Dynamic Valuation and Hedging of a Book

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Outline

- Defining the Book
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- Unhedged Valuation
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- Hedging Problem
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- Relationship to BSPIDE
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- Explicit Driver Constructions
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- Spatially Inhomogeneous Bilateral Gamma Processes
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- Maximum Likelihood Estimation for Spot Average Ratio Dependence
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- Implementation of Hedge for a Sample Book
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- Markovian model using Average and Spot Average Ratio
- Implementation of Hedge for a Sample Book
- Analysis of Results
Definition of Book

- For linear valuation operators, values and hedges may be determined for each component and then aggregated across all asset positions to get the hedge position for the book.

The functions may be constructed from a knowledge of positions and we presume access to such functions. They define the book to be valued and hedged.
For linear valuation operators, values and hedges may be determined for each component and then aggregated across all asset positions to get the hedge position for the book.

For nonlinear valuation operators this additivity is not valid and one must work at the level of the book directly.

For a vanilla options book at each traded maturity there are long and short option positions at each strike. The net result is that the book has an obligation to the receipt of a function of the underlying at this maturity $t_i$ of the cash flow $c_i(S(t_i), t_i)$, for $t_i \leq T$.

The longest maturity under consideration is $T$. The functions may be constructed from a knowledge of positions and we presume access to such functions. They define the book to be valued and hedged.
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The net result is that the book has an obligation to the receipt of a function of the underlying at this maturity $t_i$ of the cash flow $c_i(S(t_i), t_i)$, for $t_i < T$. The longest maturity under consideration is $T$.

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- The functions may be constructed from a knowledge of positions and we presume access to such functions.
- They define the book to be valued and hedged.
For stock positions in dollar terms of \( a(S(u), u) \) at time \( u \) and discount rate \( r \) the present value of such flows to date are

\[
\sum_{t_i \leq t} c_i (S(t_i), t_i) e^{-rt_i} + \int_0^t \int_{-\infty}^\infty e^{-ru} a(S(u_\_), u) (e^x - 1) \mu(dx, du)
\]
For stock positions in dollar terms of $a(S(u), u)$ at time $u$ and discount rate $r$ the present value of such flows to date are

$$\sum_{t_i \leq t} c_i (S(t_i), t_i) e^{-rt_i} + \int_0^t \int_{-\infty}^\infty e^{-ru} a(S(u_), u) (e^x - 1) \mu(dx, du)$$

where $\mu(dx, du)$ is the integer valued random measure associated with the finite variation jumps in the log price relative of the stock price.
For valuation we follow the $\mathcal{G} \times \textit{expectations}$ approach introduced in Peng (2006) with valuations defined by nonlinear expectations that are unique viscosity solutions to equations of the form

$$V_t = \mathcal{G}(V) - rV$$
For valuation we follow the $G - expectations$ approach introduced in Peng (2006) with valuations defined by nonlinear expectations that are unique viscosity solutions to equations of the form

$$V_t = G(V) - rV$$

for a nonlinear operator $G$ for boundary conditions

$$V(S, 0) = c_T(S)$$
$$V(S, t_i) = V(S, t_{i-}) + c_i(S), \ 0 < t_1 < \cdots < t_N = T.$$
Unhedged Valuation

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- for a nonlinear operator $\mathcal{G}$ for boundary conditions

$$V(S, 0) = c_T(S)$$
$$V(S, t_i) = V(S, t_{i-}) + c_i(S), \quad 0 < t_1 < \ldots < t_N = T.$$  

- The result is a nonlinear valuation at time $t$ of the future cash flows yet to be realized, when the spot at time $t$ is at level $S$.  

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Value and Hedge Book
The operator is defined in terms of measure distortions that distort the compensator of $\mu$ that we take to possibly be dependent on the level of $S$ and in the bilateral gamma class.
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We write

$$\nu(S, x)dxdt = \left( \frac{c_p(S)}{x} \exp \left( -\frac{x}{b_p(S)} \right) 1_{x>0} + \frac{c_n(S)}{|x|} \exp \left( -\frac{|x|}{b_n(S)} \right) 1_{x<0} \right) dxdt$$
The operator is defined in terms of measure distortions that distort the compensator of $\mu$ that we take to possibly be dependent on the level of $S$ and in the bilateral gamma class.

