Convergence to the Mean Field Game Limit: A Case Study

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Joint Work with

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Outline

1. Introduction

2. $n$-Player Game

3. Mean Field Game

4. Convergence: Extremal Equilibria

5. Convergence: General Equilibria
Mean Field Games

- Nash equilibria for $n \to \infty$ players (non-atomic game)
- Anonymous: Interaction through empirical distribution of states
Connecting Mean Field Game and \( n \)-Player Game

Convergence Forward:

- Cardaliaguet–Delarue–Lasry–Lions, \ldots: \( (\text{closed-loop} \) \( n \)-player equilibria converge to mean field equilibrium. Based on master equation, under classical solution/monotonicity condition/uniqueness

- Lacker, Fischer, Carmona–Delarue–Lacker, \ldots: \( (\text{open-loop} \) \( n \)-player equilibria converge to weak mean field equilibria. Includes mixtures. Based on compactness. Holds for closed-loop in certain settings (Lacker, Cardaliaguet–Rainer).

Convergence Backward:


- Hopefully these are close/similar to actual equilibria
Our Main Question

Our question: Are mean field equilibria limits of $n$-player equilibria? (Especially when there is more than one.)
I.e., are they “justified” by the $n$-player game?

Parallel work:

- Cecchin–Dai Pra–Fischer–Pelino study a two-state game with unique $n$-player equilibria, these converge to a mean field equilibrium as expected; however, a second, less plausible mean field solution can appear for certain parameter values and this solution is not a limit.
- Delarue–Foguen Tchuendom study several approaches of selecting an equilibrium in a linear-quadratic mean field game with multiple equilibria, including the convergence of $n$-player equilibria. Different approaches are shown to select different equilibria.
Games of Optimal Stopping (Timing)

- Agents aim to stop optimally
- Interaction through proportion of players that have already stopped

Guiding idea: bank-run models as in Diamond–Dybvig

N., Carmona–Delarue–Lacker, Bertucci, Bouveret–Dumitrescu–Tankov
Notion of Equilibrium

Full information, “open-loop”: all processes adapted to a common filtration
Agent space \((I, \mathcal{I}, \lambda)\), either \(I = \{1, \ldots, n\}\) or \(I = [0, 1]\), \(\lambda\) uniform

- Each agent \(i\) solves an optimal stopping problem: \(\tau^i\)
- Compute proportion \(\rho_t^\rightarrow = \lambda\{j \neq i : \tau^j \leq t\}\) of other agents that have stopped
- Optimal stopping problem depends on \(\rho_t^\rightarrow\): fixed point

An Nash equilibrium consists of \(\rho_t = \lambda\{i : \tau^i \leq t\}\) and \((\tau^i)_{i \in I}\)
The Single-Agent Problem

Optimal stopping problem:

\[
\sup_{\tau \in T} E \left[ e^{r\tau} 1_{\{\theta > \tau\}} \cup \{\theta = \infty\} \right].
\]

- \( r \) is an interest rate
- \( \theta \) is the default of the bank
- \( \theta \) comes as a surprise, but has an observed subjective intensity \( \gamma^i \)
- First jump of a Cox process: \( \theta \overset{\text{law}}{=} \inf \{ t : \int_0^t \gamma^i_s \, ds = \Exp(1) \} \).
Specification in this Talk

- Intensities

\[ \gamma_t^i = Y_t^i + c\rho_t^{-i}, \quad \rho_t^{-i} = \lambda\{j \neq i : \tau^j \leq t\} \]

- \( Y_t^i \) are i.i.d., increasing, right-continuous processes
- \( F_t(y) := P\{ Y_t^i \leq y \} \) the continuous c.d.f. at time \( t \)
- Solution of single-agent problem:

\[ \tau^i = \inf\{ t : Y_t^i + c\rho_t^{-i} \geq r \} \quad \text{(assume } < \infty) \]

