is the space of all continuous, \( \mathbb{F} \)-adapted, processes \( \xi = \{ \xi_t \} \) such that \( \| \xi \|_{C_T} \in L^p(\mathbb{P}; \Omega) \). We will often drop “\( \mathbb{P} \)” from the subscript/superscript when the context is clear.

### 2.1 Formulation of the control problem

We now give a more precise description of our stochastic control problem. We first reformulate the SDEs (1.2) in terms of the standard McKean-Vlasov SDE. Again we consider only the case \( b = 0 \), and we assume further that \( \hat{\sigma} = 1 \) for simplicity.

We begin by introducing some notations. Let \( \mathcal{F}_t \) be the state process and \( Y \) the observation process. We denote the “filtered” state process by \( U^X_t = \mathbb{P}[X_t | \mathcal{F}^Y_t], \ t \geq 0 \). We note that (as we show in Lemma 3.2 below), the process \( U^X_t \) is continuous, thus we denote its law under \( \mathbb{P} \) on \( C_T \) by \( \mu^X_t = \mathbb{P} \circ [U^X_t]^{-1} \in \mathcal{P}(C_T) \). Next, let \( P_t(\varphi) = \varphi(t), \ \varphi \in C_T, \ t \geq 0, \) be the projection mapping, and define \( \mu_t^X = \mu^X_t \circ P_t^{-1} \). Then, for any \( \varphi \in C_T \), and \( u \in \mathbb{R} \), we can write

\[
\mathbb{E}[\sigma(s, \varphi, \mathcal{F}_s, \mathbb{E}[X_s | \mathcal{F}^Y_s], u)] = \int \sigma(t, \varphi, \mathcal{F}_s, y, u) \mu^X_t(dy) \overset{\Delta}{=} \sigma(t, \varphi, \mathcal{F}_s, \mu^X_t, u).
\]

We should note that since the dynamics \( X \) is non-observable, the decision of the controller can only be made based on the information observed from the process \( Y \). Therefore, it is reasonable to assume that the control process \( u \) is \( \mathbb{F}^Y = \{ \mathcal{F}^Y_t \}_{t \geq 0} \) adapted (or progressively measurable). Furthermore, for each given control, we do not require the state-observation SDEs to have a strong solution on a prescribed probability space (our well-posedness result does not guarantee this point). We thus consider a “weak formulation”. More precisely, we consider pairs \( (\mathbb{P}, u) \), where \( \mathbb{P} \in \mathcal{P}(\Omega) \), \( u \in L^2(\mathbb{P}; [0, T]) \), for the following SDE:

\[
X_t = x + \int_0^t \mathbb{E}[\sigma(s, \varphi, \mathcal{F}_s, \mathbb{E}[X_s | \mathcal{F}^Y_s], z)]_{\varphi = \mathcal{F}_s, \mathcal{F}^Y_s} dB^1_s \overset{(2.4)}{=} x + \int_0^t \mathbb{E}[\sigma(s, \varphi, \mathcal{F}_s, y, u_s)] dB^1_s = x + \int_0^t \sigma(t, X_s, \mu_t, u_s) dB^1_s,
\]

where \( \mu_t(\cdot) \overset{\Delta}{=} \mathbb{P} \circ \mathbb{E}[X_t | \mathcal{F}^Y_t]^{-1}(\cdot) \) is the distribution, under \( \mathbb{P} \), of the conditional expectation of \( X_t \), given \( \mathcal{F}^Y_t \); and the observation process \( Y \) satisfies the SDE:

\[
Y_t = \int_0^t h(s, X_s) ds + B^2_t, \quad \text{ (2.5)}
\]

where \( B^2 \) is a standard Brownian motion, independent of \( B^1 \), under \( \mathbb{P} \). We now define the set of admissible controls. Let \( U \) be a convex subset of \( \mathbb{R}^k \). For simplicity, assume \( k = 1 \).
Definition 2.1. A pair \((P, u) \in \mathcal{P}(\Omega) \times L^2_F(P; [0, T])\) is called an “admissible control” if
(i) \(u_t \in U\), for all \(t \in [0, T]\),
(ii) There exist processes \((X, Y) \in L^2_F(P; [0, T])\) satisfying SDEs (2.4) and (2.5); and
(iii) \(u \in L^\infty - F_Y(P; [0, T])\).

We shall denote the set of all admissible controls by \(\mathcal{U}_{ad}\). For simplicity we often write \(u \in \mathcal{U}_{ad}\), and denote the associated probability measure \(P\) by \(P^u\), for \(u \in \mathcal{U}_{ad}\).

Remark 2.2. We should note that since the probability measure \(P^u\) varies with the control, the main difficulty of SDE (2.4) lies in its “closed-looped” nature, that is, the dependence of the law of conditional expectation \(\mu_t = \mu^u_t = P^u \circ E^{P^u}[X^u_t | F^Y_t]^{-1}\) of \(X^u_t\) knowing \(Y\). Moreover, the requirement that \(u\) is \(F_Y\)-adapted adds an additional seemingly “circular” nature to the problem. Thus the well-posedness of the problem is by no means obvious, and will be discussed in details in the next section.

To conclude our control problem let us introduce our cost functional. We shall consider this following “mean-field” type cost functional, which reflects the nature of the problem. For any \(u \in \mathcal{U}_{ad}\), we define
\[
J(t, x; u) \triangleq E^{Q^0} \left\{ \int_t^T f(s, X^u_s, \mu^u_s, u_s) ds + \Phi(X^u_T, \mu^u_T) \right\} \\
= E^{Q^0} \left\{ \int_t^T E^{P^u} \left[ f(s, \varphi_s \wedge s, E^{P^u}[X^u_s | F^Y_s], u) \right]_{\varphi = X^u_s, u = u_s} ds \right. \\
+ E^{P^u} \left[ \Phi(x, E^{P^u}[X^u_T | F^Y_T]) \right]_{x = X^u_t} \right\}. \tag{2.6}
\]

We denote the value function as
\[
V(t, x) \triangleq \min_{u \in \mathcal{U}_{ad}} J(t, x; u). \tag{2.7}
\]

Throughout this paper we shall assume that all the control processes take values in the convex set \(U \subseteq \mathbb{R}^k\), and for simplicity we shall assume that \(k = 1\). Furthermore, we shall make use of the following Standing Assumptions.

Assumption 2.3. (i) The mappings \((t, \varphi, x, y, z) \mapsto \sigma(t, \varphi, x, y, z), h(t, x), f(t, \varphi, x, y, z), \) and \(\Phi(x, y)\) are bounded and continuous, for \((t, \varphi, x, y, z) \in [0, T] \times \mathbb{C}_T \times \mathbb{R} \times \mathbb{R} \times U;\)
(ii) The partial derivatives $\partial_\varphi \sigma$, $\partial_\omega \sigma$, $\partial_\varphi f$, $\partial_\omega f$, $\partial_\varphi h$, $\partial_{\varphi} \Phi$, $\partial_{\omega} \Phi$ are bounded and continuous, for $(\varphi, \omega, x, y, z) \in \mathbb{C}_T \times \mathbb{R} \times \mathbb{R} \times U$, uniformly in $t \in [0, T]$;

(iii) The mappings $\varphi \mapsto \Phi(t, \varphi, x, y, z, f(t, \varphi, x, y, z))$, as functionals from $\mathbb{C}_T$ to $\mathbb{R}$, are Frechét differentiable. Furthermore, there exists a family of measures $(\mu_t)_{t \in [0, T]}$, uniformly in $t \in [0, T]$, satisfying

$$|D_{\varphi} \sigma(t, \varphi, x, y, z)(\psi)| + |D_{\varphi} f(t, \varphi, x, y, z)(\psi)| \leq \int_0^T |\psi(s)| \ell(t, ds), \quad \psi \in \mathbb{C}_T, \quad (2.8)$$

uniformly in $(t, \varphi, y, z)$;

(iv) The mapping $y \mapsto y \partial_{\varphi} \sigma(t, \varphi, x, y, z)$ is uniformly bounded, uniformly in $(t, \varphi, z)$;

(v) The mapping $x \mapsto x \partial_{\varphi} h(t, x)$ is uniformly bounded, uniformly in $t \in [0, T]$.

(vi) The mappings $x \mapsto x \partial_{\varphi} h(t, x), x^2 \partial_{\omega} h(t, x)$ are uniformly bounded, uniformly in $t \in [0, T]$.

We should note that some of these assumptions are merely technical, and can certainly be improved. But we prefer not to dwell on such technicalities and focus on the main ideas instead.

**Remark 2.4.** Note that if $(t, \varphi, y, z) \mapsto \phi(t, \varphi, x, y, z)$ is a function defined on $[0, T] \times \mathbb{C}_T \times \mathbb{R} \times \mathbb{R}$ satisfying Assumption 2.3-(i), (ii), then for any $\mu \in \mathcal{P}_2(\mathbb{C}_T)$, we can define a function on the space $[0, T] \times \Omega \times \mathbb{C}_T \times \mathcal{P}_2(\mathbb{C}_T) \times U$:

$$\tilde{\phi}(t, \omega, \varphi, x, y, z, \mu) \triangleq \int_\mathbb{R} \phi(t, \varphi, x, y, z) \mu_t(dy), \quad (2.9)$$

where $\mu_t = \mu \circ P^{-1}_t$ and $P_t(\varphi) \triangleq \varphi(t), (t, \varphi) \in [0, T] \times \mathbb{C}_T$. Then, $\tilde{\phi}$ must satisfy the following Lipschitz condition:

$$|\tilde{\phi}(t, \varphi^1, \mu^1, z^1) - \tilde{\phi}(t, \varphi^2, \mu^2, z^2)| \leq K \left\{ \|\varphi^1 - \varphi^2\|_{\mathbb{C}_t} + W_2(\mu^1, \mu^2) + |z^1 - z^2| \right\}, \quad (2.10)$$

where $\| \cdot \|_{\mathbb{C}_t}$ is the sup-norm on $\mathbb{C}([0, t])$ and $W_2(\cdot, \cdot)$ is the Wasserstein-2 metric.

**Remark 2.5.** The Frechét derivatives $D_{\varphi} \sigma$ and $D_{\varphi} f$ by definition belong to $\mathbb{C}_T \triangleq \mathcal{M}[0, T]$, the space of all finite signed Borel measures on $[0, T]$, endowed with the total variation norm $| \cdot |_{TV}$ (with a slight abuse of notation, we still denote it by $| \cdot |$). Thus the Assumption 2.3-(iii) amounts to saying that, as measures,

$$|D_{\varphi} \sigma(t, \varphi, x, y, z)(ds)| + |D_{\varphi} f(t, \varphi, x, y, z)(ds)| \leq \ell(t, ds), \quad \forall (t, \varphi, x, z). \quad (2.11)$$

This inequality will be crucial in our discussion in Section 7.
2.2 Nonlinear Filtering Revisited

We now review some basic ideas in nonlinear filtering theory, adapted to our situation. These ideas are essential for us to establish the framework on which the well-posedness of the state dynamics (2.4) can be proved. The main idea is the so-called “reference measure method” to solve this filtering problem, which is based on the following assumption.

Assumption 2.6. There exists a probability measure $Q^0$ on $(\Omega, \mathcal{F})$, such that, under $Q^0$, $(B^1, Y)$ is a 2-dimensional Brownian motion, where $Y$ is the observation process.

We should note that the existence of such reference measure is evident in the case where the underlying state-observation SDEs (2.4) and (2.5) have a solution, thanks to the Girsanov theorem. Indeed, let $u \in \mathcal{U}_{ad}$, $P^u$ be the associate probability measure, and $X^u$ be the corresponding state process. Consider the following SDE:

$$\bar{L}_t = 1 - \int_0^t h(s, X^u_s)\bar{L}_s dB^2_s = 1 + \int_0^t \bar{L}_s dZ^u_s, \quad (2.12)$$

where $Z^u_t = -\int_0^t h(s, X^u_s)dB^2_s$. We denote its solution by $\bar{L}^u_t$. Then under appropriate conditions on $h$, both $Z^u$ and $\bar{L}^u$ are $P^u$-martingales, and $L^u$ is the Doléan-Dade stochastic exponential:

$$\bar{L}_t^u = \exp \left\{ Z^u_t - \frac{1}{2} (Z^u)_t^2 \right\} = \exp \left\{ -\int_0^t h(s, X^u_s)dB^2_s - \frac{1}{2} \int_0^t |h(s, X^u_s)|^2 ds \right\}. \quad (2.13)$$

Furthermore, if we define $dQ^0 \triangleq \bar{L}^u_T dP^u$, then the Girsanov theorem states that $Y_t = B^2_t + \int_0^t h(s, X^u_s)ds$, $t \geq 0$, is a Brownian motion under $Q^0$ and is independent of $B^1$. Thus $Q^0$ is a reference measure.

We note that in what follows we will use more often the inverse Girsanov kernel $L^u_t \triangleq [\bar{L}^u_t]^{-1}$, $t \geq 0$. That is,

$$L^u_t \triangleq [\bar{L}^u_t]^{-1} = \exp \left\{ \int_0^t h(s, X^u_s)dY_s - \frac{1}{2} \int_0^t |h(s, X^u_s)|^2 ds \right\}, \quad t \in [0, T]. \quad (2.14)$$

We observe that $L^u$ is a $Q^0$-martingale and $dP^u = L^u_T dQ^0$.

We now start from the reference measure $Q^0$. For fixed control $(P, u)$, we denote $L = L_u$ by (2.14). Then $L$ satisfies the SDE on $(\Omega, \mathcal{F}, Q^0)$:

$$L_t = 1 + \int_0^t h(s, X_s)L_s dY_s, \quad t \in [0, T], \quad (2.15)$$
where $X = X^u$. An important ingredient that we are going to use often is the SDEs for the “normalized conditional probability,” known as the Kushner-Stratonovic or Fujisaki-Kallianpur-Kunita (FKK) equation. Let us denote

$$S_t \triangleq \mathbb{E}^{Q^0}[L_t X_t | \mathcal{F}^Y_t], \quad S^0_t \triangleq \mathbb{E}^{Q^0}[L_t | \mathcal{F}^Y_t], \quad t \geq 0. \quad (2.16)$$

Since under $Q^0$ the process $(B^1, Y)$ is a Brownian motion, the $\sigma$-field $\mathcal{F}^Y_{t,T}$ and $\mathcal{F}^Y_t \vee \mathcal{F}^B_{t}^1$ are independent, where $\mathcal{F}^Y_{t,T} \triangleq \sigma\{Y_s : t \leq s \leq T\}$. It is standard to show that (in light of (2.15)) $S$ and $S^0$ satisfy the following SDEs:

$$S^0_t = 1 + \int_0^t \mathbb{E}^{Q^0}[h(s, X_s) L_s | \mathcal{F}^Y_s]dY_s, \quad t \geq 0. \quad (2.17)$$

and

$$S_t = x + \int_0^t \mathbb{E}^{Q^0}[L_s X_s h(s, X_s) | \mathcal{F}^Y_s]dY_s, \quad t \geq 0, \quad (2.18)$$

Furthermore, let $U_t \triangleq U[I](t) = \mathbb{E}^{P^u}[X_t | \mathcal{F}^Y_t], \quad t \geq 0$. Then, by the Bayes formula (also known as the Kallianpur-Striebel formula, see, e.g., [1]) we have

$$U_t = \frac{\mathbb{E}^{Q^0}[L_t X_t | \mathcal{F}^Y_t]}{\mathbb{E}^{Q^0}[L_t | \mathcal{F}^Y_t]} = \frac{S_t}{S^0_t}, \quad t \geq 0. \quad (2.19)$$

A simple application of Itô’s formula and some direct computation then lead to the following Fujisaki-Kallianpur-Kunita (FKK) equation:

$$dU_t = \left[\mathbb{E}^{P^u}[X_t h(t, X_t) | \mathcal{F}^Y_t] - \mathbb{E}^{P^u}[X_t | \mathcal{F}^Y_t] \mathbb{E}^{P^u}[h(t, X_t) | \mathcal{F}^Y_t]\right]dY_t$$

$$+ \left\{\mathbb{E}^{P^u}[X_t | \mathcal{F}^Y_t] \mathbb{E}^{P^u}[h(t, X_t) | \mathcal{F}^Y_t] - \mathbb{E}^{P^u}[X_t h(t, X_t) | \mathcal{F}^Y_t] \mathbb{E}^{P^u}[h(t, X_t) | \mathcal{F}^Y_t]\right\} dt$$

$$= \{U[h](t)U_t - U[Ih](t)U[h](t)\} dt + \{U[Ih](t) - U[h](t)U_t\} dY_t. \quad (2.20)$$

In fact, one can easily show that

$$S_t = U_t \exp\left\{\int_0^t \mathbb{E}^{P^u}[h(s, X_s) | \mathcal{F}^Y_s]dY_s - \frac{1}{2} \int_0^t \mathbb{E}^{P^u}[h(s, X_s) | \mathcal{F}^Y_s]2 ds\right\}$$

$$= U_t \exp\left\{\int_0^t U[h](s) dY_s - \frac{1}{2} \int_0^t [U[h](s)]^2 ds\right\}. \quad (2.21)$$

3 Well-posedness of the State-Observation Dynamics

In this and next sections we investigate the well-posedness of the filtering system (2.4) and (2.5) for any given $(\mathbb{P}^u, u) \in \mathcal{U}_{ad}$. We first give a slightly more generic formulation
of the SDE, which will include (2.4) and (2.5) as special case. Note that in this case the probability measure $\mathbb{P}^u$ will be fixed, we shall denote it as $\mathbb{P}^0$. We shall assume that $(B^1, B^2)$ is the canonical process under $\mathbb{P}^0$ (as opposed to the reference measure $\mathbb{Q}^0$) throughout this section.

