Pricing Variance Swaps on Time-Changed Markov Processes

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Intro to variance swaps

- Let $F$ be forward price of an asset.
- Assume $F_t > 0$ for all $t$.
- Define the process $X := \log F$.
- The floating leg of a variance swap pays (to the long side)
  \[
  \sum_{t_i \in [0,T]} (X_{t_{i+1}} - X_{t_i})^2. \tag{1}
  \]
- As sampling frequency increases: (1) $\xrightarrow{\mathbb{P}} [X]_T$.
- A swap whose floating leg pays $[X]_T$ is called a continuously monitored variance swap.
- Using risk-neutral pricing, the fair strike of a VS is $\mathbb{E} [X]_T$
- Question: how to compute $\mathbb{E} [X]_T$?
Non-parametric approach (with no jumps)

Suppose \( F_t = \exp(X_t) \) with

\[
dX_t = -\frac{1}{2} \sigma_t^2 dt + \sigma_t dW_t.
\]

In this setting, Neuberger (1990) and Dupire (1993) show VS priced by log contract:

\[
E[X]_T = -2 E(X_T - X_0) = -2 E \log(F_T/F_0).
\]

Quick proof:

\[
E[X]_T = E \int_0^T \sigma_i^2 dt
\]

\[
= -2 E \int_0^T dX_t + 2 E \int_0^T \sigma_t dW_t
\]

\[
= -2 E (X_T - X_0).
\]
Synthetic European contracts

As shown in Carr and Madan (1998), if $h \in C^2(\mathbb{R}^+)$ then for any $\kappa \in \mathbb{R}^+$ we have

$$h(F_T) = h(\kappa) + h'(\kappa)\left((F_T - \kappa)^+ - (\kappa - F_T)^+\right)$$

$$+ \int_0^\kappa h''(K)(K - F_T)^+ dK + \int_\kappa^\infty h''(K)(F_T - K)^+ dK.$$

Taking expectations, we have

$$\mathbb{E} h(F_T) = h(\kappa) + h'(\kappa)\left(C(T, \kappa) - P(T, \kappa)\right)$$

$$+ \int_0^\kappa h''(K)P(T, K) dK + \int_\kappa^\infty h''(K)C(T, K) dK,$$

where $P(T, K)$ and $C(T, K)$ are European put and call prices.
Synthetic log contract and VIX

To price a VS, take $h(F) = -2 \log(F/F_0)$.

$$
\mathbb{E} [X]_T = -2 \mathbb{E} \log(F_T/F_0) \\
= \int_0^{F_0} \frac{2}{K^2} P(T, K) dK + \int_{F_0}^{\infty} \frac{2}{K^2} C(T, K) dK,
$$

Discretized version of (2) is used to construct VIX, the CBOE’s 30-day forward looking measure of volatility.

Note: equation (2) prices VS correctly only when $X$ experiences no jumps.
Non-parametric pricing of VS with jumps

Suppose $F_t = \exp(Y_{\tau_t})$ where $\tau$ is a continuous stochastic clock (possibly correlated with $Y$) and $Y$ is a Lévy process

$$dY_t = b \, dt + \sigma \, dW_t + \int_{\mathbb{R}} zd\tilde{N}_t(dz),$$

$$d\tilde{N}_t(dz) = dN_t(dz) - \mu(dz)dt,$$

$$b = -\frac{1}{2} \sigma^2 - \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz).$$

In this setting Carr, Lee, and Wu (2011) show

$$\mathbb{E} [X]_T = -Q \mathbb{E} (X_T - X_0) = -Q \mathbb{E} \log(F_T/F_0).$$

The multiplier $Q$ depends only on the Lévy process $Y$ – not on the clock

$$Q = \frac{\sigma^2 + \int_{-\infty}^{\infty} z^2 \nu(dz)}{\sigma^2/2 + \int_{-\infty}^{\infty} (e^z - 1 - z) \nu(dz)}$$

Note: if $\nu \equiv 0$ then $Q = 2$ (recover result of Neuberger/Dupire).
Features and limitations of time-changed Lévy processes

Features

▷ Allows for jumps, stochastic volatility and leverage effect
▷ Multiplier $Q$ depends only on background Lévy process – not time-change.
▷ Includes many popular models (e.g., Heston, Exponential Lévy, etc.) in a single framework.

