Large Deviations from the Hydrodynamic Limit for a System with Nearest Neighbor Interactions

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Background

- Let \( \{Y_i\}_{i \geq 1} \) be a sequence of \( \mathbb{R}^d \)-valued iid zero mean random variables with common probability law \( \rho \).

- Let \( S_n = \sum_{i=1}^{n} Y_i \). Then \( S_n/n \to 0 \) a.s. by LLN.

- **Large Deviation Principle:** For \( c > 0 \)
  \[
  \mathbb{P}(|S_n| > nc) \approx \exp\{-n \inf\{l(y) : |y| \geq c\}\},
  \]
  where for \( y \in \mathbb{R}^d \),
  \[
  l(y) = \sup_{\alpha \in \mathbb{R}^d} \{\langle \alpha, y \rangle - \log \int_{\mathbb{R}^d} \exp(\langle \alpha, y \rangle) \rho(dy)\}.
  \]
Large Deviation Principle.

**Definition.** Consider a sequence \( \{X^\varepsilon\}_{\varepsilon > 0} \) of \( \mathcal{E} \) valued r.vs.

- \( I : \mathcal{E} \to [0, \infty] \) is a rate function on \( \mathcal{E} \) if for each \( M < \infty \), \( \{x \in \mathcal{E} : I(x) \leq M\} \) is compact.

- \( \{X^\varepsilon\} \) is said to satisfy the large deviation principle on \( \mathcal{E} \) (as \( \varepsilon \to 0 \)) with rate function \( I \) if:
  - For each closed \( F \subset \mathcal{E} \)
    \[
    \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq -\inf_{x \in F} I(x).
    \]
  - For each open \( G \in \mathcal{E} \)
    \[
    \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq -\inf_{x \in G} I(x).
    \]

Formally, for small \( \varepsilon \):

\[
\mathbb{P}(X^\varepsilon \in A) \approx \exp \left\{ -\frac{\inf_{x \in A} I(x)}{\varepsilon} \right\}, \quad A \in \mathcal{B}(\mathcal{E}).
\]
Stochastic Control Connection (Fleming 1978)

Consider a small noise $n$-dimensional SDE:

$$dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t))dW(t), \ X^\varepsilon(0) = x.$$ 

- $b, \sigma$ suitable coefficients... $W$ a f.d. BM.

- Let $G \subset \mathbb{R}^n$ be bounded open. Let $x \in G$ and $\tau^\varepsilon = \inf\{t : X^\varepsilon(t) \in \partial G\}$.

- Interested in $\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^\varepsilon(\tau^\varepsilon) \in N)$, where $N \subset \partial G$. 
Formally, with $\Phi$ a nonnegative $C^2$ function, $\Phi(x) \approx M1_{\mathcal{N}_c}(x)$, $M$ a large scaler,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(X^\varepsilon(\tau^\varepsilon) \in \mathcal{N}) \approx \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_x \left\{ e^{-\Phi(X^\varepsilon(\tau^\varepsilon))}/\varepsilon \right\}.$$ 

Then $g^\varepsilon(x) = \mathbb{E}_x \left\{ e^{-\Phi(X^\varepsilon(\tau^\varepsilon))}/\varepsilon \right\}$ solves

$$\begin{cases} 
\mathcal{L}^\varepsilon g^\varepsilon(x) = 0, & x \in G \\
g^\varepsilon(x) = e^{-\Phi(x)}/\varepsilon, & x \in \partial G 
\end{cases}$$

where $\mathcal{L}^\varepsilon g = \frac{\varepsilon}{2} \text{Tr}(\sigma D^2 g \sigma') + b \cdot \nabla g$.

Interested in asymptotics of $-\varepsilon \log g^\varepsilon$. 

