Three-dimensional Brownian motion and its applications to trading

Alexander Lipton

Bank of America & University of Oxford

April 2014
“The Dutch call the option business “opsies,” a term derived from the Latin word optio, which means choice, because the payer of the premium has the choice of delivering the shares to the acceptor of the premium or demanding them from him.”

de la Vega, Joseph (1688) Confusion de Confusiones, Amsterdam.
What is Mathematical Finance?

Mathematical Finance is one of several competing scientific disciplines aimed at understanding the behavior of financial markets. There are other disciplines aiming to achieve similar goals, such as

- Economics;
- Theory of Finance;
- Econometrics;
- Statistics;
- Econo-physics;
- Actuarial science;
- Etc.

Practitioners of each of these disciplines claim to possess a unique set of tools and a special angle to deal with financial markets. There is a lot of friendly (and not so friendly) competition among them.
Mathematical Finance covers a large variety of financial instruments

Important asset classes and topics of interest:

- Equities;
- Non-risky debt;
- Risky debt;
- Forex;
- Commodities;
- Derivatives written on the above assets;
- Structure of financial exchanges and market intermediation (including market making);
- Asset and liability management and theory of investing;
- Risk management;
- Capital calculations, etc.
Constituent parts

- Investors;
- Pension funds;
- Asset managers;
- Banks;
- Insurance companies;
- Hedge funds;
- Various supporting players, such as custodians, credit rating agencies, etc.
Purposes of investing

- Wealth preservation;
- Wealth creation;
- Life-time wealth transfer;
- Intergenerational wealth transfer.

"The importance of money essentially flows from it being the link between the present and the future." John Maynard Keynes
Mathematical models

- Stochastic nature of financial time series and their non-stationarity;
- Different time scales of modern finance: from milliseconds to decades;
- Different intellectual sources for model building: probability theory, numerical methods, statistics, economics (the weakest link);
- Main difficulty: "fallacy of the historical data series";
- Main purposes of the models: investing, trading, risk management, speculation;
- Important point: risk-neutral probability vs. real-world probability pricing of derivatives.
Brownian motion (BM)

- 1D BM $W_t$ on a positive semi-axis;
- 2D BM $(W^1_t, W^2_t)$, $dW^1_t dW^2_t = \rho^{12} dt$, in a positive quadrant;
- 3D BM $(W^1_t, W^2_t, W^3_t)$, $dW^i_t dW^j_t = \rho^{ij} dt$, in a positive octant.

Our strategy is to construct the corresponding Green’s function and apply it to solve the relevant financial math problems.

In 1D case it is very simple, in 2D case it is not too complicated, in 3D case it is very difficult.
One-dimensional case, Green’s function on positive semi-axis

Green’s function can be constructed via the method of images:

\[
G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi \tau}} \left( e^{-\frac{(y'-y_0)^2}{2\tau}} - e^{-\frac{(y'+y_0)^2}{2\tau}} \right).
\]

Alternatively, Green’s function can be constructed via the eigenfunction expansion method:

\[
G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(k y_0) \sin(k y) e^{-\frac{k^2 \tau}{2}} dk.
\]

Needless to say that these expressions are in agreement with each other.
Finite interval $0 \leq y \leq L$

Green’s function can be constructed via the method of images:

$$G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi\tau}} \sum_{n=-\infty}^{\infty} \left( e^{-\frac{(y'-y_0+2nL)^2}{2\tau}} - e^{-\frac{(y'+y_0+2nL)^2}{2\tau}} \right).$$

Alternatively, Green’s function can be constructed via the eigenfunction expansion method:

$$G(\tau, y_0, y') = \frac{2}{L} \sum_{n=1}^{\infty} \sin \left( \frac{2\pi n}{L} y_0 \right) \sin \left( \frac{2\pi n}{L} y' \right).$$

Needless to say that these expressions are in agreement with each other. Close relation to the Poisson summation formula!
Two-dimensional case

General pricing equation for the value function $V(t, x, y)$ in the positive quadrant:

$$V_t + \frac{1}{2} V_{xx} + \frac{1}{2} V_{yy} + \rho_{xy} V_{xy} - \varrho V = 0.$$ 

It is supplied with appropriate boundary and terminal conditions at $x = 0$, $y = 0$, $t = T$.