Possible limitations

▷ In time-changed Lévy approach, the multiplier $Q$ should be constant

$$Q = \frac{-\mathbb{E} \left[ \log F \right]_T}{\mathbb{E} \log(F_T/F_0)}.$$ 

Evidence from Carr, Lee, and Wu (2011) suggests $Q$ is not constant in time or across maturities.
Time-changed Markov Processes

Suppose \( F \) is modeled by

\[
F_t = \exp(Y_{\tau_t})
\]

where \( \tau \) is a continuous stochastic clock \underline{possibly correlated} with \( Y \) and \( Y \) is any continuous time scalar Markov process

\[
dY_t = b(Y_t) \, dt + a(Y_t) \, dW_t + \int_{\mathbb{R}} z \, d\tilde{N}_t(Y_t- \, , d\zeta),
\]

Here, \( d\tilde{N}_t(Y_t-, \, d\zeta) \) is a compensated Poisson random measure with state-dependent jumps

\[
d\tilde{N}_t(Y_t-, \, d\zeta) = dN_t(Y_t-, \, d\zeta) - \mu(Y_t-, \, d\zeta) dt,
\]

and the drift \( b(Y_t) \) is fixed by \( a \) and \( \mu \) so that \( F \) is a martingale

\[
b(Y_t) = -\frac{1}{2} a^2(Y_t) - \int_{\mathbb{R}} \mu(Y_t-, \, d\zeta) \left( e^\zeta - 1 - \zeta \right).
\]
The generator of $Y$

Note that $Y$ has generator

$$\mathcal{A} = \frac{1}{2} a^2(y) \left( \partial^2 - \partial \right) + \int_{\mathbb{R}} \mu(y, dz) \left( \theta_z - 1 - z \partial \right)$$

$$- \int_{\mathbb{R}} \mu(y, dz) \left( e^z - 1 - z \right) \partial,$$

where $\theta_z$ is the shift operator: $\theta_z f(y) = f(y + z)$.

Note, for analytic $f$, we have

$$e^{z \partial} f(y) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \partial^n f(y) = f(y + z).$$

Formally, then, we re-write the generator $\mathcal{A}$ as follows:

$$\mathcal{A} = \frac{1}{2} a^2(y) \left( \partial^2 - \partial \right) + \int_{\mathbb{R}} \mu(y, dz) \left( e^{z \partial} - 1 - z \partial \right)$$

$$- \int_{\mathbb{R}} \mu(y, dz) \left( e^z - 1 - z \right) \partial.$$
The quadratic variation process \([Y]\)

Note that \(d[Y]_t\) is given by

\[
d[Y]_t = \left( a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt + \int_{\mathbb{R}} z^2 d\tilde{N}(Y_{t-}, dz). \]

Hence

\[
\mathbb{E} d[Y]_t = \mathbb{E} \left( a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt.
\]

Likewise, for \(G \in \text{dom}(\mathcal{A})\) we have

\[
dG(Y_t) = \mathcal{A}G(Y_t) dt + \text{martingale}.
\]

Hence

\[
\mathbb{E} dG(Y_t) = \mathbb{E} \mathcal{A}G(Y_t) dt.
\]
Suppose we can find a function $G$ such that

$$-A \, G(y) = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz).$$

Then, using results from the previous page, we have

$$\mathbb{E} \left[ Y \right]_{\tau_T} = \mathbb{E} \int_0^{\tau_T} d[Y]_t$$

$$= \mathbb{E} \int_0^{\tau_T} \left( a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt$$

$$= - \mathbb{E} \int_0^{\tau_T} A \, G(Y_t) dt$$

$$= - \mathbb{E} \int_0^{\tau_T} dG(Y_t) + \mathbb{E} \text{martingale}$$

$$= - \mathbb{E} \, G(Y_{\tau_T}) + G(Y_0).$$
Recall $F_t = \exp(Y_{\tau_t})$

Continuity of time-change $\tau$ implies: $[\log F]_T = [Y]_{\tau_T}$. Hence

$$\mathbb{E} [\log F]_T = \underbrace{-\mathbb{E} G(\log F_T)}_{A} + \underbrace{G(\log F_0)}_{C}.$$  

- **A**: Fair strike of a variance swap.  
- **B**: Value of a European contract with payoff: $-G(\log F_T)$.  
- **C**: Value of $G(\log F_0)$ zero-coupon bonds.

