Stochastic Games with Delay: a Toy Model

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Networking
Fish Schooling
Literature Review

- **Mean Field Games:**
  - R. Carmona, F. Delarue [2018].
  - C. Wu and J. Zhang [2018].
  - etc.

- **Control Problems with Delay:**
  - Y. Alekal, P. Brunovsky, DH. Chyung, and EB. Lee [1971].
  - RB Vinter and RH Kwong [1981].
  - F. Gozzi and C. Marinelli [2004].
  - S. Peng and Z. Yang [2009].
  - Y. Saporito and J. Zhang [2018].
  - etc.

- **Stochastic Games with Delay:**
  - J.-P. Fouque, Z. Zhang [2018].
  - etc.
Introduction: Linear Quadratic Stochastic Games
Linear Quadratic Game

Bank \( i \) for \( i = 1, \cdots, N \) is borrowing from and lending to a central bank:

\[
dX^i_t = \alpha^i_t \, dt + \sigma \, dW^i_t, \quad X^i_0 = \xi^i.
\]

where

- \( X^i_t \) represents the log-monetary reserves of the \( i \)th bank,
- \( W^i_t \) are independent standard Brownian motions,
- \( \sigma > 0 \), the diffusion coefficients are constant and identical,
- Bank \( i \) controls its rate of borrowing (\( \alpha^i_t > 0 \))/lending (\( \alpha^i_t < 0 \)) to a central bank through the control \( \alpha^i_t \).
The Cost Functional

Bank $i$ wants to minimize

$$J^i(\alpha) = \mathbb{E} \left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\},$$

with running cost

$$f_i(x, \alpha^i) = \frac{1}{2} (\alpha^i)^2 + \frac{\epsilon}{2} (x - x^i)^2,$$

and terminal cost

$$g_i(x) = \frac{c}{2} (x - x^i)^2.$$

Value function

$$V^i(t, x) = \inf_{\alpha} J^i(\alpha).$$
Solving for an Exact Nash Equilibrium

Definition
A set of admissible strategy profiles $\hat{\alpha} = (\hat{\alpha}^1, \cdots, \hat{\alpha}^N) \in \mathbb{A}^N$ is said to be a Nash equilibrium for the game if:

$$\forall i \in \{1, \cdots, N\}, \forall \alpha^i \in \mathbb{A}^i, \quad J^i(\hat{\alpha}) \leq J^i(\alpha^i, \hat{\alpha}^{-i}),$$

where $(\alpha^i, \hat{\alpha}^{-i})$ stands for the strategy profile $(\hat{\alpha}^1, \cdots, \hat{\alpha}^{i-1}, \alpha^i, \hat{\alpha}^{i+1})$, in which the player $i$ chooses the strategy $\alpha^i$ while the others keep the original ones $\hat{\alpha}^j$.

- **Probabilistic Approach** ($N$-coupled Forward-Backward SDEs)
- **PDE Approach** ($N$-coupled Hamilton-Jacobi-Bellman (HJB) PDEs)
- This is an example of Mean Field Game (MFG) studied extensively by P.L. Lions and collaborators, R. Carmona and F. Delarue, ...
Stochastic Game/Mean Field Game with Delay
Stochastic Game with Delay

Banks are borrowing from and lending to a central bank and money is returned at maturity $\tau$:

$$dX_t^i = \left[\alpha_t^i - \alpha_{t-\tau}^i\right] dt + \sigma dW_t^i, \quad i = 1, \ldots, N$$

where $\alpha^i$ is the control of bank $i$ which wants to minimize

$$J^i(\alpha) = \mathbb{E}\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T)\right\},$$

$$f_i(x, \alpha^i) = \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(x - x_i)^2,$$

$$g_i(x) = \frac{c}{2} (x - x_i)^2,$$

$$X_0^i = \xi_i, \quad \alpha_0^i = 0, \quad t \in [-\tau, 0).$$

Case $\tau = 0$: no lending/borrowing $\rightarrow$ no liquidity.

Case $\tau = T$: no return/delay $\rightarrow$ full liquidity.
Mean Field Game with Delay

- Mean field game theory is the study of strategic decision making in very large populations of small interacting agents, i.e., a game with infinite many indistinguishable players.