- Unique e.g. if \( Y^i \) is strictly increasing
- Assume all agents use this stopping rule

Multiplicity of Equilibria:

- If everybody stops, you also want to stop (and vice versa)
- “Strategic complementarity”
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Equilibria of the $n$-Player Game

- If $\rho_n$ is an $n$-player equilibrium and $\rho_n(t)(\omega) = k/n$, then
  \[
  \#\{ Y^i_t(\omega) + c \cdot \frac{k-1}{n} \geq r \} = k \quad \text{and} \quad \#\{ Y^i_t(\omega) + c \cdot \frac{k}{n} < r \} = n - k
  \]

- This is also sufficient for the existence of $\rho_n$
Minimal and Maximal Equilibria

**Theorem:** There exists an $n$-player equilibrium $\rho_n^m$ such that

$$\rho_n^m(t) = \frac{k}{n} \iff \begin{cases} \#\{Y_t^i + c \cdot \frac{k}{n} \geq r\} = k \\ \#\{Y_t^i + c \cdot \frac{k - l}{n} \geq r\} \geq k - l + 1, \quad 1 \leq l \leq k. \end{cases}$$

This equilibrium is **minimal**: $\rho_n^m(t) \leq \rho_n(t) \forall \ n$-player equilibrium $\rho_n$.

- Similarly, there exists a **maximal** equilibrium $\rho_n^M$.
- The set of all equilibria $\rho_n(t) = \#\{i : \tau^i \leq t\}/n$ can be constructed recursively:
Minimal and Maximal Equilibria

**Theorem:** There exists an $n$-player equilibrium $\rho^m_n$ such that

$$\rho^m_n(t) = \frac{k}{n} \iff \begin{cases} \#\{Y_t^i + c \cdot \frac{k}{n} \geq r\} = k \\ \#\{Y_t^i + c \cdot \frac{k - l}{n} \geq r\} \geq k - l + 1, \quad 1 \leq l \leq k. \end{cases}$$

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- Similarly, there exists a **maximal** equilibrium $\rho^M_n$
- The set of all equilibria $\rho_n(t) = \#\{i : \tau^i \leq t\}/n$ can be constructed recursively:
Recursive Construction

1. Suppose that at time $\tau_0$, a group $K \subsetneq I$ of agents has already stopped. Then every remaining agent $i \notin K$ examines her criterion $\theta^i_K = \inf \{ t : Y^i_t + c \cdot \frac{\#K}{n} \geq r \}$.

   If $\theta^i_K \leq \tau_0$, then player $i$ must stop immediately. We add $i$ to the set $K$ and repeat 1. until no further players are forced to stop. (Order does not matter.)

2. A group $J \subseteq K^c$ may be able to stop together. Indeed, suppose that $\theta^j_K = \inf \{ t : Y^i_t + c \cdot \frac{\#K + \#J - 1}{n} \geq r \}$ satisfies $\theta^j_K \leq \tau_0$ for all $i \in J$. Then it is optimal for all these agents to stop together, but they do not have to. If they stop, we add $J$ to $K$ and repeat from 1.
3. After all remaining groups of agents have decided whether to stop at time $\tau_0$, we increment time until there exists a group or individual agent wanting to stop, and start again at 1.

- Multiplicity of equilibria arises because of the choices taken by the groups $J$.
- “Always no” leads to $\rho_n^m$, “always yes” leads to $\rho_n^M$. 
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Mean Field Game Equilibria

- Note $\rho^{-i}(t) = \rho(t)$ and recall $\tau^i = \inf \{ t : Y^i_t + c\rho(t) \geq r \}$

- Fix $t \geq 0$. If $\rho(t)$ is an equilibrium,

\[
\rho(t) = \lambda \{ i : \tau^i \leq t \} = \lambda \{ i : Y^i_t + c\rho(t) \geq r \}
\]
\[
= P \{ Y^i_t + c\rho(t) \geq r \}
\]
\[
= P \{ Y^i_t \geq r - c\rho(t) \}
\]
\[
= 1 - F_t(r - c\rho(t))
\]