We shall consider the following system of SDEs:

$$
\begin{align*}
X_t &= x + \int_0^t b(s, \cdot, X_{\wedge s}, \mu^X_s)ds + \int_0^t \sigma(s, \cdot, X_{\wedge s}, \mu^X_s)dB^1_s; \\
Y_t &= \int_0^t h(s, X_s)ds + B^2_t,
\end{align*}
$$

where $\mu^X_t = \mathbb{P}^0 \circ [\mathbb{E}^{\mathbb{P}^0}[X_t|\mathcal{F}_t]]^{-1}$. We note that SDE (3.1) is exactly the control-observation system (2.4) for the given control $u \in \mathcal{U}_{ad}$, in which the coefficients $b, \sigma$ should be understood as the random fields defined by

$$
\phi^u(t, \omega, \varphi, \mu_t) \triangleq \int_\mathbb{R} \phi(t, \varphi, y, u_t(\omega))\mu_t(dy),
$$

where $\phi = b, \sigma$. Thus, in light of Remark 2.4, we shall assume that the coefficients $b, \sigma$ in (3.1) satisfy the following assumptions that are slightly weaker than Assumption 2.3, but sufficient for our purpose in this section.

**Assumption 3.1.** The coefficients $b, \sigma : \Lambda \times \mathcal{C}_T \times \mathcal{P}_2(\mathcal{C}_T) \mapsto \mathbb{R}$ enjoy the following properties:

(i) For fixed $(\varphi, \mu) \in \mathcal{C}_T \times \mathcal{P}_2(\mathcal{C}_T)$, the mapping $(t, \omega) \mapsto (b, \sigma)(t, \omega, \varphi, \mu)$ is a $\mathbb{F}$-progressively measurable process;

(ii) For fixed $t \in [0, T]$, and $\mathbb{P}^0$-a.e. $\omega \in \Omega$, there exists $K > 0$, independent of $(t, \omega)$, such that for all $(\varphi^1, \mu^1), (\varphi^2, \mu^2) \in \mathcal{C}_T \times \mathcal{P}_2(\mathcal{C}_T)$, it holds that

$$
|\phi(t, \omega, \varphi^1, \mu^1) - \phi(t, \omega, \varphi^2, \mu^2)| \leq K [\sup_{t \in [0, T]} |\varphi^1_t - \varphi^2_t| + W_2(\mu^1, \mu^2)],
$$

for $\phi = b, \sigma$, respectively. 

In the rest of the section we shall still assume $b = 0$, as it does not add difficulties. For a continuous $\mathbb{F}$-adapted process $X$ let us denote $U_t^X \triangleq \mathbb{E}^{\mathbb{P}^0}[X_t|\mathcal{F}_t^Y]$, $t \geq 0$. (We note that $U_t^X$ should be understood as the “optional projection” of $X$ onto $\mathbb{P}^Y$!) We first check that $U_t^X$ is indeed a continuous process.

**Lemma 3.2.** If $X$ has continuous paths, $\mathbb{P}^0$-a.s., then $U_t^X$ admits a continuous version.
Proof. First note that by Bayes formula (2.19) we can write \( U_t^{X|Y} = \frac{E^{0}[L_t X_t | F_t^Y]}{E^{0}[L_t | F_t^Y]} = \frac{S_t}{S^0_t} \), where \( S^0 \) and \( S \) satisfy (2.17) and (2.18), respectively. Clearly, the representations (2.17) and (2.18) indicate that both \( S^0 \) and \( S \) have continuous paths, thus \( U_t^{X|Y} \) must have a continuous version.

We now define \( \mu_t^{X|Y}(\cdot) = P^0 \circ [U_t^{X|Y}]^{-1}(\cdot) \), and \( \mu_t^{X|Y}(\cdot) = P^0 \circ [U_t^{X|Y}]^{-1}(\cdot) \), for any \( t \geq 0 \). Lemma 3.2 then implies that \( \mu^{X|Y} \in \mathcal{P}_2(\mathbb{C}_T) \), justifying the definition of SDE (3.1). In what follows when the context is clear, we shall omit “\( X \mid Y \)” from the superscript.

We note that the special circular nature of SDE (3.1) between its solution and its law of the conditional expectation (whence the underlying probability) makes it necessary to specify the meaning of a solution. We have the following definition.

**Definition 3.3 (Weak Solution).** An eight-tuple \((\Omega, \mathcal{F}, P, \mathcal{F}, X, Y, B^1, B^2)\) is called a solution to the filtering equation (3.1) if

(i) \((\Omega, \mathcal{F}, P; \mathcal{F})\) is a filtered probability space;

(ii) \((B^1, B^2)\) is a 2-dimensional \(\mathbb{F}\)-Brownian motion under \(P\);

(iii) \((X, Y)\) is an \(\mathcal{F}\)-adapted continuous process such that (3.1) holds for all \(t \in [0, T]\), \(P\)-almost surely.

To prove the well-posedness we shall use the Schauder fixed point theorem. To this end we consider the following subset of \(\mathcal{P}_2(\mathbb{C}_T)\):

\[
\mathcal{E} = \left\{ \mu \in \mathcal{P}_2(\mathbb{C}_T) \mid \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |y|^4 \mu_t(dy) < \infty \right\}.
\]  

(3.4)

In the above \(\mu_t = \mu \circ P_t^{-1} \in \mathcal{P}(\mathbb{R})\), and \(P_t(\varphi) = \varphi(t), \varphi \in \Omega\), is the projection mapping. Clearly \(\mathcal{E}\) is a convex subset of \(\mathcal{P}_2(\mathbb{C}_T)\).

We now construct a mapping \(\mathcal{T} : \mathcal{E} \mapsto \mathcal{E}\), whose fixed point, if exists, would give a solution to the SDE (3.1). To begin with, we recall Assumption 3.1, and let \(Q^0\) be the reference measure. For simplicity we assume that \(Q^0\) is the Wiener measure on \((\Omega, \mathcal{F})\), and we denote the canonical process by \((B^1, Y)\). Thus, it is a 2-dimensional Brownian motion under \(Q^0\). For any \(\mu \in \mathcal{E}\) we consider the SDE on the space \((\Omega, \mathcal{F}, Q^0)\):

\[
X_t = x + \int_0^t \sigma(s, X_{\wedge s}, \mu_s) dB^1_s, \quad t \geq 0.
\]  

(3.5)

Note that as the distribution \(\mu\) is given, (3.5) is an “open-loop” SDE with “functional Lipschitz” coefficient, thanks to Assumption 3.1. Thus, there exists a unique (strong) solution to (3.5), which we denote by \(X = X^\mu\).
Now, using $X^\mu$ we define the process $L^\mu = \{L^\mu_t\}_{t \geq 0}$ as in (2.14) on probability space $(\Omega, \mathcal{F}, Q^0)$, and then we define the probability $d\mathbb{P}^\mu \triangleq L^\mu_T dQ^0$. By the Kallianpur-Striebel formula (2.19) we can define a process

$$U^\mu_t \triangleq \mathbb{E}^{\mathbb{P}^\mu}[X_t^\mu|\mathcal{F}_T] = \frac{\mathbb{E}^{Q^0}[L^\mu_t X_t^\mu|\mathcal{F}_T]}{\mathbb{E}^{Q^0}[L^\mu_t|\mathcal{F}_T]} = \frac{S^\mu_t}{S^\mu_{t,0}}, \quad t \geq 0. \tag{3.6}$$

where $S^\mu_t \triangleq \mathbb{E}^{Q^0}[L^\mu_t X_t^\mu|\mathcal{F}_T], S^\mu_{t,0} \triangleq \mathbb{E}^{Q^0}[L^\mu_t|\mathcal{F}_T], t \geq 0$, and then we denote

$$\mathcal{I}(\mu) \triangleq \nu^\mu = \mathbb{P}^\mu \circ [U^\mu]^{-1} \in \mathcal{P}(C_T). \tag{3.7}$$

Our task is to show that the solution mapping $\mathcal{I} : \mathcal{E} \rightarrow \mathcal{P}_2(C_T)$ enjoys the following properties:

**Theorem 3.4.** The solution mapping $\mathcal{I} : \mathcal{E} \rightarrow \mathcal{P}_2(C_T)$ enjoys the following properties:

1. $\mathcal{I}(\mathcal{E}) \subseteq \mathcal{E}$;
2. $\mathcal{I}(\mathcal{E})$ is compact under the 2-Wasserstein metric.
3. $\mathcal{I} : \mathcal{P}_2(C_T) \rightarrow \mathcal{P}_2(C_T)$ is continuous under the 2-Wasserstein metric.

**Proof.**

(1) Given $\mu \in \mathcal{E}$ we need only show that

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} |y|^4 \nu^\mu_t(dy) < \infty. \tag{3.8}$$

To see this we note that for $t \in [0, T]$, by Jensen’s inequality,

$$\int_{\mathbb{R}^d} |y|^4 \nu^\mu_t(dy) = \int_{\mathbb{R}^d} |y|^4 \mathbb{P}^\mu \circ [U^\mu]^{-1}(dy) = \mathbb{E}^{\mathbb{P}^\mu}[[|X_t^\mu|\mathcal{F}_T]^4] \leq \mathbb{E}^{\mathbb{P}^\mu}[|X_t^\mu|^4].$$

Since under $Q^0$, $B^1$ is also a Brownian motion, it is thus standard to argue that, as $X^\mu$ is the solution to the SDE (3.5), it holds that

$$\sup_{0 \leq t \leq T} \mathbb{E}^{Q^0}[|X^\mu_t|^2] \leq C(1 + |x|^2n), \quad \text{for all } n \in \mathbb{N}. \tag{3.9}$$

Furthermore, noting that the process $L^\mu$ is an $L^2$-martingale under $Q^0$, we have

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |y|^4 \nu^\mu_t(dy) \leq \sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{P}^\mu}[|X_t^\mu|^4] = \sup_{0 \leq t \leq T} \mathbb{E}^{Q^0}[L^\mu_t |X^\mu_t|^4] \leq \left(\mathbb{E}^{Q^0}[|L^\mu_T|^2]\right)^{\frac{1}{2}} \sup_{0 \leq t \leq T} \mathbb{E}^{Q^0}[|X^\mu_t|^8]\right]^\frac{1}{2} < \infty,$$

thanks to (3.9). In other words, $\nu^\mu = \mathcal{I}(\mu) \in \mathcal{E}$, proving (1).

(2) We shall prove that for any sequence $\{\mu^n_t\} \subseteq \mathcal{E}$, there exists a subsequence, denoted by $\{\mu^n_t\}$ itself, such that $\lim_{n \to \infty} \mathcal{I}(\mu^n) = \nu$ in 2-Wasserstein metric, for some $\nu \in \mathcal{I}(\mathcal{E})$. 

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In light of the equivalence relation (2.2), we shall first argue that the family \( \{ \mathcal{F}(\mu^n) \}_{n \geq 1} \) is tight. To this end, recall that

\[
U^n_t = \mathbb{E}^n[X^n_t | \mathcal{F}_t] = \frac{S^n_t}{S^n_{t^0}},
\]

where \( S^n_t \triangleq \mathbb{E}^Q[L^n_t X^n_t | \mathcal{F}_t], S^n_{t^0} \triangleq \mathbb{E}^Q[L^n_{t^0} X^n_{t^0}], \) \( t \geq 0, \) and \( dP^n \triangleq L^n_{t^0} dQ^0. \) It then follows from the FKK equation (2.20) that

\[
dU^n_t = \left\{ \mathbb{E}^n[X^n_t h(t, X^n_t) | \mathcal{F}_t] - \mathbb{E}^n[X^n_t | \mathcal{F}_t] \mathbb{E}^n[h(t, X^n_t) | \mathcal{F}_t] \right\} dY_t \]

\[
+ \left\{ \mathbb{E}^n[X^n_t | \mathcal{F}_t] \mathbb{E}^n[h(t, X^n_t) | \mathcal{F}_t] - \mathbb{E}^n[X^n_t h(t, X^n_t) | \mathcal{F}_t] \right\} dt.
\]

Now denote \( B^{2,n} \triangleq Y_t - \int_0^t h(s, X^n_{s, t}) ds. \) Then \( (B^1, B^{2,n}) \) is a 2-dimensional standard \( \mathbb{P}^n \)-Brownian motion. Furthermore, since \( h \) is bounded, so is \( \mathbb{E}^n[h(t, X^n_{\cdot, t}) | \mathcal{F}_t] \). We thus have the following estimate:

\[
\mathbb{E}^n[|U^n_t - U^n_s|^4] \leq C \mathbb{E}^n\left[ \left( \int_s^t (\mathbb{E}^n[|X^n_r|^2 | \mathcal{F}_r] ds \right)^2 \right]
\]

\[
\leq C \mathbb{E}^n\left[ \sup_{0 \leq s \leq t} \mathbb{E}^n[|X^n_r|^2 | \mathcal{F}_r] \right] |t - s|^2
\]

\[
\leq C \mathbb{E}^n\left[ \sup_{0 \leq s \leq t} \mathbb{E}^n\left[ \sup_{0 \leq r \leq T} |X^n_r|^2 | \mathcal{F}_r \right]^2 \right] |t - s|^2
\]

\[
\leq C \mathbb{E}^n\left[ \sup_{0 \leq s \leq t} |X^n_s|^4 \right] |t - s|^2 \leq C |t - s|^2.
\]

Thus, as \( U^n_0 = x \), the sequence of continuous processes \( \{U^n\} \) is relatively compact (cf. e.g., Ethier-Kurtz [13]). Therefore, the sequence of their laws \( \{ \mathcal{F}(\mu^n) \triangleq \mathbb{P}^n \circ [U^n]^{-1}, n \geq 1 \} \subseteq \mathcal{P}(\mathcal{C}_T) \) is tight. Consequently, we can find a subsequence, we may assume itself, that converges weakly to a limit \( \nu \in \mathcal{P}_2(\mathcal{C}_T) \). Furthermore, for each \( n \geq 1 \), we apply the Jensen, Burkholder-Davis-Gundy, and Hölder inequalities to get, with \( \nu^n \triangleq \mathcal{F}(\mu^n), \)

\[
\int_{\mathcal{C}_T} ||\varphi||^2 \nu^n(d\varphi) = \mathbb{E}^n[||U^n||^2_{\mathcal{C}_T}] = \mathbb{E}^n\left[ \sup_{0 \leq t \leq T} |X^n_t|^2 | \mathcal{F}_t \right]^2
\]

\[
\leq \mathbb{E}^n\left[ \sup_{0 \leq t \leq T} \mathbb{E}^n\left[ \sup_{0 \leq r \leq T} |X^n_r|^2 | \mathcal{F}_r \right]^2 \right]
\]

\[
\leq 3 \mathbb{E}^n\left[ \sup_{0 \leq r \leq T} |X^n_r|^3 \right]^{2/3} = 3 \mathbb{E}^Q[L^n_T \sup_{0 \leq r \leq T} |X^n_r|^3]^{2/3}
\]

\[
\leq 3 \mathbb{E}^Q[(L^n_T)^4]^{1/6} \left[ \mathbb{E}^Q\left[ \sup_{0 \leq r \leq T} |X^n_r|^4 \right] \right]^{1/2} < +\infty.
\]

But noting that \( h \) is bounded, one deduces from (3.9) that

\[
\sup_{n \geq 1} \int_{\mathcal{C}_T} ||\varphi||^2 \nu^n(d\varphi) < \infty.
\]
This, together with the fact that \( \nu^n = \mathcal{F}(\mu^n) \xrightarrow{w} \nu \), implies that \( W_2(\nu^n, \nu) \to 0 \), and \( \nu \in \mathcal{E} \), as \( n \to \infty \), where \( W_2(\cdot, \cdot) \) is the Wasserstein metric on \( \mathcal{P}(\mathbb{C}_T) \). This proves (2).