- All players are rational, i.e., each player tries to minimize their cost against the mass of other players.

- The running cost and terminal cost only depend on $i$th player’s state $x^i$ and the empirical distribution of $(x^j)_{j \neq i}$.

- As $N \to \infty$, denote $m_t = \int_R x d\mu_t(x)$,

  \[
  f(X_t, \mu_t, \alpha_t) = \frac{1}{2} (\alpha_t)^2 + \frac{\epsilon}{2} (m_t - X_t)^2,
  \]

  \[
  g(X_T, \mu_T) = \frac{c}{2} (m_T - X_T)^2.
  \]
Probabilistic Approach
Forward-Advanced-Backward SDEs

**Theorem.** The strategy \( \hat{\alpha} \) given by

\[
\hat{\alpha}_t = -Y_t + E^{F_t}(Y_{t+\tau})
\]

is a open-loop Nash equilibrium where \((X, Y, Z)\) is the unique solution to the following system of FABSDEs:

\[
\begin{align*}
X_t &= \xi + \int_0^t (\hat{\alpha}_s - \hat{\alpha}_{s-\tau}) \, ds + \sigma W_t, \quad t \in [0, T], \\
Y_t &= c (X_T - m_T) + \int_t^T \epsilon (X_s - m_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T], \\
Y_t &= 0, \quad t \in (T, T + \tau],
\end{align*}
\]

where the processes \( Z_t \) are adapted and square integrable, and \( E^{F_t} \) denotes the conditional expectation with respect to the filtration generated by the Brownian motions.
Outline of the Proof

Proof.
Let $\alpha' \in A$ be a generic admissible control, and $X' = X^{\alpha'}$ the corresponding controlled state.

$$J(\hat{\alpha}) - J(\alpha') = E\left\{ \int_0^T (f(X_t, \mu_t, \hat{\alpha}_t) - f(X'_t, \mu'_t, \alpha'_t)) dt \right. \\
\left. + g(X_T, \mu_T) - g(X'_T, \mu'_T) \right\}.$$ 

Since $g$ is L-convex in $(x, \mu)$,

$$E(g(X_T, \mu_T) - g(X'_T, \mu'_T)) \leq E[(\partial_x g(X_T, \mu_T) + \tilde{E}[\partial_{\mu} g(\tilde{X}_T, \mu_T)(X_T)]) \cdot (X_T - X'_T)]$$

$$= E[Y_T(X_T - X'_T)].$$
Outline of the Proof

Applying Itô’s formula, we have

\[
E( Y_T (X_T - X'_T))
\]

\[
= E \left[ \int_0^T (X_t - X'_t) dY_t + \int_0^T Y_t d(X_t - X'_t) \right]
\]

\[
= E \int_0^T \left\{ -\epsilon(X_t - m_t)(X_t - X'_t) + Y_t \left( \hat{\alpha}_t - \alpha'_t - (\hat{\alpha}_{t-\tau} - \alpha'_{t-\tau}) \right) \right\} dt
\]

\[
= E \int_0^T \left\{ -\epsilon(X_t - m_t)(X_t - X'_t) + (Y_t - E^{\mathcal{F}_t}(Y_{t+\tau})) \left( \hat{\alpha}_t - \alpha'_t \right) \right\} dt.
\]

due to the change of time,

\[
E \int_0^T Y_t (\hat{\alpha}_{t-\tau} - \alpha'_{t-\tau}) dt = E \int_{-\tau}^{T-\tau} Y_{s+\tau} (\hat{\alpha}_s - \alpha'_s) ds
\]

\[
= E \int_0^T Y_{s+\tau} (\hat{\alpha}_s - \alpha'_s) ds = E \int_0^T E^{\mathcal{F}_s}(Y_{s+\tau}) (\hat{\alpha}_s - \alpha'_s) ds.
\]

since \(\hat{\alpha}_t = \alpha'_t = 0\) for \(t \in [-\tau, 0)\) and \(Y_t = 0\) for \(t \in (T, T + \tau]\).
Outline of the Proof

