Default Contagion in Random Financial Block Models

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Systemic risk: risk that in case of an adverse (local) shock substantial parts of the system default due to contagion effects.

Aim: measurement and management of default contagion in financial systems in terms of statistical network characteristics.

Tool: asymptotic analysis of default contagion in weighted, directed, inhomogeneous random graphs.

- In the spirit of [Amini, Cont, Minca, 13], [Gai, Kapadia, 10], [Hurd, 16], [Detering, M.-B., Panagiotou, Ritter, 15,16]
Random financial network

Stylized features of financial networks:

- strong heterogeneity (core/periphery structure)
- the graph is directed
- there are weights on the edges
- the graph is sparse: number of edges linear in network size
- The networks are large
Random financial network

**Strong heterogeneity:**

- in degrees (connectivity)
  - degree distribution might not have second moments ([Boss et al., 2004], [Cont et al., 2013], [Craig and van Peter, 2014])

- in financial exposures

- in capital endowments

- assortative structure ([Hurd, 16])
  - in connectivity
  - in financial exposures
Random financial network

- [Amini, Cont, Minca, 13], [Gai, Kapadia, 10], [Hurd, 16]: configuration model
  - degree distribution needs to have 2nd moment (no core/periphery)
  - exposures depend only on creditor
  - no flexible assortativity

- [Detering, M.-B., Panagiotou, Ritter, 15,16]: inhomogeneous random graph
  - degree distribution without 2nd moment possible
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Random financial network

Figure: Inhomogeneous random graph with Pareto-distributed degrees with shape parameter (a) 3.5 (bounded second moment) respectively (b) 2.5 (unbounded second moment). Node sizes scale with the corresponding degree.
Random financial network

The directed, weighted block random graph:

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- We construct a random network with edge-weights (financial exposures) in \([R]\), \(R \in \mathbb{N}\).

- Each bank \(i \in [n]\) equipped with a set of non-negative vertex-weights: for all \(r \in [R], \alpha \in [T]\)
  
  - \(w_i^{-,r,\alpha}\) describes tendency of bank \(i\) to develop incoming edges of weight \(r\) from institutions of type \(\alpha\)
  
  - \(w_i^{+,r,\alpha}\) describes tendency of bank \(i\) to develop outgoing edges of weight \(r\) to institutions of type \(\alpha\)
Random financial network

- A directed edge from $i$ to $j$ of weight $r \in [R]$ appears with probability

$$p_{i,j}^r := \begin{cases} R^{-1} \land n^{-1} w_i^{+,r,\alpha_j} w_j^{-,r,\alpha_i}, & i \neq j, \\ 0, & i = j. \end{cases}$$

- edge between banks appear independently
- no multiple edges of different weights between banks
- edge from $i$ to $j$ of weight $r \in [R]$: $i$ owes $j$ the amount $r$
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Each bank $i \in [n]$ is equipped with a capital level (net worth) $c_i \in \mathbb{N}_0 \cup \{\infty\}$. 
Default contagion

Default cascades

- The set of initially defaulted banks:

\[ \mathcal{D}_0 = \{ i \in [n] \mid c_i \leq 0 \} \]

- Default cascade \( \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \ldots \subseteq \mathcal{D}_{n-1} \) triggered by \( \mathcal{D}_0 \):

\[ \mathcal{D}_k = \left\{ i \in [n] : c_i \leq \sum_{r=1}^{R} r \sum_{j \in \mathcal{D}_{k-1}} X_{j,i}^r \right\} \]

where \( X_{j,i}^r \) is 1 if there is an edge of weight \( r \) going from \( j \) to \( i \) and 0 otherwise.

