Optimal Investment, Derivative Demand & Arbitrage under Price Impact

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Joint work with S. Robertson (BU) and K. Spiliopoulos (BU)

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Outline

1. Motivation & goals
2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks
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Large Investor’s Price Impact

- **Large** in the sense that their orders substantially change MMs’ inventory and hence their quoted prices. → *causing price impact.*
- MMs are risk averse.
- Given MMs price-quoting, each investor places his flow of orders, aiming to increase his individual expected utility.

**Goal No. 1**

*Find the continuous-time optimal investment strategy under price-impact.*

- A key step: Optimal investment problem *upon market impact* can be written as a constrained optimal investment problem in a *fictitious market without market impact.*
  - Impose conditions that make the constraint set *non-binding.*
  - Exploit this representation to *solve the problem* (when possible).
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What about derivative pricing and demand under price impact?

→ Hedging costs are not linear anymore.
→ Standard arbitrage-free arguments should be revisited.
→ Even if there is a derivative price that creates arbitrage, the induced gains are limited, due to price impact.
→ Since, investors are utility maximizers, they may optimally ignore an arbitrage!

Goal No. 3

Could these arbitrage prices arise endogenously?

→ Indeed, through a partial equilibrium argument in segmented markets of the underlying assets.
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We begin with \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\), where \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) is the natural filtration of a \(d\)-dimensional Brownian Motion and \(T > 0\) the terminal time.

A random vector \(\Psi \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R}^d)\) denotes the payoff of the tradeable assets.

There are \(M\) risk averse market makers (MMs) that quote prices for \(\Psi\) at any time \(t \in [0, T]\).

The utility function of \(k\)th MM for terminal wealth is denoted by \(U_k\) and his endowment by \(\Sigma^k_0 \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R})\), \(k = 1, 2, ..., M\) and

**Standing assumptions on utilities:** strict concavity, increasing, smooth on whole \(\mathbb{R}\) with bounded absolute risk aversion.
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Market Makers’ Pricing Rule

- Let \( \{ Q_t \}_{t \in [0, T]} \) denote the aggregate order flow to MMs.
- Let \( X_t(Q_t)_{t \in [0, T]} \) be the cash balance (price) asked by all the MMs at time \( t \).
- The way \( X_t(Q_t) \) is determined is the following:

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| is redistributed among MMs in a Pareto optimal way and |
| each MM remains at indifference, i.e., there is no increase on the expected utility by entering into the trading of \( \Psi \).

✓ When all MMs have exponential utility: \( \rightarrow X_t(Q_t) \) is the indifference pricing of the representative MM with exponential utility and endowment \( \Sigma_0 := \sum_{k=1}^{M} \Sigma_k \) and some risk aversion \( \gamma \).
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M. Anthropelos (Un. of Piraeus)
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Some notation

Standing Assumption

For all \( q \in \mathbb{R} \), \( \mathbb{E}[e^{-\gamma \Sigma_0 + q|\psi|}] < \infty \).

Under the above assumption, \( N_t(q) := \mathbb{E} \left[ e^{-\gamma \Sigma_0 - \gamma q' \psi} \big| \mathcal{F}_t \right], \quad t \leq T, \)

is a strictly positive martingale, and by martingale representation we may write

\[
\frac{N_t(q)}{N_0(q)} = \mathcal{E} \left( \int_0^t H_s(q)' dB_s \right), \quad t \leq T,
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for some adapted process \( H(q) \) such that \( \int_0^T |H_t(q)|^2 dt < \infty \).

Then, define the class of processes

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Q \in \mathcal{A}_{Pl} := \left\{ Q \text{ adapted s.t. } \int_0^T |H_t(Q_t)|^2 dt < \infty \right\},
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A useful representation

Consider a large investor who submits order flow \( \{Q_t\}_{t \in [0,T]} \) to the MM(s).

Let \( \{V_t(Q_t)\}_{t \in [0,T]} \) be his gains process, i.e. \( V_t(Q_t) \) represents the cash amount that he gets if he sells at time \( t \) his cumulative orders. For instance,

\[
V_T(Q_T) = -X_T(Q_T) + Q_T' \Psi.
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Based on the results of Bank & Kramkov ['15], we get the following:

Proposition

For \( Q \in A_{Pl} \), and for all \( t \in [0, T] \), the gains process takes the form

\[
V_t(Q_t) = \frac{1}{\gamma} \int_0^t \left( H_s(Q_s) - H_s(0) \right)'(dB_s - H_s(0)ds) - \frac{1}{2\gamma} \int_0^t |H_s(Q_s) - H_s(0)|^2 ds
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Investor’s investment problem, under endowment \( \Sigma_1 \)

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u(x; \Sigma_1) := \sup_{Q \in A_{Pl}} \mathbb{E}[U(x + V_T(Q) + \Sigma_1)].
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A fictitious related market

Define

$$\frac{dS_t}{S_t} = \lambda_t dt + dB_t, \quad t \in [0, T],$$

for an adapted $d$-dimensional process $\lambda$, such that $\int_0^T |\lambda_t|^2 dt < \infty$.

