THE REPRESENTATION THEORY OF PROFINITE ALGEBRAS
INTERACTIONS WITH CATEGORY THEORY,
ALGEBRAIC TOPOLOGY AND COMPACT GROUPS

by

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0.1 ABSTRACT

In the present text, we examine current trends in the theory of profinite algebras, and applications, connections and interactions with other fields of mathematics. The thesis consists of one introductory and motivational chapter and two parts afterwards, each consisting of three chapters. Each chapter has its own introduction detailing the results and explaining the context of the work, and each of the chapters 2-7 is based on the research in the papers [IM, I1, I2, I3, I4, I5], and partially on [I]. The basics of the mathematical theories involved here are most of the time omitted and explained briefly, and the reader is referred to the literature; we concentrate more on the original part of the research, which is more than 90% of this text.

In the first chapter, we present the generals of the representation theory of profinite algebras, as dual of coalgebras, and the support for the category of finite dimensional representations of an arbitrary algebra. We also present a summary of the results, as well as various interconnections with other fields of mathematics, such as Hopf Algebras, Category Theory, Locally Compact Groups, Algebraic Topology, Homological Algebra.

The first part - chapters 2-4 - concern a type of problem called splitting problem. Given abelian categories $\mathcal{A} \subseteq \mathcal{C}$ with suitable properties, define the $\mathcal{A}$-torsion functor $t : \mathcal{C} \to \mathcal{A}$ as $t(X) =$ the largest subobject of $X$ belonging to $\mathcal{A}$; the splitting problem asks when is $t(X)$ a direct summand of $X$ for all $X$. We solve this problem for $\mathcal{C} =$ category of (finitely generated) modules over a profinite algebra and $\mathcal{A} =$ the subcategory of rational modules, and also for $\mathcal{A} =$ the category of semiartinian modules (chapter 4), which gives a positive answer to a conjecture of Faith for this case.

The second part concerns the development of the theory of infinite dimensional (quasi)
Frobenius algebras, which are the dual of (Quasi)co-Frobenius coalgebras. We prove various theorems regarding the left and right (quasi)co-Frobenius coalgebras, which explain why these are a generalization of the finite dimensional Frobenius algebras, and also reveal how they appear as a left-right symmetric concept. They turn out to have a very interesting “abstract integral theory” which generalize that from Hopf algebras and compact groups. Moreover, these nontrivial generalizations have applications to proving many of the foundational results in Hopf algebras, and have many connections to compact groups. We give categorical results which reveal all the connections between these notions and their finite dimensional counterparts, as well as the previously unknown connections between these notions and various categorical properties.
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M. C. Iovanov
Chapter 1

Introduction

Let $K$ be a field. Throughout this work we shall write $\otimes$ without a subscript and will understand the tensor product over $K$. The classical definition of a $K$-algebra $A$ can be axiomatized in the following way: $A$ is a $K$-vector space together with two $K$-linear maps $m : A \otimes A \to A$ and $u : K \to A$ such that $m(m \otimes \operatorname{Id}_{A}) = m(\operatorname{Id}_{A} \otimes m)$ and $m(u \otimes \operatorname{Id}_{A}) = m(\operatorname{Id}_{A} \otimes u) = \operatorname{Id}_{A}$. Equivalently, this means that the following diagrams are commutative:

\[
\begin{array}{c}
A \otimes A \otimes A \\
\downarrow \quad \downarrow \\
A \otimes A \\
\downarrow \quad m \\
A
\end{array}
\quad m \otimes \operatorname{Id}_{A}
\]

and

\[
\begin{array}{c}
A \otimes A \\
\downarrow \\
A
\end{array}
\quad m
\]

The first diagram is nothing else than the associativity axiom for the multiplication, while the second one says that $1 \cdot a = a \cdot 1 = a$ for $a \in A$, i.e. $1 = u(1_{K})$ is a unit for $A$. The advantage of this definition is that it is a categorical one, and it can be dualized as well as generalized and considered in any suitable category. The dual notion is that of a coalgebra: this is a triple $(C, \Delta, \varepsilon)$ with $C$ a $K$-vector space, $\Delta : C \to C \otimes C$ (called comultiplication) and $\varepsilon : C \to K$ (called counit) $K$-linear maps, such that
(Id\(_C\) \(\otimes\) \(\Delta\))\(\Delta\) = (\(\Delta\) \(\otimes\) Id\(_C\))\(\Delta\) and (\(\varepsilon\) \(\otimes\) Id\(_C\))\(\Delta\) = Id\(_C\) = (Id\(_C\) \(\otimes\) \(\varepsilon\))\(\Delta\). The first equation is called **coassociativity** and the second one is called the **counit property**. In categorical terms, this is expressed by the following diagrams which are the above with the arrows reversed, that is, the above two diagrams considered in the dual to the category of vector spaces (for the reader familiar with the categorical language, one says that a coalgebra in a monoidal category \(C\) is an algebra in the dual category \(C^{\text{op}}\)):

![Diagram 1](image1)

![Diagram 2](image2)