Quantity B can be constructed from $T$-maturity calls/puts a la Carr and Madan (1998).

- To price a VS, we must solve OIDE:

$$-AG(y) = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz).$$
Define: \( H := \partial G \) so that \( G(y) = \int H(y) dy \). Then \( H \) solves

\[
-\frac{\mathcal{A}}{\partial} H = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz),
\]

where

\[
\frac{\mathcal{A}}{\partial} = \frac{1}{2} a^2(y) (\partial - 1) + \int_{\mathbb{R}} \mu(y, dz) \left( \frac{e^{z\partial} - 1 - z\partial}{\partial} \right)
- \int_{\mathbb{R}} \mu(y, dz) \left( e^z - 1 - z \right)
\]

and

\[
e^{z\partial} - 1 - z\partial \bigg/ \partial := \sum_{n=2}^{\infty} \frac{1}{n!} z^n \partial^{n-1}.
\]
Example 1: jump-intensity proportional to local variance

Introduce $\gamma(y) > 0$. Assume

$$a^2(y) = \gamma^2(y) \sigma^2, \quad \mu(y, dz) = \gamma^2(y) \nu(dz),$$

Easy to check that $H(y)$ is given by

$$H = Q := \frac{\sigma^2 + I_2}{\sigma^2/2 + I_0}, \quad G = Q y,$$

where

$$I_0 := \int_{\mathbb{R}} \nu(dz) (e^z - 1 - z), \quad I_n := \int_{\mathbb{R}} \nu(dz) z^n, \quad n \geq 2.$$

Note: this case includes Time-changed Lévy case (take $\gamma(y) = 1$). Thus, we recover result of Carr, Lee, and Wu (2011):

$$\mathbb{E}[\log F]_T = -\mathbb{E} G(\log F_T) + G(\log F_0) = -Q \log(F_T/F_0).$$

Neuberger-Dupire $\subset$ Carr-Lee-Wu $\subset$ Carr-Lee-Lorig.
Limiting cases

The following **limiting cases** are useful:

- **No Jumps**: \( \nu \equiv 0, \quad H = 2, \)
- **Pure Jumps**: \( \sigma = 0, \quad H = \frac{I_2}{I_0} =: Q_0. \)

We can use these limiting cases to build other exact solutions.
Example 2: building around pure jump solution

Introduce $e_c(y) = e^{cy}$ and $\delta \geq 0$. Assume

$$a^2(y) = \delta \sigma^2(y), \quad \mu(y, dz) = \eta(y)\nu(dz), \quad \frac{\sigma^2(y)}{2 \eta(y)} = e_c(y).$$

Then $H$ solves

$$0 = \delta e_c (A_1 H + 2) + (A_0 H + I_2), \quad (3)$$

where we have defined operators $A_0$ and $A_1$

$$A_0 = \int_{\mathbb{R}} \nu(dz) \left( \frac{e^{z\partial} - 1 - z\partial}{\partial} \right) - \int_{\mathbb{R}} \nu(dz) (e^z - 1 - z),$$

$$A_1 = \partial - 1.$$
Assume $H$ is power series in $\delta$

$$H = \sum_{n=0}^{\infty} \delta^n H_n. \quad (4)$$

Insert expansion (4) into OIDE (6) and collect terms of like order in $\delta$

- $O(\delta^0)$: $\mathcal{A}_0 H_0 = -I_2$,
- $O(\delta)$: $\mathcal{A}_0 H_1 = -e_c (\mathcal{A}_1 H_0 + 2)$,
- $O(\delta^n)$: $\mathcal{A}_0 H_n = -e_c \mathcal{A}_1 H_{n-1}$, \quad $n \geq 2$.

