Insiders’ Hedging in a Stochastic Volatility Model with Informed Traders of Multiple Levels

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Market Microstructure

Market microstructure

- **micro vs. macro** ← surface of the land vs. earth from the space

- HFD, UHFD, algorithmic trading
- transaction cost, fees, taxes, regulations
- financial engineering vs. economics
  ← Who determines the price?
- information, liquidity (or liquidation)
- CAPM, Nash equilibrium etc.
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**Information Asymmetry**

**Background**

- In a market, different traders have **different levels of information**.
- Even when two traders have the exactly same information, they may interpret the information in different ways, or make different decisions.
- **Information is modeled by a filtration** in mathematical finance theory.
- A trader with more information has a larger filtration than a trader with less information.
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Traders

- **Insider** (informed trader): a trader with more (exclusive) information or better interpretation skill of the public information.
- **Honest Trader** (uninformed trader): a trader with only public information.
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How to model?

- We introduce an information process.
- This exclusive information often causes bigger movements than those usual diffusion can explain, and it is natural to involve this information to jump terms.
- jump in the price process itself? jump in the volatility term? jump size? jump timing(intensity)?
Introduction

Filtration

- Let $\mathcal{F}$ be the filtration generated by the market. It is an honest trader’s filtration.
- An insider has a larger filtration $\mathcal{G}$ available only to insiders.
- $\mathcal{F} \subset \mathcal{G}$.
- Kyle (1985), Amendinger (2000), Biagini and Oksendal (2005) assumed that the $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ for some fixed random variable $L$. (usually a future price)
- Hu and Oksendal(.) studied a model that more and more additional information is available to the investor as time goes by. They used a sequence of random variables available only to insiders as additional information at certain points of times. (scheduled announcements)
- We generalize these studies to the case with $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(X_s, 0 \leq s \leq t)$, where the additional information $X$ given to insiders is not a single random variable nor a discrete sequence of random variables, but a diffusion process.
Research on Information Effects

Q: Obviously, an informed trader should do better in the market. But how can we mathematically explain and support this? More specifically, how can we find an optimal hedging strategy and pricing for an informed trader?
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- Kang and Lee (Stochastics, vol 86, (6), 889-905, 2014): jump size only
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Earning announcement and earning jumps

- Lee and Leung
- deterministic time jump
- learning procedure $\rightarrow$ Brownian bridge
Research on Information Effects

Q: Other information issues

more finance* papers

  → Dominant markets, staggered openings, and price discovery
  → Spillover effect, leading-following interaction
  → How potent are news reversals?: Evident from futures markets
  → Surprise!
Lee and Song (2007)

\[ dS_t = f(S_{t-})dB_t + g(S_{t-})dR_t + h(S_{t-})dt \]

\[ 0 \leq t \leq T \]

where

- \( R_t = \sum_{n=1}^{N_t} U_n \)
- \( N_t - \int_0^t \lambda(X_s)ds \) is a local martingale under \( \mathbb{P} \)
- \( X \), which is a firm specific information available only to insiders, satisfies the stochastic differential equation \( dX_t = \alpha(X_t)dt + \beta(X_t)dB_t^X \) for \( 0 \leq t \leq T \).
- \( B' \) is another standard Brownian motion under \( \mathbb{P} \) such that \([B, B^X]_t = \rho t\).
- Correlation \( \rho \) between two Brownian motions \( B \) and \( B^X \) explains the level of exclusive information.
- \( U_n \) is i.i.d and has a pdf \( \nu \) on \((-1, 1)\)
- \( U_n \) denotes the jump sizes of \( S_t \) and has mean 0 and a finite second moment \( \sigma^2 \).
Kang and Lee (2014)

\[ dS_t = S_t - (\mu dt + \sigma dB_t + dR_t), \quad 0 \leq t \leq T \]  \tag{2}

where

- \( B_t \) is a standard Brownian motion.
- \( R_t = \sum_{0 < s \leq t} \theta(X_s)1(\Delta N_s = 1) \)

where \( \theta(\cdot) \) is an increasing function and \(-1 < \theta(x) < \frac{\sigma^2}{\mu} \).
- \( N_t \) is a Poisson counting process with rate \( \lambda \) under \( \mathbb{P} \). \( \hat{N}_t := N_t - \lambda t \) is a martingale under \( \mathbb{P} \).
- \[ dX_t = \alpha(X_t)dt + \beta(X_t)dB^X_t, \quad X_0 = x_0. \]

where \( B^X \) is a standard Brownian motion with \([B, B^X]_t = \rho t \).
Distribution of Jump Sizes for $\alpha(x) = 0, \beta(x) = 1$.