Changes of independent and dependent variables

$$U(t, x, y) = e^{\varrho(T-t)} V(t, x, y),$$

$$\alpha = x, \quad \beta = (-\rho_{xy}x + y)/\bar{\rho}_{xy},$$

$$\alpha = r\sin \varphi, \quad \beta = r\cos \varphi,$$

$$\bar{\rho}_{xy} = \sqrt{1 - \rho_{xy}^2}.$$ 

Final form of the pricing equation:

$$U_t + \frac{1}{2} \left( U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} \right) = 0.$$ 

The new domain is an angle $0 \leq \varphi \leq \omega$, where \( \omega = \arccos(-\rho_{xy}) \).
Green’s function via the eigenfunction expansion method

Green’s function solves the forward equation:

\[ G_{\tau} - \frac{1}{2} \left( G_{r' r'} + \frac{1}{r'} G_{r'} + \frac{1}{r'^2} G_{\varphi' \varphi'} \right) = 0. \]

Initial condition:

\[ G(0, r', \varphi') = \frac{1}{r_0} \delta(r' - r_0) \delta(\varphi' - \varphi_0). \]

Boundary conditions:

\[ G(\tau, r', 0) = 0, \quad G(\tau, r', \varpi) = 0, \quad G(\tau, 0, \varphi') = 0, \quad G(\tau, r', \varphi') \rightarrow 0 \quad \text{as} \quad r' \rightarrow \infty. \]

Solution obtained through the eigenfunction expansion method:

\[ G(\tau, r_0, r', \varphi_0, \varphi') = 2e^{-\frac{r'^2 + r_0^2}{2\tau}} \sum_{n=1}^{\infty} l_{\nu_n} \left( \frac{r' r_0}{\tau} \right) \sin(\nu_n \varphi') \sin(\nu_n \varphi_0), \]

where \( \nu_n = \frac{n\pi}{\varpi} \).
Green’s function via the method of images

Define $\psi = \phi' - \phi_0$, $s_\pm = \text{sign} (\pi \pm \psi)$, and $f(p, q)$, $h(p, q)$, $p \geq 0$, $-\infty < q < \infty$,

$$f(p, q) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-p (\cosh(2q\zeta) - \cos(q))} \frac{d\zeta}{\zeta^2 + \frac{1}{4}}$$

$$h(p, q) = \frac{1}{2} [s_+ f(p, \pi + q) + s_- f(p, \pi - q)].$$

We can define a non-periodic solution of the heat equation as follows

$$H(\tau, r_0, r', \phi_0, \phi') = \frac{1}{2\pi \tau} e^{-\frac{r'^2 + r_0^2 - 2\cos(\psi)r'r_0}{2\tau}} h\left(\frac{r'r_0}{\tau}, \psi\right),$$

and obtain Green’s function through the method of images:

$$G(\tau, r_0, r', \phi_0, \phi') = \sum_{n=-\infty}^{\infty} \left[ H(\tau, r_0, r', \phi_0 + 2n\omega, \phi') - H(\tau, r_0, r', -\phi_0 + 2n\omega, \phi') \right].$$
Three-dimensional case

General pricing equation for the value function $V(t, x, y)$ in the positive quadrant:

$$V_t + \frac{1}{2} V_{xx} + \frac{1}{2} V_{yy} + \frac{1}{2} V_{zz} + \rho_{xy} V_{xy} + \rho_{xz} V_{xz} + \rho_{yz} V_{yz} - \rho V = 0.$$  

It is supplied with appropriate boundary and terminal conditions at $x = 0$, $y = 0$, $z = 0$, $t = T$.

Changes of independent and dependent variables (Cholesky decomposition)

$$U(t, x, y, z) = e^{\rho(T-t)} V(t, x, y, z),$$

$$\alpha = x, \quad \beta = (-\rho_{xy} x + y) / \tilde{\rho}_{xy},$$

$$\gamma = \left( (\rho_{xy} \rho_{yz} - \rho_{xz}) x + (\rho_{xy} \rho_{xz} - \rho_{yz}) y + \tilde{\rho}_{xy}^2 z \right) / \tilde{\rho}_{xy} \chi,$$

$$\alpha = r \sin \varphi \sin \theta, \quad \beta = r \cos \varphi \sin \theta, \quad \gamma = r \cos \theta,$$

$$\tilde{\rho}_{xy} = \sqrt{1 - \rho_{xy}^2}, \quad \chi = \sqrt{1 - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 + 2 \rho_{xy} \rho_{xz} \rho_{yz}}.$$
Three-dimensional pricing problem