By construction, there is a unique measure $\mathbb{Q}_0$ on $\mathcal{F}_T$ under which $S$ is a martingale. $\mathbb{Q}_0$ has density

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}}|_{\mathcal{F}_T} = \mathcal{E} \left( - \int_0^T \lambda'_t dB_t \right).$$

Self-financing trading strategies are denoted by $\pi$ (proportions of wealth) and the induced wealth process' dynamics

$$\frac{dX_t(\pi)}{X_t(\pi)} = \pi'_t (\lambda_t dt + dB_t), \quad t \in [0, T].$$

With initial wealth $X_0 = e^{\gamma x}$ the terminal wealth is

$$X_T(\pi) = \exp \left( \gamma x + \int_0^T \pi'_t (dB_t + \lambda_t dt) - \frac{1}{2} \int_0^T |\pi_t|^2 dt \right).$$
Some simple observations

Recall that large investor’s gain process is

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\[ X_T(\pi) = e^{\gamma x + \gamma V_T(Q)} \implies x + V_T(Q) = \frac{1}{\gamma} \log(X_T(\pi)) \]

- For \( Q \in A_{Pl} \) we can construct \( \pi \). For the reverse we need \( \pi_t \) to belong in the random constraint set \( K^0_t \), where

\[ K_t := \{ H_t(q) : q \in \mathbb{R}^d \}, \quad K^0_t := \{ H_t(q) - H_t(0) : q \in \mathbb{R}^d \} \]

- Therefore, we define the acceptable strategies

\[ A := \left\{ \pi \text{ adapted} : \int_0^T |\pi_t|^2 dt < \infty \right\}, \quad A_C := \left\{ \pi \in A : \pi_t \in K^0_t, \ t \leq T \right\} \]
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A representation of optimal investment problem

Define the utility field \( \tilde{U}(w, \omega) : (0, \infty) \times \Omega \) by

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\tilde{U}(w, \omega) := U \left( \frac{1}{\gamma} \log(w) + \Sigma_1(\omega) \right),
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and the value functions

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\tilde{u}_C(x; \Sigma_1) := \sup_{\pi \in A_C} \mathbb{E}[\tilde{U}(X_T(\pi), \Sigma_1) | X_0 = e^{\gamma x}],
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With the above notation

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An indicative example

- Let $U(x) = -e^{-\alpha x}$ and define

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\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{e^{-\alpha \Sigma_1}}{\mathbb{E}[e^{-\alpha \Sigma_1}]}.
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Proposition

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u(0; \Sigma_1) = \tilde{u}_C(0; \Sigma_1) = \frac{\alpha}{\gamma} \mathbb{E}[e^{-\alpha \Sigma_1}] \left( \sup_{\pi \in \mathcal{A}_C} \tilde{\mathbb{E}} \left[ \frac{1}{p} (X_T(\pi))^p | X_0 = 1 \right] \right)
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On constrained problem

- In the related literature on utility maximization under random constrains the standard assumption is that constrained set is convex and closed.
- However, here $\mathcal{K}_t^o$ is typically neither convex nor closed!

**Bachelier model**

Let

$$
\Sigma_0 = \int_0^T f'_t dB_t \quad \text{and} \quad \Psi = \int_0^T \psi'_t dB_t,
$$

where $f \in L^2([0, T]; \mathbb{R}^d)$ and $\psi \in L^2([0, T]; \mathbb{R}^{d \times d})$. Then,

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Hence, if $\psi_t$ is invertible, $\mathcal{K}_t = \mathcal{K}_t^o = \mathbb{R}^d$ with $\pi_t = H_t(Q_t) - H_t(0)$ implying $Q_t = - (\gamma \psi_t)^{-1} \pi_t$. 
On constrained problem

- In the related literature on utility maximization under random constrains the standard assumption is that constrained set is convex and closed.
- However, here $\mathcal{K}_t^0$ is typically neither convex nor closed!

Bachelier model

Let

$$\Sigma_0 = \int_0^T f_t' dB_t \quad \text{and} \quad \Psi = \int_0^T \psi_t' dB_t,$$

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Solving the optimal investment problem

Impose the standing assumption and that large investor has exponential utility. Then,

\[
e^{-\frac{\alpha \gamma}{\alpha + \gamma} (\Sigma_1 + \Sigma_0)} \mathbb{E} \left[ e^{-\frac{\alpha \gamma}{\alpha + \gamma} (\Sigma_1 + \Sigma_0)} \right] = \mathcal{E} \left( \int_0^T M'_t dB_t \right),
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for some adapted process \( M \) with \( \int_0^T |M_t|^2 dt < \infty \).

