Sensitivity analysis of the expected utility maximization problem with respect to model perturbations

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based on joint work with

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Outline

Overview

Expected utility maximization
   Existence and uniqueness
   Stability and asymptotics

The model of perturbed markets
   Structure of perturbation/relation to random endowment
   Abstract version

Analysis
   1-d duality for a 2-d problem
   First order
   Second order

Risk-tolerance wealth process
   Definition and basic properties
   Connection to the second-order asymptotics

Summary
The starting point

- consider the perturbation analysis in Larsen, Mostovyi and Žitković
- do a similar analysis with a general rather than power utility
The mathematics

- present a method to approximate
  1. value functions to second order
  2. optimizers to the first order
- stochastic control problems which are convex, but not convex with respect to a parameter
- abstract version (over random variables)
- back to the original model, write approximation of strategies as Kunita-Watanabe decomposition under risk tolerance wealth process as numeraire
Utility Maximization Problem

Agent
- initial wealth
- utility

Market
- stock
- bank account
- no arbitrage
- frictionless

Initial wealth $x$

Controlling investment $H$

Wealth at time $t$

$$x + \int_0^t H_u dS_u$$

Value function:

$$u(x) \triangleq \max_{X \in \mathcal{X}(x)} \mathbb{E} [U(X_T)].$$
Utility Maximization Problem

Utility function $U$:

- $(0, \infty) \to \mathbb{R}$: strictly increasing, strictly concave, $C^1$,
- Satisfies the Inada conditions:

$$\lim_{x \to 0} U'(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} U'(x) = 0.$$ 

Standard results in the literature:

- existence and uniqueness of solutions,
- properties of the value function,
- properties of the solutions.

Merton, Cox, Huang, Karatzas, Lehoczky, Shreve, Xu, Kramkov, Schachermayer...

Under certain (quite weak) conditions, the optimal $\hat{H}$ and $\hat{X}$ exist and are unique.
Stability and asymptotics

- stability: continuous dependence on parameters (goes back to Hadamard)
- asymptotics: higher order dependence (needs differentiability structure on parameters)
Stability and asymptotics: literature

Existing results (small fraction):

- dependence on $x$: Kramkov and Schachermayer (1999, 2003)

- dependence on $U$ (and/or $\mathbb{P}$): Jouini and Napp (2004), Carasus and Rasonyi (2005), Larsen (2006), Kardaras and Žitković (2011),


Our model: the family of markets

(from Larsen, Mostovyi, Žitković)

A family of markets is parametrized by $\delta$. Every market consist of a stock and a bond. (Return of) the stock price process evolves as

$$dS_t^\delta \triangleq (\lambda_s + \delta \nu_s) d\langle M \rangle_s + dM_t$$

(see Hulley and Schweizer (2010), Delbaen and Schachermayer (1995));

The price process of the bond equals to 1 at all times.

**Goal:** study dependence on $\delta$. 
Primal problem

Define

\[ \mathcal{X}(x, \delta) \triangleq \left\{ X : X_t = x + \int_0^t H_u dS_u^\delta, \ t \in [0, T] \text{ and } X \geq 0 \right\}, \]

\[ x > 0. \]

A utility function \( U : (0, \infty) \to \mathbb{R} \) is strictly increasing, strictly concave, two times continuously differentiable on \((0, \infty)\) and there exist positive constants \( c_1 \) and \( c_2 \), such that

\[ c_1 \leq A(x) \triangleq -\frac{U''(x)x}{U'(x)} \leq c_2, \]

and define the value function as:

\[ u(x, \delta) \triangleq \sup_{X \in \mathcal{X}(x, \delta)} \mathbb{E}[U(X_T)], \quad (x, \delta) \in (0, \infty) \times \mathbb{R}. \]
Mathematical goal

How to establish an expansion with respect to $\delta$ of

- the value function $u(x, \delta)$ (second order),
- the corresponding trading strategy (first order)?

Remark

Dual problem can be helpful.
Dual problem

\[ V(y) \triangleq \sup_{x>0} (U(x) - xy), \quad y > 0, \]
\[ -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}, \quad \text{if} \quad y = U'(x). \]

Let \( \mathcal{Y}(y, \delta) \) be a set of nonnegative supermartingales such that:

1. \( Y_0 = y \),
2. \( (X_t Y_t)_{t \in [0, T]} \) is a supermartingale for every \( X \in \mathcal{X}(1, \delta) \).