- \( \mathcal{D}_n := \mathcal{D}_{n-1} \) is the final default cluster in the network generated by the fundamental defaults \( \mathcal{D}_0 \).
In the following we will focus on the final fraction of defaulted banks after contagion as systemic risk indicator:

$$\frac{|D_n|}{n}$$
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Asymptotic analysis for $n \to \infty$ of the default fraction via generalized bootstrap percolation in our weighted, directed block random graph.
For each network size $n$, the (random) financial network model is characterized by the empirical distribution of $(w_i^- r, \alpha_i, w_i^+ r, \alpha_i, c_i, \alpha_i)_{i=1,...,n}$; i.e. by a corresponding random vector

$$(W_n^- r, \alpha_n, W_n^+ r, \alpha_n, C_n, A_n)$$

For $n \to \infty$ we assume

$$(W_n^- r, \alpha_n, W_n^+ r, \alpha_n, C_n, A_n) \rightsquigarrow (W^- r, \alpha, W^+ r, \alpha, C, A)$$

Further we assume

$$\mathbb{E}[W_n^- r, \alpha] \to \mathbb{E}[W^- r, \alpha] < \infty \quad \text{and} \quad \mathbb{E}[W_n^+ r, \alpha] \to \mathbb{E}[W^+ r, \alpha] < \infty$$
Asymptotic default fraction

Define $f^{r,\alpha,\beta} : \mathbb{R}_{+,0}^V \rightarrow \mathbb{R}$, $(r, \alpha, \beta) \in V := R \times T^2$, and $g : \mathbb{R}_{+,0}^V \rightarrow \mathbb{R}_{+,0}$ by

$$f^{r,\alpha,\beta}(z) = \mathbb{E} \left[ W^{+,r,\alpha} \psi_C \left( \sum_{\gamma \in [T]} W^{-,1,\gamma} z^{1,\beta,\gamma}, \ldots, \sum_{\gamma \in [T]} W^{-,R,\gamma} z^{R,\beta,\gamma} \right) \mathbf{1}\{A = \beta\} \right] - z^{r,\alpha,\beta},$$

$$g(z) = \sum_{\beta \in [T]} \mathbb{E} \left[ \psi_C \left( \sum_{\gamma \in [T]} W^{-,1,\gamma} z^{1,\beta,\gamma}, \ldots, \sum_{\gamma \in [T]} W^{-,R,\gamma} z^{R,\beta,\gamma} \right) \mathbf{1}\{A = \beta\} \right]$$

where

$$\psi_l(x_1, \ldots, x_R) := \mathbb{P} \left( \sum_{s \in [R]} sX_s \geq l \right)$$

for independent Poisson random variables $X_s \sim \text{Poi}(x_s)$. 

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**Default Contagion in Random Financial Block Models**
Define

\[ S := \bigcap \{ z \in \mathbb{R}^V_{+,0} : f^{r,\alpha,\beta}(z) \geq 0 \} \]

Let \( S_0 \) denote the largest connected subset of \( S \) containing \( 0 \).

Let \( z^* \) and \( \hat{z} \) be the largest and smallest joint root in \( S_0 \) of all the functions \( f^{r,\alpha,\beta} \), \( (r, \alpha, \beta) \in V \), respectively.
Theorem: Let $\hat{z}$ and $z^*$ be the smallest respectively largest joint root in $S_0$ of the functions $\{f^{r,\alpha,\beta}\}_{(r,\alpha,\beta)\in V}$. Then

$$g(\hat{z}) + o_p(1) \leq n^{-1}|D_n| \leq g(z^*) + o_p(1).$$

In particular, if $\hat{z} = z^*$, then

$$n^{-1}|D_n| = g(\hat{z}) + o_p(1).$$
Identifying systemic risk for a given network with characteristics \((W^-,r,\alpha, W^+,r,\alpha, C, A)\):

- Initially no defaulted banks: \(P(C = 0) = 0\)
- Then shock system: some vertices default ex post
  - f.ex. banks default independently with probability \(\epsilon > 0\)
  - or only banks of certain types are affected by default
- \(\Rightarrow\) new threshold sequence \(C_s\) with \(P(C_s = 0) > 0\)
- Compute final default fraction \(n^{-1}|D_s|\) after contagion in shocked system
- Systemic risk: small local shock spreads to substantial parts of the system
Measure of systemic risk: resilience/non-resilience

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Default Contagion in Random Financial Block Models
Measure of systemic risk: resilience/non-resilience

Identifying systemic risk for a given network with characteristics \((W\!, r, \alpha, W^+\!, r, \alpha, C, A)\):

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  ▶ f.ex. banks default independently with probability \(\epsilon > 0\)
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  ▶ ...