We need to study the operator $\mathcal{A}_0$ and its inverse
The operator $A_0$

$A_0$ is a pseudo-differential operator ($\Psi$DO)

$$A_0 = \int_{\mathbb{R}} \nu(dz) \left( e^{z\partial} - 1 - z\partial \right) - \int_{\mathbb{R}} \nu(dz) (e^z - 1 - z).$$

$\Psi$DO's are characterized by their action on oscillating exponentials

$$A_0 \psi_\lambda = \phi_\lambda \psi_\lambda, \quad \psi_\lambda := \frac{1}{\sqrt{2\pi}} e^{i\lambda y}.$$

where $\phi_\lambda$, called the symbol of $A_0$, satisfies ($\partial \rightarrow i\lambda$)

$$\phi_\lambda = \int_{\mathbb{R}} \nu(dz) \left( \frac{e^{i\lambda z} - 1 - i\lambda z}{i\lambda} \right) - \int_{\mathbb{R}} \nu(dz) (e^z - 1 - z).$$

The inverse operator $A_0^{-1} = \frac{1}{A_0}$ is given by

$$\frac{1}{A_0} \cdot = \int_{\mathbb{R}} d\lambda \frac{1}{\phi_\lambda} \langle \psi_\lambda, \cdot \rangle \psi_\lambda, \quad \langle u, v \rangle := \int_{\mathbb{R}} dy \overline{u}(y) v(y),$$
Using definition of $\frac{1}{A_0}$ we find $H_0 = Q_0$ and for $n \geq 1$:

$$H_n = (Q_0 - 2) \int \cdots \int d\lambda_n \frac{\psi_{\lambda_n}}{\phi_{\lambda_n}} \langle \psi_{\lambda_1}, e_c \rangle \times$$

$$\prod_{k=1}^{n-1} d\lambda_k \frac{-\chi_{\lambda_k}}{\phi_{\lambda_k}} \langle \psi_{\lambda_{k+1}}, e_c \psi_{\lambda_k} \rangle,$$

(5)

where $\chi_{\lambda} = i\lambda - 1$ is symbol of $A_1 = \partial - 1$.

Noting that

$$\langle \psi_{\lambda}, e_c \rangle = \sqrt{2\pi} \delta(\lambda + ic) \quad \text{and} \quad \langle \psi_\mu, e_c \psi_{\lambda} \rangle = \delta(\mu - \lambda + ic),$$

equation (5) becomes

$$H_n = (Q_0 - 2) \sqrt{2\pi} \frac{\psi_{-\text{inc}}}{\phi_{-\text{inc}}} \prod_{k=1}^{n-1} \left( \frac{-\chi_{-ikc}}{\phi_{-ikc}} \right) \quad n \geq 1.$$
Using $H = \sum_n \delta^n H_n$ we have

$$H = Q_0 + (Q_0 - 2) \sum_{n=1}^\infty a_n (\delta e_c)^n, \quad a_n = \frac{1}{\phi - \text{inc}} \prod_{k=1}^{n-1} \frac{-\chi - \text{inc}}{\phi - ikc}.$$  

If the measure $\nu$ is such that

1. $\int_{\mathbb{R}} (e^{ncz} - 1 - nc z) \nu(dz) < \infty$ for all $n \in \mathbb{N}$,
2. $\lim_{n \to \infty} \frac{n^2 c^2}{\int_{\mathbb{R}} (e^{ncz} - 1 - nc z) \nu(dz)} = 0$,

then the coefficients $a_n$ satisfy

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0,$$

and the series converges (and the radius of convergence is $\mathbb{R}$)
Finally, using $G(y) = \int H(y)dy$, the function that prices the variance swap is

$$G = Q_0 y + \sum_{n=1}^{\infty} \delta^n G_n,$$

$$G_n = (Q_0 - 2) \frac{en c}{nc \cdot \phi_{inc}} \prod_{k=0}^{n-1} \left( \frac{-\chi_{-ikc}}{\phi_{-ikc}} \right), \quad n \geq 1.$$

In figures 1 and 2 we plot $Q_0 \log(F_T/F_0)$ and

$$h(F_T) := -G(\log F_T) + G(\log F_0) + A(F_T - F_0),$$

as a function of $F_T$

- The constant $A$ is chosen so that $h(F_T)$ has the same slope as $-Q_0 \log(F_T/F_0)$ at $F_T = F_0$.
- Forward contracts $(F_T - F_0)$ have no value since $\mathbb{E} F_T = F_0$. 
Figure 1: We plot $h(F_T)$ as a function of $F_T$ (solid blue). For comparison we also plot $-Q_0 \log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = 0.23$, $\delta = 0.22$ and jumps are distributed with a Dirac mass $\nu \sim \delta z_0$ with $z_0 = 1.0$. 
Figure 2: We plot $h(F_T)$ as a function of $F_T$ (solid blue). For comparison we also plot $-Q_0 \log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = -0.21$, $\delta = 1.00$ and jumps are distributed with a Dirac mass $\nu \sim \delta_{z_0}$ with $z_0 = -1.0$. 
Example 3: building around no-jump solution