The dual value function is

\[ v(y, \delta) \triangleq \inf_{Y \in \mathcal{Y}(y, \delta)} \mathbb{E} [V(Y_T)], \quad (y, \delta) \in (0, \infty) \times \mathbb{R}. \]
Lemma
For every $\delta \in \mathbb{R}$, we have

\[
\begin{align*}
\mathcal{Y}(1, \delta) &= \mathcal{Y}(1, 0) \mathcal{E} (-\delta \nu \cdot S^0), \\
\mathcal{X}(1, \delta) &= \mathcal{X}(1, 0) \frac{1}{\mathcal{E} (-\delta \nu \cdot S^0)}.
\end{align*}
\]

Remark
Looks like a multiplicative (and non-linear) random endowment.
Abstract theorems

In the spirit of Kramkov-Schachermayer (99), consider the sets $\mathcal{C}$ and $\mathcal{D}$ polar in $L^0_+$:

**Assumption**

*Both $\mathcal{C}$ and $\mathcal{D}$ contain a strictly positive element and

\[ \xi \in \mathcal{C} \quad \text{iff} \quad \mathbb{E} [\xi \eta] \leq 1 \quad \text{for every } \eta \in \mathcal{D}, \]

*as well as

\[ \eta \in \mathcal{D} \quad \text{iff} \quad \mathbb{E} [\xi \eta] \leq 1 \quad \text{for every } \xi \in \mathcal{C}. \]
Primal and dual problems for 0-model

We set

\[ C(x, 0) \triangleq xC \quad \text{and} \quad D(x, 0) \triangleq xD, \quad x > 0. \]

Now we can state the abstract primal and dual problems as

\[ u(x, 0) \triangleq \sup_{\xi \in C(x, 0)} \mathbb{E}[U(\xi)], \quad x > 0, \]

\[ v(y, 0) \triangleq \inf_{\eta \in D(y, 0)} \mathbb{E}[V(\eta)], \quad y > 0. \]
Abstract version for $\delta$-models

For some random variables $F$ and $G \geq 0$, we set

$$L^\delta \triangleq \exp \left( -(\delta F + \frac{1}{2} \delta^2 G) \right),$$

$$C(x, \delta) \triangleq C(x, 0) \frac{1}{L^\delta} \quad \text{and} \quad D(y, \delta) \triangleq D(y, 0)L^\delta, \quad \delta \in \mathbb{R}.$$ 

The abstract versions of the perturbed optimization problems:

$$u(x, \delta) \triangleq \sup_{\xi \in C(x, \delta)} \mathbb{E} \left[ U(\xi) \right] = \sup_{\xi \in C(x, 0)} \mathbb{E} \left[ U \left( \frac{\xi}{L^\delta} \right) \right], \quad x > 0, \delta \in \mathbb{R},$$

$$v(y, \delta) \triangleq \inf_{\eta \in D(y, \delta)} \mathbb{E} \left[ V(\eta) \right] = \inf_{\eta \in D(y, 0)} \mathbb{E} \left[ V \left( \eta L^\delta \right) \right], \quad y > 0, \delta \in \mathbb{R}.$$
The approach

Follows Henderson (2002).

- find lower bound up to second order for $u$
- upper bound up to second order for $v$
- "match" them

Matching one-sided bounds can be found using quadratic optimization problems:

- (Kramkov, S. (2006))
Lack of convexity in $\delta$

The value functions $-u$ and $v$ are convex in $x, y$.

But for the parametrized family of markets, we do not have convexity in $\delta$. 
No-convexity in $\delta$, cont’d

Can use convexity only in direction of $x, y$. For

$$y = u_x(x, \delta), \quad u(x, \delta) - xy = v(y, \delta)$$

Even if we fix $x$ and vary $\delta$ alone, $y$ depends on $\delta$: need to approximate at least $v$ in both directions $(y, \delta)$.