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▶ Compute final default fraction \(n^{-1}\lvert D^s_n \rvert\) after contagion in shocked system

▶ Systemic risk: *small local shock spreads to substantial parts of the system*
Theorem (resilient system): Assume that $S_0 = \{0\}$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that $n^{-1}|D_n^s| \leq \epsilon$ w.h.p. for every ex post default shock satisfying $\mathbb{P}(C_s = 0) < \delta$.

- Small shocks remain local!
  - Final default fraction goes to 0 when shock goes to 0!
Measure of systemic risk: resilience/non-resilience

Default Contagion in Random Financial Block Models
Theorem (non-resilient system): Assume that $S_0 \neq \{0\}$, i.e. $z^\ast \neq 0$. Consider any ex post default shock which is independent of type $A$, vertex-weights $W^{\pm,r,\alpha}$, and capital $C$. Then w.h.p.

$$n^{-1} |D_n^s| \geq g(z^\ast) > 0$$

- Any small shock spreads to a substantial part (linear fraction)!
  Lower bound independent of shock size!
Theorem (non-resilient system): Assume that $S_0 \neq \{0\}$, i.e. $z^* \neq 0$. Consider any ex post default shock which is independent of type $A$, vertex-weights $W^{\pm, r, \alpha}$, and capital $C$. Then w.h.p.

$$n^{-1}|D_n^s| \geq g(z^*) > 0$$

- Any small shock spreads to a substantial part (linear fraction)!
- Lower bound independent of shock size!

*In the paper we provide more general analysis that allows only certain parts of system (certain types of banks) to be hit by ex post default.*
Measure of systemic risk: resilience/non-resilience

Default Contagion in Random Financial Block Models
Example: influence of non-resilient subsystem

- Global system consisting of two subsystems \((R=1, T=2)\) given by
  \[
  (W^{\pm,1}, W^{\pm,2}, C, A)
  \]

- Subsystem of banks of type 1 assumed to be non-resilient.

- Subsystem of banks of type 2 assumed to be resilient.
Now apply ex post shock to subsystem 1: $\mathbb{P}(C_s = 0, A = 1) > 0$

- Assume there are some banks in subsystem 1 lending to banks in the non-resilient sub-system 1:

  $$ W^{+,2}|_{A=1} > 0 \text{ and } \mathbb{P}(W^{-,1} > 0, C < \infty, A = 2) > 0 $$

- Then every howsoever small shock to the non-resilient subsystem 1 spreads to a lower bounded fraction of finally defaulted banks in subsystem 2 as well.
Applications

Now apply ex post shock to subsystem 2: \( \mathbb{P}(C_s = 0, A = 2) > 0 \)

- Assume there are some banks in non-resilient subsystem 1 lending to banks in the resilient subsystem 1:

  \[ W^{+,1}_{|A=2} > 0 \quad \text{and} \quad \mathbb{P}(W^{-,2} > 0, C < \infty, A = 1) > 0 \]

- Then every howsoever small shock to the resilient subsystem 2 spreads to a lower bounded fraction of finally defaulted banks in subsystem 2.

- That is, by connecting to the non-resilient subsystem 1 the a priori resilient subsystem 2 becomes non-resilient as well.
Even connecting a priori resilient subsystems may result in a global non-resilient system!

But:

**Proposition:** Consider a global financial network consisting of $T$ resilient subnetworks ($R=1$ for simplicity) and assume that there exists a constant $K < \infty$ such that for all $\alpha \neq \beta \in \{1, \ldots, T\}$

$$W^{\pm, \beta}|_{A=\alpha} \leq KW^{\pm, \alpha}|_{A=\alpha}$$

Then the global system is still resilient.
Example: counterparty dependent exposures

- Core/periphery network ($T = 2$) with 2 possible exposure sizes ($R = 2$)
- $p = 1/3$ of all banks have type 1 (core), $1 - p = 2/3$ banks have type 2 (periphery)
- each possible edge appears with probability $4/n$
- all banks have capital $C = 2$
Counterparty-dependent exposures:

- Exposures between core banks are of size 2, all other exposures are of size 1

Creditor-dependent exposures:

- assign size 2 with probability $p$ to any exposure of a core bank, all other exposures are of size 1.

Proposition: The network with counterparty-dependent exposures is non-resilient, while the network with creditor-dependent exposures is resilient.
Thank you!
References


Artzner, Delbaen, Eber, Heath: Coherent measures of risk, Math. Fin., 1999


