Problem 11

1. \[ E[X] = A \cdot P(\text{pick 1st envelope}) + B \cdot P(\text{pick 2nd envelope}) \]
   \[ P(\text{pick 1st envelope}) = P(\text{pick 2nd envelope}) = \frac{1}{2} \]
   \[ \therefore E[X] = \frac{1}{2}(A+B) \]

I think there does exist a better strategy. The reason is that the person has the right of selection \( \xrightarrow{\text{(keep or exchange)}} \) and he can get the whole information of one check. These two things have big values. However, the first strategy doesn't consider them.

2. \[ E[X] = P(\text{pick 1st}) \cdot \left[ A \cdot P(\text{accept} | \text{pick 1st}) + B \cdot P(\text{reject} | \text{pick 1st}) \right] \]
   \[ + P(\text{pick 2nd}) \cdot \left[ A \cdot P(\text{reject} | \text{pick 2nd}) + B \cdot P(\text{accept} | \text{pick 2nd}) \right] \]
   \[ = \frac{1}{2} \left[ A \cdot (1 - \frac{1}{A}) + B \cdot \frac{1}{A} \right] + \frac{1}{2} \left[ A \cdot \frac{1}{B} + B \cdot (1 - \frac{1}{B}) \right] \]
   \[ = \frac{A+B}{2} + \frac{(A-B)^2}{2AB} \]

Since \((A-B)^2 \geq 0, A > 0, B > 0\), so that \( E[X]_{2\text{nd strategy}} \geq E[X]_{1\text{st strategy}} \)

equal only if \( A = B \).

3. \[ P(\text{get } B) = P(\text{pick } A) \cdot P(\text{reject} | \text{pick } A) + P(\text{pick } B) \cdot P(\text{accept} | \text{pick } B) \]
   \[ = \frac{1}{2} + \frac{1}{2A} - \frac{1}{2B} \]

Since \( B > A \)

so \( P(\text{get } B) > \frac{1}{2} \).
Problem 12.

\[ P(\text{have a gun}) = 0.4, \quad P(\text{no guns}) = 0.6, \]

\[ P(\text{die | no guns}) = 0.1, \quad P(\text{survive | no guns}) = 0.9 \]

\[ P(\text{die | have a gun}) = 0.8, \quad P(\text{survive | have a gun}) = 0.2 \]

\[ P(\text{have a gun | survive}) = \frac{P(\text{survive | have a gun}) P(\text{have a gun})}{P(\text{survive})} \]

\[ P(\text{survive}) = P(\text{survive | no guns}) P(\text{no guns}) + P(\text{survive | have a gun}) P(\text{have a gun}) \]

\[ \therefore P(\text{have a gun | survive}) = \frac{0.2 \times 0.4}{0.9 \times 0.6 + 0.2 \times 0.4} \approx 0.129 \]

Problem 13.

(1) \[ E[Y|N=n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = nE[X] = 6n \]

(2) \[ E[Y] = E[CN]E[X] = 6 \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} k = 6\lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = 6\lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \]

\[ = 6\lambda e^{-\lambda} \cdot e^\lambda = 6\lambda = 600 \text{.} \]

(3) \[ E[Y^2|N=n] = E[(\sum_{i=1}^{n} X_i)(\sum_{j=1}^{n} X_j)] \overset{\text{independence and identical}}{=} nE[X^2] + (n^2-n)E[X]^2 \]

\[ E[Y^2] = E[CN]Var(X) + E[N^2]E[X]^2 \]


(4) Let \( Y_i \) be the amount collected on the \( i \)th charity.

\[ P\left( \sum_{i=1}^{50} X_i \leq 2600 \right) = P\left( \frac{\sum_{i=1}^{50} X_i - 3000}{40\sqrt{50}} \leq \frac{-400}{40\sqrt{50}} \right) \leq P(Z \leq -12) \approx 0.001 \]
Problem 14.