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log transform: Let \( J^\varepsilon = -\varepsilon \log g^\varepsilon \). Then \( J^\varepsilon \) solves

\[
\frac{\varepsilon}{2} \text{Tr}(\sigma D^2 J^\varepsilon \sigma') + H(x, \nabla J^\varepsilon) = 0
\]

where

\[
H(x, p) = \min_{v \in \mathbb{R}^n} [L(x, v) + p \cdot v], \quad x \in G, \quad p \in \mathbb{R}^n
\]

and \( L(x, v) = \frac{1}{2}(b(x) - v)'[\sigma(x)\sigma'(x)]^{-1}(b(x) - v) \).

\( J^\varepsilon \) can be characterized as the value function of the stochastic control problem:

\[
J^\varepsilon(x) = \inf_{u \in \mathcal{A}} \mathbb{E}_x \left\{ \int_0^{\tilde{\tau}^\varepsilon} L(\tilde{X}^\varepsilon(t), u(t))dt + \Phi(\tilde{X}^\varepsilon(\tilde{\tau}^\varepsilon)) \right\}
\]

\[
d\tilde{X}^\varepsilon(t) = u(t)dt + \sqrt{\varepsilon}\sigma(X^\varepsilon(t))dW(t), \quad \tilde{X}^\varepsilon(0) = x
\]
Stochastic Control Connection (ctd.)

One can argue $J^\varepsilon \to J$, where $J(x)$ is the value function of the deterministic control problem:

$$J(x) = \inf_{\phi, \theta} \left[ \int_0^\theta L(\phi(t), \dot{\phi}(t)) dt + \Phi(\phi(\theta)) \right],$$

where $\inf$ is over all abs. cts. $\phi$ such that $\phi(0) = x$, and $\theta = \inf \{ t : \phi(t) \in \partial G \}$.

LDP and Laplace Principle.

- LDP is equivalent to Laplace principle if the state space is Polish (Varadhan(1966), Bryc(1990)):
  - A collection of $\mathcal{E}$ valued random variables $\{X^\varepsilon\}$ is said to satisfy Laplace principle with rate function $I$, if for all $h \in C_b(\mathcal{E})$
    \[
    \lim_{\varepsilon \to 0} -\varepsilon \log \mathbb{E}\left\{ \exp\left[ -\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right\} = \inf_{x \in \mathcal{E}} \{ h(x) + I(x) \}.
    \]
- From Donsker-Varadhan:
    \[
    -\varepsilon \log \mathbb{E}\left\{ \exp\left[ -\frac{1}{\varepsilon} h(X^\varepsilon) \right] \right\} = \inf_{Q \in \mathcal{P}(\mathcal{E})} \left[ \int h(x) dQ(x) + R(Q \| P^\varepsilon) \right].
    \]
LDP and Laplace Principle.

- Goal is to show the convergence of variational expressions:

\[
\inf_{Q \in \mathcal{P}(E)} \left[ \int h(x) dQ(x) + R(Q \| P^\varepsilon) \right] \xrightarrow{\varepsilon \to 0} \inf_{x \in \mathcal{E}} \{ h(x) + I(x) \}.
\]
Some Settings where the Approach Works.

- Small Noise SPDE (B., Dupuis and Maroulas (2008)).
- Stochastic Flows of Diffeomorphisms (B., Dupuis and Maroulas (2010)).
- Finite and Infinite Dimensional Jump-Stochastic Dynamical Systems with Small Noise (B., Chen and Dupuis (2013)).
- Moderate deviation principles for SDE w/ Jumps in Finite and Infinite Dimensions (B., Dupuis and Ganguly (2016)).
- Component Size Large Deviations for Configuration Model (Bhamidi, B., Dupuis and Wu (2017)).
- Multiscale jump-diffusions – Large Deviations from Stochastic Averaging Principle (B., Dupuis and Ganguly (2017)).
- Weakly Interacting Diffusions – Large and Moderate Deviations (B., Dupuis and Fischer (2012), B. and Wu (2016)).
A System with Nearest Neighbor Interactions.