Introduce $e_c(y) = e^{cy}$ and $\delta \geq 0$. Assume

$$a^2(y) = \sigma^2(y), \quad \mu(y, dz) = \delta \eta(y) \nu(dz), \quad \frac{2 \eta(y)}{\sigma^2(y)} = e_c(y).$$

Then $H$ solves

around no jump : \quad 0 = \delta e_c (A_0 H + I_2) + (A_1 H + 2), \quad (6)

Compare to

around pure jump : \quad 0 = (A_0 H + I_2) + \delta e_c (A_1 H + 2)

Just reverse roles of:

$A_0 \leftrightarrow A_1$ \quad and \quad 2 \leftrightarrow I_2,$

$\phi \lambda \leftrightarrow \chi \lambda$
Reversing roles of $\phi$ and $\chi$ (symbols of $\mathcal{A}_0$ and $\mathcal{A}_1$ resp.)

$$G = 2y + \sum_{n=1}^{\infty} \delta^n G_n,$$

$$G_n = (2I_0 - I_2) \frac{e_{nc}}{nc \cdot \chi_{-inc}} \prod_{k=0}^{n-1} \left( \frac{-\phi_{-ikc}}{\chi-ikc} \right), \quad n \geq 1.$$ 

Conditions for convergence:

- $\int_{\mathbb{R}} (e^{ncz} - 1 - ncz) \nu(dz) < \infty$ for all $n \in \mathbb{N}$,
- $\lim_{n \to \infty} \frac{\int_{\mathbb{R}} (e^{ncz} - 1 - ncz) \nu(dz)}{n^2 c^2} = 0$ (reciprocal of prev. cond.).

In figures 3 and 4 we plot $2 \log(F_T/F_0)$ and

$$h(F_T) := -G(\log F_T) + G(\log F_0) + A(F_T - F_0),$$

as a function of $F_T$. The constant $A$ is chosen so that $h(F_T)$ has the same slope as $-2 \log(F_T/F_0)$ at $F_T = F_0$. 
Figure 3: We plot \( h(F_T) \) as a function of \( F_T \) (solid blue). For comparison we also plot \(-2 \log(F_T/F_0)\) (dashed black). In this Figure, \( F_0 = 10.0, c = 0.39, \delta = 1.25 \) and \( \nu = \delta z_0 \) (Dirac measure) with \( z_0 = -1.50 \). Negative jumps raise value of VS relative to two log contracts.
Figure 4: We plot $h(F_T)$ as a function of $F_T$ (solid blue). For comparison we also plot $-2 \log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = -1.05$, $\delta = 1.00$ and $\nu = \delta z_0$ (Dirac measure) with $z_0 = 1.75$. Positive jumps lower value of VS relative to two log contracts.
Example 4: subordinate diffusions

Model forward price as

\[ F_t = \exp(Y_{\tau_t}^\phi), \quad Y_t^\phi = Y_{T_t} \]

where \( \tau \) is a continuous time-change, \( Y \) is a diffusion absorbed at endpoints \( L < R \)

\[ dY_t = -\frac{1}{2}\sigma^2(Y_t)dt + \sigma(Y_t)dW_t, \]

and \( T \) is a Lévy subordinator

\[ dT_t = b \, dt + \int_0^\infty zdN_t^\rho(dz), \quad \mathbb{E}dN_t^\rho(dz) = \rho(dz)dt. \]

The process \( Y^\phi \) experiences jumps because the subordinator \( T \) jumps.
Some spectral theory

The generator of $Y$, given by

$$\mathcal{A} = \frac{1}{2} \sigma^2(y)(\partial^2 - \partial),$$

$$\text{dom}(\mathcal{A}) = \{ f \in C^2([L, R]) : f(L) = f(R) = 0 \},$$

is self-adjoint on $L^2([L, R], m)$ where $m$ is the speed measure of $\mathcal{A}$

$$\langle f, \mathcal{A}g \rangle_m = \langle \mathcal{A}f, g \rangle_m, \quad m(y) = \frac{e^{-y}}{\sigma^2(y)}.$$