The expectation of jump size is given by $E[\theta(X_0 + \sqrt{T}Z)]$ where $T$ and $Z$ follow independent exponential with rate $\lambda$ and standard normal distribution respectively. $\theta(x) = \frac{2}{\pi} \arctan(x)$

Figure: Jump distributions for different $X_0$'s.
Comparison with Honest Trader’s Strategy

We assume that an honest trader believes the Black-Scholes model. The first number in a cell denotes the expected total cost of the informed trader, and the second number denotes that of the honest trader. \( E[(C_T - C_0)^2] \) denotes the expected total cost, which will be explained in 3 slides. (A smaller number is better!)

**Table: \( E[(C_T - C_0)^2] \), \( \rho = -0.5 \)**

<table>
<thead>
<tr>
<th>Vol Ratio</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.104438, 0.751323</td>
<td>1.028191, 1.189337</td>
<td>1.851046, 1.792153</td>
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<tr>
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<td>1.537828, 1.686250</td>
<td>4.415904, 4.045074</td>
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<td>120</td>
<td>0.721526, 0.922125</td>
<td>1.702788, 2.112400</td>
<td>2.645012, 1.369176</td>
</tr>
</tbody>
</table>

**Table: \( E[(C_T - C_0)^2] \), \( \rho = 0.0 \)**

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<td>0.693646, 2.347789</td>
<td>1.366413, 2.494248</td>
<td>1.683573, 1.805261</td>
</tr>
</tbody>
</table>
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Table: $E[(C_T - C_0)^2]$, $\rho = 0.5$

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</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.611927, 1.590476</td>
<td>0.289834, 0.699058</td>
<td>0.814961, 1.101575</td>
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<td>1.052195, 1.639205</td>
<td>1.729693, 1.940148</td>
</tr>
<tr>
<td>120</td>
<td>1.072975, 1.680556</td>
<td>1.362013, 1.727360</td>
<td><strong>1.875722, 1.793749</strong></td>
</tr>
</tbody>
</table>
Park and Lee (2016?)

- 
  \[ dS_t = \mu_0 S_t dt + \sigma S_t dB_t + S_t dR_t, \quad S_0 = s \]

- 
  \[ dX_t = \alpha(X_t) dt + \beta(X_t) dB_t^X, \quad X_0 = 1 \]

where \( W^X \) is a standard Brownian motion.

- Define
  \[ R_t = \int_0^t \int_{-\infty}^\infty y p^R(X_s, dy, ds), \]

where \( p^R(X_t, dy, dt) \) is a random measure on \( R \times [0, T] \).

- Also, we assume that there exists a compensated measure \( m_1(X_t, dt) \) such that
  \[ E[\int_0^T C_s dR_t] = E[\int_0^T \int_R C_s(y)m_1(X_s, dy, ds)] \]

for all nonnegative \( \mathcal{F}_t \)-adapted processes \( C_t \).
Multi-Level Traders

Idea

- multiple information processes $\rightarrow$ vector process
- several levels within informed traders
- hard to model a price process with multiple jumps $\rightarrow$ volatility factors in stochastic volatility model

Basics

- We consider a market with one risky asset $(S_t)$ and one riskless asset which would be assumed 1.
- Portfolio: a pair of processes $(\xi_t, \eta_t)$, $V_t = \xi_t S_t + \eta_t$
- Contingent claim: $H = H(S_T)$ at time $T$.
- Cost process of a portfolio $(\xi_t, \eta_t)$: $C_t = V_t - \int_0^t \xi_u dS_u$, $0 \leq t \leq T$
Hedging (replicating)

A (perfect) hedging portfolio (strategy) for a contingent claim $H(S_T)$ should satisfy the following two conditions.

