Capital allocation under Fundamental Review of Trading Book

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Fundamental Review of Trading Book (FRTB)

Basel Committee on Banking Supervision

STANDARDS

Minimum capital requirements for market risk

January 2016
Basel 2, 2.5 and FRTB

Basel 2 and 2.5

- 10 days P&L of different risk positions are aggregated
Basel 2, 2.5 and FRTB

Basel 2 and 2.5

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- Liquidity is not taken into account
Basel 2, 2.5 and FRTB

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- Value-at-Risk (VaR)
Basel 2, 2.5 and FRTB

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- Value-at-Risk (VaR)
  Incentive to take skewed risk, not sub-additive
Basel 2, 2.5 and FRTB

Basel 2 and 2.5

- 10 days P&L of different risk positions are aggregated
  Liquidity is not taken into account

- Value-at-Risk (VaR)
  Incentive to take skewed risk, not sub-additive

FRTB sets out revised standards for minimum capital requirements for market risk

- Incorporate the risk of market illiquidity
- An Expected Shortfall (ES) measure
- Constrain the capital-reducing effects of hedging
Structure and implementation

- Standardized approach (SA), Internal models approach (IMA)
- QIS shows that capital charge increases 128% in SA and 54% in IMA (average over 44 banks)
- Model approval down to desk level
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- Standardized approach (SA), Internal models approach (IMA)
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- Model approval down to desk level

Implementation timeline (Picture from EY)
Consulting firm Oliver Wyman estimates that banks need to spend $5 billion to get ready for FRTB.
Impact of FRTB

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“... one certain thing about the process is that capital requirements will rise. This is going to be life-threatening for some trading desks, as heads of divisions assess whether it is economical to be in certain businesses.” — Bloomberg News
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Capital charge in SA is very expensive.

IMA requires 90 or more times of calculations than the current rule.
Outline

- FRTB ES and its properties
- Capital allocation
- Two allocation methods under FTRB
- Simulation analysis
Risk factor and liquidity horizon bucketing

P&L of a risk position is attributed to

\[ \{ \text{RF}_i : 1 \leq i \leq 5 \} = \{ \text{CM, CR, EQ, FX, IR} \} \]

\[ \{ \text{LH}_j : 1 \leq j \leq 5 \} = \{ 10, 20, 40, 60, 120 \} \]
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BCBS (2016) 181(k)

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<td>Precious metals and non-ferrous metals price</td>
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<td>Other commodities price</td>
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Risk profile

Loss: Negative of P&L

Consider a portfolio of $N$ risk positions. $1 \leq n \leq N \sim X_n(i,j)$: loss (over 10 days) attributed to RF $i$ and LH $j$

$\sum_{i,j} \sim X_n(i,j)$: total loss (over 10 days) of the risk position $n$

Liquidity horizon adjusted loss:

$X_n(i,j) = \sqrt{LH_j - LH_{j-1} \cdot 5 \sum_{k=i}^j \sim X_n(i,k), 1 \leq i, j \leq 5}$

We record the liquidity horizon bucketing by a $5 \times 5$ matrix: $X_n = \{X_n(i,j)\}$ with $1 \leq i, j \leq 5$

and call the matrix the risk profile of position $n$.

The risk profile of a portfolio is $X = \sum_n X_n$. 
Risk profile

Loss: Negative of P&L

Consider a portfolio of \( N \) risk positions. \( 1 \leq n \leq N \)

\( \tilde{X}_n(i, j) \): loss (over 10 days) attributed to RF\(_i\) and LH\(_j\)

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Liquidity horizon adjusted loss:

\[
X_n(i, j) = \sqrt{\frac{\text{LH}_j - \text{LH}_{j-1}}{10}} \sum_{k=j}^{5} \tilde{X}_n(i, k), \quad 1 \leq i, j \leq 5
\]
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The risk profile of a portfolio is

\[
X = \sum_n X_n.
\]
\[
X_n(i, 1) \cdot \cdots \cdot X_n(i, 5) \times \sqrt{\frac{10-0}{10}} \times \sqrt{\frac{120-60}{10}}
\]
The FRTB expected shortfall for portfolio loss attributed to $RF_i$ is

$$\text{ES}(X(i)) = \sqrt{\sum_{j=1}^{5} \text{ES}(X(i,j))^2},$$

where $\text{ES}(X(i,j))$ is the expected shortfall of $X(i,j)$ calculated at the 97.5% quantile.
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**Example:** Consider a portfolio with only one risk position whose is loss is concentrated on RF$_i$ with LH$_5 = 120$.