$\Rightarrow$ Fixed point equation for $u = \rho(t)$:

\[
F_t(r - cu) = 1 - u
\]
Theorem: A real function $\rho : \mathbb{R}_+ \rightarrow [0, 1]$ is a mean field game equilibrium if and only if it is increasing, right-continuous and

$$F_t(r - c\rho(t)) = 1 - \rho(t), \quad t \geq 0.$$ 

There exist minimal and maximal equilibria $\rho^m, \rho^M$. 
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Limits of $n$-Player Equilibria

**Theorem:**
Let $t \geq 0$ and $\mathcal{U}(t) = \{u : 1 - u = F_t(r - cu)\}$. If $(\rho_n)_{n \geq 1}$ are $n$-player equilibria, $(\rho_n(t))$ is asymptotically concentrated on $\mathcal{U}(t)$.

(I.e., any weak cluster point of $(\rho_n(t))$ is concentrated on $\mathcal{U}(t)$.)

**Corollary:**
If the mean field game has a unique equilibrium, any sequence of $n$-player equilibria converges to it.

- “Limits of $n$-player equilibria are (randomized) mean field equilibria”
- Converse?
Limit of the Minimal $n$-Player Equilibria

Obvious guess: $\rho^m_n \to \rho^m$ in a suitable sense

Lemma: Let $t \geq 0$. The equation $u + F_t(r - cu) = 1$ has the solutions:
A Bad Case

Example: Let $r = c = 1$ and let $Y^i_t$ be i.i.d. increasing processes such that
\[ \text{Law}(Y^i_t) = \frac{1}{2} \delta_{\frac{1}{2}} + \frac{1}{2} \delta_2 \]
for all $0 \leq t < T$ (and $Y^i_t > r$ later). Then
\[ \text{Law}(\rho^m_n(t)) \to \frac{1}{2} \delta_{\frac{1}{2}} + \frac{1}{2} \delta_1, \quad t < T. \]

- Here $\rho^m(t) \equiv \frac{1}{2}$ and $\rho^{mrt}(t) \equiv 1$
- The limit is a mixture of these equilibria

Corollary: $\rho^m(t)$ is not the limit of $n$-player equilibria
Bad Case with Density

**Example:** As above, but with density $f(y) = 4 \mathbf{1}_{[\frac{3}{8}, \frac{1}{2}]}(y) + \mathbf{1}_{[\frac{3}{2}, 2]}(y)$.

- Again, $\rho^m(t) \equiv \frac{1}{2}$ and $\rho^{mrt}(t) \equiv 1$
- The limit is a mixture of these equilibria
The Good (and Generic) Case

**Theorem:** Assume that \( \rho^m(t) \) is not a local max, for a dense set of \( t \).

Then the minimal \( n \)-player equilibrium \( \rho^m_n \) “Fatou converges” in probability to the minimal mean field equilibrium \( \rho^m_+ \).

- Assumption is “generic”
- Cannot have convergence at every \( t \)
- Right-continuity might be a philosophical matter in the first place
- Similar result for the maximal equilibrium
Interior Equilibria

\[ u + F_t(r - cu) \]

- We exclude the “tangential” case (positive and negative examples)

**Increasing-Transversal Equilibria:**

**Theorem:** Let \( \rho \) be a mean field equilibrium. Suppose that for all \( t \) in a dense subset \( D \subseteq \mathbb{R}_+ \), the solution \( x := \rho(t) \) is increasing-transversal. Then there exist \( n \)-player equilibria \( (\rho_n)_{n \geq 1} \) which Fatou converge in probability to \( \rho \).
Decreasing-Transversal Equilibria