1. **Self-financing:**

   $$V_t = \xi_t S_t + \eta_t = \xi_0 S_0 + \eta_0 + \int_0^t \xi_u dS_u$$

2. **Perfect match at maturity:** $H(S_T) = V_T$

For a self-financing portfolio, the cost process $C_t = V_t - \int_0^t \xi_u dS_u = \xi_0 S_0 + \eta_0 = C_0$ is a constant for all $t$.
A (prefect) hedging portfolio(strategy) for a contingent claim \( H(S_T) \) should satisfy the following two conditions.

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- A complete market is a market where every contingent claim has a hedging portfolio. (ex. Black-Scholes model)
- On the other hand, in an incomplete market, there is no strategy which satisfies both conditions.
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- On the other hand, in an incomplete market, there is no strategy which satisfies both conditions.

Q: Then what is a ’good’ hedging strategy in an incomplete market?
Model

\[ dS_t = \mu S_t dt + f(Y_t) S_t dW_t^{(0)}, \quad (3) \]
\[ dY_t^{(i)} = \alpha_i(t, Y_t^{(i)}) dt + \beta_i(t, Y_t^{(i)}) dW_t^{(i)} + \gamma_i(t, Y_t^{(i)}) dR_t^{(i)}, \quad i = 1, \ldots, n. \quad (4) \]

on a \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\) where \(P\) is the empirical probability measure, and \(Y = (Y^{(1)}, \ldots, Y^{(n)})\).

- \(R_t^{(i)} = \sum_{j=1}^{N_t^{(i)}} U_j^{(i)}\).
- \(U_j^{(i)}\) : i.i.d. random variables with densities \(\nu_i\),
- \(E[U_j^{(i)}] = 0\) and \(E[|U_j^{(i)}|^2] = \eta_i^2\).
- \(N^{(i)}\) : a Poisson process with bounded intensity \(\lambda_i\).
- \(\rho_{ij}\) : correlation between \(W^{(i)}\) and \(W^{(j)}\)

- different types of information: scheduled, randomly arriving, continuous etc.
Basic Assumptions

Notations

- $S_t$ is the solution vector $(S_t, Y_t^{(1)}, \cdots, Y_t^{(n)})$.
- $M_t$ denotes the martingale part of $S_t$ i.e.

$$M_t = \left( \int_0^t fS_s dW_s^{(0)}, \int_0^t (\beta_1 dW_s^{(1)} + \gamma_1 dR_s^{(1)}), \cdots, \int_0^t (\beta_n dW_s^{(n)} + \gamma_n dR_s^{(n)}) \right)$$

Basic assumptions

- spot rate of interest $r = 0$ and no dividend.
- The volatility function $f$ is always positive.
- $S_t$ is a $\mathcal{H}^2$ special semimartingale with the canonical decomposition $S_t = M_t + A_t$ and $M_t$ is a square-integrable martingale under $P$. In other words,

$$\| [M, M]_T^{1/2} \|_{L^2}^2 < \infty$$

$$\| \int_0^T |\alpha_i(t, Y_t^{(i)})| dt \|_{L^2}^2 < \infty, \quad i = 1, \cdots, n.$$
Minimal Martingale Measure

· pricing point of view, the second fundamental theorem
· useful to find the Föllmer-Schweizer decomposition
Minimal Martingale Measure

- pricing point of view, the second fundamental theorem
- useful to find the Föllmer-Schweizer decomposition

Definition

A martingale measure $Q$ which is equivalent to $P$ is called *minimal* if $Q = P$ on $\mathcal{F}_0$, and if any square-integrable $P$-martingale $L$ that satisfies $\langle L, M \rangle = 0$ remains a martingale under $Q$, where $M$ is the martingale part of $S$ in the canonical decomposition under $P$. 
Minimal Martingale Measure

· pricing point of view, the second fundamental theorem
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Theorem

Let

$$X_t = \int_0^t \frac{\mu}{f(Y_s)} dW_s^{(0)},$$

and assume that $E[e^{2X_t}] < \infty$ for every $t \leq T$. Then,

$$Z_t = 1 - \int_0^t Z_{s-} dX_s$$

is a $P$-martingale and the probability measure $Q$ defined by $dQ = Z_T dP$ is the minimal martingale measure of $S$. 
Idea of the Proof:

- Doob Meyer Decomposition of $M_t$
- Girsanov-Meyer theorem
- Kunita-Watanabe inequality
- Uniqueness of SDE
- Stochastic Exponential
- condition on a local martingale to be a true martingale
Lemma