$$\tilde{X}(i,j) = 0, \quad j = 1, \ldots, 4,$$
FRTB ES

The FRTB expected shortfall for portfolio loss attributed to RF\textsubscript{i} is

\[ \text{ES}(X(i)) = \sqrt{\sum_{j=1}^{5} \text{ES}(X(i, j))^2}, \]

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**Example:** Consider a portfolio with only one risk position whose is loss is concentrated on RF\textsubscript{i} with \(\text{LH}_5 = 120\).

\[ \tilde{X}(i, j) = 0, \quad j = 1, \ldots, 4, \quad \tilde{X}(i, 5) \sim N(0, \sigma^2) \]

Then the ES over 120 days is \(\sqrt{\frac{120}{10}} \sigma \text{ES}(N(0, 1))\).

On the other hand, \(X(i, j) = \sqrt{\frac{\text{LH}_j - \text{LH}_{j-1}}{10}} \tilde{X}(i, 5), 1 \leq j \leq 5\). Then

\[ \text{ES}(X(i)) = \sqrt{\sum_{j=1}^{5} \frac{\text{LH}_j - \text{LH}_{j-1}}{10} \text{ES}(\tilde{X}(i, 5))^2} = \sqrt{\frac{120}{10}} \sigma \text{ES}(N(0, 1)). \]
Stress period scaling

\( \text{ES}^{F,C}(X(i)) \): current 12-month, full set of risk factors

\( \text{ES}^{R,C}(X(i)) \): current 12-month, reduced set of risk factors

\( \text{ES}^{R,S}(X(i)) \): stress period, reduced set of risk factors

Restriction: \( \text{ES}^{R,C}(X(i)) \geq 75\% \text{ES}^{F,C}(X(i)) \).
Stress period scaling

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Restriction: \( \text{ES}^{R,C}(X(i)) \geq 75\% \text{ES}^{F,C}(X(i)) \).

FRTB ES capital charge BSBC (2016) 181 (d):

\[
\text{IMCC}(X(i)) = \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \text{ES}^{F,C}(X(i)), \quad 1 \leq i \leq 5.
\]
Capital charge for modellable risk factors

Unconstrained portfolio:

\[ X_n(6, j) = \sum_{i=1}^{5} X_n(i, j), \quad X(6, j) = \sum_{n} X_n(6, j). \]

We add \( X(6, \cdot) \) as the 6-th row of \( 5 \times 5 \) matrix, and call it extended risk profile.

IMCC(\( X(6) \)) is calculated similarly as before.

**IMCC:** BCBS (2016) 189:

The aggregate capital charge for modellable risk factors is

\[ \text{IMCC}(X) = \rho \text{IMCC}(X(6)) + (1 - \rho) \sum_{i=1}^{5} \text{IMCC}(X(i)), \]

where \( \rho = 0.5. \)
Properties of IMCC

Proposition

(i)  \((Positive \ homogeneity)\) \(IMCC(aX) = a \ IMCC(X),\ a \geq 0.\)

(ii)  \((Sub-additivity \ for \ ES)\) If \(ES((X + Y)(i,j)) \geq 0,\) then

\[
ES((X + Y)(i)) \leq ES(X(i)) + ES(Y(i)).
\]