By the spectral theorem, the operator $g(\mathcal{A})$ is defined as

$$g(\mathcal{A}) = \sum_n g(\lambda_n) \langle \psi_n, \cdot \rangle_m \psi_n, \quad \mathcal{A}\psi_n = \lambda_n \psi_n.$$

In particular

resolvent : $$(\mathcal{A} - z)u = h \quad \Rightarrow \quad u = \frac{1}{\mathcal{A} - z} h,$$

semigroup : $u(t, y) = \mathbb{E}_y h(Y_t) \quad \Rightarrow \quad u(t, y) = e^{t\mathcal{A}} h(y).$$
$\mathcal{A}^\phi$ – the generator of $Y^\phi_t = Y_{T_t}$

The subordinator $T$ is characterized by its Laplace exponent

$$E e^{\lambda T_t} = e^{t \phi(\lambda)}, \quad \phi(\lambda) = b \lambda + \int_0^t \rho(ds)(e^{s \lambda} - 1).$$

We compute the semigroup $e^{t \mathcal{A}^\phi}$ of $Y^\phi$ as follows

$$e^{t \mathcal{A}^\phi} h(y) = \mathbb{E}_y h(Y^\phi_t) = \mathbb{E} \mathbb{E}_y[h(Y_{T_t})|T_t] = \mathbb{E} e^{T_t \mathcal{A}^\phi} h(y) = e^{t \phi(\mathcal{A})} h(y).$$

Therefore, the generator $\mathcal{A}^\phi$ is given by

$$\mathcal{A}^\phi = \lim_{t \to 0} \frac{1}{t} \left( e^{t \mathcal{A}^\phi} - 1 \right) = \lim_{t \to 0} \frac{1}{t} \left( e^{t \phi(\mathcal{A})} - 1 \right) = \phi(\mathcal{A}),$$

And the resolvent is given by

$$\frac{1}{\mathcal{A}^\phi - z} = \frac{1}{\phi(\mathcal{A}) - z}.$$
The function $G$ that prices the VS solves

$$A^\phi G(y) = h(y),$$

$$h(y) = -b \sigma^2(y) - \int_{L-y}^{R-y} \mu(y, dz) z^2,$$

$$\mu(y, dz) = \int_0^\infty \rho(ds) p_Y(s, y, y + z) dz,$$

$$p_Y(s, y, y + z) = e^{tA} \delta_{y+z}(y).$$

The solution can be written down directly:

$$G = \frac{1}{A^\phi} h = \frac{1}{\phi(A)} h = \sum_n \frac{1}{\phi(\lambda_n)} \langle \psi_n, h \rangle_m \psi_n.$$

Specific solutions are computed by solving $A\psi_n = \lambda_n \psi_n$ and computing $\phi(\lambda)$. 
Simple example

Let background process $Y$ have dynamics

$$dY_t = -\frac{1}{2} \sigma^2 dt + \sigma dW_t \quad \Rightarrow \quad A = \frac{1}{2} \sigma^2 (\partial^2 - \partial).$$

We need to solve eigenvalue problem

$$A \psi_n = \lambda_n \psi_n, \quad \psi_n(L) = \psi_n(R) = 0.$$ 

The solution is

$$\psi_n(y) = e^{y/2} \sqrt{\frac{\sigma^2}{R-L}} \sin \left( \alpha_n(y - L) \right), \quad \alpha_n = \frac{n\pi}{R-L},$$

$$\lambda_n = -\frac{\sigma^2}{2} \left( \alpha_n^2 + \frac{1}{4} \right), \quad n \in \mathbb{N}.$$

Let the Lévy density of the subordinator $T$ be exponential

$$\rho(ds) = C e^{-\eta s} ds, \quad \Rightarrow \quad \phi(\lambda) = b\lambda + \frac{C\lambda}{\eta^2 - \eta\lambda},$$
Figure 5: We plot $h(F_T) = -G(\log F_T) + G(\log F_0) + A(F_T - F_0)$, (solid blue) and $-Q \log(F_T/F_0)$ (dashed black) as a function of $F_T$. The Lévy measure $\rho$ of the subordinator is exponential: $\rho(ds) = C e^{-\eta s} ds$. In this Figure, $\sigma = 1$, $b = 0$ (i.e., no diffusion component), $C = 1$, $\eta = 1$, $L = -2$, $R = 1$, $F_0 = e^{Y_0 \phi} = 1$, $Q = 2$. 
Quick recap

We have shown that, in the Time-change Markov process setting, a VS has the same value as a European option with payoff $h(F_T) := -G(\log F_T) + G(\log F_0)$.