(3) We now check that the mapping \( \mathcal{F} : \mathcal{E} \to \mathcal{E} \) is continuous. To this end, for each \( \mu \in \mathcal{E} \), we consider the following SDE on the probability space \( (\Omega, \mathcal{F}, \mathbb{Q}^0) \):

\[
\begin{align*}
\left\{ \begin{array}{ll}
dX_t = \sigma(t, X_{\Delta t}, \mu_t)dB^1_t, & X_0 = x; \\
dB^2_t = dY_t - h(t, X_t)dt, & B^2_0 = 0; \\
dL_t = h(t, X_t)L_t dY_t, & L_0 = 1.
\end{array} \right.
\tag{3.15}
\end{align*}
\]

Now let \( \{\mu^n\} \subseteq \mathcal{E} \) be any sequence such that \( \mu^n \to \mu \), as \( n \to \infty \), in the 2-Wasserstein metric, and denote by \( (X^n, B^{n,2}, L^n) \) the corresponding solutions to (3.15). Define

\[
\sigma^n(t, \omega_{\Delta t}) \triangleq \sigma(t, \omega_{\Delta t}, \mu^n_t), \quad (t, \omega) \in [0, T] \times \Omega.
\]

Then by Assumption 3.1-(ii), the \( \sigma^n \)'s are functional Lipschitz deterministic functions, with Lipschitz constant independent of \( n \). This and standard SDE arguments lead to that, as \( n \to \infty \),

\[
\mathbb{E}^0\{ \sup_{0 \leq t \leq T} \|X^n_t - X_t\|^p + \sup_{0 \leq t \leq T} \|L^n_t - L_t\|^p \} \to 0, \quad \text{in} \ L^p(\mathbb{Q}^0), \quad p \geq 1.
\tag{3.16}
\]

We deduce that \( U^n_t = \mathbb{E}^{\mu^n}[X^n_t|\mathcal{F}^Y_t] = S^n_t/S^n_{t,0} \) converges in probability under \( \mathbb{Q}^0 \) to

\[
\frac{d\mathbb{P}}{d\mathbb{Q}^0}[L_t X_t|\mathcal{F}^Y_t] = \mathbb{E}^0[X_t|\mathcal{F}^Y_t], \quad \text{where} \ d\mathbb{P} \stackrel{\triangle}{=} L_T d\mathbb{Q}^0.
\]

Now for any \( \psi \in C_b(\mathbb{R}) \), letting \( n \to \infty \) we have

\[
\langle \psi, \mathcal{F}(\mu^n)_t \rangle = \mathbb{E}^{\mu^n}[\psi(\mathbb{E}^{\mu^n}[X^n_t|\mathcal{F}^Y_t])] = \mathbb{E}^0[L^n_T \psi(\mathbb{E}^{\mu^n}[X^n_t|\mathcal{F}^Y_t])]
\rightarrow \mathbb{E}^0[L_T \psi(\mathbb{E}^0[X_t|\mathcal{F}^Y_t])] = \mathbb{E}^0[\psi(\mathbb{E}^0[X_t|\mathcal{F}^Y_t])]
\tag{3.17}
\]

This implies that \( \nu_t = \mathbb{P} \circ [\mathbb{E}^0[X_t|\mathcal{F}^Y_t]]^{-1} = \mathcal{F}(\mu)_t \), for all \( t \in [0, T] \). With the same argument one shows that, for any \( 0 \leq t_1 < t_2 < \cdots < t_k < \infty \),

\[
\mathcal{F}(\mu^n)_{t_1, \ldots, t_k} \triangleq \mathbb{P} \circ (\mathbb{E}^0[X^n_{t_1}|\mathcal{F}^Y_{t_1}], \ldots, \mathbb{E}^0[X^n_{t_k}|\mathcal{F}^Y_{t_k}])^{-1} \xrightarrow{d} \nu_{t_1, \ldots, t_k}, \quad \text{as} \ n \to \infty.
\]

That is, the finite dimensional distributions of \( \mathcal{F}(\mu^n) \) converge to those of \( \nu \), and as \( \{\mu^n\}_{n \geq 1} \) is tight by part (2), we conclude that \( \mathcal{F}(\mu^n) \xrightarrow{w} \nu \) in \( \mathcal{P}(\mathbb{C}_T) \). This, together with (3.13), further shows that \( W_2(\mathcal{F}(\mu^n), \mathcal{F}(\mu)) \to 0 \), as \( n \to \infty \), proving the continuity of \( \mathcal{F} \), whence (3). The proof is now complete.

As a direct consequence of Theorem 3.4, we have the following existence result for the SDE (3.1).
Corollary 3.5. Assume (H1) and (H2). Then SDE (3.1) has at least one solution in the sense of Definition 3.3.

Proof. We first note that the result of Theorem 3.4 is actually stronger than what the Schauder fixed-point theorem requires: (i) $\mathcal{E}$ is convex, (ii) $\mathcal{T}$ is continuous and $\mathcal{T}(\mathcal{E}) \subseteq \mathcal{E}$; and (iii) $\mathcal{T}(\mathcal{E}) \subset K$, for some compact $K$ in $\mathcal{P}_2(C_T)$ (cf. e.g., [14]). Since $\mathcal{E}$ is obviously convex, Theorem 3.4 thus implies that there exists a fixed-point $\nu \in \mathcal{P}_2(C_T)$ such that $\mathcal{T}(\nu) = \nu$, thanks to the Schauder’s fixed point theorem.

We note that the existence of the fixed point $\mu$ amounts to saying that SDE (3.15) has a solution on the probability space $(\Omega, \mathcal{F}, Q_0)$, with $\mu = \mu_X|Y = P \circ [U]^{-1}$, and $U_t = \mathbb{E}^P[X_t|\mathcal{F}_t^Y], t \geq 0$, where $dP = L_T dQ_0$ by construction. But this in turn defines a solution of (3.1) on the probability space $(\Omega, \mathcal{F}, P)$, thanks to the Girsanov transformation. However, since under $P$, $(B^1, B^2)$ constructed in (3.15) is a Brownian motion, $(\Omega, \mathcal{F}, P, X, Y, B^1, B^2)$ defines a (weak) solution of SDE (3.1).

4 Uniqueness

In this section we investigate the uniqueness of the solution to SDE (3.1). We note that the general uniqueness for the weak solution for this problem is quite difficult, we will contend ourselves with a version that is relatively more amendable.

To begin with, we recall Assumption 2.3 again, and let $Q_0$ be the reference probability measure. Then, under $Q_0$, $(B^1, Y)$ is a Brownian motion. We shall argue that, for each $u \in \mathcal{U}_{ad}$, the solution to the SDE

$$
\begin{aligned}
    dX^u_t &= \sigma(t, X^u_{\mathcal{A}_t}, \mu^u_{\mathcal{F}_t^Y}, u_t)dB^1_t, \quad X^u_0 = x;
    
    dB^2_t &= dY_t - h(t, X^u_t)dt, \quad B^2_0 = 0;
    
    dL^u_t &= h(t, X^u_t)L^u_t dY_t, \quad L_0 = 1.
\end{aligned}
$$

is pathwisely unique under $Q_0$.

Remark 4.1. 1) The pathwise uniqueness of the solution to (4.1) on $(\Omega, \mathcal{F}, Q^0)$ does not imply the weak uniqueness of the original SDEs (2.4) and (2.5). In fact, if there are two weak solutions $(\Omega, \mathcal{F}, \mathbb{P}^i, B^{1,i}, B^{i,2}, X^i, Y^i), i = 1, 2$, on the canonical space $(\Omega, \mathcal{F})$, then by defining $dQ^i = L^i_t d\mathbb{P}^i, i = 1, 2$, we see that $(B^{i,1}, Y^i)$ is the Brownian motions under $Q^i, i = 1, 2$, respectively. However, our uniqueness will be under the assumption that $Q^1 = Q^0 = Q^2$, and $(B^{1,1}, Y^1) = (B^{1,2}, Y^2)$ under $Q^0$, which is not necessarily true in general.
2) In a control problem context, such a uniqueness result amounts to saying that, given an observation process $Y$ such that $(B^1, Y)$ is a Brownian motion under the reference measure $\mathbb{Q}^0$, then thanks to Assumption 2.3, for any $(\mathbb{Q}^0, u) \in \mathcal{U}_{ad}$, the state-observation dynamics under $\mathbb{Q}^0, (X^u, Y, L^u)$, is pathwisely unique, as the solution to (4.1). In what follows we shall call the solution to (4.1) the $\mathbb{Q}^0$-dynamics of the system. 

We shall prove the desired uniqueness result by establishing the following fundamental estimates which will be useful in our future discussions. Since all controlled dynamics are constructed via the reference probability space $(\Omega, \mathcal{F}, \mathbb{Q}^0)$, we shall consider only their $\mathbb{Q}^0$-dynamics, which is the solution to (4.1). Recall the space $L^p(\mathbb{Q}^0; L^2([0, T]))$, $p > 1$, and the norm $\| \cdot \|_{p,2,\mathbb{Q}^0}$ defined by (2.3). We have the following important result.

**Proposition 4.2.** Assume that Assumption 2.3 is in force. Let $u, v \in \mathcal{U}_{ad}$ be given. Then, for any $p > 2$, there exists a constant $C_p > 0$, such that the following estimates hold:

$$
\mathbb{E}^0 \left[ \sup_{0 \leq s \leq T} \left( |X^u_s - X^v_s|^2 + |L^u_s - L^v_s|^2 + |X^u_s L^u_s - X^v_s L^v_s|^2 \right) \right] \leq C \| u - v \|_{2,2,\mathbb{Q}^0}^2; \quad (4.2)
$$

$$
\mathbb{E}^0 \left[ \sup_{0 \leq s \leq T} |X^u_s - X^v_s|^p \right] \leq C_p \| u - v \|_{p,2,\mathbb{Q}^0}^p. \quad (4.3)
$$

**Proof.** We split the proof into several steps. Throughout this proof we let $C > 0$ be a generic constant, depending only on the bounds and Lipschitz constants of the coefficients and the time duration $T > 0$, and it is allowed to vary from line to line.

**Step 1 (Estimate for $X$).** First let us denote, for any $u \in \mathcal{U}_{ad}$ and $\mu \in \mathcal{P}_2(\mathbb{C}_T)$,

$$
\sigma^u(t, \varphi, \mathcal{A}_t, \mu_t) \triangleq \int_{\mathbb{R}} \sigma(t, \varphi, \mathcal{A}_t, y, u_t) \mu_t(dy), \quad (t, \varphi) \in [0, T] \times \mathbb{C}_T, \quad (4.4)
$$

and $\mu^u_t \triangleq \mu^{X^u|Y} \circ P^{-1}_t = \mathbb{P}^u \circ (\mathbb{E}^{P^u}[X^u|\mathcal{F}^Y])^{-1}$, $t \geq 0$. Then, we have

$$
\left| \sigma^u(t, X^u_{\mathcal{A}_t}, \mu^u_t) - \sigma^v(t, X^v_{\mathcal{A}_t}, \mu^v_t) \right| 
\leq \left| \int_{\mathbb{R}} \sigma(t, X^u_{\mathcal{A}_t}, y, u_t) \mu^u_t(dy) - \int_{\mathbb{R}} \sigma(t, X^v_{\mathcal{A}_t}, y, v_t) \mu^v_t(dy) \right| 
\leq C \left\{ |u_t - v_t| + \sup_{0 \leq s \leq t} |X^u_s - X^v_s|^2 + \left| \int_{\mathbb{R}} \sigma(t, X^u_{\mathcal{A}_t}, y, v_t) |\mu^u(dy) - \mu^v(dy)| \right| \right\}. \quad (4.5)
$$

Next, let us denote $S^u_t = \mathbb{E}^0[L^u_t X^u_t|\mathcal{F}^Y]$ and $S^{u,0}_t = \mathbb{E}^0[L^u_t|\mathcal{F}^Y]$, and define $S^u_t$, $S^{u,0}_t$ in a
similar way. By (2.19) and the fact that \(d\mathbb{P}^u = L_t^u d\mathbb{Q}^0\), we see that

\[
\left| \int_\mathbb{R} \sigma(t, X^u_{\lambda,t}, y, v_t) \left[ \mu^u(dy) - \mu^v(dy) \right] \right|
\]

\[
= \left| \mathbb{E}^u[\sigma(t, \varphi_{\lambda,t}, \mathbb{E}^u[X^u_t|F^Y_t], u)] - \mathbb{E}^v[\sigma(t, \varphi_{\lambda,t}, \mathbb{E}^v[X^v_t|F^Y_t], u)] \right|_{\varphi=X^v,u=v_t} \quad (4.6)
\]

\[
= \left| \mathbb{E}^0 \left\{ L_t^u \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t}{S^v_{t,0}} u) - L_t^v \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t}{S^v_{t,0}} u) \right\} \right|_{\varphi=X^v,u=v_t} \quad (4.7)
\]

and

\[
I_2 = \left| \mathbb{E}^0 \left\{ S_t^v \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t}{S^v_{t,0}} u) - \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t}{S^v_{t,0}} u) \right\} \right|_{\varphi=X^v,u=v_t}.
\]

Clearly, we have

\[
I_2 \leq C \mathbb{E}^0 \left\{ S_t^v \left| \frac{S^u_t}{S^v_{t,0}} - S^v_{t,0} \right| \right\} \leq C \mathbb{E}^0 \left| L_t^u X^u_t - L_t^v X^v_t \right|.
\]

To estimate \(I_1\), we write \(\hat{\sigma}(t, \omega, \varphi_{\lambda,t}, y, z) = y \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t(\omega)}{y}, z)\). Since

\[
\partial_y \hat{\sigma}(t, \omega, \varphi_{\lambda,t}, y, z) = \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t(\omega)}{y}, z) - \frac{S^u_t(\omega)}{y} \partial_y \sigma(t, \varphi_{\lambda,t}, \frac{S^u_t(\omega)}{y}, z),
\]

we see that \(y \mapsto \partial_y \hat{\sigma}(t, \varphi_{\lambda,t}, y, z)\) is uniformly bounded thanks to Assumption 2.3-(iv). Thus we have

\[
I_1 \leq C \left\| \partial_y \hat{\sigma} \right\|_\infty \mathbb{E}^0 \left| S_t^v - S_t^{v,0} \right| \leq C \mathbb{E}^0 \left| L_t^u - L_t^v \right|.
\]

Now note that (4.1) implies that

\[
X^u_t - X^u_t = \int_0^t \left[ \sigma^u(s, X^u_{\lambda,s}, \mu^u_s) - \sigma^v(s, X^v_{\lambda,s}, \mu^v_s) \right] dB^1_s.
\]

Combining (4.5)–(4.9), we see that

\[
\mathbb{E}^0 \left[ \sup_{0 \leq s \leq t} |X^u_s - X^v_s|^p \right] \leq C \mathbb{E}^0 \left\{ \left[ \int_0^t \sup_{r \in [0,s]} |X^u_r - X^v_r|^2 + |u_s - v_s|^2 \right]^{p/2} + |L^u_s - L^v_s|^2 + (\mathbb{E}^0[L^u_s X^u_s - L^v_s X^v_s]|2)^{p/2} \right\}.
\]

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Applying the Gronwall inequality we obtain that
\[
\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |X^u_s - X^v_s|^2 \right] \leq C\mathbb{E}^0\left\{ \left[ \int_0^t \left( |u_s - v_s|^2 + \mathbb{E}^0[|L^u_s - L^v_s|^2] \right) \right]^p/2 \right\},
\]
(4.11)