Convexity of $f$ in $(x, \mu, \alpha)$, we deduce

$$J(\hat{\alpha}) - J(\alpha') \leq E \int_0^T \left[ \left( \partial_x f(X_t, \mu_t, \hat{\alpha}_t) + \tilde{E}[\partial_\mu f(\tilde{X}_t, \mu_t, \hat{\alpha}_t)(X_t) \right)(X_t - X'_t) \right]$$

$$+ \partial_\alpha f(X_t, \hat{\alpha}_t)(\hat{\alpha}_t - \alpha'_t) \right] dt + E[Y_T(X_T - X'_T)]$$

$$= E \int_0^T \left\{ \epsilon(X_t - m_t)(X_t - X'_t) + (\partial_\alpha f(X_t, \mu_t, \hat{\alpha})) (\hat{\alpha}_t - \alpha'_t) \right\} dt$$

$$+ E \int_0^T \left\{ - \epsilon(X_t - X'_t)(X_t - m_t) + (Y_t - E^{\mathcal{F}_t}[Y_{t+\tau}]) (\hat{\alpha}_t - \alpha'_t) \right\} dt.$$  

$$= E \int_0^T \left( \hat{\alpha}_t - \alpha'_t \right) \times \left[ \hat{\alpha}_t + (Y_t - E^{\mathcal{F}_t}(Y_{t+\tau})) \right] dt$$

$$= 0$$
Theorem. The strategy $\hat{\alpha}$ given by

$$\hat{\alpha}_t = -Y_t + E^{\mathcal{F}_t}(Y_{t+\tau})$$

is a open-loop Nash equilibrium where $(X, Y, Z)$ is the unique solution to the following system of FABSDEs:

$$X_t = \xi + \int_0^t (\hat{\alpha}_s - \hat{\alpha}_{s-\tau}) \, ds + \sigma W_t, \quad t \in [0, T],$$

$$Y_t = c(X_T - m_T) + \int_t^T \epsilon(X_s - m_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T],$$

$$Y_t = 0, \quad t \in (T, T + \tau],$$

where the processes $Z_t$ are adapted and square integrable, and $E^{\mathcal{F}_t}$ denotes the conditional expectation with respect to the filtration generated by the Brownian motions.
Existence, no Uniqueness

No simple explicit formula for the optimal strategy $\hat{\alpha}$. 
Recurrent Neural Network

Long-Short Term Memory module: LSTM

\[ Y_t \approx \phi(t, (W_s)_{0 \leq s \leq t} | \Theta_t) \]

\[ \mathbb{E}[Y_{t+\tau} | \mathcal{F}_t] \approx \psi(t, (W_s)_{0 \leq s \leq t} | \Lambda_t) \]

\[ Z_t \approx \chi(t, (W_s)_{0 \leq s \leq t} | \Gamma_t) \]

\[ f_t = \sigma_g(W_f x_t + U_f h_{t-1} + b_f) \]

\[ i_t = \sigma_g(W_i x_t + U_i h_{t-1} + b_i) \]

\[ o_t = \sigma_g(W_o x_t + U_o h_{t-1} + b_o) \]

\[ c_t = f_t \circ c_{t-1} + i_t \circ \sigma_c(W_c x_t + U_c h_{t-1} + b_c) \]

\[ h_t = o_t \circ \sigma_h(c_t) \]
Algorithm

- **Time discretization.** $h = T/N$, $D = \tau/h$.
  $$-	au = t_{-D} \leq \cdots \leq t_{-1} \leq t_0 = 0 = t_0 \leq t_1 \leq \cdots \leq t_N \leq T.$$  
- **Initial states.** $X_0 = 0$, $Y_0 \approx \phi(0, W_0|\Theta_0)$, $E[Y_{\tau}|F_0] \approx \psi(0, W_0|\Lambda_0)$, $Z_0 \approx \chi(0, W_0|\Gamma_0)$. $\alpha_0 = -Y_0 + E[Y_{\tau}|F_0]$.
- **Euler–Maruyama method.**
  $$X_{t_{k+1}} = X_{t_k} + (\alpha_{t_k} - \alpha_{t_{k-D}}) \cdot h + \sigma \Delta W_{t_{k+1}}, \text{ where } \Delta W_{t_{k+1}} \sim N(0, h).$$
  $$\tilde{Y}_{t_{k+1}} = Y_{t_k} - \epsilon X_{t_k} \cdot h + Z_{t_k} \Delta W_{t_{k+1}}.$$  
  $$Y_{t_{k+1}} \approx \phi(t_{k+1}, (W_s)_{0 \leq s \leq t_{k+1}}|\Theta_{t_{k+1}}).$$
- **Loss**
  $$\text{Loss} = \sum_{m=1}^{M} \sum_{k=1}^{N} (Y_{t_k}^m - \tilde{Y}_{t_k}^m)^2 + \sum_{m=1}^{M} (Y_{t_N}^m - cX_{T_N}^m)^2 + \sum_{m=1}^{M} \sum_{k=0}^{N-D} (Y_{t_{k+D}}^m - E[Y_{t_{k+D}}^m|F_{t_k}])^2.$$  
- **Apply stochastic gradient descent (SGD) to minimize loss and update parameters.**
Results