Pricing problem:

\[ U_t + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial}{\partial r} (rU) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi \varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right) \right] = 0, \]

Computational domain (defined parametrically as function of \( \omega \)):

\[ r > 0, \]
\[ 0 \leq \varphi \leq \omega, \]
\[ \varphi (\omega) = \arccos \left( \frac{1 - \rho_{xy} \omega}{\sqrt{1 - 2 \rho_{xy} \omega + \omega^2}} \right), \]
\[ \theta (\omega) = \arccos \left( \frac{\rho_{xy} \rho_{xz} - \rho_{yz} + (\rho_{xy} \rho_{yz} - \rho_{xz}) \omega}{\sqrt{\rho_{xy} (\rho_{xz}^2 + 2 \omega (\rho_{xz} \rho_{yz} - \rho_{xy}) \omega + \rho_{yz}^2 \omega^2)}} \right), \]
\[ \theta = \Theta (\varphi). \]
Figure:
Green’s function satisfies the forward equation:

\[
G_{\tau} - \frac{1}{2} \left[ \frac{1}{r'} \frac{\partial^2}{\partial r'^2} (r' G) + \frac{1}{r'^2} \left( \frac{1}{\sin^2 \theta'} G_{\phi \phi'} + \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left( \sin \theta' G_{\theta'} \right) \right) \right] = 0,
\]

with initial condition:

\[
G(0, r', \phi', \theta') = \frac{1}{r_0^2 \sin \theta_0} \delta (r' - r_0) \delta (\phi' - \phi_0) \delta (\theta' - \theta_0),
\]

and boundary conditions:

\[
G(\tau, r', 0, \theta') = G(\tau, r', \phi_0, \theta') = G(\tau, r', \phi', 0) = 0,
\]

\[
G(\tau, r', \phi', \Theta(\phi')) = G(\tau, 0, \phi', \theta') = 0, \quad G(\tau, r', \phi', \theta') \xrightarrow{r' \to \infty} 0.
\]
Eigenfunction expansion for Green’s function

- Construct Green’s function by using eigenfunction expansion method.
- Separation of variables yields \( G(\tau, r', \varphi', \theta') = g(\tau, r') \Psi(\varphi', \theta') \).
- Similar to 2D case, the radial part has the form
  \[
g(\tau, r') = e^{-\frac{r'^2 + r_0^2}{2\tau}} \frac{1}{\tau \sqrt{r' r_0}} \sqrt{\Lambda^2 + 1/4} \left( \frac{r' r_0}{\tau} \right).
  \]
- The angular part solves the following 2D eigenvalue problem:
  \[
  \frac{1}{\sin^2 \theta'} \Psi_{\varphi' \varphi'} + \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} (\sin \theta' \Psi_{\theta'}) = -\Lambda^2 \Psi,
  \]
  \[
  \Psi(0, \theta') = 0, \quad \Psi(\varphi, \theta') = 0, \quad \Psi(\varphi', 0) = 0, \quad \Psi(\varphi', \Theta(\varphi')) = 0.
  \]
Solution of 2D eigenvalue problem

- The corresponding eigenvalue problem is solved via the finite element method (FEM);
- The spherical domain is mapped onto the domain $\Omega$ in the $(\varphi, \theta)$ plane;
- The variational (weak) formulation of the problem is used:

$$
\int_{\Omega} \frac{1}{\sin \theta'} \Psi_{\varphi'} \overline{\Psi}_{\varphi'} d\Omega + \int_{\Omega} \sin \theta' \Psi_{\theta'} \overline{\Psi}_{\theta'} d\Omega = \Lambda^2 \int_{\Omega} \sin \theta' \Psi \overline{\Psi} d\Omega.
$$

- The domain $\Omega$ is triangulated by using iterative algorithm to construct adaptive mesh;
- After each iteration the Delaunay triangulation method is used.
Figure 8: $\rho_{xy} = 0, \quad \rho_{xz} = 0, \quad \rho_{yz} = 0$

Figure 9: $\rho_{xy} = 0.8, \quad \rho_{xz} = 0.2, \quad \rho_{yz} = 0.5$
Mesh, source Lipton & Savescu

Figure:

(a) First iteration mesh

(b) Mesh after 100 iterations
Details of FEM

- Dimension of the space in which eigenvectors are searched for is equal to the number of free points in the mesh;
- Basis in this space is represented by basis functions $\Phi_i$, $1 \leq i \leq n$, linear in each triangle;
- The variational problem is approximated by a linear system
  \[ K\Psi = \Lambda^2 M\Psi. \]
- Here $K = (K_{ij})$ is the stiffness matrix, $M = (M_{ij})$ is the mass matrix:
  \[ K_{ij} = \int_{\Omega} \nabla \Phi_i (A \nabla \Phi_j) \, d\Omega, \]
  \[ M_{ij} = \int_{\Omega} \Phi_i \Phi_j \sin \theta' \, d\Omega, \]
  \[ A = \begin{pmatrix} 1 & 0 \\ \frac{\sin \theta'}{\sin \theta'} & \sin \theta' \end{pmatrix}. \]
Eigenvectors, source Lipton & Savescu,

\[ \rho_{xy} = 80\%, \rho_{xz} = 20\%, \rho_{yz} = 50\% \]

Figure:
Green’s function

Eigenfunction expansion for Green’s function:

\[ G(\tau, r', \varphi', \theta') = \sum_{n=1}^{\infty} C_n g_n(\tau, r') \Psi_n(\varphi', \theta'). \]

Coefficients \( C_n \) - can be computed by imposing the initial condition:

\[ G(0, r', \varphi', \theta') = \frac{1}{r_0^2 \sin \theta_0} \delta(r' - r_0) \delta(\varphi' - \varphi_0) \delta(\theta' - \theta_0), \]

and we obtain \( C_n = \Psi_n(\varphi_0, \theta_0) \).

Final formula for Green’s function:

\[ G(\tau, r_0, r', \varphi_0, \varphi', \theta_0, \theta') = e^{-\frac{r'^2+r_0^2}{2\tau}} \sum_{n=1}^{\infty} \frac{1}{\tau \sqrt{r' r_0}} \sqrt{\Lambda_n^2+\frac{1}{4}} \left( \frac{r' r_0}{\tau} \right) \Psi_n(\varphi_0, \theta_0) \Psi_n(\varphi', \theta'). \]
Historically, "HFT" technologies were opposed to: (A) Racing pigeons; (B) Telegraph; (C) Telephone; (D) Radio; (E) Screen trading; etc.

Yet, various HFT strategies persisted in spite of these objections. Here is an interesting example.

In 15th century Florence state built galleys to send goods to London and Bruges. These were leased to the highest bidder, who, in turn, subleased them to others. An auction lasted until a candle burned out. Since everyone tried to put their bid as late as possible, rules were changed. Now an auction would end with the clock on the tower of Palazzo della Signoria, which was audible but not visible.
LOB

LIMIT SELL ORDERS

SUPPLY

MARKET SELL ORDER

BEST ASK

BEST BID

MARKET BUY ORDER

LIMIT BUY ORDERS

DEMAND

Figure:

A Lipton (Bank of America & University of C)

Three-dimensional Brownian motion

04/21 26 / 44
Algorithmic traders in high frequency (HF) electronic markets can be roughly divided into the following categories:

- Market makers, who provide liquidity and try to capture bid-ask spread;
- Systematic traders and arbitrageurs, who try to profit from price dislocations and statistical relationships among different prices;
- Agency brokers, who execute large trades for clients and earn fees.

We are interested in the agency broker point of view.
Figure:

- Three-dimensional Brownian motion
A simple model

We model the number of shares $q^b$ and $q^a$ posted at a simple stochastic process in a positive quadrant:

$$\left( dq^b, dq^a \right) = \left( dW^b, dW^a \right),$$

where $W^b, W^a$ are correlated Brownian motions.

To capture the joint dynamics of the bid and ask queues and trade arrival, we introduce another stochastic process to model the arrival of trades on the near side of the book:

$$\left( dq^b, dq^a, d\phi \right) = \left( dW^b, dW^a, dW^\phi \right).$$

In what follows, we follow a recent paper "Trade arrival dynamics and quote imbalance in a limit order book" by A. Lipton, U. Pesavento, M. Sotiropoulos. References to prior art are mentioned in the end.
By using Cholesky-style transformations, we can write the exit probability problem on the computational interval $[0, \omega]$ in the form

$$P_{\phi\phi}(\phi) = 0,$$

$$P(0) = 0, \quad P(\omega) = 1.$$

Its solution is straightforward

$$P(\phi) = \frac{\phi}{\omega}.$$

When expressed in the original $(x, y)$ coordinates, this probability has the form

$$P(x, y) = \frac{1}{2} \left( 1 - \frac{\arctan \left( \sqrt{\frac{1+\rho}{1-\rho}} \frac{y-x}{y+x} \right)}{\arctan \left( \sqrt{\frac{1+\rho}{1-\rho}} \right)} \right).$$
2D-Hitting Probability

Figure: Hitting probability in 2D.