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$$

The spatially inhomogeneous compensating measure is then

$$
\nu(S, A) = \int_A \nu(S, x) \, dx
$$
The law of one price coupled with no arbitrage implies that the value function $V(X)$ satisfies

$$V(aX + bY) = aV(X) + bV(Y).$$
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Hence the valuation function is linear.
One Price Issues

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- Hence the valuation function is linear.
- The set of acceptable risks are those with a positive value and this is a very large convex set.
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The set of acceptable risks are those with a positive value and this is a very large convex set.

Useful as it may be, it renders optimization useless and defines risk acceptability too generously.
Graph of Arbitrages, Positive Alpha, and Acceptable Opportunities

- Set of Positive Values
- Set of Acceptable Outcomes
- Arbitrages

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When risk acceptability is contracted from a half space to a proper convex cone containing the nonnegative outcomes that are certainly acceptable at zero cost, then the acceptable risks become all outcomes $X$ satisfying

$$E^Q[X] \geq 0, \text{ for all } Q \in \mathcal{M}$$
Conservative valuation in two price economies

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- where $\mathcal{M}$ is a collection of test probabilities or scenarios.
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Selling $X$ at $b(X)$ or buying it for $a(X)$ requires that

$$X - b(X) \text{ and } a(X) - X$$
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Selling $X$ at $b(X)$ or buying it for $a(X)$ requires that $X - b(X)$ and $a(X) - X$ are acceptable.

The best and bid and ask prices are

$$b(X) = \inf_{Q \in \mathcal{M}} E^Q[X]$$

$$a(X) = \sup_{Q \in \mathcal{M}} E^Q[X]$$
The maximization objective for risk exposure evaluation should be concave.
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We now ask why one should scale value with the scale of the outcome?
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We now ask why one should scale value with the scale of the outcome?

If the value is market based and contemplated positions are small relative to the size of the market then the value of twice the outcome should be twice the value.
Conservative valuation and probability distortions

- Probabilities of rare or extreme events are not to be trusted and should be distorted upwards for losses and downwards for gains to induce risk aversion and the absence of gain enticement.

Let $\Psi(u)$ be a concave distribution function on the unit interval and evaluate the value of outcome $X$ with distribution function $F(x)$ and density $f(x)$ as

$$b(X) = \int_0^1 \psi(0) F(x) f(x) \, dx$$

$\Psi(u)$ for $u$ near zero lifts the weighting on losses while $\Psi(u)$ for $u$ near unity reduces the weighting on gains.
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- Let $\Psi(u)$ be a concave distribution function on the unit interval and evaluate the value of outcome $X$ with distribution function $F(x)$ and density $f(x)$ as

$$b(X) = \int x\Psi'(F(x)) f(x) dx$$
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$\Psi'(u)$ for $u$ near zero lifts the weighting on losses while $\Psi'(u)$ for $u$ near unity reduces the weighting on gains.
The lower price formed from distorted expectations is associated with $\mathcal{M}$ the set of supporting measures being all measures $Q$ such that

$$Q(A) \leq \Psi(P(A)),$$

for all $A$. 
Bid Ask Distortions

![Graph showing Bid and Ask Distortions with probability on the x-axis and distorted probability on the y-axis.](image-url)
Event probabilities require a knowledge of the frequency of events of interest relative the frequency of all other possible events.
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- Taking the view that events of interest happen occasionally while nothing of interest happens all the time we work with just the numerator as an infinite measure.
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Taking the view that events of interest happen occasionally while nothing of interest happens all the time we work with just the numerator as an infinite measure.

The outcomes of interest are integrable with respect to the infinite measure and hence constant outcomes are inadmissible.

All outcomes are bounded away from zero on sets of finite measure.