\[ T = 10, \tau = 1, h = 0.1, \epsilon = 1, c = 1, M = 2560. \]
Results

$$T = 10, \tau = 1, h = 0.1, \epsilon = 1, c = 1, M = 2560.$$
PDE Approach
Infinite-dimensional HJB Approach

- Denote $\mathbf{H} := L^2([-\tau, 0]; \mathbb{R})$.
- Given $z := (z_0, z_1) \in \mathbb{R} \times \mathcal{H}$, where $z_0 \in \mathbb{R}$, and $z_1 \in \mathcal{H}$. The inner product on $\mathbb{R} \times \mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$, and it is defined by

$$\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^{0} z_1(s) \tilde{z}_1(s) ds.$$ 

- Therefore, the new state is denoted by $Z_t = (Z_{0,t}, Z_{1,t}(s)), s \in [-\tau, 0]$, which corresponds to $(X_t, \alpha_{t-\tau-s})$, i.e., the states and the past of the strategies in our case.
Infinite-dimensional HJB Approach

In order to use the **dynamic programming principle** for stochastic game in search of a **closed-loop Nash equilibrium**, at time $t \in [0, T]$, given the initial state $Z_0 = z$, one representative bank chooses the control $\alpha$ to minimise its objective function $J(t, z, \alpha)$.

$$J(t, z, \alpha) = \mathbb{E}\left\{ \int_t^T f(Z_{0,s}, \mu_{0,s}, \alpha_s)dt + g(Z_{0,T}, \mu_{0,T}) \mid Z_t = z \right\},$$

The value function $V(t, z)$ is

$$V(t, z) = \inf_{\alpha} J(t, z, \alpha).$$

subject to

$$dZ_t = (AZ_t + B\alpha_t)dt + GdW_t.$$
Coupled HJB Equations

The value functions $V(t, z)$ is the unique solution (in a suitable sense) of the following HJB equations:

$$
\partial_t V + \frac{1}{2} \text{Tr}(Q \partial_{zz} V) + \langle Az, \partial_z V \rangle + H_0(\partial_z V) = 0,
$$

$$
V(T) = g(Z_0, T, \mu_0, T),
$$

where

$$
Q = G \ast G, \quad G : z_0 \rightarrow (\sigma z_0, 0),
$$

$$
A : (z_0, z_1(\gamma)) \rightarrow (z_1(0), -\frac{dz_1(\gamma)}{d\gamma}) \quad \text{a.e.,} \quad \gamma \in [-\tau, 0],
$$

$$
H_0(p) = \inf_{\alpha} [\langle B\alpha, p \rangle + f(z_0, \alpha)], \quad p \in \mathbb{R} \times H,
$$

$$
B : u \rightarrow (u, -\delta_{-\tau}(\gamma)u), \quad \gamma \in [-\tau, 0].
$$
Forward Kolmogorov Equation

Next, since we “lift” the original non-Markovian optimization problem into an infinite dimensional Markovian control problem. We are able to write the corresponding generator, which is denoted by $(L_t)_{t \in [0, \tau]}$,

$$L\varphi(t, z) = \langle (AZ + B\hat{\alpha}), \partial_z \varphi \rangle + \frac{1}{2} Tr(G^* G \partial_{zz} \varphi).$$