- **Ginzburg-Landau in Finite Volume**: For $t \in [0, T]$ and $i = 1, \ldots, N$

  \[
  dX_i^N(t) = \frac{N^2}{2} \left[ \phi' \left( X_{i-1}^N(t) \right) - 2\phi' \left( X_i^N(t) + \phi' \left( X_{i+1}^N(t) \right) \right) \right] dt \\
  + N \left[ dB_i(t) - dB_{i+1}(t) \right]
  \]

- $\{1/N, \ldots, (N - 1)/N, 1\}$ is the periodic lattice. I.e. identify $X_{N+1}^N$ with $X_1^N$.

- $\{B_i(t)\}_{i=1}^{\infty}$ are independent standard one-dimensional Brownian motions given on some probability space $(\mathcal{V}, \mathcal{F}, \mathbb{P})$. 
A System with Nearest Neighbor Interactions.

- $\phi : \mathbb{R} \to \mathbb{R}$ is $C^2$ and

$$\int_{\mathbb{R}} \exp(-\phi(x))dx = 1, \ M(\lambda) \doteq \int_{\mathbb{R}} \exp(\lambda x - \phi(x)) < \infty$$

for all $\lambda \in \mathbb{R}$, and for all $\sigma < \infty$

$$\int_{\mathbb{R}} \exp(\sigma |\phi'(x)| - \phi(x))dx < \infty.$$
Invariant measure for $X^N$: $\Phi^N(dx) \equiv \Phi(dx_1)\Phi(dx_2)\ldots \Phi(dx_N)$ where $\Phi(dx) \equiv e^{-\phi(x)}dx$.

Consider $X^N$ with $X^N(0) \sim \Phi^N$ and process $\mu^N$ with values in $\mathcal{M}_S$ (the space of signed measures on the unit circle $S$):

$$\mu^N(t, d\theta) \equiv \frac{1}{N} \sum_{i=1}^{N} X_i^N(t)\delta_{i/N}(d\theta).$$

A LLN for $\mu^N$ shown in Guo-Papanicolaou-Varadhan (1988) and a LDP proved in Donsker-Varadhan (1989).

A new proof...
Remarks

- The original proof [DW(1989)] requires control on exponential moments and exponential probability estimates. This approach has been extended to many different systems.

- Exponential estimates are the hardest parts of the proof.

- The new proof uses stochastic control representations and weak convergence methods.

- Proof techniques similar to that for LLN analysis. No exponential estimates are invoked.

- Key Technical Step: Suitable Regularity of Densities of Controlled Processes. Bounds on certain Dirichlet Forms.
Main Result

- Let $M^l_S$ be elements in $M_S$ with total variation bounded by $l$.

- Let for $l \in \mathbb{N}$, $\Omega_l \equiv C([0, T] : M^l_S)$ the Polish space of continuous paths of signed measures with total variation bounded by $l$.

- Then $\Omega \equiv C([0, T] : M_S) = \bigcup_{l \in \mathbb{N}} C([0, T] : M^l_S) = \bigcup_{l \in \mathbb{N}} \Omega_l$. This space is equipped with the direct limit topology.

- Theorem $\{\mu^N\}$ satisfies a LDP in $C([0, T] : M_S)$ with rate function $I$. 
Rate Function

Let
\[ \rho(\lambda) \doteq \log M(\lambda), \quad \lambda \in \mathbb{R}, \quad h(x) \doteq \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \rho(\lambda) \}. \]

Let \( \tilde{\Omega} \) be the collection of all \( \mu \) in \( \Omega \) such that for all \( t \), \( \mu(t, d\theta) = m(t, \theta)d\theta \) and \( m \) satisfies
\[
\int_{[0, T] \times S} \left[ h(m(t, \theta)) + \left[ h'(m(t, \theta)) \right]_\theta^2 \right] dt d\theta < \infty,
\]