On sets of infinite measure outcomes converge towards zero reflecting the view that only nothing happens all the time.
Introduction two increasing functions on the positive half line, $G^+$ concave and above the identity and $G^-$ convex and below the identity with both zero at zero.

The measure distorted lower value of outcome $X$ on an infinite measure space with measure $\mu$ is defined by

$$b(X) = \int_0^{\infty} G^+(\mu(X < a)) \, da + \int_0^{\infty} G^-(\mu(X > a)) \, da$$

The upper value is given by

$$a(X) = \int_0^{\infty} G^-(\mu(X < a)) \, da + \int_0^{\infty} G^+(\mu(X > a)) \, da$$
Introduce two increasing functions on the positive half line, $G^+$ concave and above the identity and $G^-$ convex and below the identity with both zero at zero.

The measure distorted lower value of outcome $X$ on an infinite measure space with measure $\mu$ is defined by

$$b(X) = - \int_0^\infty G^+ (\mu (X < -a)) \, da + \int_0^\infty G^- (\mu (X > a)) \, da$$
Introduce two increasing functions on the positive half line, $G^+$ concave and above the identity and $G^-$ convex and below the identity with both zero at zero.

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The lower valuation using measure distortions is

\[ b(X) = \inf_{\tilde{\nu} \in \mathcal{M}} \int X(\omega) \tilde{\nu}(d\omega) \]

\[ a(X) = \sup_{\tilde{\nu} \in \mathcal{M}} \int X(\omega) \tilde{\nu}(d\omega) \]

\[ \text{where } \tilde{\nu} \text{ just if } G(\nu(A)) \equiv \tilde{\nu}(A) + \nu(A), \text{ for all } A \text{ with } \nu(A) < \infty. \]
The lower valuation using measure distortions is

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b(X) = \inf_{\tilde{\nu} \in \mathcal{M}} \int X(\omega)\tilde{\nu}(d\omega)
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a(X) = \sup_{\tilde{\nu} \in \mathcal{M}} \int X(\omega)\tilde{\nu}(d\omega)
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where \(\tilde{\nu} \in \mathcal{M}\) just if

\[
G^-(\nu(A)) \leq \tilde{\nu}(A) \leq G^+(\nu(A)), \text{ for all } A \text{ with } \nu(A) < \infty.
\]
Measure Distortions
Hedged Valuation

- For two measure increasing distortions $G^+, G^-$ concave/convex, above/below the identity and hedge policy $a(S, t)$ we define the operator

$$G(V) = -\int_0^\infty G^+ \left( \nu(S, \begin{bmatrix} a(S, t) (e^x - 1) \\ +V(Se^x, t) - V(S, t) \end{bmatrix}^- > w \right) dw$$

$$+ \int_0^\infty G^- \left( \nu(S, \begin{bmatrix} a(S, t) (e^x - 1) \\ +V(Se^x, t) - V(S, t) \end{bmatrix}^+ > w \right) dw$$
Hedged Valuation

- For two measure increasing distortions $G^+, G^-$ concave/convex, above/below the identity and hedge policy $a(S, t)$ we define the operator

$$G(V) = -\int_0^\infty G^+ \left( v(S, \left[ a(S, t)(e^x - 1) + V(Se^x, t) - V(S, t) \right]^- > w \right) \, dw$$

$$+ \int_0^\infty G^- \left( v(S, \left[ a(S, t)(e^x - 1) + V(Se^x, t) - V(S, t) \right]^+ > w \right) \, dw$$

- The hedge selection defines

$$a(S, t) = \underset{a}{\text{arg max}}$$

$$\left( -\int_0^\infty G^+ \left( v(S, \left[ a(S, t)(e^x - 1) + V(Se^x, t) - V(S, t) \right]^- > w \right) \, dw$$

$$+ \int_0^\infty G^- \left( v(S, \left[ a(S, t)(e^x - 1) + V(Se^x, t) - V(S, t) \right]^+ > w \right) \, dw \right).$$
The nonlinear valuation may be related to the solution of Backward Stochastic Partial Integro-Differential Equations.
Connections to BSPIDE’s