Summary: better provide joint expansion for both

- $(x, \delta)$ for $u$
- $(y, \delta)$ for $v$
The 0-model:
If \( u \) is finite at some point

(i) \( u(x, 0) < \infty \), for every \( x > 0 \), and \( v(y, 0) > -\infty \), for every \( y > 0 \).
The functions \( u \) and \( v \) are Legendre conjugate
\[
\begin{align*}
    v(y, 0) &= \sup_{x>0} \left( u(x, 0) - xy \right), \quad y > 0, \\
    u(x, 0) &= \inf_{y>0} \left( v(y, 0) + xy \right), \quad x > 0.
\end{align*}
\]

(ii) The functions \( u \) and \( -v \) are continuously differentiable on \((0, \infty)\), strictly concave, strictly increasing and satisfy the Inada conditions
\[
\begin{align*}
    \lim_{x \downarrow 0} u_x(x, 0) &= \infty, \\
    \lim_{y \downarrow 0} (-v_x(y, 0)) &= \infty, \\
    \lim_{x \uparrow \infty} u_x(x, 0) &= 0, \\
    \lim_{y \uparrow \infty} (-v_y(y, 0)) &= 0.
\end{align*}
\]

(iii) For every \( x > 0 \) and \( y > 0 \), the solutions \( \hat{X}(x, 0) \) and \( \hat{Y}(y, 0) \) exist and are unique and, if \( y = u'(x) \), we have
\[
\hat{Y}_T(y) = U' \left( \hat{X}_T(x) \right), \quad \mathbb{P}\text{-a.s.}
\]

(iv) \( \hat{X}(x) \hat{Y}(y) \) is a strictly positive, uniformly integrable martingale.
Assumption on perturbations

- First, we set:

\[
\frac{d \mathbb{R}(x, 0)}{d \mathbb{P}} \triangleq \frac{\hat{X}_T(x, 0) \hat{Y}_T(y, 0)}{xy}.
\]

- Let \(x > 0\) be fixed. There exists \(c > 0\), such that

\[
\mathbb{E}^{\mathbb{R}(x, 0)} \left[ \exp \left( c(|\nu \cdot S^0_T| + \langle \nu \cdot S^0 \rangle_T) \right) \right] < \infty.
\]
Let us assume that $c_1 > 1$, i.e. that relative-risk aversion of $U$ is strictly greater than 1 (relative risk aversion uniformly exceeds 1).

A sufficient condition for the previous slide Assumption to hold is the existence of some positive exponential moments under $\mathbb{P}$.
First-order analysis

Theorem (Envelope)

Let $x > 0$ be fixed and assumptions above hold. Then we have

- There exists $\delta_0 > 0$ such that for every $\delta \in (-\delta_0, \delta_0)$, we have
  
  $$u(z, \delta) \in \mathbb{R} \text{ and } v(z, \delta) \in \mathbb{R}, \quad z > 0.$$  

- The first-order derivatives are
  
  $$u_\delta(x, 0) = v_\delta(y, 0) = xy \mathbb{E}^{\mathbb{R}(x,0)}[\nu \cdot S^0_T], \quad y = u_x(x, 0).$$

- The value functions $u$ and $v$ are continuous at $(x, 0)$ and $(y, 0)$, respectively.

Remark

$u_\delta(x, 0)$ and $v_\delta(y, 0)$ are linear in $\nu$.  

Second-order analysis

Let $S^{X(x,0)}$ be the price process of the traded securities under the numéraire $\frac{\hat{X}(x,0)}{x}$, i.e.

$$S^{X(x,0)} = \left( \frac{x}{\hat{X}(x,0)}, \frac{xS^0}{\hat{X}(x,0)} \right).$$

For every $x > 0$, let $H^2_0(\mathbb{R}(x,0))$ denote the space of square integrable martingales under $\mathbb{R}(x,0)$, such that

$$\mathcal{M}^2(x,0) \triangleq \left\{ M \in H^2_0(\mathbb{R}(x,0)) : M = H \cdot S^{X(x,0)} \right\},$$

$$\mathcal{N}^2(y,0) \triangleq \left\{ N \in H^2_0(\mathbb{R}(x,0)) : MN \text{ is } \mathbb{R}(x,0)\text{-martingale for every } M \in \mathcal{M}^2(x,0) \right\}, \quad y = u_x(x,0).$$
Auxiliary minimization problems (for $u_{xx}$ and $v_{yy}$)

Let us set

$$a(x, x) \triangleq \inf_{M \in M^2(x, 0)} \mathbb{E}^{\mathbb{R}(x, 0)} \left[ A(\hat{X}_T(x, 0))(1 + M_T)^2 \right],$$

$$b(y, y) \triangleq \inf_{N \in N^2(y, 0)} \mathbb{E}^{\mathbb{R}(x, 0)} \left[ B(\hat{Y}_T(y, 0))(1 + N_T)^2 \right],$$

where $y = u_x(x, 0)$,

$$A(x) = -\frac{U''(x)x}{U'(x)} \quad \text{and} \quad B(y) = -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}.$$
Second-order derivatives with respect to $x$ and $y$