Forward Kolmogorov Equation

$$\partial_t \nu = \int_{-\tau}^0 \partial_{z_1} \left( \frac{d}{ds} z_1 \nu \right) ds - \int_{-\tau}^0 \partial_{z_1} (z_1 \nu)(\delta_0(s) - \delta_{-\tau}(s))ds$$

$$+ \partial_{z_0} \{(\partial_{z_0} V - [\partial_{z_1} V](-\tau))\nu\}$$

$$- \int_{-\tau}^0 \partial_{z_1} \{(\partial_{z_0} V - [\partial_{z_1} V](-\tau))\nu\} \delta_{-\tau}(s)ds + \frac{1}{2} \sigma^2 \partial_{z_0z_0} \nu,$$

$$\nu_0 = P(\xi, \phi(s)_{s \in [-\tau, 0]}).$$
Derivative in $P(H)$

### Definition
We say that $F : P(H) \to H$ is $C^1$ if there exists an operator $\frac{\delta F}{\delta \nu} : P(H) \times H \to H$ such that for any $\mu_1$ and $\mu'_1 \in P(H)$

$$\lim_{\epsilon \to 0^+} \frac{F(\mu_1 + \epsilon(\mu'_1 - \mu_1)) - F(\mu_1)}{\epsilon} = \int_H \frac{\delta F}{\delta \mu_1}(\mu_1, y_1) d(\mu'_1 - \mu_1)(y_1).$$

### Definition
If $\frac{\delta F}{\delta \mu_1}(\mu_1, y_1)$ is of class $C^1$ with respect to $y_1$, the marginal derivative $D_{\mu_1} F : P(H) \times H \to H$ is defined in the sense of Fréchet derivative:

$$D_{\mu_1} F(\mu_1, y_1) := D_{y_1} \frac{\delta F}{\delta \mu_1}(\mu_1, y_1).$$
Remark

Usually we will encounter a map $U : P(H) \rightarrow \mathbb{R}$. In this case, $U$ can be expressed in a form of composition $\tilde{U} \circ F$, where $\tilde{U} : H \rightarrow \mathbb{R}$, and $F : P(H) \rightarrow H$, i.e., $U = (\tilde{U} \circ F)(\mu_1)$.

If $\frac{\delta F}{\delta \mu_1}$ is $C^1$ with respect to $y_1$, and $\tilde{U}$ is Fréchet differentiable, then

$\frac{\delta U}{\delta \mu_1} : P(H) \times H \rightarrow H$, and $D_{\mu_1} U : P(H) \times H \rightarrow H$ are defined by

$$\frac{\delta U}{\delta \mu_1}(\mu_1, y_1) := (D_F \tilde{U}) \left( \frac{\delta F}{\delta \mu_1} \right), \text{ and } D_{\mu_1} U(\mu_1, y_1) := \left( D_F \tilde{U} \right) \left( D_{\mu_1} F \right).$$
The Master Equation

For any \((t_0, \nu_0) \in [0, T] \times \mathcal{P}(\mathbb{R} \times \mathbb{H})\), we define

\[
U(t_0, \cdot, \nu_0) := V(t_0, \cdot),
\]

where \((V, \nu)\) is a classical solution to the system of forward-backward equations. Then \(U\) must satisfy the following master equation

\[
\begin{align*}
\partial_t U(t, z_0, z_1, \nu) + & \frac{1}{2} \sigma^2 \partial_{z_0 z_0} U(t, z_0, z_1, \nu) + \frac{1}{2} \sigma^2 \int_{\mathbb{R}} \partial_{y_0} D_{\mu_0} U(t, z_0, z_1, \nu, y_0) d\mu_0(y_0) \\
+ & \int_{-\tau}^{0} z_1 \frac{d}{ds} \partial_{z_1} U(t, z_0, z_1, \nu) ds + \int_{-\tau}^{0} \int_{\mathbb{H}} y_1 \frac{d}{ds} [D_{\mu_1} U(t, z_0, z_1, \nu, y_1)](s) d\mu_1(y_1) ds \\
- & \int_{\mathbb{R} \times \mathbb{H}} [\partial_{y_0} U(t, y_0, y_1, \nu) - [\partial_{y_1} U(t, y_0, y_1, \nu)](-\tau)] D_{\mu_0} U(t, z_0, z_1, \nu, y_0) d\nu(y) \\
+ & \int_{\mathbb{R} \times \mathbb{H}} [\partial_{y_0} U(t, y_0, y_1, \nu) - [\partial_{y_1} U(t, y_0, y_1, \nu)](-\tau)] [D_{\mu_1} U(t, z_0, z_1, \nu, y_1)](-\tau) d\nu(y) \\
- & \frac{1}{2} (\partial_{z_0} U(t, z_0, z_1, \nu) - [\partial_{z_1} U(t, z_0, z_1, \nu)](-\tau))^2 + \frac{\epsilon}{2} \left( \int_{\mathbb{R}} y_0 d\mu_0(y_0) - z_0 \right)^2 = 0,
\end{align*}
\]