Let \( \mathcal{P}_*(\mathbb{R} \times S) \) be all \( \pi \in \mathcal{P}(\mathbb{R} \times S) \) such that
\[
\pi(dx \ d\theta) = \pi_1(dx | \theta)d\theta,
\]
with
\[
m_0(\theta) = \int_{\mathbb{R}} x \pi_1(dx | \theta), \quad \int_S h(m_0(\theta))d\theta < \infty.
\]
Rate Function

For \( u \in L^2([0, T] \times S : \mathbb{R}) \) and \( \pi \in \mathcal{P}_*(\mathbb{R} \times S) \) let \( \mathcal{M}_\infty(u, \pi) \) be all \( \mu \in \tilde{\Omega} \), s.t. \( \mu(t, d\theta) = m(t, \theta)d\theta \), and \( m \) solves weakly

\[
\partial_t m(t, \theta) = \frac{1}{2} \left[ h'(m(t, \theta)) \right]_{\theta\theta} - \partial_\theta u(t, \theta), \quad m(0, \theta) = m_0(\theta)
\]

Letting \( \pi_0(dx \ d\theta) = \Phi(dx)d\theta \), define \( I : \Omega \to [0, \infty] \) by

\[
I(\mu) = \inf_{\{(u, \pi) : \mu \in \mathcal{M}_\infty(u, \pi)\}} \left[ \frac{1}{2} \int_0^T \int_S |u(s, \theta)|^2 d\theta ds + R(\pi\|\pi_0) \right]
\]

for \( \mu \in \tilde{\Omega} \), and set \( I(\mu) = \infty \) otherwise.
Main Steps in Proof

- **Compact level sets:** $I$ is a rate function on $\Omega$, namely for every $M < \infty$ 
  \[ \{ \mu \in \Omega : I(\mu) \leq M \} \text{ is compact.} \]

- **Laplace upper bound:** For all $F \in C_b(\Omega)$
  \[ \limsup_{N \to \infty} -\frac{1}{N} \log \mathbb{E}[\exp(-NF(\mu^N))] \geq \inf_{\mu \in \Omega} \{ F(\mu) + I(\mu) \}. \]

- **Laplace lower bound:** For all $F \in C_b(\Omega)$
  \[ \liminf_{N \to \infty} -\frac{1}{N} \log \mathbb{E}[\exp(-NF(\mu^N))] \leq \inf_{\mu \in \Omega} \{ F(\mu) + I(\mu) \}. \]
Variational Representation


Let \((\bar{V}, \bar{F}, \bar{P})\) be a probability space with an \(N\)-dimensional Brownian motion, \(B^N = (B_1, \ldots B_N)\), and a \(\mathbb{R}^N\)-valued random variable \(\bar{X}^N(0)\) independent of \(B^N\) and with probability law \(\Pi^N\).

Let \(\{\bar{F}_t\}\) be any filtration satisfying the usual conditions such that \(B^N\) is a \(\{\bar{F}_t\}\)-Brownian motion and \(\bar{X}^N(0)\) is \(\bar{F}_0\) measurable.

Let \(K_{\Pi^N} = (\bar{V}, \bar{F}, \{\bar{F}_t\}, \bar{P}, \bar{X}^N(0), B^N)\) and let

\[ A^N(K_{\Pi^N}) \doteq \{\psi : \psi = (\psi_i)_{i=1}^N, \psi_i \text{ is simple and } \bar{F}_t \text{ adapted}\}. \]
Variational Representation

For a $\psi^N \in \mathcal{A}^N(\mathcal{K}_{\Pi^N})$, let

$$\bar{B}_i^N(t) = B_i(t) + \int_0^t \psi_i^N(s) ds, \ t \in [0, T], \ i = 1, \ldots N.$$ 

Let

$$d\bar{X}_i^N(t) = \frac{N^2}{2} \left[ \phi' \left( \bar{X}_{i-1}^N(t) \right) - 2\phi' \left( \bar{X}_i^N(t) + \phi' \left( \bar{X}_{i+1}^N(t) \right) \right) \right] dt 
+ N \left[ d\bar{B}_i(t) - d\bar{B}_{i+1}(t) \right]$$