- Assume that $F_t$ admits a continuous density $f_t$
- Call a solution $x$ of $u + F_t(r - cu) = 1$ strongly decreasing-transversal if $\partial_u|_{u=x}[u + F_t(r - cu)] < 0$; i.e.,
  $$\alpha := cf_t(r - cx) > 1.$$ 

**Theorem:** Let $\rho$ be a mean field equilibrium and suppose that the complement of $\{t \geq 0 : \rho(t) \text{ is strongly decreasing-transversal}\}$ is not dense. Then there does not exist a sequence of $n$-player equilibria $\rho_n$ Fatou converging to $\rho$ in probability.
Decreasing-Transversal Equilibria: Static Result

**Lemma:** Fix \( t \geq 0 \) and let \( x \in [0, 1] \) satisfy \( x + F_t(r - cx) = 1 \). If \( x \) is strongly decreasing-transversal, then

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} P(\exists \text{n-player equilibrium } \varepsilon\text{-close to } x) < 1.
\]

Bounds depend on \( \alpha := cf_t(r - cx) = 1 - \text{slope} \)

Dotted: \( \frac{e^{-\alpha}}{|1-\alpha|} \)

Dashed: \( \frac{1-\theta}{\alpha-1} \)

where \( \theta \in (0, 1) \) is defined by \( \theta e^{-\theta} = \alpha e^{-\alpha} \).

Solid: \( \frac{e^{-\alpha}}{\alpha-1} \left(1+2\sqrt{\frac{2}{|a_0|}} \{1-\Phi(\sqrt{2|a_0|})\}\right) \)

where \( a_0 := 1 - \alpha + \log(\alpha) < 0 \),

\( \Phi \) standard normal c.d.f.
Crossings of Empirical C.D.F.

- Relaxing the equilibrium condition results in different problem:
- **Crossings** between a certain empirical c.d.f. (related to $F_t$) with the theoretical uniform c.d.f.
- **Nair–Shepp–Klass** studied the distribution of such crossings
- Their result is used to obtain the dashed bound

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**ON THE NUMBER OF CROSSINGS OF EMPirical DISTRIBUTION FUNCTIONS**

**By Vijayan N. Nair, Lawrence A. Shepp and Michael J. Klass**

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Let $F$ and $G$ be two continuous distribution functions that cross at a finite number of points $-\infty < t_1 < \cdots < t_k < \infty$. We study the limiting behavior of the number of times the empirical distribution function $G'_n$ crosses $F$ and the number of times $G'_n$ crosses $F'_n$. It is shown that these variables can be represented, as $n \to \infty$, as the sum of $k$ independent geometric random variables whose distributions depend on $F$ and $G$ only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. The technique involves approximating $F'_n(t)$ and $G'_n(t)$ locally by Poisson processes and using renewal-theoretic arguments. The implication of the results to an algorithm for determining stochastic dominance in finance is discussed.
Expected Number of Equilibria Near $x$

**Proposition:** Fix $t \geq 0$ and let $x \in (0, 1)$ satisfy $x + F_t(r - cx) = 1$. Let $\alpha := cf_t(r - cx) \neq 1$. Then

$$
\lim_{n \to \infty} E[\#n\text{-player equilibria close to } x] = \frac{e^{-\alpha}}{|1 - \alpha|}.
$$

- Solutions occur in a window of size $a_n/\sqrt{n}$ for any $a_n \to \infty$
- Implies the dotted bound

**Lower Bound:**
- Uses the above bound and a second-moment argument
- In particular, $\lim \inf_{n \to \infty} P(\exists n\text{-player equilibria close to } x) > 0$
- $x$ is part of a mixture which is itself a limit of $n$-player equilibria
Conclusion

- $n$-Player equilibria converge to randomized mean field game equilibria
- Randomization may happen even for natural choices like the minimal equilibrium
- Not all mean field game equilibria are limits of $n$-player equilibria

- Identification in other games?

Thank you
Conclusion

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