- The nonlinear valuation may be related to the solution of Backward Stochastic Partial Integro-Differential Equations.
- For this purpose we work on a filtered probability space associated with a pure jump Markov process with finite variation jump compensators for the logarithm of a positive $k$ dimensional Markov process $X$. 

\[
\nu(X(t),t,x) \, dx \, dt \\
+ \int_0^t R_{nf} \, (\exp(I)(x)) \, (1_k) \, \nu(X(s),s,x) \, dx \, ds
\]

where $(\exp(I)(x)) = (e^x, \ldots, e^k)^T$, and $1_k$ is the $k$ dimensional vector with all entries unity, and $e^{N(dx \, ds)}$ is a compensated jump measure.
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$$\nu(X(t), t, x) dx dt$$

- with

$$X(t) = \int_{(0,t] \times \mathbb{R}^k \setminus \{0\}} ((\exp \circ I)(x) - 1_k) \nu(X(s), s, x) dx ds$$

$$+ \int_{(0,t] \times \mathbb{R}^k \setminus \{0\}} ((\exp \circ I)(x) - 1_k) \tilde{N}(dx \times ds)$$
Connections to BSPIDE’s

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- The compensator has the form

$$\nu(X(t), t, x) dx dt$$

- with

$$X(t) = \int_{(0,t] \times \mathbb{R}^k \setminus \{0\}} \left( (\exp \circ l)(x) - 1_k \right) \nu(X(s), s, x) dx ds$$

$$+ \int_{(0,t] \times \mathbb{R}^k \setminus \{0\}} \left( (\exp \circ l)(x) - 1_k \right) \widetilde{N}(dx \times ds)$$

- where $(\exp \circ l)(x) = (e^{x_1}, \ldots, e^{x_k})^T$, $1_k$ is the $k$ dimensional vector with all entries unity, and $\widetilde{N}(dx \times ds)$ is a compensated jump measure.
On this probability space let $\chi$ be a terminal random variable and let $B_t$ be a lower prudential valuation for $\chi$ associated with a driver function $g(t, z)$ defined on $(0, T] \times \mathcal{L}^2(\nu(X(t), t, x)dx)$ that is positive, homogeneous and convex in $z$. 

The Backward Stochastic Partial Integro-Differential Equation (BSPIDE) solves for $(B_t, Z_t)$ for $t < T$ the equation

$$B_t + Z_T g(s, Z_s) ds + \int_0^T g_s(x) e_N(\nu(x)dx) = \chi.$$
On this probability space let $\chi$ be a terminal random variable and let $B_t$ be a lower prudential valuation for $\chi$ associated with a driver function $g(t, z)$ defined on $(0, T] \times L^2(\nu(X(t), t, x)dx)$ that is positive, homogeneous and convex in $z$.

The Backward Stochastic Partial Integro-Differential Equation (BSPIDE) solves for $(B_t, Z_t)$ for $t < T$ the equation

$$B_t + \int_t^T g(s, Z_s)ds + \int_{(0, t] \times \mathbb{R}^k \setminus \{0\}} Z_s(x) \tilde{N}(ds \times dx) = \chi.$$
In this Markovian context the lower prudential valuation can be written as

\[ B_t = V(t, X(t)) \]
BSPIDE and Valuation

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BSPIDE and Valuation

- In this Markovian context the lower prudential valuation can be written as
  \[ B_t = V(t, X(t)) \]
- where
  \[ V(T, X(T)) = \chi \]
- and \( V \) solves the semilinear partial integro-differential equation
  \[
  \dot{V} + \mathcal{K} V(t, x) - g(t, \mathcal{D} V_{t,x}) = 0 \\
  \mathcal{K} V(t, x) = d_t^T \nabla V \\
  + \int_{\mathbb{R}^k \setminus \{0\}} (\mathcal{D} V_{t,x} - \nabla V(t, x)^T x (e^y - 1)) \nu(dy) \\
  \mathcal{D} V_{t,x} = V(t, xe^y) - V(t, x) \\
  d_t = \int_{\mathbb{R}^k \setminus \{0\}} x (e^y - 1) \nu(x, t, dy)
  \]
The lower prudential value is related to risk acceptability via

\[ B_t = \inf_{Q \in S_g} E^Q [\chi | \mathcal{F}_t]. \]
Valuations and Risk Acceptability