(iii)  \((Sub-additivity \ for \ IMCC)\) If

\[
\frac{ES^{R,S}((X + Y)(i))}{ES^{R,C}((X + Y)(i))} \leq \min \left\{ \frac{ES^{R,S}(X(i))}{ES^{R,C}(X(i))}, \frac{ES^{R,S}(Y(i))}{ES^{R,C}(Y(i))} \right\},
\]

and \(ES^{F,C}((X + Y)(i,j)) \geq 0,\) then

\[
IMCC((X + Y)(i)) \leq IMCC(X(i)) + IMCC(Y(i)).
\]
Profit and sub-additivity

Example:
Consider X and Y concentrating on RF$_i$ and LH$_j$.

\[ \mathbb{P}(X(i, j) = -1) = \mathbb{P}(X(i, j) = 0) = 0.5, \quad (X + Y)(i, j) \equiv -1. \]

Then ES(X(i)) = ES(Y(i)) = 0, but ES((X + Y)(i, j)) \geq 0 is violated,

\[ \text{ES}((X + Y)(i)) = \left| \text{ES}((X + Y)(i, j)) \right| = |-1| = 1 > \text{ES}(X(i)) + \text{ES}(Y(i)). \]
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\[ ES((X + Y)(i)) = |ES((X + Y)(i, j))| = |-1| = 1 > ES(X(i)) + ES(Y(i)). \]

We propose to floor each ES(X(i, j)) at zero (not required by FRTB)

\[ ES^+(X(i)) = \sqrt{\sum_{j=1}^{5} ES^+(X(i, j))^2}, \]

where \( ES^+(X(i, j)) = \max\{ES(X(i, j)), 0\} \).

The resulting FRTB ES is sub-additivity and positive homogeneous.
Capital allocation

Consider a portfolio of $N$ risk positions with losses $L_1, \ldots, L_N$. The total loss is $L = \sum_{n=1}^{N} L_n$. $ho$ is a risk measure.

An allocation is a map $\text{Law}(L_1, \ldots, L_N) \rightarrow \mathbb{R}^N$:

$$\text{Law}(L_1, \ldots, L_N) \mapsto \rho(L_n | L), \quad \text{for each } n,$$

such that

$$\sum_{n=1}^{N} \rho(L_n | L) = \rho(L).$$
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Banks need allocations to calculate return on risk-adjusted capital (RORAC):

$$-\frac{\mathbb{E}[L_n]}{\rho(L_n | L)}.$$

RORAC evaluates the capital efficiency of each position.
Euler allocation principle

Let $v_1, \ldots, v_N$ be a sequence of numbers and $L^v = \sum_{i=1}^N v_n L_n$.

Per-unit Euler allocation is

$$\rho(L_n | L)(v) := \frac{\partial}{\partial v_n} \rho(L^v).$$

Setting all $v_n = 1$, we denote the allocation to $L_n$ as $\rho(L_n | L)$.

If $\rho$ is homogeneous of degree 1, Euler’s theorem for homogeneous functions implies

$$\rho(L^v) = \sum_n v_n \frac{\partial}{\partial v_n} \rho(L^v).$$

Setting $v = 1$, we have the full allocation property.
Pros and Cons of Euler allocation

Tasche (1999) shows that

\[ \frac{\partial}{\partial v_n} \left( \frac{-\mathbb{E}[L^v]}{\rho(L^v)} \right) \begin{cases} > 0, & \text{if} \quad \frac{-\mathbb{E}[L_i]}{\rho(L_i | L)(v)} > \frac{-\mathbb{E}[L^v]}{\rho(L^v)} \\ < 0, & \text{if} \quad \frac{-\mathbb{E}[L_i]}{\rho(L_i | L)(v)} < \frac{-\mathbb{E}[L^v]}{\rho(L^v)} \end{cases} \]

Denault (2001) uses corporative game (Shapley (1953) and Aumann-Shapley (74)) to show that the Euler allocation is the “fair” allocation.

There is no sub-portfolio whose total capital charge is less than the sum of the capital allocations of its components.
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There is no sub-portfolio whose total capital charge is less than the sum of the capital allocations of its components.