**Step 2 (Estimate for L).** We first note that, for \( t \in [0, T] \),
\[
|L^u_t h(t, X^u_t) - L^v_t h(t, X^v_t)| = \left| L^u_t h\left( t, \frac{L^u_t X^u_t}{L^u_t} \right) - L^v_t h\left( t, \frac{L^v_t X^v_t}{L^v_t} \right) \right|
\]
\[
\leq \left| L^u_t h\left( t, \frac{L^u_t X^u_t}{L^u_t} \right) - L^u_t h\left( t, \frac{L^v_t X^u_t}{L^v_t} \right) \right| + \left| L^v_t h\left( t, \frac{L^u_t X^v_t}{L^v_t} \right) - L^v_t h\left( t, \frac{L^v_t X^v_t}{L^v_t} \right) \right|.
\]
(4.12)

To estimate the second term above we define, as before, \( \tilde{h}(t, \omega, x) \triangleq x h(t, L^u_t(\omega) X^u_t(\omega)) \). Then, similar to (4.8), one shows that \( x \mapsto \partial_x \tilde{h}(t, \omega, x) \) is uniformly bounded, thanks to Assumption 2.3-(v). Consequently, we have
\[
\left| L^u_t h\left( t, \frac{L^u_t X^u_t}{L^u_t} \right) - L^v_t h\left( t, \frac{L^u_t X^u_t}{L^v_t} \right) \right| \leq \| \partial_x \tilde{h} \|_{\infty} |L^u_t - L^v_t|.
\]
(4.13)

Now, combining (4.12) and (4.13) we obtain
\[
|L^u_t h(t, X^u_t) - L^v_t h(t, X^v_t)| \leq C(|L^u_t - L^v_t| + |L^u_t X^u_t - L^v_t X^v_t|).
\]
(4.14)

Therefore, noting that \( L^u_t = 1 + \int_0^t h(s, X^u_s) L^u_s dY_s \), we deduce from (4.14) and Gronwall’s inequality that
\[
\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |L^u_s - L^v_s|^2 \right] \leq C\mathbb{E}^0\left[ \int_0^t |L^u_s X^u_s - L^v_s X^v_s|^2 ds \right], \quad \mathbb{P}^0\text{-a.s., } 0 \leq t \leq T.
\]
(4.15)

**Step 3 (Estimate for L_t X_t).** It is clear from (4.11) and (4.15) that it suffices to find the estimate of \( L^u_t X^u_t - L^v_t X^v_t \) in terms of \( u - v \). To see this we note that
\[
L^u_t X^u_t = x + \int_0^t L^u_s X^u_s h(s, X^u_s) dY_s + \int_0^t L^u_s \mathbb{E}^v[\sigma(s, \varphi, \lambda, s, \mathbb{E}^u[X^u_s | F^Y_s], v)]|_{v = u} dB^1_{s}. \]
(4.16)

Now define \( \tilde{h}(t, x) \triangleq x h(t, x) \). Then it is easily seen that as \( h \) satisfies Assumption 2.3-(vi), \( \tilde{h} \) satisfies Assumption 2.3-(v). Thus, similar to (4.14) we have
\[
|L^u_s X^u_s h(s, X^u_s) - L^v_s X^v_s h(s, X^v_s)| = |L^u_s \tilde{h}(s, X^u_s) - L^v_s \tilde{h}(s, X^v_s)| \leq C(|L^u_s - L^v_s| + |L^u_s X^u_s - L^v_s X^v_s|).
\]
(4.17)
On the other hand, for any $u \in \mathcal{U}_{ad}$, recalling Remark 2.4 for the notations $\sigma^u$ and $\mu^u$, we have,

$$\Delta_t^{u,v} \triangleq \left| L^u_s \mathbb{E}^u[\sigma(s, \varphi_{\cdot,s}, \mathbb{E}^u[X^u_s|\mathcal{F}^Y_s], z)]|_{\varphi = x^u_{s}} - L^v_s \mathbb{E}^v[\sigma(s, \varphi_{\cdot,s}, \mathbb{E}^v[X^v_s|\mathcal{F}^Y_s], z)]|_{\varphi = x^v_{s}} \right|.$$

Then, following the a similar argument as in Step 1 we have

$$\Delta_t^{u,v} \leq C L^u_t (\mathbb{E}^0[|L^u_t - L^v_t|^2] + \mathbb{E}^0[|X^u_t L^u_t - X^v_t L^v_t|^2]) + C \mathbb{E}^0[(L^u_t)^2|u_t - v_t|^2].$$

Taking expectation we see that

$$\mathbb{E}^0[|\Delta_t^{u,v}|^2] \leq C (\mathbb{E}^0[|L^u_t - L^v_t|^2] + \mathbb{E}^0[|X^u_t L^u_t - X^v_t L^v_t|^2]) + C \mathbb{E}^0[(L^u_t)^2|u_t - v_t|^2].$$

(4.18)

Now, combining (4.16)–(4.18), for $p > 2$ we can find $C_p > 0$ such that

$$\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |L^u_s X^u_s - L^v_s X^v_s|^2 \right] \leq C \mathbb{E}^0\left[ \int_0^t |L^u_s X^u_s h(s, X^u_s) - L^v_s X^v_s h(s, X^v_s)|^2 ds \right] + C \mathbb{E}^0 \int_0^t |\Delta_s^{u,v}|^2 ds.$$

(4.19)

$$\leq C_p \left\{ \mathbb{E}^0 \left[ \left( \int_0^t |u_s - v_s|^2 ds \right)^{p/2} \right] \right\}^{2/p} + C \mathbb{E}^0 \int_0^t |L^u_s - L^v_s|^2 ds + C \mathbb{E}^0 \int_0^t |L^u_s X^u_s - L^v_s X^v_s|^2 ds.$$

Hence, applying Gronwall’s inequality we obtain

$$\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |L^u_s X^u_s - L^v_s X^v_s|^2 \right] \leq C_p \|u - v\|_{p,2,Q^0}^2 + C \mathbb{E}^0 \int_0^t |L^u_s - L^v_s|^2 ds.$$  

(4.20)

Combining (4.20) with (4.15) and applying the Gronwall inequality again, we conclude that

$$\mathbb{E}^0\left\{ \sup_{0 \leq s \leq t} |L^u_s - L^v_s|^2 \right\} \leq C_p \|u - v\|_{p,2,Q^0}^2.$$  

(4.21)

This, together with (4.11) and (4.20), implies (4.2). (4.3) then follows easily from (4.2) and (4.10), proving the Proposition.

A direct consequence of Proposition 4.2 is the following uniqueness result for the solution to (4.1).

**Corollary 4.3.** Assume that Assumption 2.3 holds. Then the solution to SDE (4.1) is pathwisely unique.

**Proof.** Setting $u = v$ in Proposition 4.2 we obtain the result. 

\[ \square \]
5 A Stochastic Control Problem with Partial Observation

We are now ready to study the stochastic control problem with partial observation. We first note that in theory for each \((P^u, u) \in \mathcal{U}_{ad}\) our state-observation dynamics \((X^u, Y^u)\) lives on probability space \((\Omega, \mathcal{F}, P^u)\), which varies with control \(u\). We shall consider their \(Q^0\)-dynamics so that our analysis can be carried out on a common probability space, thanks to Assumption 2.3. Therefore, in what follows for each \((P^u, u) \in \mathcal{U}_{ad}\) we consider only the \(Q^0\)-dynamics \((X^u, Y, L^u)\), which satisfies the following SDE:

\[
\begin{align*}
&dX^u_t = \sigma^u(t, X^u_t, \mu^u_t)dB^1_t, \quad t \geq 0, \quad X^u_0 = x; \\
&dB^2_t = dY_t - h(t, X^u_t)dt, \quad t \geq 0, \quad B^2_0 = 0; \\
&dL^u_t = h(t, X^u_t)L^u_t dY_t, \quad t \geq 0, \quad L^u_0 = 1,
\end{align*}
\]

(5.1)

where \((B^1, Y)\) is a \(Q^0\)-Brownian motion, and \(dP^u = L^u_T dQ^0\). For simplicity, we denote \(E^u[\cdot] = E^{P^u}[\cdot]\) and \(E^0[\cdot] = E^{Q^0}[\cdot]\).

**Remark 5.1.** A convenient and practical way to identify admissible control is to simply consider the space \(L_{\infty}^{\mathcal{F}}(Q^0; [0, T])\) (recall Definition 2.1), which is independently well-defined, thanks to Assumption 2.3. It is easy to check that \(u \in L_{\infty}^{\mathcal{F}}(Q^0; [0, T])\) if and only if \(u \in L_{\infty}^{\mathcal{F}}(P^u; [0, T])\), where \(P^u\) is the probability measure constructed via (5.1). Therefore in what follows by \(u \in \mathcal{U}_{ad}\) we mean that \(u \in L_{\infty}^{\mathcal{F}}(Q^0; [0, T])\).

We recall that for \(u \in \mathcal{U}_{ad}\) and \(\mu \in \mathcal{P}_2(C_T)\), the coefficient \(\sigma^u\) in (5.1) is defined by (4.4). Thus we can write the cost functional as

\[
J(u) \triangleq E^0\left\{ \Phi(X_T^u, \mu_T^u) + \int_0^T f^u(s, X_s^u, \mu_s^u) ds \right\}.
\]

(5.2)

An admissible control \(u^* \in \mathcal{U}_{ad}\) is said to be optimal if

\[
J(u^*) = \inf_{u \in \mathcal{U}_{ad}} J(u).
\]

(5.3)

We remark that the cost functional \(J(\cdot)\) involves the law of the conditional expectation of the solution in a nonlinear way. Therefore such a control problem is intrinsically "time-inconsistent" and, thus, the dynamic programming approach in general does not apply. For this reason, we shall consider only the necessary condition of the optimal solution, that is, Pontryagin’s Maximum Principle.

To this end, we let \(u^* \in \mathcal{U}_{ad}\) be an optimal control, and consider the convex variations of \(u^*:\)

\[
u^\theta,v_t := u^*_t + \theta(v_t - u^*_t), \quad t \in [0, T], \quad 0 < \theta < 1, \quad v \in \mathcal{U}_{ad}.
\]

(5.4)
Here, we assume that $v, v \in L^\infty_{ad}([0,T])$. Since $U$ is convex, $u^{\theta,v} \in U$, for all $t \in [0,T]$, $v \in \mathcal{V}_{ad}$, and $\theta \in (0,1)$. We denote $(X^{\theta,v}, Y, L^{\theta,v})$ to be the corresponding $Q^0$-dynamics that satisfies (5.1), with control $u^{\theta,v}$. Applying Proposition 4.2 ((4.2) and (4.3)) and noting that $Y$ is a Brownian motion under $Q^0$, we get, for $p > 2$,

$$\lim_{\theta \to 0} \mathbb{E}^0 \left[ \sup_{0 \leq t \leq T} \left| X_t^{\theta,v} - X_t^{u^*} \right|^2 \right] \leq C_p \lim_{\theta \to 0} \left\| u^{\theta,v} - u^* \right\|_{p,2,Q^0}^2 = 0; \quad (5.5)$$

$$\lim_{\theta \to 0} \mathbb{E}^0 \left[ \sup_{0 \leq t \leq T} \left| L_t^{\theta,v} - L_t^{v} \right|^2 \right] = 0. \quad (5.6)$$

In the rest of the section we shall derive, heuristically, the “variational equations” which play a fundamental role in the study of Maximum Principle. The complete proof will be given in the next section. For notational simplicity we shall denote $u = u^*$, the optimal control, from now on, bearing in mind that all discussions will be carried out for the $Q^0$-dynamics, therefore on the same probability space.

Now for $u^1, u^2 \in \mathcal{V}_{ad}$, let $(X^1, L^1)$ and $(X^2, L^2)$ denote the corresponding solutions of (5.1). We define $\delta X = \delta X^{1,2} = \delta X^{u^1,u^2} \triangleq X^{u^1} - X^{u^2}$ and $\delta L = \delta L^{1,2} = \delta L^{u^1,u^2} \triangleq L^{u^1} - L^{u^2}$, and will often drop the superscript “$1,2$” if the context is clear. Then $\delta X$ and $\delta L$ satisfy the equations:

$$\begin{cases}
\delta X_t = \int_0^t [\sigma^{u^1}(s,X_{s,t};\mu^1_s) - \sigma^{u^2}(s,X_{s,t};\mu^2_s)]dB^1_s, \\
\delta L_t = \int_0^t [L^1_{s}\circ h(s,X^1_s) - L^2_{s}\circ h(s,X^2_s)]dY_s.
\end{cases} \quad (5.7)$$

As before, let $U_t^i \triangleq \mathbb{P}^{u^i}[X_t^i|\mathcal{F}_t^Y]$ and $\mu_t^i = \mathbb{P}^{u^i} \circ [U_t^i]^{-1}$, $t \geq 0$, $i = 1, 2$. We can easily check that

$$\begin{align*}
\sigma^{u^1}(t,X^1_{t\lambda},\mu^1_t) - & \sigma^{u^2}(t,X^2_{t\lambda},\mu^2_t) \\
= & \mathbb{E}^0 \left\{ L^1_t \sigma(t,\varphi^{1\lambda}_{t\lambda},U^1_t,z^1) - L^2_t \sigma(t,\varphi^{2\lambda}_{t\lambda},U^2_t,z^2) \right\} \bigg|_{\varphi^1 = X^1, \varphi^2 = X^2, z^1 = u^1_t, z^2 = u^2_t} \\
= & \mathbb{E}^0 \delta L^1_{t\lambda} \sigma(t,\varphi^{1\lambda}_{t\lambda},U^1_t,z^1) \quad (5.8)
\end{align*}$$

$$\begin{align*}
+ & L^2_t \left[ \int_0^1 D_{\varphi} \sigma(t,\varphi^{1\lambda}_{t\lambda} + \lambda(\varphi^{1\lambda}_{t\lambda} - \varphi^{2\lambda}_{t\lambda}),U^1_t,z^1)(\varphi^1 - \varphi^2)d\lambda, \\
+ & \int_0^1 \partial_{U} \sigma(t,\varphi^{1\lambda}_{t\lambda},U^1_t + \lambda(U^1_t - U^2_t),z^1)d\lambda \cdot (U^1_t - U^2_t) \\
+ & \int_0^1 \partial_{z} \sigma(t,\varphi^{1\lambda}_{t\lambda},U^1_t,z^2 + \lambda(z^1 - z^2))d\lambda \cdot (z^1 - z^2) \bigg] \bigg|_{\varphi^1 = X^1, \varphi^2 = X^2, z^1 = u^1_t, z^2 = u^2_t}.
\end{align*}$$

Here the integral involving the Frechet derivative $D_{\varphi} \sigma$ is in the sense of Bochner. Now let $u^1 = u^{\theta,v}$ and $u^2 = u^* = u$, and denote

$$\delta_{\theta} X \triangleq \delta_{\theta} X^{u,v} = \frac{X^{\theta,v} - X^u}{\theta}, \quad \delta_{\theta} L \triangleq \delta_{\theta} L^{u,v} = \frac{L^{\theta,v} - L^u}{\theta}, \quad \delta_{\theta} U \triangleq \delta_{\theta} U^{u,v} = \frac{U^{\theta,v} - U^u}{\theta}. \quad 22$$
Combining (5.7) and (5.8) we have

\[
\delta_\theta X_t = \int_0^t \left\{ \mathbb{E}^0 \left\{ \delta_\theta L_s \cdot \sigma(s, \varphi_{\land s}, U^\theta_s, z^1) \right\} \bigg|_{\varphi^1=\mathcal{X}^\theta, v, z^1=\mathcal{U}^\theta} + [D\sigma]_s^{\theta,u,v}(\delta_\theta X_{\land s}) + \mathbb{E}^0 \left\{ B^{\theta,u,v}(s, \varphi_{\land s}, z^1) \delta_\theta U_s \right\} \bigg|_{z^1=\mathcal{U}^\theta} + C^{\theta,u,v}_\sigma(s)(v_s - u_s) \right\} dB_s^1, \tag{5.9}
\]

where

\[
[D\sigma]^{\theta,u,v}_t(\psi) = \mathbb{E}^0 \left\{ L^\theta_t \int_0^1 D_\varphi \sigma(t, \varphi_{\land t}, \lambda(\varphi^1_{\land t} - \varphi^2_{\land t}), U^\theta_t, z^1)(\psi)d\lambda \right\} \bigg|_{\varphi^1=\mathcal{X}^\theta, \varphi^2=\mathcal{X}^u, z^1=\mathcal{U}^\theta},
\]

\[
B^{\theta,u,v}(t, \varphi_{\land t}, z^1) = L^\theta_t \int_0^1 \partial_\varphi \sigma(t, \varphi^2_{\land t}, U^\theta_t + \lambda(U^\theta_t - U^u_t), z^1)d\lambda,
\]

\[
C^{\theta,u,v}_\sigma(t) = \mathbb{E}^0 \left\{ \left[ L^\theta_t \int_0^1 \partial_\varphi \sigma(t, \varphi_{\land t}^2, U^\theta_t, z^2 + \lambda(z^1 - z^2))d\lambda \right] \right\} \bigg|_{\varphi^1=\mathcal{X}^u, z^1=\mathcal{U}^\theta, z^2=\mathcal{U}^u}.
\]