- The lower prudential value is related to risk acceptability via
  \[ B_t = \inf_{Q \in S^g} E^Q [\chi | \mathcal{F}_t]. \]

- Here $S^g$ consists of all measures $Q$ for which the representation $M$ of the stochastic logarithm satisfies a condition related to $g$.

\[ \zeta = \frac{dQ}{dP} \]
\[ \tilde{\zeta} = \mathcal{E}(M), \; \mathcal{E} \text{ is stochastic exponential} \]
\[ M = \int_{(0,T] \times \mathbb{R}^k \setminus \{0\}} H_s(y) \tilde{N}(ds \times dy) \]
The lower prudential value is related to risk acceptability via

\[ B_t = \inf_{Q \in S^g} E^Q \left[ \chi \mid \mathcal{F}_t \right]. \]

Here \( S^g \) consists of all measures \( Q \) for which the representation \( M \) of the stochastic logarithm satisfies a condition related to \( g \).

\[ \xi = \frac{dQ}{dP}, \quad \xi = \mathcal{E}(M), \quad \mathcal{E} \text{ is stochastic exponential} \]

\[ M = \int_{(0,T] \times \mathbb{R}^k \setminus \{0\}} H_s(y) \bar{N}(ds \times dy) \]

with

\[ \int z(y) H_s(y) \nu(X(s), s, y) dy \leq g(s, z), \quad z \in \mathcal{L}^2 \left( \nu(X(s), s, y) dy \right). \]
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The Valuation Driver

- For the moment we ignore the hedge and just consider valuation.
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The distorted variation we work with is

\[ GW = - \int_0^\infty G^+ \left( \nu(X, [W(Xe^x, t) - W(X, t)]^- > w) \right) dw + \int_0^\infty G^- \left( \nu \left( X, [W(Xe^x, t) - W(X, t)]^+ > w \right) \right) dw. \]
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\[ + \int_0^\infty G^- \left( v \left( X, [W(Xe^x, t) - W(X, t)]^+ > w \right) \right) dw. \]

- The variation on the other hand is

\[ V(W) = - \int_0^\infty \left( v(X, [W(Xe^x, t) - W(X, t)]^- > w) \right) dw \]
\[ + \int_0^\infty -v \left( X, [W(Xe^x, t) - W(X, t)]^+ > w \right) dw \]
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and let

\[ g(t, z) = \int_0^\infty \Gamma^+ (v(X, [z(y)^- > w])) \, dw + \int_0^\infty \Gamma^- (v(X, [z(y)^+ > w])) \, dw \]
The semilinear equation

We then write

$G W = V W - g(t, Z)$
The semilinear equation

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\[ G \dot{W} = \nabla W - g(t, Z) \]

or equivalently

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- We then write

\[ \mathcal{G} \mathcal{W} = \mathcal{V} \mathcal{W} - g(t, Z) \]

- or equivalently

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- The semilinear equation in \( \mathcal{W} \) is on noting \( \mathcal{K} \mathcal{W} = \mathcal{V} \mathcal{W} \) that

\[ \dot{\mathcal{W}} + \mathcal{K} \mathcal{W} - g(t, \mathcal{D} \mathcal{W}_{t,x}) = 0 \]
We now write that

\[ e^{-rt} V(t, x) + \int_t^T g(s, Z_s) ds + \int_{(t, T] \times \mathbb{R}^k \setminus \{0\}} Z_s(y) \tilde{N}(ds \times dy) = \chi. \]
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Furthermore we have that

\[ e^{-rt} V(t, X(t)) = \inf_{Q \in \mathcal{S}_g} E_Q[e^{-rT} \chi | \mathcal{F}_t]. \]
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But for a genuine Markov context we must establish a dependence of the law of motion on observables.