- auxiliary minimization problems admit unique solutions $M^0(x, 0)$ and $N^0(y, 0)$;
- the value functions are two-times differentiable and
  
  \[
  u_{xx}(x, 0) = -\frac{y}{x} a(x, x), \\
  v_{yy}(y, 0) = \frac{x}{y} b(y, y);
  \]

- $u_{xx}$ and $v_{yy}$ are linked via
  
  \[
  u_{xx}(x, 0) v_{yy}(y, 0) = -1, \\
  a(x, x)b(y, y) = 1;
  \]

- the optimizers to auxiliary problems satisfy
  
  \[
  A(\hat{X}_T(x, 0))(1 + M^0_T(x, 0)) = a(x, x)(1 + N^0_T(y, 0)).
  \]
Auxiliary minimization problem (for $u_{\delta\delta}$ and $v_{\delta\delta}$)

With

$$F \triangleq \nu \cdot S^0_T \quad \text{and} \quad G \triangleq \nu^2 \cdot \langle M \rangle_T,$$

we consider the following minimization problems.

\[
a(d, d^{'}) \triangleq \inf_{M \in M^2(x,0)} \mathbb{E}_{(x,0)} \left[ A(\hat{X}_T(x,0))(M_T + xF)^2 - 2xFM_T - x^2(F^2 + G) \right] ,
\]

\[
b(d, d^{'}) \triangleq \inf_{N \in N^2(y,0)} \mathbb{E}_{(x,0)} \left[ B(\hat{Y}_T(y,0))(N_T - yF)^2 + 2yFN_T - y^2(F^2 - G) \right] .
\]
Structure of $u_{x\delta}$ and $v_{y\delta}$

Denoting by $M^1(x, 0)$ and $N^1(y, 0)$ the unique solutions the auxiliary problems above, we set

$$\begin{align*}
a(x, d) & \triangleq \mathbb{E}_{(x, 0)} \left[ A(\hat{X}_T(x, 0))(1 + M^0_T(x, 0))(xF + M^1_T(x, 0)) \\
& \quad - xF(1 + M^0_T(x, 0)) \right], \\
b(y, d) & \triangleq \mathbb{E}_{(x, 0)} \left[ B(\hat{Y}_T(y, 0))(1 + N^0_T(y, 0))(N^1_T(y, 0) - yF) \\
& \quad + yF(1 + N^0_T(y, 0)) \right].
\end{align*}$$
Theorem (Mostovyi., S.)

Let $x > 0$ be fixed. Let the assumptions above hold and $y = u_x(x, 0)$. Define

$$H_u(x, 0) \triangleq -\frac{y}{x} \begin{pmatrix} a(x, x) & a(x, d) \\ a(x, d) & a(d, d) \end{pmatrix},$$

$$H_v(y, 0) \triangleq \frac{x}{y} \begin{pmatrix} b(y, y) & b(y, d) \\ b(y, d) & b(d, d) \end{pmatrix}.$$

Then, the value functions $u$ and $v$ admit the second-order expansions around $(x, 0)$ and $(y, 0)$, respectively,

$$u(x + \Delta x, \delta) = u(x, 0) + (\Delta x \quad \delta) \nabla u(x, 0) + \frac{1}{2}(\Delta x \quad \delta) H_u(x, 0) \begin{pmatrix} \Delta x \\ \delta \end{pmatrix} + o(\Delta x^2 + \delta^2),$$

$$v(y + \Delta y, \delta) = v(y, 0) + (\Delta y \quad \delta) \nabla v(y, 0) + \frac{1}{2}(\Delta y \quad \delta) H_v(y, 0) \begin{pmatrix} \Delta y \\ \delta \end{pmatrix} + o(\Delta y^2 + \delta^2).$$
Theorem (Mostovyi, S.)