where \(\mu_0\) and \(\mu_1\) are the marginal law for \(Z_0\) and \(Z_1\) respectively.
Explicit Solution of the Master Equation with Delay

It turns out that this master equation can be solved explicitly by making the following ansatz.

We define $m_0 := \int_{\mathcal{R}} y_0 d\mu_0(y_0)$ and $m_1 := \int_{\mathcal{H}} y_1 d\mu_1(y_1)$ for convenience, then

$$U(t, z_0, z_1, \nu) = E_0(t)(m_0 - z_0)^2 - 2(m_0 - z_0) \int_{-\tau}^{0} E_1(t, -\tau - s)(m_1 - z_1)ds$$

$$+ \int_{-\tau}^{0} \int_{-\tau}^{0} E_2(t, -\tau - s, -\tau - r)(m_1 - z_1)(m_1 - z_1)dsdr + E_3(t).$$
Explicit Solution of the Master Equation with Delay

We compute the partial derivatives needed in the master equation explicitly, we have

\[
\partial_t U = \frac{dE_0(t)}{dt}(m_0 - z_0)^2 - 2(m_0 - z_0) \int_{-\tau}^{0} \frac{\partial E_1(t, -\tau - s)}{\partial t}(m_1 - z_1)ds \\
+ \int_{-\tau}^{0} \int_{-\tau}^{0} \frac{\partial E_2(t, -\tau - s, -\tau - r)}{\partial t} (m_1 - z_1)(m_1 - z_1)dsdr + \frac{dE_3(t)}{dt},
\]

\[
\partial z_0 U = -2E_0(t)(m_0 - z_0) + 2 \int_{-\tau}^{0} E_1(t, -\tau - s)(m_1 - z_1)ds,
\]

\[
\partial z_1 U = 2E_1(t, -\tau - s)(m_0 - z_0) - 2 \int_{-\tau}^{0} E_2(t, -\tau - s, -\tau - r)(m_1 - z_1)dr,
\]

\[
D_{\mu_0} U = 2E_0(t)(m_0 - z_0) - 2 \int_{-\tau}^{0} E_1(t, -\tau - s)(m_1 - z_1)ds,
\]

\[
D_{\mu_1} U = -2E_1(t, -\tau - s)(m_0 - z_0) + 2 \int_{-\tau}^{0} E_2(t, -\tau - s, -\tau - r)(m_1 - z_1)dr,
\]

\[
\partial z_0z_0 U = 2E_0(t),
\]
Collecting \((m_0 - z_0)^2\) terms, \((m_0 - z_0)(m_1 - z_1)\) terms, \((m_1 - z_1)^2\) terms, and constant terms, we obtain that the function \(E_i, i = 0, \cdots, 3\), satisfy the system of PDEs:

\[
\frac{dE_0(t)}{dt} - 2(E_0(t) + E_1(t, 0))^2 + \frac{\epsilon}{2} = 0,
\]

\[
\frac{\partial E_1(t, s)}{\partial t} - \frac{\partial E_1(t, s)}{\partial s} - 2(E_0(t) + E_1(t, 0))(E_1(t, s) + E_2(t, 0, r)) = 0,
\]

\[
\frac{\partial E_2(t, s, r)}{\partial t} - \frac{\partial E_2(t, s, r)}{\partial s} - \frac{\partial E_2(t, s, r)}{\partial r} - 2(E_1(t, s) + E_2(t, s, 0))(E_1(t, r) + E_2(t, r, 0)) = 0,
\]