Disintegrate $\Pi^N \in \mathcal{P}(\mathbb{R}^N)$, as

$$\Pi^N(dx) = \Pi_1(dx_1) \Pi_2(dx_2|x_1) \ldots \Pi_N(dx_N|dx_1, \ldots, dx_{N-1}) = \prod_{i=1}^N \Phi_i^N(x, dx_i),$$

and with $\bar{X}^N(0)$ distributed as $\Pi^N$, let $\Phi_i^N(dz) = \Phi_i^N(\bar{X}^N(0), dz)$. 
Variational Representation

Let $F \in C_b(\Omega)$. Then for all $N \in \mathbb{N}$

$$-rac{1}{N} \log \mathbb{E} \exp(-NF(\mu_N)) = \inf_{\Pi_N, \mathcal{K}_\Pi} \inf_{\psi_N \in \mathcal{A}_N(\mathcal{K}_\Pi)} \mathbb{E}_{\Pi_N} \left[ \frac{1}{N} \sum_{i=1}^{N} \left( R(\Phi_i^N \| \Phi) + \frac{1}{2} \int_0^T |\psi_i^N(s)|^2 ds \right) + F(\bar{\mu}_N) \right].$$
Laplace Upper Bound

- Fix $F \in C_b(\Omega)$ and let $\epsilon \in (0, 1)$. Choose for each $N \in \mathbb{N}$, $\Pi^N \in \mathcal{P}(\mathbb{R}^N)$, a system $\mathcal{K}_{\Pi^N}$ and $\psi^N \in \mathcal{A}^N(\mathcal{K}_{\Pi^N})$ such that

$$-\frac{1}{N} \log \mathbb{E} \exp(-NF(\mu^N)) \geq \mathbb{E}_{\Pi^N} \left[ \frac{1}{N} \sum_{i=1}^{N} \left( R(\Phi_i^N \| \Phi) + \frac{1}{2} \int_0^T |\psi_i^N(s)|^2 ds \right) + F(\bar{\mu}^N) \right] - \epsilon.$$

- Since $F$ is bounded, there is a $C \in (0, \infty)$ such that

$$\sup_{N \in \mathbb{N}} \mathbb{E}_{\Pi^N} \left( \frac{1}{N} \sum_{i=1}^{N} R(\Phi_i^N \| \Phi) \right) \leq C, \sup_{N \in \mathbb{N}} \mathbb{E}_{\Pi^N} \left( \frac{1}{2N} \sum_{i=1}^{N} \int_0^T |\psi_i^N(s)|^2 ds \right) \leq C.$$

- By a localization argument, we can assume that for every $N$

$$\frac{1}{2N} \sum_{i=1}^{N} \int_0^T |\psi_i^N(s)|^2 ds \leq C \text{ a.s.}$$
Consequences of Bounded Costs

Suppose

\[ \sup_{N \in \mathbb{N}} \bar{P}_{\Pi N} \left( \frac{1}{N} \sum_{i=1}^{N} R(\Phi_i^N \| \Phi) \right) \leq C_0, \quad \sup_{N \in \mathbb{N}} \left( \frac{1}{2N} \sum_{i=1}^{N} \int_{0}^{T} |\psi_i^N(s)|^2 \, ds \right) \leq C_0. \]

**Lemma 1** Let \( \bar{Q}_{\Pi N} (t) = \mathcal{L}(\bar{X}^N(t)) \). Then, there exists \( C_T \in (0, \infty) \) s.t. for \( t \in [0, T] \),

\[ H_N(t) = R(\bar{Q}_{\Pi N}(t) \| {\Phi}^N) \leq C_T N \quad \text{for all} \quad N \in \mathbb{N}. \]

Let \( V_i = \partial_i - \partial_{i+1} \) and a positive \( f \) on \( \mathbb{R}^N \) that is continuously differentiable along \( V_1, \ldots, V_N \), define

\[ I_N(f) = 4D_N(\sqrt{f}) = \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{(V_i f(x))^2}{f(x)} \Phi^N(dx). \]