We will use the **Sweedler notation** (sometimes also called **sigma notation**), which is explained in the following. Let \(c \in C\) and write

\[
\Delta c = \sum_i c_{i(1)} \otimes c_{i(2)},
\]

with the subscripts \((1)\) and \((2)\) indicating positions in the tensor product. In the Sweedler notation, we omit the index \(i\) and write more simply

\[
\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}.
\]

In fact, when no confusion can arise, we frequently suppress the summation sign and the parantheses in the subscripts, writing simply \(\Delta(c) = c_1 \otimes c_2\). But the reader should always be aware that this is short hand for (1.1) - the summation is implicit and the subscripts \(1, 2\) indicate only the tensor factors and do not identify specific elements \(c_1, c_2 \in C\). In this formula, one can apply the comultiplication \(\Delta\) to the first position, and the notation
for that would then be $\Delta(c_{(1)}) = \sum (c_{(1)}) \otimes c_{(1)(2)}$; similarly, for the elements in the second position $\Delta(c_{(2)}) = \sum (c_{(2)}) \otimes c_{(2)(2)}$. Using Sweedler notation, the axioms of the coassociativity and the counit property become:

$$\sum \sum c_{(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} \quad (\text{coassociativity})$$

$$\sum (c_{(1)}) \epsilon(c_{(2)}) = c = \sum (c_{(1)}) \epsilon(c_{(2)}) \quad (\text{counit property})$$

Because of this, if we put $\Delta^{(2)} = (\text{Id}_C \otimes \Delta)\Delta = (\Delta \otimes \text{Id}_C)\Delta$ and we write $\Delta^{(2)}(c) = \sum (c_{(1)}) \otimes c_{(2)} \otimes c_{(3)}$, then $\sum (c) c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = \sum \sum c_{(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$. More generally, we write $\Delta^{(n)} = (I^p \otimes \Delta \otimes I^{n-1-p}) \circ \Delta_{n-1}$, and use the Sweedler notation $\Delta^{(n)}(c) = \sum (c_{(1)}) \otimes c_{(2)} \otimes \ldots \otimes c_{(n+1)}$.

In a way dual to the definition of modules over an algebra $A$, one can define comodules over a coalgebra $C$. More specifically, a left module over an algebra $A$ is a $K$-vector space $M$ together with a $K$-linear map $\mu: A \otimes M \rightarrow M$ satisfying the commutative diagrams:

$$
\begin{align*}
A \otimes A \otimes M & \xrightarrow{\text{Id}_A \otimes \mu} A \otimes M \\
& \quad \downarrow \mu \\
& A \otimes M \xrightarrow{\mu} M
\end{align*}
\begin{align*}
A \otimes M & \xrightarrow{\mu} M \\
& \quad \downarrow \mu \\
& K \otimes M \xrightarrow{\mu} M
\end{align*}
\begin{align*}
& \quad \downarrow \mu \\
& \quad \downarrow \mu \\
& M
\end{align*}

By dualizing this definition, we obtain the notion of a right comodule over a coalgebra $C$: this is a $K$-vector space $M$ together with a $K$-linear map called coaction $\rho: M \rightarrow M \otimes C$ such that the following diagrams commute:

$$
\begin{align*}
M \otimes C \otimes C & \xleftarrow{\rho \otimes \text{Id}_C} M \otimes C \\
& \quad \downarrow \rho \\
& M \otimes C \xrightarrow{\rho} M
\end{align*}
\begin{align*}
& \quad \downarrow \rho \\
& \quad \downarrow \rho \\
& M \otimes K \xrightarrow{\rho} M
\end{align*}
\begin{align*}
& \quad \downarrow \rho \\
& \quad \downarrow \rho \\
& M
\end{align*}
The Sweedler notation convention for comodules is then \( \rho(m) = \sum_{(m)} m_0 \otimes m_1 \), or more briefly, \( \rho(m) = m_0 \otimes m_1 \). The 0 position indicates elements in the comodule \( M \), and the other positions indicate the number of times some comultiplication (of the comodule or of the coalgebra) has been applied. The above commutative diagrams are then translated to \( m_{00} \otimes m_{01} \otimes m_1 = m_0 \otimes m_1 \otimes m_2 \) and \( m = m_0 \varepsilon(m_1) \). In analogy to morphisms of modules over an algebra, one defines a morphism of comodules over a coalgebra \( f : (M, \rho_M) \to (N, \rho_N) \) to be a \( K \)-linear map \( f : M \to N \) satisfying \((f \otimes \text{Id})\rho_M = \rho_N f\); equivalently, using Sweedler notation one has \( f(m_0) \otimes m_1 = f(m)_0 \otimes f(m)_1 \). This determines a category (right comodules over the coalgebra \( C \)), and this is denoted by \( \mathcal{M}^C \) or Comod-\( C \).

For now, the above definitions are only motivated by categorical reasons: a coalgebra is just an algebra in the category dual to that of vector spaces. However, there are many places where coalgebras and their corepresentations (comodules) occur naturally, such as representation theory of groups and algebras, algebraic topology, compact groups, quantum groups and category theory, as we shall see in what follows.