Under the minimal martingale measure $Q$,

\[
\tilde{W}_t^{(0)} := W_t^{(0)} + \int_0^t \frac{\mu}{f(Y_s)} ds,
\]

\[
\tilde{W}_t^{(i)} := W_t^{(i)} + \rho_{oi} \int_0^t \frac{\mu}{f(Y_s)} ds \quad i = 1, 2, \ldots, n
\]

are Brownian motions under $Q$. Thus $S$ satisfies SDEs

\[
dS_t = f(Y_t) S_t d\tilde{W}_t^{(0)} \\

\]

\[
dY_t^{(i)} = (\alpha_i(t, Y_t^{(i)}) - \mu \rho_{oi} \frac{\beta_i(t, Y_t^{(i)})}{f(Y_s)}) dt + \beta_i(t, Y_t^{(i)}) d\tilde{W}_t^{(i)} + \gamma_i(t, Y_t^{(i)}) dR_t^{(i)} \quad (9)
\]

under measure $Q$. 

Let $p_{R(i)}(dt, dy_i)$ be the random measure associated to the jump process $R^{(i)}$ under $P$. Then, the compensated measure of $R^{(i)}$ under $Q$ is given by

$$p_{\tilde{R}(i)} = p_{R(i)}(dt, dy_i) - \lambda_i \nu_i(dy_i)dt$$

→ characteristics of semimartingale
→ Girsanov’s theorem for random measures
→ conditional expectation with respect to predictable $\sigma$-field
Local Risk Minimization Strategy

Value process

- $\xi_t$: the amount of the underlying asset
- $\eta_t$: the amount of the money market account
- $V_t$: the value process of a portfolio $(\xi, \eta)$ defined by $V_t = \xi_t S_t + \eta_t$

Cost process

- $C_t$: the cost process defined by $C_t = V_t - \int_0^t \xi_t dS_t$

Local risk minimization strategy in an incomplete market (Föllmer and Schweizer)

- Local risk minimization strategies $\xi_t$: The cost process $C$ is a square integrable martingale orthogonal to $M$, i.e. $\langle C, M \rangle_t = 0$ where $M$ is the martingale part of $S$ under $\mathcal{P}$. 
A sufficient condition for the existence

The existence of an optimal strategy is equivalent to a decomposition

$$H = V_0 + \int_0^T \xi_u^H \, dS_u + L_T^H$$

where $L_t^H$ is a square integrable martingale orthogonal to $M_t$. For such a decomposition, the associated optimal strategy $(\xi_t, \eta_t)$ is given by $\xi_t = \xi_t^H$, $\eta_t = V_t - \xi_t S_t$, where $V_t = V_0 + \int_0^t \xi_u^H \, dS_u + L_t^H$. 
Local Risk Minimization Strategy

Computation of the optimal strategy

Suppose that $V_t = E^Q[H(S_T)|\mathcal{G}_t]$ has a decomposition

$$V_t = V_0 + \int_0^t \xi_u^H dS_u + L_t$$

where $L_t$ is a square integrable $\mathbb{P}$ martingale such that $\langle L, M \rangle_t = 0$ under $\mathbb{P}$. Then $\xi_t^H$ is given by

$$\xi_t^H = \frac{d\langle V, S \rangle}{d\langle S, S \rangle}.$$  \hspace{1cm} (11)

where the conditional quadratic variations are calculated under $\mathbb{P}$.

· role of the minimal martingale measure($L_t$)
Different traders

Different Traders

- A level $k$ trader: a trader with information $Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}$, $k = 1, 2, \ldots, n$
- A level $n$ trader: a fully informed trader
- A level 0 trader: honest trader, uninformed trader, noise trader, liquidity trader

Filtration

- $\mathcal{G}^{(k)}_t = \sigma\{(S_s, Y^{(1)}_s, \ldots, Y^{(k)}_s), 0 \leq s \leq t\}$
- $\mathcal{G}^{(0)}_t \subset \mathcal{G}^{(1)}_t \subset \cdots \subset \mathcal{G}^{(n)}_t \subset \mathcal{F}_t$
Local Risk Minimization Strategy for a Fully Informed Trader

Consider a European style contingent claim $H(S_T) \in \mathbb{L}^2(P)$

The fully informed trader

Let $V_t^{(n)} = E^Q[H(S_T)|\mathcal{G}_t^{(n)}]$ be a price process of a fully informed trader.