However,

- Euler allocation is unstable.
- Euler allocation induces large negative allocations.
Two steps allocation for FRTB

Given liquidity horizon adjusted risk profiles \( \{X_n\}_{1 \leq n \leq N} \),

**Step 1:** allocate to each \( X_n(i, j) \)

\[
\rho(X_n(i, j) \mid X).
\]

**Step 2:** allocate to each \( \tilde{X}_n(i, k) \)

\[
\rho(\tilde{X}_n(i, k) \mid X_n(i, j)), \quad k \geq j.
\]

Then aggregate

\[
\rho(\tilde{X}_n(i, k) \mid X) = \sum_{j=1}^{k} \rho(X_n(i, k) \mid X_n(i, j)).
\]
Euler allocation under FRTB

Let $\nu = \{\nu_1, \ldots, \nu_n\}$ be real numbers.

Let $X^\nu_j(i) = \sum_n X_{n}^{\nu_n,j}(i)$, where

$X_{n}^{\nu_n,j}(i) = (X_n(i, 1), \ldots, X_n(i, j - 1), \nu_nX_n(i, j), X_n(i, j + 1), \ldots, X_n(i, 5)).$

For each RF $i$, we define the Euler allocation for FRTB ES as

$$\text{ES}(X_n(i, j) \mid X(i)) := \frac{\partial}{\partial \nu_n} \text{ES}(X^\nu_j(i)) \bigg|_{\nu = 1},$$

where $\nu = 1$ means all $\nu_n = 1$. 
Euler allocation under FRTB

Let \( v = \{v_1, \ldots, v_n\} \) be real numbers

Let \( X^{v,j}(i) = \sum_n X^{v_n,j}(i) \), where

\[
X^{v_n,j}(i) = (X_n(i, 1), \ldots, X_n(i, j-1), v_nX_n(i, j), X_n(i, j+1), \ldots, X_n(i, 5)).
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For each RF \( i \), we define the Euler allocation for FRTB ES as

\[
ES(X_n(i, j) \mid X(i)) := \left. \frac{\partial}{\partial v_n} ES(X^{v,j}(i)) \right|_{v=1},
\]

where \( v = 1 \) means all \( v_n = 1 \).

Lemma

\[
ES(X_n(i, j) \mid X(i)) = \frac{ES(X(i, j))}{ES(X(i))} \left. \frac{\partial}{\partial v_n} ES(X^{v}(i, j)) \right|_{v=1},
\]

where \( X^v(i, j) = \sum_n v_nX_n(i, j) \).
Euler allocation of FRTB ES is a scaled version of the Euler allocation of regular ES.
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• Euler allocation of regular ES can be calculated by scenario-extraction (Trashe (1999)):

\[
\frac{\partial}{\partial v_n} \text{ES}(X^v(i,j)) \bigg|_{v=1} = \mathbb{E} \left[ X_n(i,j) \mid X(i,j) \geq \text{VaR}(X(i,j)) \right].
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Euler allocation of FRTB ES is a scaled version of the Euler allocation of regular ES.

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If each $X(i,j)$ is floored at zero, then

$$\text{ES}^+(X_n(i,j) \mid X(i)) = \left\{ \begin{array}{ll} \frac{\text{ES}^+(X(i,j))}{\text{ES}^+(X(i))} \mathbb{E}[X_n(i,j) \mid X(i,j) \geq \text{VaR}(X(i,j))] & \text{if } \text{ES}(X(i,j)) > 0 \\ 0 & \text{otherwise} \end{array} \right.$$
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$$\text{ES}^+(X_n(i,j) | X(i)) = \begin{cases} \frac{\text{ES}^+(X(i,j))}{\text{ES}^+(X(i))} \mathbb{E}[X_n(i,j) | X(i,j) \geq \text{VaR}(X(i,j))] & \text{if } \text{ES}(X(i,j)) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Euler allocation of IMCC

$$\text{IMCC}^E(X_n(i,j) | X) := 0.5 \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \text{ES}^{F,C}(X_n(i,j) | X(i)).$$

It is a full allocation.
Negative allocations

Hedging among different RFs or LHs does not lead to negative allocations.

Example:

Consider two loss $Y$ with RF $i$ and $Z$ with RF $k$, $i \neq k$, $Y, Z \sim N(0, \sigma)$, and $Y + Z = 0$.