One can compute the value of this option $\mathbb{E} h(F_T)$ using co-terminal calls and puts

$$\mathbb{E} h(F_T) = h(\kappa) + h'(\kappa) \left( C(T, \kappa) - P(T, \kappa) \right)$$

$$+ \int_0^\kappa h''(K) P(T, K) dK + \int_\kappa^\infty h''(K) C(T, K) dK,$$

For certain special cases, we can also compute $\mathbb{E} h(F_T)$ directly from model parameters.

This will allow us to show how the ratio $\frac{-\mathbb{E} \log F_T}{\mathbb{E} \log(F_T/F_0)}$ varies as a function of $F_0$. 
Special case 1: European option pricing

Introduce \( \omega > 0 \) and \( \delta > 0 \). Let

\[
a^2(y) = 2 \omega^2, \quad \mu(y, dz) = \delta \omega^2 e_c(y) \nu(dz),
\]

This model falls under the “building around no jumps” setting of example 3. Thus, we know the function \( G \) that prices the VS. We wish to find \( u(t, y) := \mathbb{E}_y G(Y_t) \). From the KBE we have

\[
( - \partial_t + \mathcal{A} ) u = 0, \quad u(0, y) = G(y).
\]

where \( \mathcal{A} \) is the infinitesimal generator of \( Y \).
The generator $\mathcal{A}$ of $Y$

The process $Y$ has generator

$$\mathcal{A} = \delta e_c \mathcal{L}_0 + \mathcal{L}_1$$

with

$$\mathcal{L}_0 = \omega^2 \int_{\mathbb{R}} \nu(dz) \left( e^{z\partial} - 1 - z\partial \right) - \omega^2 \int_{\mathbb{R}} \nu(dz) \left( e^{z} - 1 - z \right) \partial,$$

$$\mathcal{L}_1 = \omega^2 \left( \partial^2 - \partial \right).$$

$\mathcal{L}_0$ and $\mathcal{L}_1$ are $\Psi$DOs with symbols $\Phi_\lambda$ and $\chi_\lambda$ respectively

$$\Phi_\lambda = \omega^2 \int_{\mathbb{R}} \nu(dz) \left( e^{i\lambda z} - 1 - i\lambda z \right)$$

$$- \omega^2 \int_{\mathbb{R}} \nu(dz) \left( e^{z} - 1 - z \right) i\lambda,$$

$$\chi_\lambda = \omega^2 \left( -\lambda^2 - i\lambda \right).$$
Assume $u$ is power series in $\delta$ (same game as for $H$)

$$u = \sum_{n \geq 0} \delta^n u_n.$$ 

Insert expansion for $u$ and $A = \delta e_c L_0 + L_1$ into KBE and collect like powers of $\delta$.

$$O(\delta^0) : \quad (-\partial_t + L_1)u_0 = 0, \quad u_0(0, y) = G(y),$$

$$O(\delta^n) : \quad (-\partial_t + L_1)u_n = -e_c L_0 u_{n-1}, \quad u_n(0, y) = 0.$$ 

The formal solution is

$$O(\delta^0) : \quad u_0(t, y) = e^{tL_1} G(y)$$

$$O(\delta^n) : \quad u_n(t, y) = \int_0^t ds \, e^{(t-s)L_1} e^{cy} L_0 u_{n-1}(s, y),$$

where the semigroup of operators $P_t := e^{tL_1}$ is given by

$$e^{tL_1} \cdot \psi = \int_\mathbb{R} d\lambda \, e^{tx_\lambda} \langle \psi_\lambda, \cdot \rangle \psi_\lambda.$$
Using $\langle \psi_\mu, e_c L_0 \psi_\lambda \rangle = \Phi_\lambda \delta(\lambda - \mu - ic)$, one can find an explicit expression for $u_n$:

$$u_n(t, y) = \int_{\mathbb{R}} d\lambda \left( \sum_{k=0}^{n} \frac{e^{t X_{\lambda-ikc}}}{\prod_{j \neq k}(X_{\lambda-ikc} - X_{\lambda-ijc})} \right) \ldots$$

$$\times \left( \prod_{k=0}^{n-1} \Phi_{\lambda-ikc} \right) \langle \psi_\lambda, G \rangle \psi_{\lambda-inc}(y).$$