Theorem

The local risk minimization strategy is given by

$$
\xi_{t,H}^{n} = \frac{\partial V^{(n)}}{\partial S_t} + \sum_{i=1}^{n} \rho_0 i \beta_i(t, Y^{(i)}) \frac{\partial V^{(n)}}{\partial y_i} \frac{f(Y_t) S_t}{f(Y_t) S_t}.
$$

(12)
Local Risk Minimization Strategy for a Fully Informed Trader

Idea of the proof

- Expand $V_t^{(n)} = E^Q[H(S_T) | \mathcal{G}_t^{(n)}]$ using the Markov property and Ito’s formula.
- How to change the jumps in terms of integrals? → no common jumps!
- $V_t^{(n)}$ is a $Q$ martingale, so the drift term of the expansion should be 0. This gives us the pricing differential equation as well as the representation of $V_t^{(n)}$.
- Calculate the Radon-Nikodym derivative to get $\xi_t^{n,H}$, using properties of the predictable version of quadratic variation.
Underlying Dynamic of a Level $k$ Trader

Let $\sigma_0 := E^{Q}[f(Y_t)] \geq 0$. Then price process becomes

$$\frac{1}{S_t} dS_t = \sigma_0 d\tilde{W}_t^0 + \tilde{f}(Y_t) d\tilde{W}_t^0 \quad (13)$$

where $\tilde{f} := f - \sigma_0$.

Underlying of a level $k$ trader

They can’t observe all the information. So, $\tilde{f}(Y_t)$ is not their volatility function. Define

$$\tilde{f}_k(Y_t^{(1)}, \cdots, Y_t^{(k)}) := \tilde{f}(Y_t^{(1)}, \cdots, Y_t^{(k)}, \tilde{y}_{k+1}, \cdots, \tilde{y}_n) \quad k = 1, \cdots, n$$

and $\tilde{f}_k := 0$ if $k = 0$. Here, $(\tilde{y}_{k+1}, \cdots, \tilde{y}_n)$ is a constant vector. So a level $k$ trader’s price process (??) becomes

$$\frac{1}{S_t} dS_t = \sigma_0 d\tilde{W}_t^0 + \tilde{f}_k(Y_t) d\tilde{W}_t^0 \quad (14)$$
Cost Process of a Level $k$ Trader

Cost process of a level $k$ trader

- $V^{(k)}(t, S_t)$: the value process of a level $k$ trader.
- $(\xi^{(k)}, \eta^{(k)})$: the portfolio of a level $k$ trader.
- $C^{(k)}$: the cost process of a level $k$ trader defined by $C^{(k)}_t = V^{(k)}_t - \int_0^t \xi^{(k)}_s dS_s$
Local Risk Minimization Strategy for a Level $k$ Trader

Theorem

The local risk minimization strategy for a level $k$ trader is given by

$$\xi^{k,H}_t = \frac{\partial V^{(k)}}{\partial \tilde{S}_t} + \sum_{i=1}^{k} \rho_{0i} \beta_i(t, Y^{(i)}) \frac{\partial V^{(k)}}{\partial y_i}.\quad (15)$$

Note that the level 0 trader case corresponds to the B.S. hedging strategy $\frac{\partial V^{(k)}}{\partial \tilde{S}_t}$. 

Kiseop Lee (Department of Statistics, Purdue University Mathematical Finance Seminar University of Southern California)
The Optimal Choice for a Level $k$ Trader

Assumption

- A level $k$ trader wants to reduce the error in hedging.
- A level $k$ trader has to choose a proper $f_k \rightarrow$ choose proper values for $(\tilde{y}_{k+1}, \cdots, \tilde{y}_n)$

Error function

- $\Theta := V^{(k)} - V^{(n)} :$ an error function of a level $k$ trader.

Theorem

Assume that $V^{(n)}(t, s)$ are in $C^{1,2}$. Then there exists a constant $C$ which depends on a contingent claim $H(S_T)$ such that we have

$$E^Q[|V_t^{(k)} - V_t^{(n)}|] \leq C E^Q[\int_t^T |f_k(Y_s) - f(Y_s)|^2 ds]^{1/2}$$

(16)
The Optimal Choice for a Level $k$: Trader

Optimal condition

$$E^Q[f_k(Y_s) - f(Y_s)] = 0, \text{ for } t \leq s \leq T \quad (17)$$

Example (A special case)

We assume that $Y_t$ is a $Q$-martingale and $f$ is a linear function $f = \sum_{i=1}^{n} c_i y_i$, where $c_i > 0$.