(i) The values of quadratic optimizations

\[
\begin{pmatrix}
an(x, x) & 0 \\
an(x, d) & -\frac{x}{y}
\end{pmatrix} \begin{pmatrix}
b(y, y) & 0 \\
b(y, d) & -\frac{y}{x}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
y a(d, d) + \frac{x}{y} b(d, d) = a(x, d) b(y, d).
\]

(ii) The optimizers of the quadratic problems are related

\[
U''(\hat{X}_T(x, 0)) \hat{X}_T^0(x, 0) \left( \begin{array}{c} M^0_T(x, 0) + 1 \\ M^1_T(x, 0) + xF \end{array} \right) = - \left( \begin{array}{cc} a(x, x) & 0 \\ a(x, d) & -\frac{x}{y} \end{array} \right) \hat{Y}_T^0(y, 0) \left( \begin{array}{c} N^0_T(y, 0) + 1 \\ N^1_T(y, 0) - yF \end{array} \right),
\]

\[
V''(\hat{Y}_T(y, 0)) \hat{Y}_T(y, 0) \left( \begin{array}{c} 1 + N^0_T(y, 0) \\ -yF + N^1_T(y, 0) \end{array} \right) = \left( \begin{array}{cc} b(y, y) & 0 \\ b(y, d) & -\frac{y}{x} \end{array} \right) \hat{X}_T(x, 0) \left( \begin{array}{c} 1 + M^0_T(x, 0) \\ xF + M^1_T(x, 0) \end{array} \right).
\]

(iii) The product of any of \( \hat{X}(x, 0), \hat{X}(x, 0) M^0(x, 0), \hat{X}(x, 0) M^1(x, 0) \)

and any of \( \hat{Y}(y, 0), \hat{Y}(y, 0) N^0(y, 0), \hat{Y}(y, 0) N^1(y, 0) \) is a \( \mathbb{P} \)-martingale.
Derivatives of the optimizers

Theorem (Mostovyi, S.)

Let us set

\[ X'_T(x, 0) \triangleq \frac{\hat{X}_T(x, 0)}{x} (1 + M^0_T(x, 0)), \quad Y'_T(x, 0) \triangleq \frac{\hat{Y}_T(y, 0)}{y} (1 + N^0_T(y, 0)), \]

and

\[ X_d^T(x, 0) \triangleq \frac{\hat{X}_T(x, 0)}{x} (M^1_T(x, 0) + xF), \quad Y_d^T(y, 0) \triangleq \frac{\hat{Y}_T(y, 0)}{y} (N^1_T(y, 0) - yF). \]

Then, we have

\[
\lim_{|\Delta x| + |\delta| \to 0} \frac{1}{|\Delta x| + |\delta|} \left| \hat{X}_T(x + \Delta x, \delta) - \hat{X}_T(x, 0) - \Delta x X'_T(x, 0) - \delta X_d^T(x, 0) \right| = 0,
\]

\[
\lim_{|\Delta y| + |\delta| \to 0} \frac{1}{|\Delta y| + |\delta|} \left| \hat{Y}_T(y + \Delta y, \delta) - \hat{Y}_T(y, 0) - \Delta y Y'_T(y, 0) - \delta Y_d^T(y, 0) \right| = 0,
\]

where the convergence takes place in \( \mathbb{P} \)-probability.
Approximation of the optimal trading strategies

**Observation:** because the "random endowment" is multiplicative, proportions work better.

With

\[ M^R = S^0 - \hat{\pi}(x, 0) \cdot \langle M \rangle, \]

let

\[ \gamma^0 \cdot M^R = \frac{M^0(x, 0)}{x} \quad \text{and} \quad \gamma^1 \cdot M^R = \frac{M^1(x, 0)}{x}, \]

and

\[ \sigma_\varepsilon \triangleq \inf \{ t \in [0, T] : |M^0_t(x, 0)| \geq \frac{x}{\varepsilon} \text{ or } \langle M^0_t(x, 0) \rangle_t \geq \frac{x}{\varepsilon} \}, \]

\[ \tau_\varepsilon \triangleq \inf \{ t \in [0, T] : |M^1_t(x, 0)| \geq \frac{x}{\varepsilon} \text{ or } \langle M^1_t(x, 0) \rangle_t \geq \frac{x}{\varepsilon} \}, \quad \varepsilon > 0, \]

as well as

\[ \gamma^{0,\varepsilon} = \gamma^0 1_{[0,\sigma_\varepsilon]} \quad \text{and} \quad \gamma^{1,\varepsilon} = \gamma^1 1_{[0,\tau_\varepsilon]}, \quad \varepsilon > 0. \]
Approximation of the optimal trading strategies