**Lemma 2** For \( t \in [0, T] \) and \( N \in \mathbb{N} \), \( \bar{X}^N(t) \) has a density \( \bar{p}_N(t, \cdot) \) w.r.t \( \Phi^N \), which is continuously differentiable, once in time and twice along \( V_1, \ldots, V_N \), and satisfies for some \( C \in (0, \infty) \):

\[ I_N \left( \frac{1}{T} \int_{0}^{T} \bar{p}_N(s, \cdot) \, ds \right) \leq \frac{C}{N} \quad \text{for all} \quad N \geq 1. \]
Consequences of Bounds on RE and Dirichlet Forms

- \{\mu^N\} is a tight sequence of \(\Omega\)-valued random variables.

- \{\nu^N\} is a tight sequence of \(\mathcal{P}(\mathbb{R} \times S)\) valued r.v., where

\[
\nu^N(dx \, d\theta) = \sum_{i=1}^{N} \Phi^N_i(dx) \mathbb{I}_{(i/N, (i+1)/N]}(\theta) d\theta.
\]

- Define \(u^N\) as

\[
u^N(dx \, d\theta) = \sum_{i=1}^{N} \psi^N_i(t) \mathbb{I}_{((i-1)/N, i/N]}(\theta), \quad (t, \theta) \in [0, T] \times S.
\]

Then \(\{u^N\}\) is a sequence of r.v. with values in

\[
S_{C_0} = \left\{ u \in L^2([0, T] \times S) : \int_{[0, T] \times S} |u(t, \theta)|^2 \, d\theta \, dt \leq C_0 \right\}.
\]

With the weak topology on the Hilbert space \(S_{C_0}\) is compact and thus \(\{u^N\}_{N \in \mathbb{N}}\) is a tight sequence of \(S_{C_0}\)-valued random variables.

- Suppose \((\mu^N, u^N, \nu^N)\) converges in distribution along a subsequence to \((\mu, u, \nu)\). Then \(\mu \in \mathcal{M}_\infty(u, \nu)\), a.s.
Completing the Proof of Upper Bound.

\[
\liminf_{N \to \infty} \frac{-1}{N} \log \mathbb{E} \exp \left( -NF \left( \mu^N \right) \right) + \epsilon \\
\geq \liminf_{N \to \infty} \mathbb{E}_{\Pi^N} \left( F \left( \bar{\mu}^N \right) + \frac{1}{N} \sum_{i=1}^{N} \left( R(\Phi_i^N \| \Phi) + \frac{1}{2} \int_0^T |\psi_i^N(s)|^2 ds \right) \right) \\
= \liminf_{N \to \infty} \mathbb{E}_{\Pi^N} \left( F \left( \bar{\mu}^N \right) + R(\nu^N \| \pi_0) + \frac{1}{2} \int_0^T \int_S |u_N(s, \theta)|^2 dsd\theta \right) \\
\geq \mathbb{E} \left( F(\bar{\mu}) + R(\nu \| \pi_0) + \frac{1}{2} \int_0^T \int_S |u(s, \theta)|^2 dsd\theta \right) \\
\geq \mathbb{E} [F(\bar{\mu}) + I(\bar{\mu})] \geq \inf_{\mu \in \Omega} [F(\mu) + I(\mu)],
\]

where the next to last inequality follows from \( \bar{\mu} \in M_\infty(u, \nu) \) a.s.
Laplace Lower Bound.