- $u(t, y) := \mathbb{E}_y G(Y_t) = \sum_n \delta^n u_n$ is price of option with no time-change ($\tau_t = t$)

- To price option with independent time-change $\tau$, simply condition on $\tau_t$

$$v(t, y) := \mathbb{E}_y G(Y_{\tau_t}) = \mathbb{E} \mathbb{E}_y [G(Y_{\tau_t}) | \tau_t] = \mathbb{E} u(\tau_t, y).$$

Analytic formulas result as long as Laplace transform $\mathbb{E} e^{\lambda \tau_t}$ is known (e.g. $\tau_t = \int_0^t c_s ds$ where $c$ is CIR).
Figure 6: A plot of \( \left( \frac{\mathbb{E}_y [\log F]_T}{-\mathbb{E}_y \log (F_T/F_0)} \right) \) as a function of \( F_0 = e^y \) (blue).

\[ F_0 \to 0 : \quad \frac{\mu(y, \mathbb{R})}{a(y)} \to 0 \quad \left( \frac{\mathbb{E}_y [\log F]_T}{-\mathbb{E}_y \log (F_T/F_0)} \right) \to 2, \]

\[ F_0 \to \infty : \quad \frac{\mu(y, \mathbb{R})}{a(y)} \to \infty \quad \left( \frac{\mathbb{E}_y [\log F]_T}{-\mathbb{E}_y \log (F_T/F_0)} \right) \to Q_0 (= e!). \]
Special case 2: European option pricing

Return to the subordinated diffusion setting

\[ F_t = \exp(Y_{\tau_t}^\phi), \quad Y_t^\phi = Y_{T_t} \]

where \( \tau \) is a continuous time-change, \( Y \) is a diffusion absorbed at endpoints \( L < R \) and \( T \) is a Lévy subordinator. We know the function \( G \) that prices the VS. We also know how to compute

\[ u(t, y) = \mathbb{E}_y G(Y_t^\phi) = e^{tA^\phi} G(y) = \sum_n e^{t\phi(\lambda_n)} \langle \psi_n, G \rangle_m \psi_n(y). \]

If the time-change \( \tau \) is independent of \( Y^\phi \) and we know its Laplace transform \( \mathbb{E} e^{\lambda \tau_t} = L(t, \lambda) \), then the price can be computed by conditioning on \( \tau_t \)

\[ v(t, y) := \mathbb{E}_y G(Y_{\tau_t}^\phi) = \mathbb{E} \mathbb{E}_y [G(Y_{\tau_t}^\phi)|\tau_t] = \mathbb{E} u(\tau_t, y) = \sum_n L(t, \phi(\lambda_n)) \langle \psi_n, G \rangle_m \psi_n(y). \]
Figure 7: A plot of \( \left( \frac{\mathbb{E}_y \log F T}{-\mathbb{E}_y \log(F_T/F_0)} \right) \), as a function of \( F_0 = e^y \) for the subordinated diffusion model. We let the Lévy measure \( \rho \) of the subordinator be exponential: \( \rho(ds) = C e^{-\eta s} ds \) and we assume a deterministic clock \( \tau_t = t \). We use the following parameters: \( \sigma = 1 \), \( b = 1 \), \( C = 2.0 \), \( \eta = 1 \), \( L = -2 \), \( R = 1 \), \( T = 3 \).
1. We have shown that when $F$ is modeled as the exponential of a time-changed Markov process $F_t = \exp(Y_{\tau_t})$ the VS is priced by a European option whose payoff $G$ depends only on the dynamics of $Y$ – not on the time-change $\tau$.

2. For certain cases, we can explicitly compute the function $G$ that prices the VS.

3. When $Y$ is a Lévy process we recover the results of Carr, Lee, and Wu (2011) ($G(\log F_T) = \log F_T$).

\[ \text{Carr-Lee-Wu} \subset \text{Carr-Lee-Lorig} \]

4. When $Y$ is not a Lévy process we find that the ratio 
\[ \left( \frac{E_y [\log F]_T}{-E_y \log(F_T/F_0)} \right) \] 
depends on the value of $F_0 = e^y$. 

Review