Let us set

\[
\frac{dX_t^{\Delta x, \delta, \varepsilon}}{dS_t} = X_t^{\Delta x, \delta, \varepsilon}(\hat{\pi}_t(x, 0) + \Delta x \gamma_t^{0, \varepsilon} + \delta(\nu_t + \gamma_t^{1, \varepsilon}))dS_t,
\]

\[
X_0^{\Delta x, \delta, \varepsilon} = x + \Delta x.
\]

Note that

\[
X_t^{\Delta x, \delta, \varepsilon} = (x + \Delta x) \frac{\hat{X}(x, 0)}{x} \frac{\mathcal{E}((\Delta x \gamma_t^{0, \varepsilon} + \delta \gamma_t^{1, \varepsilon}) \cdot M^R)}{\mathcal{E}(-\delta \nu \cdot S^0)}.
\]

Theorem (Mostovyi, S.)

There exists a function \( \varepsilon = \varepsilon(\Delta x, \delta) \), such that

\[
\mathbb{E}\left[U\left(X_T^{\Delta x, \delta, \varepsilon(\Delta x, \delta)}\right)\right] = u(x + \Delta x, \delta) - o(\Delta x^2 + \delta^2).
\]
Risk-tolerance wealth process

Definition
For $x > 0$ and $\delta \in \mathbb{R}$, the risk-tolerance wealth process is a maximal wealth process $R(x, \delta)$, such that

$$R_T(x, \delta) = -\frac{U'(\hat{X}_T(x, \delta))}{U''(\hat{X}_T(x, \delta))}.$$ 

Remark
This process was introduced in Kramkov and S. (2006) in the context of asymptotic analysis of utility-based prices.
Theorem (Kramkov and S. (2006))

The following assertions are equivalent:

(1) The risk-tolerance wealth process \( R(x, 0) \) exists.

(2) The value function \( u \) admits the expansion quadratic expansion at \((x, 0)\) and \( u_{xx}(x, 0) = -\frac{Y}{X} a(x, x) \) satisfies

\[
\frac{(u_x(x, 0))^2}{u_{xx}(x, 0)} = \mathbb{E} \left[ \frac{\left( U'(\hat{X}_T(x, 0)) \right)^2}{U''(\hat{X}_T(x, 0))} \right],
\]

\[
u_{xx}(x, 0) = \mathbb{E} \left[ U''(\hat{X}_T(x, 0)) \left( \frac{R_T(x, 0)}{R_0(x, 0)} \right)^2 \right].
\]

(3) The value function \( v \) admits the quadratic expansion at \((y, 0)\) and \( v_{yy}(y, 0) = \frac{X}{Y} b(y, y) \) satisfies

\[
y^2 v_{yy}(y, 0) = \mathbb{E} \left[ \left( \hat{Y}_T(y, 0) \right)^2 V''(\hat{Y}_T(y, 0)) \right] = xy \mathbb{E}^{R(x, 0)} \left[ B(\hat{Y}_T(y, 0)) \right].
\]
Theorem (..Continued)

In addition, if these assertions are valid, then the initial value of $R(x)$ is given by

$$R_0(x, 0) = -\frac{u_x(x, 0)}{u_{xx}(x, 0)} = \frac{x}{a(x, x)},$$

the product $R(x, 0) Y(y, 0) = (R_t(x, 0) Y_t(y, 0))_{t \in [0, T]}$ is a uniformly integrable martingale and

$$\lim_{\Delta x \to 0} \frac{\hat{X}_T(x + \Delta x, 0) - \hat{X}_T(x, 0)}{\Delta x} = \frac{R_T(x, 0)}{R_0(x, 0)},$$

$$\lim_{\Delta y \to 0} \frac{\hat{Y}_T(y + \Delta y, 0) - \hat{Y}_T(y, 0)}{\Delta y} = \frac{\hat{Y}_T(y, 0)}{y},$$

where the limits take place in $\mathbb{P}$-probability.
For \( x > 0 \) and with \( y = u_x(x, 0) \), let us define
\[
\frac{d\tilde{R}(x, 0)}{d\mathbb{P}} \triangleq \frac{R_T(x, 0)\hat{Y}_T(y, 0)}{R_0(x, 0)y},
\]
and choose \( \frac{R(x, 0)}{R_0(x, 0)} \) as a numéraire, i.e., let us set
\[
S^{R(x, 0)} \triangleq \left( \frac{R_0(x, 0)}{R(x, 0)}, \frac{R_0(x, 0)S}{R(x, 0)} \right).
\]

We define the spaces of martingales
\[
\tilde{\mathcal{M}}^2(x, 0) \triangleq \left\{ M \in H_0^2(\tilde{\mathbb{R}}(x, 0)) : M = H \cdot S^{R(x, 0)} \right\},
\]
and \( \tilde{\mathcal{N}}^2(y, 0) \) it the orthogonal complement in \( H_0^2(\tilde{\mathbb{R}}(x, 0)) \).
Theorem (Mostovyi, S.)