A Lipton (Bank of America & University of Oxford)
3D-Problem formulation

- We wish to solve the following classical problem - compute the exit probability for 3D Wiener process in a positive octant with absorbing boundaries. To the best of our knowledge our solution is new.

- By using Cholesky-style transformations, we can write the exit probability problem in the computational domain \( \Omega \) in the form

\[
\frac{1}{\sin^2 \theta} P_{\phi \phi} (\phi, \theta) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta P_{\theta} (\phi, \theta)) = 0,
\]

\[
P (0, \theta) = 0, \quad P (\varpi, \theta) = 0, \quad P (\phi, \Theta (\phi)) = 1.
\]

- Introduce new variable \( \zeta = \ln \left( \tan \left( \frac{\theta}{2} \right) \right) \) and rewrite the exit problem as follows

\[
P_{\phi \phi} (\phi, \zeta) + P_{\zeta \zeta} (\phi, \zeta) = 0,
\]

\[
P (0, \zeta) = 0, \quad P (\varpi, \zeta) = 0, \quad P (\phi, Z (\phi)) = 1.
\]

- Computational domain is now a semi-infinite strip with curvilinear boundary

\[
\zeta = Z (\phi) = \ln \left( \tan \left( \frac{\Theta (\phi)}{2} \right) \right).
\]
We look for the solution of the Dirichlet problem for the Laplace equation in the form

\[ P(\phi, \zeta) = \sum_{n=1}^{\infty} c_n \sin(k_n \phi) e^{k_n \zeta}, \quad k_n = \frac{\pi n}{\omega}. \]

It is clear that each individual term is a harmonic function. We choose coefficients \( c_n \) in such a way that \( P(\phi, Z(\phi)) = 1 \). Specifically,

\[ \sum_{n=1}^{\infty} c_n \sin(k_n \phi) e^{k_n Z(\phi)} = 1. \]

Thus, we need to build a theory of Fourier series expansion with respect to the following set of (non-orthonormal) basis functions

\[ E_n(\phi) = \sin(k_n \phi) e^{k_n Z(\phi)}, \]

\[ \sum_{n=1}^{\infty} c_n E_n(\phi) = 1. \]
Introduce the following integrals

\[ J_{mn} = \int_0^\omega \sin(k_m\phi) \sin(k_n\phi) e^{(k_m+k_n)Z(\phi)} d\phi, \]

\[ I_m = \int_0^\omega \sin(k_m\phi) e^{k_mZ(\phi)} d\phi. \]

It is clear that

\[ \sum_n J_{mn} c_n = I_m, \]

\[ \overrightarrow{c} = \widehat{J}^{-1} \overrightarrow{I}. \]

In general, this matrix problem is difficult to invert on the boundary \( \zeta = Z(\phi) \), however, away from the boundary automatic regularization kicks in, and everything works very well.

When the boundary is (approximately) linear, the corresponding integrals can be found analytically.

As a by-product, we managed to find a new(?) solution of a long-standing problem of exit probabilities for three correlated Brownian motions in a positive octant.
3D-Hitting probability, source Lipton, Pesavento, Sotiropoulos

Figure: Three-dimensional Brownian motion
Figure: Three-dimensional Brownian motion

- unfavourable price move
- favourable price move
- trade at near side

Event probability vs. time.
Conclusions

- Mathematical finance is a thriving discipline which poses extremely intricate and important questions.
- It requires a diverse skill set and ability to apply both sophisticated and simple tools in an appropriate fashion.
- Its future belongs to people who are willing to analyze very big data, and are able to extract some sense out of a seemingly never ending time-series of disjoint data points.
- Successful practitioners and academic should be prepared to deal with extraordinary large data sets, industrial strength problems of mind-boggling complexity, and heavy computational burdens.
- The time of closed-form solutions is (sadly) passed once and for all, or is it?
- Come join the party!!


Bibliography - IV


The views and opinions expressed in this presentation are those of the author and do not necessarily reflect the views and opinions of Bank of America Merrill Lynch.