- Fix $F \in C_b(\Omega)$, and let $\epsilon > 0$. Choose $\bar{\mu}^* \in \Omega$ such that
  \[
  F(\bar{\mu}^*) + I(\bar{\mu}^*) \leq \inf_{\mu \in \Omega} \{ F(\mu) + I(\mu) \} + \epsilon < \infty,
  \]

- Choose $u^* \in L^2([0, T] \times S)$ and $\pi^* \in \mathcal{P}_*(\mathbb{R} \times S)$ such that $\bar{\mu}^* \in \mathcal{M}_\infty(u^*, \pi^*)$ and
  \[
  I(\bar{\mu}^*) + \epsilon \geq \frac{1}{2} \left[ \int_0^T \int_S |u^*(s, \theta)|^2 d\theta ds \right] + R(\pi^*\|\pi_0).
  \]

- Fix $\delta \in (0, 1)$ and let $u^{**} \in C^\infty$ be such that $\|u^{**} - u^*\|_2 \leq \frac{\delta}{2(1 + \|u^*\|_2)}$.

- **Lemma** There is a unique $\bar{\mu}^{**} \in \Omega$ s.t. $\bar{\mu}^{**} \in \mathcal{M}(u^{**}, \pi^*)$. Furthermore, as $\delta \to 0$, $\bar{\mu}^{**} \equiv \bar{\mu}^{**}(\delta) \to \bar{\mu}^*$.
Control Synthesis.

Let $\pi^*(dx, d\theta) = \pi_1^*(dx|\theta)d\theta$ and define

$$\Phi_i^N(dx) \doteq N \int_{(i-1)/N}^{i/N} \pi_1^*(dx|\theta)d\theta, \quad 1 \leq i \leq N,$$

Let $\bar{X}^N(0)$ be a $\mathbb{R}^N$-valued r.v. with distribution

$$\Pi^N(dx) \doteq \Phi_1^N(dx_1) \ldots \Phi_N^N(dx_N).$$

Define $\psi_i^N \in L^2([0, T]: \mathbb{R})$ as

$$\psi_i^N(t) \doteq \sum_{j=1}^{N} u^{**} \left( \frac{jT}{N}, \frac{i}{N} \right) I_{[jT/N, (j+1)T/N]}(t), \quad t \in [0, T].$$


Lemma Let $\bar{\mu}^N$ be constructed using $\Pi^N$ and $\{\psi_i^N\}$. Then

$$\lim_{N \to \infty} \frac{1}{N} \int_0^T \sum_{i=1}^N |\psi_i^N(t)|^2 dt = \int_0^T \int_S |u^{**}(t, \theta)|^2 d\theta dt,$$

$$\frac{1}{N} \sum_{i=1}^N R(\bar{\Phi}_i^N \| \Phi) \leq R(\pi^* \| \pi_0), \text{ for all } N \in \mathbb{N}$$

and $\bar{\mu}^N$ converges to $\bar{\mu}^{**}$ in distribution in $\Omega$. 
Completing the Proof of Lower Bound.

- Choose $\delta$ small s.t. $|F(\bar{\mu}^*) - F(\bar{\mu}^{**})| \leq \epsilon$.

Then

$$\limsup_{N \to \infty} -\frac{1}{N} \log \mathbb{E} \exp \left( -NF \left( \mu^N \right) \right) \leq \limsup_{N \to \infty} \mathbb{E}_{\Pi^N} \left( F \left( \bar{\mu}^N \right) + \frac{1}{N} \sum_{i=1}^{N} \left( R(\Phi_i^N \| \Phi) + \frac{1}{2} \int_0^T |\psi_i^N(s)|^2 ds \right) \right)$$

$$\leq F(\bar{\mu}^{**}) + R(\pi^* \| \pi_0) + \frac{1}{2} \int_0^T \int_S |u^{**}(s, \theta)|^2 ds d\theta$$

$$\leq F(\bar{\mu}^*) + R(\pi^* \| \pi_0) + \frac{1}{2} \int_0^T \int_S |u^*(s, \theta)|^2 ds d\theta + \epsilon + 2\delta$$

$$\leq F(\bar{\mu}^*) + I(\bar{\mu}^*) + 2\epsilon + 2\delta$$

$$\leq \inf_{\mu \in \Omega} \{ F(\mu) + I(\mu) \} + 3\epsilon + 2\delta.$$