1. \[ \mathbb{E}[X_k] = \mathbb{E}[I_k] + \mathbb{E}[I_{k+1}] = \frac{1}{2} + \frac{1}{2} = 1 \]
   \[ \mathbb{E}[X_k^2] = \mathbb{E}[I_k^2 + 2I_kI_{k+1} + I_{k+1}^2] = \mathbb{E}[I_k^2] + 2\mathbb{E}[I_k] \mathbb{E}[I_{k+1}] + \mathbb{E}[I_{k+1}^2] \]
   \[ = \frac{1}{2} + 2 \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} = \frac{3}{2} \]
   \[ \text{Var}(X_k) = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2 = \frac{1}{2} \]

2. \[ \text{Cov}(X_k, X_{k+1}) = \mathbb{E}[X_k X_{k+1}] - \mathbb{E}[X_k] \mathbb{E}[X_{k+1}] \]
   \[ = \mathbb{E}[I_k I_{k+1} + I_k^2 + I_k + I_{k+1} + I_{k+1}^2] - 1 \times 1 \]
   \[ = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} - 1 \]
   \[ = \frac{1}{4} \]
   .. \[ X_k \text{ and } X_{k+1} \text{ are not independent.} \]

3. \[ \mathbb{E} \left| Y_n \right| = \sum_{k=1}^{n} \mathbb{E}[X_k] = n \mathbb{E}[X_k] = n \]

4. Since \[ X_k \text{ and } X_{k'} \text{ are independent if } |k-k'| \geq 2 \]
   [they depend on different flips], we have
   \[ \text{Var}(Y_n) = \sum_{k=1}^{n} \text{Var}(X_k) + 2 \sum_{k=1}^{n-1} \text{Cov}(X_k, X_{k+1}) \]
   \[ = n \cdot \frac{1}{2} + 2(n-1) \cdot \frac{1}{4} \]
   \[ \text{Var}(Y_n) = n - \frac{1}{2} \]
(4) Chebichev's inequality: \( P(|Y_n - \mu| \geq \epsilon) \leq \frac{\text{Var}(Y_n)}{\epsilon^2} \) \\
\( P(|Y_n - n| \geq b) \leq \frac{(n-\frac{1}{2})}{b^2} \)

Assume \( b = \epsilon n \), where \( \epsilon \) is an arbitrary positive number.

Then, 
\( P(|Y_n - n| \geq \epsilon n) = P\left(\left|\frac{Y_n}{n} - \frac{n-\frac{1}{2}}{n}\right| \geq \epsilon\right) \leq \frac{(n-\frac{1}{2})}{\epsilon^2 n^2} \)

when \( n \to +\infty \), the above inequality becomes

\( P\left(\left|\frac{Y_n}{n} - 1\right| \geq \epsilon\right) \to 0 \), \( \forall \epsilon > 0 \)

Therefore, \( \frac{Y_n}{n} \) converges in probability to 1.

(5) We cannot apply CLT to \( Y_n \), because \( X_k \) and \( X_{k+1} \) are not independent.

**Problem 15.**

\( \mathbb{E} \) Denote: 
\( X_i \): the result of \( i \)th bet

\( Y_n \): total amount the gambler wins.

So \( Y_n = \sum_{i=1}^{n} X_i \).

\( \mu = \mathbb{E}[X_i] = -1 \times 0.7 + (-2) \times 0.2 + 10 \times 0.1 = -0.1 \)

\( \sigma^2 = \text{Var}(X_i) = 0.7 \times (-1+0.1)^2 + 0.2 \times (-2+0.1)^2 + 0.1 \times (10+0.1)^2 \)

\( = 11.49 \)

\( \sigma \approx 3.39 \).

\[ P(Y_{100} \leq 0) = P\left(\frac{Y_{100} - (-0.1) \times 100}{3.39 \times 10} \leq \frac{0 - (-0.1) \times 100}{3.39 \times 10}\right) \]

\[ = P(Z \leq 0.295) \approx 0.614. \]