Noting that \( U^\theta_t = \frac{\mathbb{E}^0[L^\theta_t X^\theta_t | F^\theta_t]}{\mathbb{E}^0[L^\theta_t | F^\theta_t]} \) and \( U^u_t = \frac{\mathbb{E}^0[L^u_t X^u_t | F^\theta_t]}{\mathbb{E}^0[L^u_t | F^\theta_t]} \), we can easily check that

\[
\delta_\theta U_t = \frac{\mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[L^\theta_t X^\theta_t | F^\theta_t] - \mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[L^u_t X^u_t | F^\theta_t]}{\mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[L^u_t X^u_t | F^\theta_t]}
\]

\[
= \frac{\mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[\delta_\theta L_t X^\theta_t + L^u_t \delta_\theta X_t | F^\theta_t] - \mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[\delta_\theta L_t | F^\theta_t] \mathbb{E}^0[L^u_t X^u_t | F^\theta_t]}{\mathbb{E}^0[L^\theta_t | F^\theta_t] \mathbb{E}^0[L^u_t | F^\theta_t]}
\]

\[
= \frac{\mathbb{E}^0[\delta_\theta L_t X^\theta_t + L^u_t \delta_\theta X_t | F^\theta_t]}{\mathbb{E}^0[L^\theta_t | F^\theta_t]} - \frac{\mathbb{E}^0[\delta_\theta L_t | F^\theta_t] U^u_t}{\mathbb{E}^0[L^u_t | F^\theta_t]}.
\]

Now, sending \( \theta \to 0 \), and assuming that

\[
K_t = K_t^{u,v} \triangleq \lim_{\theta \to 0} \delta_\theta X^u_t; \quad R_t = R_t^{u,v} \triangleq \lim_{\theta \to 0} \delta_\theta L_t^{u,v}
\]

both exist in \( L^2(\mathbb{Q}^0) \), then it follows from (5.7)-(5.11) we have, at least formally,

\[
K_t = \int_0^t \left\{ \mathbb{E}^0 \left[ R_s \sigma(s, \varphi_{\land s}, U^u_s, z) \right] \bigg|_{\varphi^1=\mathcal{X}^u, z=\mathcal{U}^u} + [D\sigma]^{u,v}_s(K_{\land s}) + \mathbb{E}^0 \left[ B^{u,v}(s, \varphi_{\land s}, z) \right] \right\} dB_s^1
\]

\[
+ \mathbb{E}^0 \left[ C^{u,v}_\sigma(s)(v_s - u_s) \right] dB_s^1,
\]

where

\[
[D\sigma]^{u,v}_t(\psi) \triangleq \mathbb{E}^0 \left\{ L^u_t D_\varphi \sigma(t, \varphi_{\land t}, U^u_t, z)(\psi) \right\} \bigg|_{\varphi^1=\mathcal{X}^u, z=\mathcal{U}^u},
\]

\[
B^{u,v}(t, \varphi_{\land t}, z) = L^u_t \partial_\varphi \sigma(t, \varphi_{\land t}, U^u_t, z),
\]

\[
C^{u,v}_\sigma(t) = \mathbb{E}^0 \left\{ L^u_t \partial_\varphi \sigma(t, \varphi_{\land t}, U^u_t, z) \right\} \bigg|_{\varphi^1=\mathcal{X}^u, z=\mathcal{U}^u}.
\]

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Observing also that $U^u_t$ is $\mathcal{F}^Y_t$-measurable, we have
\[
\mathbb{E}^u\left[B^u(s, \varphi \wedge_s, z) \left( \frac{\mathbb{E}[R_s X^u_s + L^u_s K_s]}{\mathbb{E}[L^u_s]} - \frac{\mathbb{E}[R_s]}{\mathbb{E}[L^u_s]} \right) \right]_{\varphi = X^u_t, \Delta = 1} = \mathbb{E}^u\left[ \partial_y \sigma(s, \varphi \wedge_s, U^u_s, z) \mathbb{E}\left\{ (L^u_s)^{-1} R_s [X^u_s - U^u_s] + K_s \right\} \right]_{\varphi = X^u_t, \Delta = 1} \tag{5.15}
\]
Consequently, if we define
\[
\Psi(t, \varphi \wedge_t, x, y, z) \triangleq \sigma(t, \varphi \wedge_t, y, z) + \partial_y \sigma(t, \varphi \wedge_t, y, z)(x - y),
\]
then we can rewrite (5.13) as
\[
K_t = \int_0^t \left\{ \mathbb{E}^u\left[ \Psi(s, \varphi \wedge_s, X^u_s, U^u_s, z) R_s + \partial_y \sigma(s, \varphi \wedge_s, U^u_s, z) L^u_s K_s \right] \right\}_{\varphi = X^u_t, \Delta = 1} \tag{5.17}
\]
Similarly, we can formally write down the SDE for $R$:
\[
R_t = \int_0^t [R_s h(s, X^u_s) + L^u_s \partial_y h(s, X^u_s) K_s] dY^u_s, \quad t \geq 0. \tag{5.18}
\]

**Theorem 5.2.** Assume that Assumption 2.3 is in force. Then for any $u = u^* \in L_\mathbb{F}^\infty(\mathbb{Q}^0; [0, T])$, there is a unique solution $(K, R) \in \mathcal{L}_\mathbb{F}^\infty(\mathbb{Q}^0; \mathbb{C}_T)$ to SDEs (5.17) and (5.18).

**Proof.** Let $u \in L_\mathbb{F}^\infty(\mathbb{Q}^0; [0, T])$ be given. We define $F^1_t(K, R)$ and $F^2_t(K, R)$, $t \in [0, T]$, to be the right hand side of (5.17) and (5.18), respectively.

We first observe that $F^1_t(0, 0) = \int_0^t C^u \sigma(s) (v_s - u_s) dB^1_s$, and $F^2_t(0, 0) \equiv 0$, $t \in [0, T]$. Then, for any $p > 2$, it holds that
\[
\mathbb{E}^u \left[ \sup_{0 \leq s \leq t} |F^1_s(0, 0)|^p \right] \leq C_p \mathbb{E}^u \left[ \left( \int_0^t |v_s - u_s|^2 ds \right)^{p/2} \right], \quad t \in [0, T]. \tag{5.19}
\]
Now let $(K^i, R^i) \in \mathcal{L}_\mathbb{F}^\infty(\mathbb{Q}^0; \mathbb{C}_T)$, $i = 1, 2$. We define $K^i \triangleq F^1_t(K^i, R^i)$, $\tilde{R}^i \triangleq F^1_t(K^i, R^i)$, $i = 1, 2$, and $\tilde{K} \triangleq K^1 - K^2$, $\tilde{R} \triangleq R^1 - R^2$, $\hat{K} \triangleq \bar{K}^1 - \bar{K}^2$, and $\hat{R} \triangleq \bar{R}^1 - \bar{R}^2$. Then, noting that $\sigma, \partial_y \sigma, y \partial_y \sigma$, and $\partial_z \sigma$ are all bounded, thanks to Assumption 2.3, we see that
\[
|\Psi(t, \varphi \wedge_t, x, y, z)| \leq C(1 + |x|), \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^3, \varphi \in \mathbb{C}_T,
\]
where, and in what follows, $C > 0$ is some generic constant which is allowed to vary from line to line. It then follows that
\[
\mathbb{E}^0[\Psi(t, \varphi, \Lambda t, X^u_t, U^u_t, z) \hat{R}_t + \partial_y \sigma(t, \varphi, \Lambda t, U^u_t, z) L^u_t \hat{K}_t] \leq C \mathbb{E}^0[(1 + |X^u_t|)\|\hat{R}_t\| + |L^u_t \hat{K}_t|] \leq C \left[ \mathbb{E}^0[|\hat{K}_t|^2 + |\hat{R}_t|^2] \right]^{1/2}. \tag{5.20}
\]
Furthermore, since $D_\varphi \sigma$ is also bounded, we have $||D\sigma\|_{t}^u \psi \| \leq C \sup_{0 \leq s \leq t} |\psi(s)|$, for $\psi \in C_T$. Then from the definition of $\hat{K}$ and (5.20) we have, for any $p \geq 2$ and $t \in [0, T]$, \[
\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{K}_s|^p \right] \leq C_p \int_t^t \left( \mathbb{E}^0[|\hat{R}_s|^2 + |\hat{K}_s|^2] \right)^p ds + C_p \int_0^t \mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{K}_s|^2 \right] ds. \tag{5.21}
\]
On the other hand, the boundedness of $h$ and $\partial_y h$ implies that, recalling the definition of $\hat{R}$, for $p \geq 2$ and $t \in [0, T]$, \[
\left( \mathbb{E}^0\left[ \sup_{s \leq t} |\hat{R}_s|^p \right] \right)^2 \leq C_p \int_0^t \mathbb{E}^0[|\hat{R}_s|^2] ds + C_p \int_0^t \mathbb{E}^0[|L^u_t \hat{K}_s|^2] ds \tag{5.22}
\]
Combining (5.21) and (5.22) we have, for $t \in [0, T]$, \[
\mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{K}_s|^2 \right] + \left( \mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{R}_s|^p \right] \right)^2 \leq C_p \int_0^t \left( \mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{K}_r|^2 \right] + \left( \mathbb{E}^0\left[ \sup_{0 \leq s \leq t} |\hat{R}_r|^p \right] \right)^2 \right) ds.
\]
This, together with (5.19), enables us to apply standard SDE arguments to deduce that there is a unique solution $(K, R) \in L^\infty_{\mathbb{F}}(\mathbb{P}; C_T)$ of (5.17) and (5.18), such that for all $p \geq 2$, it holds that \[
\mathbb{E}^0[\|K\|_{C_T}^{2p}] + \mathbb{E}^0[\|R\|_{C_T}^{2p}] \leq C_p \|v_s - u_s\|_{L^p, Q_0}^2. \tag{5.23}
\]
We leave it the interested reader, and this completes the proof. \hfill \qed

6 Variational Equations

In this section we validate the heuristic arguments in the previous section and derive the variational equation of the optimal trajectory rigorously. Recall the processes $\delta_\theta X = \delta_\theta X^u, \delta_\theta L = \delta_\theta L^u$, and $(K, R)$ defined in the previous section. Denote \[
\eta^\theta_t \triangleq \delta_\theta X_t - K_t, \quad \tilde{\eta}^\theta_t \triangleq \delta_\theta L_t - R_t, \quad t \in [0, T]. \tag{6.1}
\]
Our main purpose of this section is to prove the following result.
Proposition 6.1. Let \((\mathbb{P}^u, u) = (\mathbb{P}^u_t, u^*) \in \mathcal{U}_{ad}\) be an optimal control, \((X^u, L^u)\) be the corresponding solution of (5.1), and let \(U^u_t = \mathbb{E}^u[X^u_t | \mathcal{F}^Y_t], \quad t \geq 0\). For any \(v \in \mathcal{U}_{ad}\), let \((K, R) = (K^{u,v}, R^{u,v})\) be the solution of the linear equations (5.17) and (5.18). Then, for all \(p > 1\), it holds that

\[
\lim_{\theta \to 0} \mathbb{E}^0[\|\eta^\theta \|_{\mathcal{C}^p_T}] = \lim_{\theta \to 0} \mathbb{E}^0\left[ \sup_{s \in [0,T]} \left| \frac{X^u_s - X^u_s}{\theta} - K_s \right|^p \right] = 0; \tag{6.2}
\]

\[
\lim_{\theta \to 0} \mathbb{E}^0[\|\tilde{\eta}^\theta \|_{\mathcal{C}^p_T}] = \lim_{\theta \to 0} \mathbb{E}^0\left[ \sup_{s \in [0,T]} \left| \frac{L^u_s - L^u_s}{\theta} - R_s \right|^p \right] = 0. \tag{6.3}
\]

The proof of Proposition 6.1 is quite lengthy, we shall split it into two parts.

[Proof of (6.3).] This part is relatively easy. We note that with a direct calculation using the equations (5.7) and (5.18) it is readily seen that \(\tilde{\eta}^\theta\) satisfies the following SDE:

\[
\tilde{\eta}^\theta_t = \int_0^t \tilde{\eta}^\theta_s h(r, X^u_r) dY_r + \int_0^t L^u_s \int_0^1 \partial_x h(r, X^u_r + \lambda(\eta^\theta_r + K_r)) \eta^\theta_r d\lambda dY_r \tag{6.4}
\]

where

\[
I^{1,\theta}_t = \int_0^t R_r(h(r, X^u_r) - h(r, X^u_r)) dY_r;
\]

\[
I^{2,\theta}_t = \int_0^t L^u_s \int_0^1 \partial_x h(r, X^u_r + \lambda(\eta^\theta_r + K_r)) K_r d\lambda dY_r - \int_0^t L^u_s \partial_x h(r, X^u_r) K_r dY_r.
\]

We claim that, for all \(p > 1\),

\[
\lim_{\theta \to 0} \mathbb{E}^u[\sup_{t \in [0,T]} |I^{1,\theta}_t|^p] = 0, \quad \lim_{\theta \to 0} \mathbb{E}^u[\sup_{t \in [0,T]} |I^{2,\theta}_t|^p] = 0. \tag{6.5}
\]

Indeed, note that \(dY_t = dB_t - h(t, X^u_t) dt\), where \(B^2\) is a \(\mathbb{P}^u\)-Brownian motion. By Proposition 4.2 and the continuity assumptions on \(h\) and \(\partial_x h\) one checks that, for all \(p \geq 2,\)

\[
\mathbb{E}^u\left\{ \sup_{t \in [0,T]} |I^{1,\theta}_t|^p \right\} \leq 2 \mathbb{E}^u\left\{ L^u_T \sup_{t \in [0,T]} \left| \int_0^t R_s[h(s, X^u_s) - h(s, X^u_s)] dY_s \right|^p \right\}
\]

\[
\leq 2 \mathbb{E}^u\left\{ L^u_T \sup_{t \in [0,T]} \left| \int_0^t R_s[h(s, X^u_s) - h(s, X^u_s)] dB^2_s \right|^p \right\}
\]

\[
+ 2 \mathbb{E}^0\left\{ L^u_T \sup_{t \in [0,T]} \left| \int_0^t R_s[h(s, X^u_s) - h(s, X^u_s)] h(s, X^u_s) ds \right|^p \right\}
\]

\[
\leq C_p \mathbb{E}^0\left\{ L^u_T \int_0^T R_s(\|X^u_s - X^u_s\|^{p \wedge 1}) ds \right\}
\]

\[
\leq C_p \left\{ \mathbb{E}^0\left[ \sup_{t \in [0,T]} |R_s|^p \right] \right\}^{\frac{1}{3}} \left\{ \mathbb{E}^0\left[ \sup_{s \in [0,T]} (|X^u_s - X^u_s|^{2 \wedge 1}) \right] \right\}^{\frac{1}{3}}
\]

\[
= C_p \|u - u^{\theta, v}\|_{p, 2, Q^0} \leq C|\theta|^{\frac{1}{3}},
\]
and

\[ E^u \left\{ \sup_{t \in [0,T]} |I_t^{2, \theta}|^p \right\} \]

\[ = E^0 \left\{ L_T^u \sup_{t \in [0,T]} \left| \int_0^t L_r^u K_r \left[ \int_0^1 \left[ \partial_x h(r, X_r^u + \lambda \theta(\eta_r^\theta + K_r)) - \partial_x h(r, X_r^u) \right] d\lambda \right] dY_r \right|^p \right\} \]

\[ \leq C_p E^0 \left\{ L_T^u \int_0^T |L_r^u|^p |K_r|^p \left[ \int_0^1 \left[ \partial_x h(r, X_r^u + \lambda \theta(\eta_r^\theta + K_r)) - \partial_x h(r, X_r^u) \right] d\lambda \right]^p dr \right\} \]

\[ \leq C_p E^0 \left\{ \int_0^T \left\| \partial_x h(r, X_r^u + \lambda \theta(\eta_r^\theta + K_r)) - \partial_x h(r, X_r^u) \right\|^p d\lambda \right\}^{1/3}. \]

Here in the above the last inequality follows from the $L^p$-estimate (5.23). Now, from (4.2), (5.17), and (5.18) we see that

\[ E^0 \left\{ \sup_{t \in [0,T]} (|\eta_t^\theta|^2 + |K_t|^2) \right\} \leq C, \quad \theta \in (0, 1). \]

Hence, since $\theta(||\eta^\theta||_{C^r} + ||K||_{C^r}) \to 0$, in probability $Q^0$, as $\theta \to 0$, the continuity of $\partial_x h$ and the Bounded Convergence Theorem then imply (6.5), proving the claim. Recalling (6.4), we see that (6.3) follows from (6.5), provided (6.2) holds, which we now substantiate.