It is unreasonable to suppose that \textit{SPY} return distributions over long periods used in estimation will show a dependence of such on the level of the index.

We consider instead a dependence on the ratio of the index level to a geometrically weighted average of past prices.
For a thousand days ending December 31, 2018 we obtain data on daily returns one day forward.
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The bilateral gamma model is then estimated separately in each bucket using digital moment estimation.
A Preliminary Investigation

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- With weights $\lambda^n$ for prices lagged $n$ days and $\lambda = 0.9$.
- We then bucket the returns based on the level of the ratio.
- The bilateral gamma model is then estimated separately in each bucket using digital moment estimation.
- From the estimated parameters one may evaluate the return drift separately in each bucket.
The Figure presents the dependence observed for the drift as a function of the ratio.
We employed a thousand days of data ending December 31, 2018 on the one day forward return and the contemporaneous spot average ratio. The ratio lies between 0.96 and 1.03 in the data and for the bilateral gamma simulated using median parameter values. On a coarse grid from 0.85 to 1.05 in five steps of 0.05 we take the candidate value of each of the four parameters at these levels for the ratio to be free parameters to be estimated. At all points including the coarse grid the actual parameters are taken from the coarse grid candidate values as a smooth function using a Gaussian Kernel smoother. This avoids having to describe up front a functional form for the dependence.
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MLE Estimation of BG dependence on Spot Average Ratio

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This minimizes the number of Fourier inversions to be computed.

The logarithm of the thousand densities evaluated at the observed returns are summed to construct the log likelihood to be maximized.
The Figure presents the dependence of the four bilateral gamma parameters on the spot average ratio.
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- The drift though positive falls with the level of the ratio.
Markov Model for the Average, $Y$, and the Spot Average Ratio, $Z$

- In a continuous formulation the average $Y(t)$ may be constructed from the Spot prices $S(t)$ by

$$Y(t) = \theta \int_{-\infty}^{t} e^{-\theta(t-u)} S(u) du.$$
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- The dynamics for the ratio are given by

$$dZ = -\theta Z(t) (Z(t) - 1) \ dt Z(t) ((e^x - 1) * \mu(dx, du))$$
The compensator for $Z$ is given by

\[ \nu(Z, dx) dt = \left( \frac{c_p(Z)}{x} \exp\left( -\frac{x}{b_p(Z)} \right) \mathbf{1}_{x > 0} + \frac{c_n(Z)}{|x|} \exp\left( -\frac{|x|}{b_n(Z)} \right) \right) dxdt \]
Hence $Z(t), Y(t)$ is a two dimensional Markov process and we seek the value and hedge policy functions

$$V(Z(t), Y(t), t)$$
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to be solved for on a three dimensional grid for $Z, Y$ and time $t$. 
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V(Z(t), Y(t), t)
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\[
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to be solved for on a three dimensional grid for \( Z, Y \) and time \( t \).

We do this on an 18 core Alienware Machine.
The Operator to be Solved

- We solve

\[ V_t = G(V) - rV - \theta Z(Z - 1)V_Z + \theta(Z - 1)Y_V. \]
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For

\[
G(V) = -\int_0^\infty G^+ \left( V(Z, [a(Z, Y, t) (e^t - 1) + V(Ze^t, Y, t) - V(Z, Y, t)]^>) > w \right) dw \\
+ \int_0^\infty G^- \left( V(Z, [a(Z, Y, t) (e^t - 1) + V(Ze^t, Y, t) - V(Z, Y, t)]^<) > w \right) dw.
\]
The boundary conditions in reversed time are

\[ V(Z, Y, 0) = C_T(YZ) \]

\[ V(Z, Y, t_i) = V(Z, Y, t_i) + c_i(Y(t_i), Z(t_i)), t_i > 0. \]
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- For cash flows being held at maturities \( t_i \) with claims to \( c_i(S) \).
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- For cash flows being held at maturities \( t_i \) with claims to \( c_i(S) \).
- The policies locally maximize \( G(V) \) over choices for hedge positions \( a \).
We take a non-uniform grid of 50 points on the average between half and twice the initial level.
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We also take a 40 point non-uniform grid on the ratio between 0.9 and 1.1.
Implementation Details