Euler of regular ES:

$$\text{ES}^R(Y|Y + Z) + \text{ES}^R(Z|Y + Z) = \text{ES}^R(Y + Z) = 0.$$ 

Then one of two allocations must be negative, say $\text{ES}^R(Y|Y + Z)$. 
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Then one of two allocations must be negative, say $ES^R(Y|Y + Z)$.

Euler of FRTB ES:
Let $X$ be the risk profile containing $X$ and $Y$. $X(i) = Y$, $X(k) = Z$, then

$$ES(Y|X(i)) = ES(Y) > 0, \quad ES(Z|X(k)) = ES(Z) > 0.$$

Even though $Y + Z = 0$, $IMCC(X) > 0$. 
Constrained Aumann-Shapley allocation

Motivated by Li, Naldi, Nisen, and Shi (2016), who combine Shapley and Aumann-Shapley allocations.

LH permutation matrix

\[
\mathcal{L} := \begin{bmatrix}
10 & 20 & 40 & 60 & 120 \\
10 & 20 & 40 & 120 & 60 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
120 & 60 & 40 & 20 & 10
\end{bmatrix}_{5! \times 5}.
\]

Let \( \mathcal{L}^{-1}(r,j) \) be the column of \( \mathcal{L} \) in which LH\(_j\) locates. e.g. \( \mathcal{L}^{-1}(2,5) = 4 \).
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120 & 60 & 40 & 20 & 10
\end{bmatrix}_{5! \times 5}.
\]

Let \( \mathcal{L}^{-1}(r,j) \) be the column of \( \mathcal{L} \) in which \( \text{LH}_j \) locates. e.g. \( \mathcal{L}^{-1}(2,5) = 4 \).

Let \( v = \{v_1, \ldots, v_n\} \),

\[
X_{v,r,j}^v(i) = \sum_n X_{n,v,r,j}^v(i),
\]

where \( X_{v,r,j}^v(i) \) is a row depending on when \( \text{LH}_j \) appears in \( r \).

For example,

\[
X_{n,v,2,5}^v(i) = (X_n(i,1), X_n(i,2), X_n(i,3), 0, v_n X_n(i,5)).
\]
Constrained Aumann-Shapley allocation

We define the **Constrained Aumann-Shapley allocation** (CAS) in the permutation \( r \) as

\[
\text{CAS}(r, X_n(i, j)) := \int_0^1 \left. \frac{\partial}{\partial v_n} \text{ES}(X^{v, r, j}(i)) \right|_{v=q} dq,
\]

where \( \text{CAS} \) is a scaled version of Euler Losses with the same LH need to be added to the same portfolio at the same time, to ensure computational efficiency.
Constrained Aumann-Shapley allocation

We define the Constrained Aumann-Shapley allocation (CAS) in the permutation $r$ as

$$\text{CAS}(r, X_n(i, j)) := \int_0^1 \frac{\partial}{\partial v_n} \text{ES}(X^{v,r,j}(i)) \bigg|_{v=q} dq,$$

Lemma

$$\text{CAS}(r, X_n(i, j)) = \eta(r, i, j) \frac{\partial}{\partial v_n} \text{ES}(X^v(i, j)) \bigg|_{v=1},$$

where

$$\eta(r, i, j) = \frac{\sqrt{\sum_{1 \leq s \leq \mathcal{L}^{-1}(r,j)} \text{ES}(X(i, \mathcal{L}(r, s)))^2} - \sqrt{\sum_{1 \leq s < \mathcal{L}^{-1}(r,j)} \text{ES}(X(i, \mathcal{L}(r, s)))^2}}{\text{ES}(X(i, j))}.$$ 

- CAS is a scaled version of Euler
- Losses with the same LH need to be added to the same portfolio at the same time, to ensure computational efficiency
CAS allocation of IMCC

We define the CAS allocation for IMCC as

\[
\text{IMCC}^C(X_n(i,j) \mid X) := 0.5 \frac{\text{ES}^{R,S}(X(i))}{\text{ES}^{R,C}(X(i))} \sum_{r=1}^{5!} \text{CAS}^{F,C}(r, X_n(i,j)).
\]

It is a full allocation.
Stress scaling adjustment

In the previous two methods, the $X_n(i,j)$ induced risk contribution is not considered in the stress scaling factor $\frac{ES_{R,S}(X(i))}{ES_{R,C}(X(i))}$.