[Proof of (6.2).] This part is more involved. We first rewrite (5.9) as follows

\[ \delta_\theta X_t = \int_0^t \left\{ E^0 \left\{ (\tilde{\eta}_s^\theta + R_s) \sigma(s, \varphi_{\wedge, s}, U_s^{\theta, v}, z) \right\} \bigg|_{\varphi = \chi^{\theta}_{u, \theta}} + [D\sigma]_{s, \wedge}^{\theta, u, v}(\eta_{s, \wedge}) + K_{s, \wedge} + B_{s, \wedge}^{\theta, u, v}(s, \varphi_{\wedge, s}, z) \delta_\theta U_s \right\} dB_s. \]

Here $[D\sigma]^{\theta, u, v}$, $B^{\theta, u, v}$, and $C^{\theta, u, v}$ are defined by (5.10). Furthermore, in light of (5.11), we can also write:

\[ \delta_\theta U_t = \frac{E^0[(\tilde{\eta}_t^\theta + R_t) X_t^{\theta, v} + L_t^u (\eta_t^\theta + K_t)]/F_t^Y}{E^0[L_t^{\theta, v}/F_t^Y]} - \frac{E^0[(\tilde{\eta}_t^\theta + R_t)/F_t^Y] U_t^u}{E^0[L_t^{\theta, v}/F_t^Y]} \]

Plugging this into (6.6) we have

\[ \delta_\theta X_t = \int_0^t \left\{ E^0 \left\{ \tilde{\eta}_s^\theta \sigma(s, \varphi_{\wedge, s}, U_s^{\theta, v}, z) \right\} \bigg|_{\varphi = \chi^{\theta}_{u, \theta}} + [D\sigma]_{s, \wedge}^{\theta, u, v}(\eta_{s, \wedge}) \right\} d\lambda \]

\[ + E^0 \left\{ B^{\theta, u, v}(s, \varphi_{\wedge, s}, z) \left[ \frac{E^0[\tilde{\eta}_s^\theta X_s^{\theta, v} + L_s^u (\eta_s^\theta + K_s)]/F_s^Y}{E^0[L_s^{\theta, v}/F_s^Y]} - \frac{E^0[\eta_s^\theta] F_s^Y}{E^0[L_s^{\theta, v}/F_s^Y]} U_s^u \right] \bigg|_{\varphi = \chi^{\theta}_{u, \theta}} \right\} dB_s \]

\[ + \int_0^t \left\{ E^0 \left\{ R_s \sigma(s, \varphi_{\wedge, s}, U_s^{\theta}, z) \right\} \bigg|_{\varphi = \chi^{\theta}_{u, \theta}} + [D\sigma]_{s, \wedge}^{\theta, u, v}(K_{s, \wedge}) + B^{\theta, u, v}(s, \varphi_{\wedge, s}, z) \left[ \frac{E^0[R_s X_s^{\theta, v} + L_s^u K_s]}{E^0[L_s^{\theta, v}/F_s^Y]} - \frac{E^0[R_s] F_s^Y}{E^0[L_s^{\theta, v}/F_s^Y]} U_s^u \right] \bigg|_{\varphi = \chi^{\theta}_{u, \theta}} \right\} dB_s \]

\[ + C^{\theta, u, v}(s)(v_s - u_s) dB_s. \]
Now, recalling (5.17) (or more conveniently, (5.13)) we have

\[
\eta_t^\theta = \delta_t X_t - K_t = \int_0^t \left\{ \mathbb{E}^0 \left\{ \eta_s^\theta \sigma(s, \varphi, \lambda_s, U_s^{\theta,u,v}, z) \right\}_{\varphi = \chi, \theta, z = u^\theta_s} + [D\sigma]_{z = u^\theta_s}^{\theta,u,v} (\eta_{\lambda_s}^\theta) \right. \\
+ \mathbb{E}^0 \left\{ B_{\theta,u,v}(s, \varphi, \lambda_s, z) \left[ \frac{\mathbb{E}^0 [\eta_s^\theta X_s^{\theta,v} + L_s^u \eta_s^\theta | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} - \frac{\mathbb{E}^0 [\eta_s^\theta | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} U_s^u \right] \right\}_{\varphi = \chi, \theta, z = u^\theta_s} \right. \\
\left. + I_t^{3,\theta,1} + I_t^{3,\theta,2} + I_t^{3,\theta,3} + I_t^{3,\theta,4} \right\} dB_s^1 \\
(6.7)
\]

where, for \( t \in [0, T] \),

\[
I_t^{3,\theta,1} \triangleq \int_0^t \mathbb{E}^0 \left\{ R_s \left[ \sigma(s, \varphi, \lambda_s, U_s^{\theta}, z^1) - \sigma(s, \varphi^{2}, \lambda_s, U_s^{\theta}, z^2) \right] \right\}_{\varphi = \chi, \theta, z = u^\theta_s} dB_s^1,
\]

\[
I_t^{3,\theta,2} \triangleq \int_0^t \mathbb{E}^0 \left\{ [D\sigma]_{z = u^\theta_s}^{\theta,u,v} (K_{\lambda_s}) - [D\sigma]_{z = u^\theta_s}^{\theta,u,v} (K_{\lambda_s}) \right\} dB_s^1,
\]

\[
I_t^{3,\theta,3} \triangleq \int_0^t \mathbb{E}^0 \left\{ B_{\theta,u,v}(s, \varphi, \lambda_s, z) \left( \frac{\mathbb{E}^0 [R_s X_s^{\theta,v} + L_s^u K_s | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} - \frac{\mathbb{E}^0 [R_s | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} U_s^u \right) \right\}_{\varphi = \chi, \theta, z = u^\theta_s} \right. \\
\left. - \mathbb{E}^0 \left\{ B_{u,v}(s, \varphi, \lambda_s, z) \left( \frac{\mathbb{E}^0 [R_s X_s^{u,v} + L_s^u K_s | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} - \frac{\mathbb{E}^0 [R_s | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} U_s^u \right) \right\}_{\varphi = \chi, \theta, z = u^\theta_s} \right\} dB_s^1,
\]

\[
I_t^{3,\theta,4} \triangleq \int_0^t \mathbb{E}^0 \left\{ C_{\theta}^{\theta,u,v}(s)(v_s - u_s) - C_{\theta}^{\theta,u,v}(s)(v_s - u_s) \right\} dB_s^1.
\]

We have the following lemma.

**Lemma 6.2.** Assume Assumption 2.3. Then, for all \( p > 1 \),

\[
\lim_{\theta \to 0} \mathbb{E}^0 \left\{ \sup_{0 \leq t \leq T} |I_t^{3,\theta,i}|^p \right\} = 0, \quad i = 1, \ldots, 4. \quad (6.9)
\]

**Proof.** We first recall that \( U_s^{\theta,v} \triangleright \mathbb{E}^0 [X_s^{\theta,v} | F_s^Y] \) and \( U_s^u \triangleright \mathbb{E}^0 [X_s^u | F_s^Y] \). Using the Kallianpur-Strieble formula we have

\[
\mathbb{E}^0 \int_0^T |U_s^{\theta,v} - U_s^u|^p \, ds \leq C_p \left\{ \mathbb{E}^0 \int_0^T \left| \frac{\mathbb{E}^0 [L_s^{\theta,v} X_s^{\theta,v} | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} - \frac{\mathbb{E}^0 [L_s^u X_s^{u,v} | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} \right|^p \, ds \right. \\
\left. + \mathbb{E}^0 \int_0^T \left| \frac{\mathbb{E}^0 [L_s^u X_s^{u,v} | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} - \frac{\mathbb{E}^0 [L_s^u X_s^{u,v} | F_s^Y]}{\mathbb{E}^0 [L_s^u | F_s^Y]} \right|^p \, ds \right\} \quad (6.10)
\]

\[
\triangleright C_p \{ J_1^1 + J_2^1 \}.
\]

We now estimate \( J_1^1 \) and \( J_2^1 \) respectively. First note that, for any \( p > 1 \), we can find a constant \( C_p > 0 \) such that for any \( \theta \in (0, 1) \) and \( u \in \mathcal{U}_{ad} \),

\[
\mathbb{E}^0 \left[ (L_s^{\theta,v})^p \right] \leq C_p.
\]
Thus, applying the Hölder and Jensen inequalities as well as Proposition 4.2, we have, for any $p > 1$, and $\theta \in (0, 1)$,
\[
\mathbb{E}^0 \int_0^T \left| \frac{\mathbb{E}^0[L_s^{\theta,v} X_s^{\theta,v} | \mathcal{F}_s^v]}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} - \frac{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} \right|^p ds \leq \mathbb{E}^0 \left\{ \frac{[L_s^{\theta,v} X_s^{\theta,v} - L_s^{\theta,v} X_s^{\theta,v}]}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} \right\} ds
\]
\[
\leq \mathbb{E}^0 \left\{ \frac{[L_s^{\theta,v} X_s^{\theta,v} - L_s^{\theta,v} X_s^{\theta,v}]}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} \right\}^{1/2} ds
\leq C_p \theta \| u - v \|_{2,2,Q^0}.
\]

Similarly, one can also argue that, for any $p > 1$, the following estimates hold:
\[
\mathbb{E}^0 \int_0^T \left| \frac{1}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} - \frac{1}{\mathbb{E}^0[L_s^{\theta,v} | \mathcal{F}_s^v]} \right|^p ds \leq C_p \theta \| u - v \|_{2,2,Q^0}, \quad \theta \in (0, 1).
\]

Clearly, (6.11) and (6.12) imply that $J_0^1 + J_0^2 \leq C_p \theta \| u - v \|_{2,2,Q^0}$, for some constant $C_p > 0$, depending only on $p$, the Lipschitz constant of the coefficients, and $T$. Therefore we have
\[
\mathbb{E}^0 \left( \int_0^T |U_s^{\theta,v} - U_s^{\theta,v}| ds \right) \leq C_p \theta \| u - v \|_{2,2,Q^0} \rightarrow 0, \quad \text{as } \theta \to 0.
\]

We can now prove (6.9) for $i = 1, \cdots, 4$. First, by Burkholder-Gundy-Davis inequality we have
\[
\mathbb{E}^0[\sup_{0 \leq t \leq T} |I_{t}^{3,\theta,1}|^2] \leq C \int_0^T \mathbb{E}^0 \left\{ R_s[\sigma(s, \varphi_{\lambda,s}^{1}, U_{s}^{\theta,v}, z^1) - \sigma(s, \varphi_{\lambda,s}^{2}, U_{s}^{u}, z^2)] \right\} \varphi_{\infty, \varphi_{\lambda,s}^{1}, \varphi_{\lambda,s}^{2}}^2 ds.
\]

Since $\sigma$ is bounded and Lipschitz continuous in $(\varphi, y, z)$, it follows from Proposition 4.2 and (6.13) that $\lim_{\theta \to 0} \mathbb{E}^0[\sup_{0 \leq t \leq T} |I_{t}^{3,\theta,1}|] = 0$. By the similar arguments using the continuity of $D_{\lambda} \sigma$ and that of $\partial_{\theta} \sigma$, respectively, it is not hard to show that, for all $p > 1$,
\[
\lim_{\theta \to 0} \mathbb{E}^0[\sup_{0 \leq t \leq T} |I_{t}^{3,\theta,2}|] = 0; \quad \lim_{\theta \to 0} \mathbb{E}^0[\sup_{0 \leq t \leq T} |I_{t}^{3,\theta,4}|] = 0.
\]

It remains to prove the convergence of $I_{t}^{3,\theta,3}$. To this end, we note that, for any $p > 1$,
\[
\mathbb{E}^0 \left[ \sup_{s \in [0,T]} \left( |R_s|^p + |K_s|^p \right) \right] \leq C_p,
\]
and by (6.13) we have, for $p > 1$,
\[
\lim_{\theta \to 0} \mathbb{E}^0 \left( \int_0^T \left| \mathbb{E}^0[B_{t}^{\theta,u,v}(s, \varphi_{\lambda,s}^{1}, z) - \mathbb{E}^0[B_{t}^{u,v}(s, \varphi_{\lambda,s}^{1}, z)] \varphi_{\infty, \varphi_{\lambda,s}^{1}, \varphi_{\lambda,s}^{2}}^2 \right|^p ds \right) = 0.
\]
This, together with (6.12), (6.13), an estimate similar to (6.11), and Proposition 4.2, yields that
\[
\lim_{\theta \to 0} \mathbb{E}[\sup_{0 \leq t \leq T} |I_t^{3,\theta,2}|] = 0,
\]
proving the lemma.