Let us assume that the risk-tolerance process $R(x, 0)$ exists. Consider the Kunita-Watanabe decomposition of the square integrable martingale

$$P_t \triangleq \mathbb{E}^{\hat{R}(x,0)} \left[ \left( A(\hat{X}_T(x, 0)) - 1 \right) xF | \mathcal{F}_t \right], \quad t \in [0, T].$$

given by

$$P = P_0 - \tilde{M}^1 - \tilde{N}^1,$$

where $\tilde{M}^1 \in \tilde{M}^2(x, 0)$, $\tilde{N}^1 \in \tilde{N}^2(y, 0)$, $P_0 \in \mathbb{R}$. 

Risk-tolerance wealth process and a Kunita-Watanabe decomposition
Theorem (..Continued)

Then, the optimal solutions $M^1(x,0)$ and $N^1(y,0)$ of the auxiliary quadratic optimization problems for $u_{\delta\delta}$ and $v_{\delta\delta}$ can be obtained from the Kunita-Watanabe decomposition (above) by reverting to the original numéraire, through the identities:

$$\tilde{M}_t^1 = \frac{\hat{X}_t(x,0)}{R_t(x,0)} M_t^1(x,0), \quad \tilde{N}_t^1 = \frac{x}{y} N_t^1(y,0), \quad t \in [0,T].$$

In addition, the Hessian terms in the quadratic expansion of $u$ and $v$ can be identified as

$$a(d,d) = \frac{R_0(x,0)}{x} \inf_{\tilde{M} \in \tilde{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ \left( \tilde{M}_T + x F \left( A \left( \hat{X}_T(x,0) \right) - 1 \right) \right)^2 \right] + C_a.$$

$$= \frac{R_0(x,0)}{x} \mathbb{E}^{\mathbb{R}(x,0)} \left[ \left( \tilde{N}_T^1 \right)^2 \right] + \frac{R_0(x,0)}{x} P_0^2 + C_a,$$

where $C_a \triangleq x^2 \mathbb{E}^{\mathbb{R}(x,0)} \left[ F^2 \frac{A(\hat{X}_T(x,0)) - 1}{A(\hat{X}_T(x,0))} - G \right].$
Theorem (..Continued)

\[ b(d, d) = \frac{R_0(x,0)}{x} \inf_{\tilde{N} \in \tilde{N}^2(y,0)} \mathbb{E}^\tilde{R}(y,0) \left[ \left( \tilde{N}_T + yF \left( A \left( \tilde{X}_T(x,0) \right) - 1 \right) \right)^2 \right] + C_b. \]

\[ = \frac{R_0(x,0)}{x} \left( \frac{y}{x} \right)^2 \mathbb{E}^\tilde{R}(y,0) \left[ \left( \tilde{M}_T^1 \right)^2 \right] + \frac{R_0(x,0)}{x} \left( \frac{y}{x} \right)^2 P_0^2 + C_b, \]

where \( C_b \triangleq y^2 \mathbb{E}^\tilde{R}(x,0) \left[ G + F^2 \left( 1 - A \left( \tilde{X}_T(x,0) \right) \right) \right]. \) The cross terms in the Hessians of \( u \) and \( v \) are identified as

\[ a(x, d) = P_0 \]

and \( b(y, d) \) is given by

\[ b(y, d) = \frac{y}{x} \frac{P_0}{a(x, x)}. \]

With these identifications, all the expansions of the value functions above hold.
Summary

- look at the simultaneous perturbations of the market price of risk and the initial wealth
- formulate quadratic optimization problems and relate the second-order approximations of both primal and dual value functions to these problems.
- in case when the risk-tolerance wealth process exists, we used it as a numéraire, and changed the measure accordingly, to identify solutions to the quadratic optimization problems above in terms of a Kunita-Watanabe decomposition.