\[
\frac{dE_3(t)}{dt} + E_0(t)\sigma^2 = 0,
\]

with boundary conditions

\[
E_0(T) = \frac{c}{2}, \quad E_1(T, s) = 0, \quad E_2(T, s, r) = 0, \quad E_2(t, s, r) = E_2(t, r, s),
\]

\[
E_1(t, -\tau) = -E_0(t), \quad E_2(t, s, -\tau) = -E_1(t, s), \quad E_3(T) = 0.
\]
Finite Dimensional Projection

Set \( u^i(t, z_0, z_1) := U(t, z_0^i, z_1^i, \nu^i) \), where \( \nu^i = \frac{1}{N-1} \sum_{k \neq i} \delta_{(z_0^k, z_1^k)} \), denotes the joint empirical measure of \( z_0 \) and \( z_1 \). The empirical measure of \( z_0 \) is given by \( \mu_0^i = \frac{1}{N-1} \sum_{k \neq i} \delta_{z_0^k} \), and the empirical measure of \( z_1 \) is given by \( \mu_1^i = \frac{1}{N-1} \sum_{k \neq i} \delta_{z_1^k} \).

By direct computation, for \( k \neq i \), and any \( N \geq 2 \),

\[
\partial_{z_0^k} u^i(t, z_0, z_1) = \frac{1}{N-1} D_{\mu_0^i} U(t, z_0^i, z_1^i, \nu^i, z_0^k),
\]

\[
\partial_{z_1^k} u^i(t, z_0, z_1) = \frac{1}{N-1} D_{\mu_1^i} U(t, z_0^i, z_1^i, \nu^i, z_1^k),
\]

\[
\partial_{z_0^k z_0^k} u^i(t, z_0, z_1) = \frac{1}{N-1} \partial_{z_0^k} [D_{\mu_0^i} U](t, z_0^i, z_1^i, \nu^i, z_0^k)
\]
\[
+ \frac{1}{(N-1)^2} D_{\mu_0^i \mu_0^i} U(t, z_0^i, z_1^i, \nu^i, z_0^k, z_0^k).
\]
Convergence of the Nash System

Proposition

For any \( i \in \{1, \cdots, N\} \), \( u^i(t, z_0, z_1) \) satisfies

\[
\partial_t u^i + \sum_{k=1}^{N} \frac{1}{2} \sigma^2 \partial z^k_0 \partial z^i_0 u^i + \sum_{k=1}^{N} \int_{-\tau}^{0} z^k_1 \frac{d}{ds} (\partial z^i_1 u^i) ds \\
- \sum_{k \neq i}^{N} \left( \partial z^k_0 u^k - [\partial z^k_1 u^k](-\tau) \right) \left( \partial z^i_0 u^i - [\partial z^i_1 u^i](-\tau) \right) \\
- \frac{1}{2} \left( \partial z^i_0 u^i - [\partial z^i_1 u^i](-\tau) \right)^2 + \frac{\epsilon}{2} (\bar{z}_0 - z^i_0)^2 + e^i(t, z) = 0,
\]

where \( \|e^i(t, z)\| < \frac{C}{N} \), with terminal condition \( u^i(T, z) = \frac{\epsilon}{2} (\bar{z}_0 - z^i_0)^2 \).

This shows that \( (u^i)_{i \in \{1, \cdots, N\}} \) is "almost" a solution to the Nash system.
Convergence of the Nash System

Let $V^i$ be the solution to the HJB equation of the $N$-player system, where $N \geq 1$ fixed, and $U$ be the solution to the master equation. Fix any $(t_0, \nu_0) \in [0, T] \times P(\mathbb{R} \times H)$. For any $z \in \mathbb{R}^N \times H^N$, let $\nu^i = \frac{1}{N-1} \sum_{j \neq i} \delta(z^i_0, z^i_1)$, then we have

$$\frac{1}{N} \sum_{i=1}^{N} |V^i(t_0, z) - U(t_0, z^i, \nu^i)| \leq CN^{-1}.$$
The End

THANKS FOR YOUR ATTENTION