Under these conditions, $E^Q[f(Y_s)] = f(E^Q[Y_s]) = f(Y_0)$. Therefore, the choice

$$(\tilde{y}_{k+1}, \cdots, \tilde{y}_n) := (E^Q[Y_{t}^{(k+1)}], \cdots, E^Q[Y_{t}^{(n)}])$$

is the minimizer.

For example, $f(y) = \sum_{i=1}^{n} y_i$ and $dY_t^{(i)} = \sqrt{Y_t^{(i)}} d\tilde{W}_t^{(1)}$ satisfy all the conditions. Therefore, $\sigma_0 := \sum_{i=1}^{n} E^Q[Y_{t}^{(i)}] = \sum_{i=1}^{n} Y_0^{(i)}$ and the optimal of $f_k$ is

$f_k := \sum_{i=1}^{k} y_i + \sum_{i=k+1}^{n} Y_0^{(i)}$. 
We consider two information processes

\[
    dY_t^{(1)} = m_1 \tilde{W}_t^{(1)} + m_2 dR_t^{(1)}
\]
\[
    dY_t^{(2)} = m_3 \tilde{W}_t^{(2)} + m_4 dR_t^{(2)},
\]

where each \( m_i \) is a given constant.

- \( R_t^{(i)} \) : uniformly distributed jumps with bounded intensities \( \lambda_1 = 4 \) and \( \lambda_2 = 2 \).
- \( \sigma_0 = 0.2, m_1 = 0.1, m_2 = 0.05, m_3 = 0.05, m_4 = 0.1, \rho_{01} = \frac{1}{4}, \rho_{02} = \frac{1}{5} \) and \( \rho_{12} = \frac{1}{20} \).
- \( Y_0^{(1)} = Y_0^{(2)} = 0 \) and volatility functions are \( f(y_1, y_2) = \sigma_0 + y_1 + y_2 \), \( f_1(y_1, y_2) = \sigma_0 + y_1 \).
Numerical Result for a Call Option

![Graph showing call option values for different levels of traders.]

**Figure:** Call Price, $\sigma_0 = 0.2$, $K = 100$, $T = 1$
Numerical Result for a Call Option

Figure: Sample Path of the Underlying
Numerical Result for a Call Option

\[
E[(C_T - C_0)^2]
\]

<table>
<thead>
<tr>
<th>Level 0 trader</th>
<th>Level 1 trader</th>
<th>Level 2 trader</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.728860384</td>
<td>3.533073956</td>
<td>2.714644221</td>
</tr>
</tbody>
</table>

**Table:** Expected total cost : \(S_0 = 100, \sigma_0 = 0.2, K = 100, T = 1, dt = 1/100\)

\[
E[(C_T - C_0)^2]
\]

<table>
<thead>
<tr>
<th>Level 0 trader</th>
<th>Level 1 trader</th>
<th>Level 2 trader</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2653</td>
<td>2.2360</td>
<td>1.8912</td>
</tr>
</tbody>
</table>

**Table:** Expected total cost : \(S_0 = 90, \sigma_0 = 0.2, K = 90, T = 1, dt = \frac{1}{50}\)

\[
E[(C_T - C_0)^2]
\]

<table>
<thead>
<tr>
<th>Level 0 trader</th>
<th>Level 1 trader</th>
<th>Level 2 trader</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6429</td>
<td>0.6136</td>
<td>0.5127</td>
</tr>
</tbody>
</table>

**Table:** Expected total cost : \(S_0 = 90, \sigma_0 = 0.2, K = 90, T = 1, dt = \frac{1}{100}\)
Summary

How the Information Works in a Trading?

- A trader with more information should do better in trading. We introduced those models in several cases. (jump size, timing, etc)
- We focused on a market with multiple levels of information processes.
- A numerical study shows mixed results. It is not clear how much advantage a trader gets by observing one more information process.

What to do next?

- more microstructure → algorithmic trading/ HFT
- other problems on information asymmetry
- uninformed or less informed trader’s learning dynamic
- real data fitting??