A key assumption

\[ M_t \in \mathcal{K}_t, \quad \forall t \in [0, T]. \]

Proposition

Under the standing and key assumptions, the constrain set is non-binding and in fact,

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u(0; \Sigma_1) = \tilde{u}_C(0; \Sigma_1) = \tilde{u}(0; \Sigma_1)
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Example I

Endowments as portfolios of $Ψ$

Let $Σ_0 = k_0'Ψ$ and $Σ_1 = k_1'Ψ$ for some $k_0, k_1 ∈ \mathbb{R}^d$. Recall that

$$N_t(q) := \mathbb{E} \left[ e^{-γΣ_0 - γq'Ψ} \mid \mathcal{F}_t \right], \quad \text{and} \quad \frac{N_t(q)}{N_0(q)} = \mathbb{E} \left( \int_0^t H_s(q)' dB_s \right)_t.$$ 

We immediately get that

$$M_t = H_t \left( \frac{αk_1 - γk_0}{α + γ} \right), \quad \text{i.e.,} \quad M_t ∈ \mathcal{K}_t, \; ∀t ∈ [0, T]$$

and since $\hat{π}_t = M_t - H_t(0)$, we also get that

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We have a similar situation when $Σ_0 = k_0'Ψ + Y_0$ and $Σ_1 = k_1'Ψ + Y_1$, where $(Y_0, Y_1)$ and $Ψ$ are independent.
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**Bachelier Model**

- Recall that $\Sigma_0 = \int_0^T f_t' dB_t$ and $\Psi = \int_0^T \psi_t' dB_t$.
- We have seen that $\mathcal{K}_0 = \mathbb{R}^d$, so the crucial assumption holds.
- Assume also that $\Sigma_1 = \int_0^T g_t' dB_t$.
- What is the optimal demand?
- We have seen that $H_t(q) - H_t(0) = -\gamma \psi_t q$, and we readily have that $\forall t \in [0, T]$

$$M_t = -\frac{\alpha \gamma}{\alpha + \gamma} (f_t + g_t) \quad \text{and} \quad \hat{\pi}_t = M_t - H_t(0) = \frac{\gamma}{\alpha + \gamma} (\gamma f_t - \alpha g_t).$$

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Outline

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2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks
Introducing a derivative contract

- Consider a single contingent claim with $\mathcal{F}_T$ measurable payoff $h$.
- MMs do not make the market of $h$.
- However, investor could hedge his positions on $h$ by trading the underlying market $\Psi$ through MMs.
- Note that if $K^0_t = \mathbb{R}^d$, $\forall t \in [0, T]$, large investor can fully hedge.
- Indeed, for every $u \neq 0$ units of $h$, there is an order flow $Q \in \mathcal{A}_{PI}$ and a (per unit) initial capital $\overline{h}(u)$ such that

$$u\overline{h}(u) + V_T(Q) = uh.$$  

- In fact, $\overline{h}(u)$ is the MM’s indifference value of selling $u$ units of $h$, given by

$$\overline{h}(u) := \frac{1}{\gamma_u} \log \left( \mathbb{E}^0 \left[ e^{-\gamma u h} \right] \right).$$

- Note that the value $\overline{h}(u)$ is increasing for $u > 0$, but not linear.
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Derivative pricing, price impact and *arbitrage*

**Arbitrage-free price for all positions**

A price $p \in \mathbb{R}$ is an arbitrage-free price for all position in $h$, when:

For all $Q \in \mathcal{A}_{PI}$ and $u \in \mathbb{R}$, if $up + V_T(Q) - uh \geq 0$ a.s., then $up + V_T(Q) = uh$ a.s.

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- If the large investor gets a price for $h$ different than $\mathbb{E}^0[h]$, an arbitrage opportunity arises thanks to his price impact.

- However, because of not linearity, the arbitrage cannot be exploited for arbitrarily large units of $h$.

- In other words, the gains from the arbitrage are limited up to a certain position $u^*$.

- Note however that large investors is a utility maximizer with hedging needs. Hence, exploiting the limited arbitrage may be less preferable than reducing the risk exposure.

- Investor may ignore certain cash in favor of a higher expected utility.

✓ But who is going to ask/bid an arbitrage price?
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Partial equilibrium in segmented markets

- Suppose that there are two large investors, labeled A and B.
- They trade with different MMs in segmented markets (possibly with different securities too).
- The large investors trade to each other the derivative $h$ at a partial equilibrium price & quantity (PEPQ), as introduced by A. and Žitković ['10].

A PEPQ of $h$ is a pair $(p^*, u^*) \in \mathbb{R}^2$ if

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Proposition

Let both large investors have exponential utility and assume that $\gamma_A \Sigma_A^0 - \gamma_B \Sigma_B^0$ and $h$ are not constants. Then,

- i. There is a unique PEPQ.
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5 Conclusive remarks
The optimal investment problem under price impact can be written as an optimal investment constrained problem without market impact.

There is a specific condition that guarantees that constrain set is non-binding and the problem can be solved.

Derivative pricing upon price impact on the underlying market differs from the standard arbitrage-free pricing.

Arbitrage is limited → Investors may optimally ignore it!

In segmented markets, arbitrage-price may arise as the equilibrium price!
The End

Thank you for your attention!