We now continue the proof of (6.2). First we rewrite (6.7) as
\[
\eta_t^\theta = \int_0^t \left\{ \mathbb{E}\left[ \eta_t^\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) \right] \right\} \bigg|_{\varphi=\chi_t^u} + [D\sigma]_{s}^{\theta,\psi}(\eta_t^\theta) + \int_0^t \mathbb{E}\left[ B^{\theta,\psi}(s, \varphi_s, z) \left( \frac{\mathbb{E}\left[ \eta_t^\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) \right]}{\mathbb{E}[L_s^{\psi}]} \frac{\mathbb{E}[L_s^{\psi}]}{U_s^{\psi}} - \frac{\mathbb{E}[\eta_t^\theta]}{\mathbb{E}[L_s^{\psi}]} U_s^{\psi} \right] \right\} \bigg|_{\varphi=\chi_t^u} \right\} dB^1_s + I_t^{3,\theta,0} + \sum_{i=1}^4 I_t^{3,\theta,i},
\]
where
\[
I_t^{3,\theta,0} = \int_0^t \mathbb{E}\left[ B^{\theta,\psi}(s, \varphi_s, z) \left( \frac{\mathbb{E}\left[ \eta_t^\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) \right]}{\mathbb{E}[L_s^{\psi}]} \frac{\mathbb{E}[L_s^{\psi}]}{U_s^{\psi}} - \frac{\mathbb{E}[\eta_t^\theta]}{\mathbb{E}[L_s^{\psi}]} U_s^{\psi} \right] \right\} \bigg|_{\varphi=\chi_t^u} \right\} dB^1_s.
\]
We note that with the same argument as before one shows that \( \lim_{\theta \to 0} \mathbb{E}[\sup_{0 \leq t \leq T} |I_t^{3,\theta,0}|] = 0 \). On the other hand, similar to (5.15) one can argue that
\[
\mathbb{E}\left[ B^{\theta,\psi}(s, \varphi_s, z) \left( \frac{\mathbb{E}\left[ \eta_t^\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) \right]}{\mathbb{E}[L_s^{\psi}]} \frac{\mathbb{E}[L_s^{\psi}]}{U_s^{\psi}} - \frac{\mathbb{E}[\eta_t^\theta]}{\mathbb{E}[L_s^{\psi}]} U_s^{\psi} \right] \right\} \bigg|_{\varphi=\chi_t^u} \right\} = \mathbb{E}\left[ \int_0^1 \partial_\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) d\lambda \right. \left. \left( \frac{\mathbb{E}\left[ \eta_t^\theta \sigma(s, \varphi_s, U_s^{\theta,\psi}, z) \right]}{\mathbb{E}[L_s^{\psi}]} \frac{\mathbb{E}[L_s^{\psi}]}{U_s^{\psi}} - \frac{\mathbb{E}[\eta_t^\theta]}{\mathbb{E}[L_s^{\psi}]} U_s^{\psi} \right) \right\} \bigg|_{\varphi=\chi_t^u} \right\}.
\]
Consequently, we have
\[
\eta_t^\theta = \int_0^t \mathbb{E}\left[ \alpha_t^\theta(\varphi_s, z) \eta_t^\theta + \beta_t^\theta(\varphi_s, z) \eta_t^\theta \right] \bigg|_{\varphi=\chi_t^u} + [D\sigma]_{s}^{\theta,\psi}(\eta_t^\theta) + \sum_{i=1}^4 I_t^{3,\theta,i},
\]
where \( I_t^{3,\theta} = \sum_{i=0}^4 I_t^{3,\theta,i} \), and
\[
|\alpha_t^\theta(\varphi_s, z)| \leq C(1 + |X_t^{\theta,\psi}| + |U_s^{\psi}|), \quad |\beta_t^\theta(\varphi_s, z)| \leq CL_s^{\psi}.
\]
Now by the Burkholder and Cauchy-Schwartz inequalities we have, for all \( p \geq 2 \) and \( t \in [0, T] \),
\[
\mathbb{E}\left[ \sup_{s \in [0,t]} \mathbb{E}[|\eta_t^\theta|^{2p}] \right] \leq C_p \left\{ \mathbb{E}\left[ \sup_{s \in [0,t]} |I_s^{3,\theta}|^{2p} \right] + \mathbb{E}\left[ \left\{ \int_0^t \left( \mathbb{E}[|\eta_t^\theta|^2] + \mathbb{E}[|\eta_t^\theta|^2] + \sup_{r \in [0,s]} |\eta_r^\theta|^2 \right) ds \right\}^p \right] \right\}.
\]

and from Gronwall’s inequality one has

$$
\mathbb{E}^{0}\left[ \sup_{s \in [0,t]} |\tilde{\eta}_s^{\theta}|^{2p} \right] \leq C_p \left\{ \mathbb{E}^{0}\left[ \|I_s^{\theta}\|^2 \right] + \int_0^t \left( \mathbb{E}^{0}[|\tilde{\eta}_s^{\theta}|^p]^2 \right) ds \right\}, \quad t \in [0, T].
$$

(6.17)

On the other hand, setting $I_t^{\theta} \overset{\triangle}{=} I_t^{1,\theta} + I_t^{2,\theta}$, $t \in [0, T]$, we have from (6.4) that, for $p \geq 2$,

$$
\mathbb{E}^{0}\left[ \sup_{s \in [0,t]} |\tilde{\eta}_s^{\theta}|^p \right] \leq C_p \left\{ \mathbb{E}^{0}[\|I_s^{\theta}\|^p] + \int_0^t \mathbb{E}^{0}[|\tilde{\eta}_s^{\theta}|^p] ds + \int_0^t \left( \mathbb{E}^{0}[|\tilde{\eta}_s^{\theta}|^{2p}] \right)^{1/2} ds \right\}, \quad t \in [0, T].
$$

The Gronwall inequality then leads to that

$$
\left( \mathbb{E}^{0}\left[ \sup_{s \in [0,t]} |\tilde{\eta}_s^{\theta}|^p \right] \right)^2 \leq C_p \left\{ \mathbb{E}^{0}\left[ \|I_s^{\theta}\|^p \right] + \int_0^t \mathbb{E}^{0}[|\tilde{\eta}_s^{\theta}|^p] ds \right\}, \quad t \in [0, T].
$$

(6.18)

Combining (6.17), (6.18), and applying Lemma 6.2 as well as the Gronwall inequality, we can easily deduce (6.2) by sending $\theta \to 0$. Consequently, (6.3) holds as well. This completes the proof. \hfill \blacksquare

7 Stochastic Maximum Principle

We are now ready to study the Stochastic Maximum Principle. The main task will be to determine the appropriate adjoint equation, which we expect to be a backward stochastic differential equation of Mean-field type. We begin with a simple analysis. Suppose that $u = u^*$ is an optimal control, and for any $v \in \mathcal{U}_{ad}$, we define $u^{\theta,v}$ by (5.4). Then we have

$$
0 \leq \frac{J(u^{\theta,v}) - J(u)}{\theta} = \frac{1}{\theta} \mathbb{E}^{0}\left\{ \mathbb{E}^{0}[L_s^{\theta,v} \Phi(x, U_s^{\theta,v})]|_{x = X_T^u} - \mathbb{E}^{0}[L_T^{\theta,u} \Phi(x, U_T^u)]|_{x = X_T^u} \right\} + \int_0^T \mathbb{E}^{0}[L_s^{\theta,v} f(s, \varphi \wedge s, U_s^{\theta,v}, z)|_{\varphi = X^{\theta,v}, z = u^{\theta,v}} - \mathbb{E}^{0}[L_s^{\theta,u} f(s, \varphi \wedge s, U_s^u, z)|_{\varphi = X^u, z = u^u}] ds \right\}.
$$

(7.1)

Now, repeating the same analysis as that in Proposition 4.2, then sending $\theta \to 0$, it follows from Proposition 4.2 and the continuity of the functions $\Phi$ and $f$ that

$$
0 \leq \mathbb{E}^{0}[K_T \xi] + \mathbb{E}^{0}[R_T \Theta] + \mathbb{E}^{0}\left\{ \int_0^T \mathbb{E}^{0}[R_s f(s, \varphi \wedge s, U_s^u, z)|_{\varphi = X^u, z = u^u}] + \mathbb{E}^{0}[\partial_s f(s, \varphi \wedge s, U_s^u, z)(X_s^u - U_s^u) R_s + L_s^u K_s)|_{\varphi = X^u, z = u^u}] + \mathbb{E}^{0}[L_s^u \partial_s f(s, \varphi \wedge s, U_s^u, z)|_{\varphi = X^u, z = u^u}] ds \right\},
$$

(7.2)
where
\[ \xi \triangleq \mathbb{E}^0[L^u_\tau \partial_x \Phi(x, U^u_\tau)|_{x=X^u_\tau} + L^u_\tau \mathbb{E}^0[\partial_y \Phi(X^u_\tau, y)]|_{y=U^u_\tau}, \]
\[ \Theta \triangleq \mathbb{E}^0[\Phi(X^u_\tau, y)]|_{y=U^u_\tau} + (X^u_\tau - U^u_\tau) \mathbb{E}^0[\partial_y \Phi(X^u_\tau, y)]|_{y=U^u_\tau}. \quad (7.3) \]

We now consider the adjoint equations that take the following form of backward SDEs on the reference space \((\Omega, \mathcal{F}, \mathbb{Q}^0)\):
\[
\begin{aligned}
dp_t &= -\alpha_t dt + d\Gamma_t + q_t dB_t^1 + q_t Y_t, \quad p_T = \xi, \\
\frac{dQ_t}{dt} &= -\beta_t dt + d\Sigma_t + M_t dB_t^1 + \tilde{M}_t Y_t, \quad Q_T = \Theta. \quad (7.4)
\end{aligned}
\]

Here the coefficients \(\alpha, \beta\) as well as the two bounded variation processes \(\Gamma\) and \(\Sigma\) are to be determined. Applying Itô’s formula and recalling the variation equations (5.17) and (5.18), we can easily derive (denote \(U^u_t = \mathbb{E}^u[X^u_t | \mathcal{F}^\gamma_t], \ t \in [0, T]\))
\[
\begin{aligned}
\mathbb{E}^0[\xi K_T] + \mathbb{E}^0[\Theta T R_T] &= \\
= & \int_0^T \left\{ - \mathbb{E}^0[K_s \alpha_s] - \mathbb{E}^0[R_s \beta_s] + \mathbb{E}^0[q_s \mathbb{E}^0[R_s \sigma(s, \varphi \cdot \land_s, U^u_s, z)]|_{\varphi=X^u_z, z=us} \right.
\quad + \mathbb{E}^0[q_s \mathbb{E}^0[\partial_y \sigma(s, \varphi \land_s, U^u_s, z)](X^u_s - U^u_s)R_s + L^u_s K_s]|_{\varphi=X^u_z, z=us} \right.
\quad + \mathbb{E}^0[q_s [D\sigma]^u_v(K \land s) + q_s C_{\sigma, v}^u(s)](v_s - u_s) + \tilde{M}_s R_s h(s, X^u_s) + \tilde{M}_s K_s L^u_s \partial_x h(s, X^u_s)] ds
\quad + \mathbb{E}^0 \left\{ \int_0^T [K_s d\Gamma_s + R_s d\Sigma_s] \right\}.
\end{aligned}
\]

where \([D\sigma]^u_v\) and \(C_{\sigma, v}^u\) are defined by (5.14).

By Fubini’s Theorem we see that
\[
\begin{aligned}
\mathbb{E}^0[q_s \mathbb{E}^0[R_s \sigma(s, \varphi \cdot \land_s, U^u_s, z)]|_{\varphi=X^u_z, z=us}] &= \mathbb{E}^0[R_s \mathbb{E}^0[q_s \mathbb{E}^0[\sigma(s, X_\land_s, y, us)]|_{y=U^u_s}]; \\
\mathbb{E}^0[q_s \mathbb{E}^0[\partial_y \sigma(s, \varphi \land_s, U^u_s, z)](X^u_s - U^u_s)R_s + L^u_s K_s]|_{\varphi=X^u_z, z=us} &= \mathbb{E}^0[q_s \mathbb{E}^0[\partial_y \sigma(s, X_\land_s, y, us)](X^u_s - U^u_s)R_s + L^u_s K_s]|_{y=U^u_s}(X^u_s - U^u_s)R_s + L^u_s K_s].
\end{aligned}
\]

Furthermore, in light of definition of \([D\sigma]^u_v\) (5.14), if we denote, for fixed \((t, \varphi, z)\),
\[ \mu_\varphi^0(t, \varphi, \land t, z)(\cdot) \triangleq \mathbb{E}^0[L^u_t D\varphi \sigma(t, \varphi \land_t, U^u_t, z)](\cdot) \in \mathcal{M}[0, T], \quad (7.7) \]
where \(\mathcal{M}[0, T]\) denotes all the Borel measures on \([0, T]\), then we can write
\[
[D\sigma]^u_v(K \land t) = \mathbb{E}^0[L^u_t D\varphi \sigma(t, \varphi \land_t, U^u_t, z)](\cdot) \triangleq \int_0^t K_r \mu_\varphi^0(t, X^u_{\land t}, ut)(dr). \quad (7.8)
\]
Let us now argue that a similar Fubini Theorem argument holds for the random measure
\( \mu^0_t(t, X_{\lambda t}, u_t)(\cdot) \). First, for a given process \( q \in L_2^p(\mathbb{Q}^0; [0, T]) \), consider the following finite variation (FV) process (in fact, under Assumption 2.3, integrable variation (IV) process):

\[
A^0_t \triangleq \int_0^T \int_0^{t \wedge s} q_s \mu^0_s(s, X_{\lambda s}, u_s)(dr) ds, \quad t \in [0, T].
\]  

(7.9)

It is easy to check, as a (randomized) signed measure on \([0, T]\), it holds \( \mathbb{Q}^0 \)-almost surely that
\( dA^0_t = \int_t^T q_s \mu^0(s, X_{\lambda s}, u_s)(dt) ds \). We note that being a “raw FV” process, the process \( A \) is not \( \mathbb{F} \)-adapted. We now consider its dual predictable projection:

\[
p\left( \int_t^T q_s \mu^0(s, X_{\lambda s}, u_s)(dt) ds \right) \triangleq d[pA^0_t], \quad t \in [0, T].
\]  

(7.10)

We remark that \( d[pA_t] \) is a predictable random measure that can be formally understood as
\[
d[pA^0_t] = \mathbb{E}^0[dA^0_t | \mathcal{F}_t] = \mathbb{E}^0 \left[ \int_t^T q_s \mu^0(s, X_{\lambda s}, u_s)(dt) ds \right] | \mathcal{F}_t, \quad t \in [0, T].
\]

Using the definition of dual predictable projection and (7.8), we see that, for the continuous process \( K \in L_2^p(\mathbb{Q}^0; \mathbb{C}_T) \),

\[
\int_0^T \mathbb{E}^0[ q_s D\sigma_s^{\alpha u}(K_{\lambda s}) ] ds = \int_0^T \mathbb{E}^0 \left[ q_s \int_0^s K_r \mu^0_r(s, X_{\lambda s}, u_s)(dr) ds \right] ds
\]

\[
= \mathbb{E}^0 \left[ \int_0^T K_r dA^0_r \right] = \mathbb{E}^0 \left[ \int_0^T K_r d[pA^0_r] \right]
\]

(7.11)

Similarly, we denote \( A^f_t \triangleq \int_0^T \int_0^{t \wedge s} \mu^f_t(s, X_{\lambda s}, u_s)(dr) ds, t \in [0, T] \); and denote its dual predictable projection by \( p\left( \int_t^T \mu^f_t(s, X_{\lambda s}, u_s)(dt) ds \right) = d[pA^f_t], t \in [0, T] \).

We now plug (6.6) and (7.11) into (7.5) to get:

\[
\mathbb{E}^0 \left[ \mathbb{K}_T + \mathbb{E}^0(\Theta R_T) \right]
\]

\[
= \mathbb{E}^0 \left\{ \int_0^T \left\{ K_s \left[ - \alpha_s + L_s^u \mathbb{E}^0 \left[ q_s \partial_y \sigma(s, X_{\lambda s}, y, u_s) \right] \bigg| y = U^u_s + M_s L_s^u \partial_y h(s, X^u_s) \right] \\
+ R_s \left[ - \beta_s + \mathbb{E}^0 \left[ q_s \sigma(s, X_{\lambda s}, y, u_s) \right] \bigg| y = U^u_s + \tilde{M}_s h(s, X^u_s) \right] + q_s C^u_{\alpha v}(v_s - u_s) \right. \\
\left. + R_s \mathbb{E}^0 \left[ q_s \partial_y \sigma(s, X_{\lambda s}, y, u_s) \right] \bigg| y = U^u_s \right) \left( X^u_s - U^u_s \right) \right\} ds + \int_0^T K_s d[pA^0_s] \right\}
\]

(7.12)

\[
+ \mathbb{E}^0 \left\{ \int_0^T [K_s d\Gamma_s + R_s d\Sigma_s] \right\},
\]

\[
= \mathbb{E}^0 \left\{ \int_0^T \left[ - K_s \tilde{\alpha}_s - R_s \tilde{\beta}_s + q_s C^u_{\alpha v}(v_s - u_s) \right] ds + K_s d[pA^0_s] + [K_s d\Gamma_s + R_s d\Sigma_s] \right\},
\]

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where

\[
\begin{align*}
\dot{\alpha}_t & \triangleq \alpha_t - L^u_t \mathbb{E}^0 \left[ q_t \partial_y \sigma(t, X_{\Lambda t}^u, y, u_t) \right] \big|_{y = U_t^u} - \tilde{M}_t L^u_t \partial_x h(t, X_t^u); \\
\dot{\beta}_t & \triangleq \beta_t - \mathbb{E}^0 \left[ q_t \sigma(t, X_{\Lambda t}^u, y, u_t) \right] \big|_{y = U_t^u} - \tilde{M}_t h(t, X_t^u) - \mathbb{E}^0 \left[ q_t \partial_y \sigma(t, X_{\Lambda t}^u, y, u_t) \right] \big|_{y = U_t^u} (X_t^u - U_t^u).
\end{align*}
\] (7.13)

Combining (7.2) and (7.12) and using the processes \( dA^\sigma, dA^f \) and their dual predictable projections, we have

\[
0 \leq \mathbb{E}^0 \left\{ \int_0^T \left[ -K_s \dot{\alpha}_s - R_s \dot{\beta}_s + q_s C^u,v(s)(v_s - u_s) \right] ds + \int_0^T \mathbb{K}_s d[\mathbb{A}^\sigma_s] \right\} (7.14)
\]

\[
+ \mathbb{E}^0 \left\{ \int_0^T \mathbb{E}^0 \left[ f(s, X_{\Lambda s}^u, y, u_s) \right] \big|_{y = U_s^u} + \mathbb{E}^0 \left[ \partial_y f(s, X_{\Lambda s}^u, y, u_s) \right] \big|_{y = U_s^u} (X_s^u - U_s^u) \right. \\
+ L_s^u K_s \mathbb{E}^0 \left[ \partial_y f(s, X_{\Lambda s}^u, y, u_s) \right] \big|_{y = U_s^u} + C^u,v(s)(v_s - u_s) \bigg] ds + \int_0^T \mathbb{K}_s d[\mathbb{A}^f_s] \right\}
\]

\[
+ \mathbb{E}^0 \left\{ \int_0^T \mathbb{K}_s d\Sigma_s + R_s d\Sigma_s \right\},
\]

where \( C^u,v(s) \triangleq \mathbb{E}^0 \left[ L_s^u \partial_z f(s, \varphi, X_{\Lambda s}^u, z) \right] \big|_{\varphi = X_s^u, z = u_s} \). Now, if we set \( \Sigma_t = 0 \), and

\[
\dot{\alpha}_t = L_t^u \mathbb{E}^0 \left[ f(t, X_{\Lambda t}^u, y, u_t) \right] \big|_{y = U_t^u} \quad \dot{\beta}_t = \mathbb{E}^0 \left[ f(t, X_{\Lambda t}^u, y, u_t) \right] \big|_{y = U_t^u} \quad \dot{d\Gamma}_t = -d[\mathbb{A}^\sigma_t] - d[\mathbb{A}^f_t],
\]

then (7.14) becomes

\[
0 \leq \mathbb{E}^0 \left\{ \int_0^T \left[ q_t \alpha^u,v(s) + C^u,v(s)(v_s - u_s) \right] ds, \quad v \in \mathcal{U}_{ad}. \right. (7.16)
\]

From this we should be able to derive the maximum principle, provided that the adjoint equation (7.4) with coefficients \( \alpha, \beta, \) and \( \Gamma \) being determined by (7.13) and (7.15) is well-defined.