- We take a non-uniform grid of 50 points on the average between half and twice the initial level.
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- The book of claims hedged have maturities under a year.
 Implementation Details

- We take a non-uniform grid of 50 points on the average between half and twice the initial level.
- We also take a 40 point non-uniform grid on the ratio between 0.9 and 1.1.
- The book of claims hedged have maturities under a year.
- The backward recursion is implemented on 20 time steps.
The integral of the compensator $\nu(Z, x)dx$ over sets $A$ is accomplished by simulation.
Integrating the compensator

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- At these points one evaluates all outcomes and the measure change to a gamma density of shape parameter 0.075.
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Values and Hedge positions are smoothed at each time step using Gaussian Process Regression.
The book on the underlier *SPY* was estimated for June 26, 2019 using information on dollar vega exposures bucketed by strike in two percentage point intervals at traded maturities.
Explicit Boundary Functions

- The book on the underlier $SPY$ was estimated for June 26, 2019 using information on dollar vega exposures bucketed by strike in two percentage point intervals at traded maturities.
- The number of options at each strike is then constructed by dividing the dollar vega in the bucket by the vega for the specific strike taken at its implied volatility.
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Summing the spot dependent option payoffs times positions delivers the functions \( c_i(S) \).
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- Summing the spot dependent option payoffs times positions delivers the functions \( c_i(S) \).
- There were 29 maturities below a year in the option book ranging from one to 358 days.
The Figure presents graphs of state contingent claims in millions of dollars for a sample of the 29 maturities.
We present a sample of graphs showing how hedge positions vary with \( Z \) for different levels of \( Y \) and time.

The blue curves are for the current level of the average with red and black for 10 dollars down and up. The positions fall with time.
Hedge Positions By $Z$ and $Y$

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- The blue curves are for the current level of the average with red and black for 10 dollars down and up.
- The positions fall with time.
We present four graphs showing the nonlinear value as a function of \( Z \) for three settings on \( Y \) for now, three, six, and nine months in.
The hedge positions were regressed on the delta, gamma, and levels of $Z$, $Y$, $Z^2$, $Y^2$ separately for each level of time maturity.
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The Table shows results for a sample of times to maturity.

<table>
<thead>
<tr>
<th>Time Left</th>
<th>Delta</th>
<th>Gamma</th>
<th>$Z$</th>
<th>$Y$</th>
<th>$Z^2$</th>
<th>$Y^2$</th>
<th>RSQ</th>
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<tbody>
<tr>
<td>1</td>
<td>1.1293</td>
<td>29.9751</td>
<td>1723.6023</td>
<td>195.7961</td>
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<td>0.9</td>
<td>1.0964</td>
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<tr>
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<tr>
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<tr>
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<td>0.4428</td>
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<td>-0.0007</td>
<td>0.8342</td>
</tr>
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</table>
The Table shows the corresponding t-stats.

<table>
<thead>
<tr>
<th>Time Left</th>
<th>Delta</th>
<th>Gamma</th>
<th>Z</th>
<th>Y</th>
<th>Z^2</th>
<th>Y^2</th>
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<td>0.6</td>
<td>19.86</td>
<td>13.64</td>
<td>8.75</td>
<td>64.65</td>
<td>-8.50</td>
<td>-62.60</td>
</tr>
<tr>
<td>0.5</td>
<td>19.28</td>
<td>13.47</td>
<td>8.65</td>
<td>58.71</td>
<td>-8.42</td>
<td>-56.99</td>
</tr>
<tr>
<td>0.4</td>
<td>18.96</td>
<td>14.29</td>
<td>8.36</td>
<td>55.32</td>
<td>-8.17</td>
<td>-53.76</td>
</tr>
<tr>
<td>0.3</td>
<td>21.73</td>
<td>13.70</td>
<td>9.45</td>
<td>50.24</td>
<td>-9.32</td>
<td>-49.09</td>
</tr>
</tbody>
</table>
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