We define the **Euler allocation with stress scaling adjustment** as

$$IMCC_{E,S}^E(X_n(i,j) \mid X(i)) := 0.5 \frac{\partial}{\partial v_n} \left[ \frac{ES_{R,S}(X^{v,j}(i))}{ES_{R,C}(X^{v,j}(i))} ES_{F,C}^{E}(X^{v,j}(i)) \right] \bigg|_{v=1}.$$

**Lemma**

$$IMCC_{E,S}^E(X_n(i,j) \mid X(i)) = 0.5 \left[ \frac{ES_{R,S}(X(i))}{ES_{R,C}(X(i))} ES_{F,C}^{E}(X_n(i,j) \mid X(i)) + \frac{ES_{F,C}(X(i))}{ES_{R,C}(X(i))} ES_{R,S}(X_n(i,j) \mid X(i)) - \frac{ES_{R,S}(X(i)) ES_{F,C}^{E}(X(i))}{ES_{R,C}^2(X(i))} ES_{R,C}(X_n(i,j) \mid X(i)) \right].$$
Simulation analysis 1

\( \tilde{X}(i,j) \), normal with mean 0 and annual volatility 30%

Risk profiles are simulated for 250 days

Independence among different days

The following correlation structure in the same day:

1. Independence
2. Strong positive correlation among RFs and LHs
3. Strong positive correlation among RFs, independent among LHs
4. Independent among RFs, strong positive correlation among LHs
Longer LH leads to larger percentage of allocation
Simulation analysis 2

Three hedging structures:

1. Strong hedging between EQ and IR
2. Strong hedging between LH₁ and LH₂
3. Strong hedging between two risk positions in the same bucket
Simulation analysis 2

Three hedging structures:

1. Strong hedging between EQ and IR
2. Strong hedging between LH$_1$ and LH$_2$
3. Strong hedging between two risk positions in the same bucket

FRTB allocations produce

- no negative allocations for hedging among different bucket
- some negative allocations for hedging in the same bucket, but with smaller magnitude
FRTB allocations are more stable

Histograms and fitted kernel densities for allocations in Case 3:
Simulation analysis 3

Loss in EQ with 40-days LH and CM with 60 days LH have 9 times of volatility in the stress period than the normal period.

Two reduced set of risk factors

Set A : Include both RFs with large variations
Set B : Exclude both RFs with large variations
Simulation analysis 3

Loss in EQ with 40-days LH and CM with 60 days LH have 9 times of volatility in the stress period than the normal period.

Two reduced set of risk factors

Set A : Include both RFs with large variations
Set B : Exclude both RFs with large variations

<table>
<thead>
<tr>
<th></th>
<th>Set A (Adj)</th>
<th>Set A (Without adj)</th>
<th>Set B (Adj)</th>
<th>Set B (Without adj)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CM.60</td>
<td>4.00%</td>
<td>2.24%</td>
<td>1.43%</td>
<td>1.43%</td>
</tr>
<tr>
<td>EQ.40</td>
<td>5.04%</td>
<td>3.26%</td>
<td>2.11%</td>
<td>2.11%</td>
</tr>
</tbody>
</table>

Table: IMCC(Set A)=11.55 and IMCC(Set B)=3.14

The choice of reduced set of risk factors has large impact on allocations.
Conclusion

Two allocation methods reduce FRTB allocations to Euler allocations
- Computational efficiency
- Easy to adapt to the current system

Simulation analysis shows
- Longer LH leads to more allocation
- Much less negative allocations
- More stable allocations
- Sensitive to the choice of reduced set of risk factors
Conclusion

Two allocation methods reduce FRTB allocations to Euler allocations

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Thanks for your attention!