**Remark 7.1.** 1) We remark that the process \( \Gamma \) in (7.15) should be considered as a mapping from the space \( L^2_F([0, T] \times \Omega) \times L^2_F([0, T] \times \Omega; U) \) to \( \mathcal{M}_F([0, T]) \), the space of all the random measures on \([0, T]\), such that

(i) \((t, \omega) \mapsto \mu(t, \omega, A)\) is \(\mathbb{F}\)-progressively measurable for all \(A \in \mathcal{B}([0, T])\);

(ii) \(\mu(t, \omega, \cdot) \in \mathcal{M}([0, T])\) is a finite Borel measure on \([0, T]\).
2) Assumption 2.3-(iii) implies that the random measure \( \mathbb{D}_\sigma[q, X^u, u](t, dt) \) satisfies the following estimate: for any \( q \in L^2_{\mathbb{P}}([0, T] \times \Omega) \) and \( u \in \mathcal{U}_{ad} \),
\[
\mathbb{E}^0 \left[ \int_0^T \left| dPA^\sigma_t \right|^2 \right] = \mathbb{E}^0 \left\{ \int_0^T \left| pF \left( \int_0^T q_s \mu_\sigma^0(s, X^u_{\lambda,t}, u_s)(dt)ds \right) \right| \right\} 
\leq \mathbb{E}^0 \left\{ \int_0^T \int_0^T |q_s| |\mu_\sigma^0(s, X^u_{\lambda,t}, u_s)(dt)ds \right\} 
\leq \mathbb{E}^0 \left\{ \int_0^T |q_s| ds \right\} \leq C\|q\|_{2,0}. \tag{7.17}
\]
The same estimate holds for \( \mathbb{D}_f[X^u, u](t, dt) \) as well.

3) Clearly, the processes \( A^\sigma \) and \( A^f \) are originated from the Frechét derivatives of \( \sigma \) and \( f \), respectively, with respect to the path \( \varphi_{\lambda,t} \). If \( \sigma \) and \( f \) are of Markovian type, then they will be absolutely continuous with respect to the Lebesgue measure. \( \blacksquare \)

We shall now validate all the arguments presented above. To begin with, we note that the choice of \( \alpha, \beta, \) and \( \Gamma \) via by (7.13) and (7.15), together with the terminal condition \( (\xi, \Theta) \) by (7.3), amounts to saying that the processes \( (p, q, \tilde{q}) \) and \( (Q, M, \tilde{M}) \) solve the BSDE:
\[
\begin{cases}
dp_t = -L^u_t \left\{ \mathbb{E}^0 \left[ \partial_y f(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} + \mathbb{E}^0 \left[ q_t \partial_y \sigma(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} 
+ M_t \partial_y h(t, X^u_t) \right\} dt - dPA^\sigma_t - dPA^f_t + q_t dB^1_t + \tilde{q}_t dY_t \\
dQ_t = -\left\{ \mathbb{E}^0 \left[ q_t \sigma(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} - \tilde{M}_t h(t, X^u_t) 
+ \mathbb{E}^0 \left[ q_t \partial_y \sigma(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} (X^u_t - U^u_t) 
+ \mathbb{E}^0 \left[ f(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} + \mathbb{E}^0 \left[ \partial_y f(t, X^u_{\lambda,t}, y, u_t) \right]_{y=U^u_t} (X^u_t - U^u_t) \right\} dt 
+ M_t dB^1_t + \tilde{M}_t dY_t,
\end{cases} \tag{7.18}
\]
Now if we denote \( \eta = (p, Q)^T \), \( W = (B^1, Y)^T \), \( \Xi = \begin{bmatrix} q & \tilde{q} \\ M & \tilde{M} \end{bmatrix} \), then we can rewrite (7.18) in a more abstract (vector) form:
\[
\begin{cases}
d\eta_t = -\left\{ A_t + \mathbb{E}^0 [G_t \Xi_t g(t, y)] \right\}_{y=U^u_t} + H_t \Xi_t h_t \right\} dt - \Gamma(\Xi)(t, dt) - \Gamma_0(t, dt) + \Xi_t dW_t, \\
\eta_T = \Upsilon,
\end{cases} \tag{7.19}
\]
where \( \Upsilon \in L^2_{\mathbb{P}}(\Omega; \mathbb{Q}^0) \); \( A, G, H \) and \( h \) are bounded, vector or matrix-valued \( \mathbb{F}^W \)-adapted processes with appropriate dimensions, \( g \) is an \( \mathbb{R}^2 \)-valued progressively measurable random
field, and \(U\) is an \(\mathbb{F}^T\)-adapted process. Moreover, the \(\mathbb{R}^2\)-valued finite variation processes \(\Gamma(\Xi)(t,dt)\) and \(\Gamma_0(t,dt)\) take the form:

\[
\Gamma(\Xi)(t,dt) = \nu\left(\int_t^T \Xi_r \mu^1_r(dt)dr\right), \quad \Gamma_0(t,dt) = \nu\left(\int_t^T \mu^2_r(dt)dr\right),
\]

(7.20)

where \(r \mapsto \mu^i_r(\cdot), i = 1, 2\) are \(\mathcal{M}[0,T]\)-valued measurable random processes satisfying, as measures with respect to the total variation norm,

\[
|\mu^1_r(dt)| + |\mu^2_r(dt)| \leq \ell(r,dt), \quad r \in [0,T], \ Q^0\text{-a.s.} \quad (7.21)
\]

We note that \(\Gamma(\Xi)(dt)\) and \(\Gamma_0(dt)\) are representing \(d[pA^\nu_T]\) and \(d[pA^0_T]\) in (7.18), respectively, and can be substantiated by (7.9) and (7.10). Furthermore, by Assumption 2.3, they both satisfy (7.21). To the best of our knowledge, BSDE (7.19) is beyond all the existing frameworks of BSDEs, and we shall give a brief proof for its well-posedness.

**Theorem 7.2.** Assume that the Assumption 2.3 is in force. Then, the BSDE (7.19) has a unique solution \((\eta, \Xi)\).

**Proof.** The proof is more or less standard, we shall only point out a key estimate. For any \((\eta^1, \Xi^1)\) and \((\eta^2, \Xi^2)\) satisfying (7.19), define \(\bar{\xi} = \xi^1 - \xi^2, \xi = \eta, \Xi\). Noting the linearity of BSDE (7.19) we see that \(\bar{\eta}\) satisfies:

\[
\bar{\eta}_t = \int_t^T \left\{ \mathbb{E}^0\left[ G_s \Xi_s g(s, y) \right] \big| y = U^\nu_s \right\} ds + \int_t^T \Gamma(\Xi)(s, ds) - M^T_t,
\]

(7.22)

where \(M^T_t \triangleq \int_t^T \Xi_s dW_s\). Therefore

\[
|\bar{\eta}_t + M^T_t|^2 \leq 2\left\{ \left| \int_t^T \left\{ \mathbb{E}^0\left[ G_s \Xi_s g(s, y) \right] \big| y = U^\nu_s \right\} ds \right|^2 + \left| \int_t^T \Gamma(\Xi)(s, ds) \right|^2 \right\}.
\]

Taking expectation on both sides above and noting that \(\mathbb{E}^0[\bar{\eta}_t M^T_t] = 0\) and

\[
\mathbb{E}^0\left\{ \left| \int_t^T \left\{ \mathbb{E}^0\left[ G_s \Xi_s g(s, y) \right] \big| y = U^\nu_s \right\} ds \right|^2 \right\} \leq C(T - t)\mathbb{E}^0\left[ \int_t^T |\Xi_s|^2 ds \right],
\]

we have

\[
\mathbb{E}^0[|\bar{\eta}_t|^2] + \mathbb{E}^0\left[ \int_t^T |\Xi_s|^2 ds \right] \leq C(T - t)\mathbb{E}^0\left[ \int_t^T |\Xi_s|^2 ds \right] + \mathbb{E}^0\left\{ \int_t^T \Gamma(\Xi)(s, ds) \right\}^2. \quad (7.23)
\]

To estimate the term involving \(\Gamma(\Xi)\) we note that (recall (7.20)) if a square-integrable process \(V\) is increasing and continuous, then so is its dual predictable projection \(pV\). Thus, by the definition of \(pV\) we have

\[
\mathbb{E}^0\left[ \left| \int_t^T d[pV_s] \right|^2 \right] = 2\mathbb{E}^0\left[ \int_t^T (pV_s - pV_t) d[pV_s] \right] = 2\mathbb{E}^0\left[ \int_t^T (pV_s - pV_t) dV_s \right]
\]

\[
\leq 2\mathbb{E}^0[(pV_T - pV_t) (V_T - V_t)] \leq 2\left( \mathbb{E}^0\left[ \left| \int_t^T d[pV_s] \right|^2 \right] \right)^{1/2} \left( \mathbb{E}^0\left[ \left| \int_t^T dV_s \right|^2 \right] \right)^{1/2}.
\]
That is,
\[ \mathbb{E}^0\left[ \int_t^T d[^p V_s] \right]^2 \leq 4 \mathbb{E}^0\left[ \int_t^T dV_s \right]^2. \] (7.24)

Applying this to \( V_t = \int_0^T \int_t^T \mathbb{E}^1 \langle u \rangle (ds) dr, t \in [0, T] \), we have
\[ \mathbb{E}^0\left[ \int_t^T \Gamma(t, \mathbb{E}^1 \langle u \rangle, ds) \right]^2 \leq \mathbb{E}^0\left[ \int_t^T \left( \int_t^T \mathbb{E}^1 \langle u \rangle (ds) dr \right)^2 \right] \leq 4 \mathbb{E}^0\left[ \int_t^T \mathbb{E}^1 \langle u \rangle (ds) dr \right]^2 \]
\[ \leq 4 \mathbb{E}^0\left[ \int_t^T \mathbb{E}^1 \langle u \rangle (ds) dr \right]^2 \]
\[ \leq C \mathbb{E}^0\left[ \int_t^T |\mathbb{E}^1 \langle u \rangle| dr \right]^2 \leq C(T - t) \mathbb{E}^0\left[ \int_0^T |\mathbb{E}^1 \langle u \rangle| ds \right]. \]
and therefore (7.23) becomes
\[ \mathbb{E}^0[|\eta_t|^2] + \mathbb{E}^0\left[ \int_t^T |\mathbb{E}^1 \langle u \rangle|^2 ds \right] \leq C(T - t) \mathbb{E}^0\left[ \int_t^T |\mathbb{E}^1 \langle u \rangle|^2 ds \right]. \] (7.25)

With this estimate, and following the standard argument one shows that BSDE (7.18) is well-posed on \([T - \delta, T]\) for some (uniform) \( \delta > 0 \). Iterating the argument one can then obtain the well-posedness on \([0, T]\). We leave the details to the interested reader. \( \blacksquare \)

We are now ready to prove the main result of this paper. Let us define the Hamiltonian: for \((\varphi, \mu) \in C_T \times \mathcal{P}(C_T)\), and \((t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R},\)
\[ \mathcal{H}(t, \omega, \varphi, \mu, z; k) = k_t(\omega) \cdot \sigma(t, \varphi, \mu, z) + f(t, \varphi, \mu, z). \] (7.26)

We have the following theorem.

**Theorem 7.3** (Stochastic Maximum Principle). Assume that the Assumptions 2.3 and 3.1 hold. Assume further that the mapping \( z \mapsto \mathcal{H}(t, \varphi, \mu, z) \) is convex. Let \( u = u^* \in \mathcal{U}_{ad} \) be an optimal control and \( X^u \) the corresponding trajectory. Then, for \( dt \times dQ^0\)-a.e. \((t, \omega) \in [0, T] \times \Omega\) it holds that
\[ \mathcal{H}(t, \omega, X_{\varphi, \mu}^u, \mu^u, u_t; q_t) = \inf_{v \in U} \mathcal{H}(t, \omega, X_{\varphi, \mu}^u, \mu^u, v; q_t), \] (7.27)
where \((p, q, \tilde{q})\) and \((Q, M, \tilde{M})\) is the unique solution of the BSDE (7.18).

**Proof.** We first recall from (5.14) that
\[ C^{u}(t) = \mathbb{E}^0 \left[ \int_t^T \partial_z f(t, \varphi, u_t, z) |_{\varphi = X^u, z = u_t} = \partial_z f(t, X^u, \mu^u, u_t); \right. \]
\[ C^{u}(t) = \mathbb{E}^0 \left\{ \int_t^T \partial_z \sigma(t, \varphi, u_t, z) |_{\varphi = X^u, z = u_t} = \partial_z \sigma(t, X^u, \mu^u, u_t). \right. \]

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Then (7.16) implies that
\[
0 \leq \mathbb{E}^0 \left[ \int_0^T \left[ q_t C^u_\sigma(t) + C^v_\sigma(t) \right] (v_t - u_t) dt \right]
\]
(7.28)

\[
= \mathbb{E}^0 \left[ \int_0^T \partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t; q_t)(v_t - u_t) dt \right].
\]

Therefore for \( dt \times dQ^0 \)-a.e. \((t, \omega) \in [0, T] \times \Omega\), and any \( v \in U\), it holds that
\[
\partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t; q_t)(v - u_t) \geq 0.
\]
(7.29)

Now, for any \( v \in U\), one has, \( dt \times dQ^0 \)-a.e. on \([0, T] \times \Omega\),
\[
\mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, v; q_t) - \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t; q_t)
\]
\[
= \int_0^1 \partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t + \lambda(v - u_t); q_t)(v - u_t) d\lambda
\]
\[
= \int_0^1 \left[ \partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t + \lambda(v - u_t); q_t) - \partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t; q_t) \right] (v - u_t) d\lambda
\]
\[
+ \partial_z \mathcal{H}(t, \omega, X_{u,t}, \mu^u_t, u_t; q_t)(v - u_t) \geq 0,
\]

Here the first integral on the right hand side above is nonnegative due to the convexity of \( \mathcal{H} \) in variable \( z \), and the last term is non-negative because of (7.29). The identity (7.27) now follows immediately.

\[ \Box \]

**Remark 7.4.** In stochastic control literature the inequality (7.28) is sometimes referred to as **Stochastic Maximum Principle in integral form**, which in many applications is useful, as it does not require the convexity assumption on the Hamiltonian \( \mathcal{H} \).

\[ \Box \]

### References