First, let us just notice that there is a perfect duality between finite dimensional algebras and finite dimensional coalgebras: for any finite dimensional algebra \( (A, m, u) \), since \( (A \otimes A)^* = A^* \otimes A^* \), the triple \( (A^*, m^*, u^*) \) has exactly the required properties defining a coalgebra, by duality through \( (\cdot)^* \). Also, if \( C \) is a coalgebra (not necessarily finite dimensional), \( C^* \) becomes an algebra with the so called convolution product defined by \( f \ast g = (f \otimes g)\Delta \), so \( (f \ast g)(c) = f(c_1)g(c_2) \). Moreover, if \( M \) is a right \( C \)-comodule, then \( M \) becomes a left \( C^* \)-module by the action \( c^* \cdot m = m_0 \otimes c^*(m_1) = \sum_{(m)} m_0[c^*(m)][1] \). This gives a functor \( \mathcal{M}^C \to C^* - \text{Mod} \) from \( C \)-comodules to \( C^* \)-modules which is a full embedding and is an equivalence of categories when \( C \) is finite dimensional. This is easily explained by the isomorphism \( (M \otimes C)^* \simeq C^* \otimes M^* \) for finite dimensional \( C \).

Recall that if \( A \) is an algebra and \( M \) is a left \( A \)-module, then the map \( A \to \text{End}_K(M) : a \mapsto a \cdot - \) is an algebra map and is called a representation of \( A \). Also, conversely, every representation \( \rho : A \to \text{End}_K(V) \) induces a module structure on \( V \) by \( av = \rho(a)(v) \). We call the category of finite dimensional representations \( \text{Rep}(A) \) and, using the foregoing correspondence, identify it with \( A - \text{fmdmod} \), the category of finite dimensional left \( A \)-modules. For any representation \( \eta : A \to \text{End}(V) \) with \( V \) a \( K \)-vector space, fix a basis \( (v_i)_{i=1,\ldots,n} \) of \( V \) and write \( \eta(a) = (\eta_{ij}(a))_{i,j} \) for the matrix representation of \( \eta(a) \) with
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respect to this basis. Let \( R(A) \) be the subspace of \( A^* \) spanned by all these functions \( \eta_{ij} \), for all finite dimensional representations \( \eta \) and all choices of bases \( \{v_i\} \). There is a very useful and revealing characterization of \( R(A) \) whose proof is available in many books, and we only sketch it here.

**Proposition 1.0.1** The following are equivalent:

(i) \( f \in R(A) \);

(ii) \( \ker(f) \) contains a two-sided ideal of finite codimension;

(iii) there are \( f_i, g_i \in A^*, i = 1, \ldots, n \) such that

\[
f(ab) = \sum_{i=1}^{n} f_i(a)g_i(b) \tag{1.2}
\]

**Proof.** (sketch)

First, note that the functions \( \eta_{ij} \) above have property (iii), since \( \eta(ab) = \eta(a)\eta(b) \) implies \( \eta_{ij}(ab) = \sum_{k=1}^{n} \eta_{ik}(a)\eta_{kj}(b) \). This proves (i)\( \Rightarrow \) (iii).

(iii)\( \Rightarrow \) (i) \( f \) has the property in 1.2, then, by a standard linear algebra consideration we may assume that the \( g_i \)'s are linearly independent in \( A^* \) and so we may find \( c_j \in A \) such that \( g_i(c_j) = \delta_{ij} \). Then \( f(abc_j) = \sum_i f_i(ab)g_i(c_j) = \sum_i f_i(a)g_i(bc_j) \) so \( f_j(ab) = \sum_i f_i(a)g_i(bc_j) \).

This shows that \( V = \text{Span} < f_i > \) is a subspace of \( A^* \) which is invariant under the left \( A \) action \( (b \ast h)(a) = h(ab) \) \((a \in A, h \in V) \). Thus, \( V \) is a finite dimensional left \( A \)-module and the functions \( g_i((−)c_j) \) are in \( R(A) \); therefore \( f_j = \sum_i f_i(a)g_i((−)c_j) \in R(A) \) so \( f = \sum_i f_i g_i(1) \in R(A) \).

(ii)\( \Rightarrow \) (iii) Follows since \( \ker(f) \) contains some two-sided ideal \( I \) of finite codimension iff \( f \) can be thought as a linear map \( f : A/I \to K \) for some such \( I \), but then, \( A/I \) is a finite dimensional algebra, \( (A/I)^* \) is a coalgebra with comultiplication \( \Delta \) and one can check that picking \( f_i, g_i \) with \( \Delta(f) = \sum_i f_i \otimes g_i \) will suffice.

(i)\( \Rightarrow \) (ii) is obvious, since \( \ker(\eta) \) has finite codimension for any finite dimensional representation \( \eta : A \to \text{End}(V) \), and thus, for any finite set \( \{\eta^{(k)}\} \) of representations, any linear combination of (finitely many) \( \eta_{ij}^{(k)} \)'s will contain \( \bigcap_k \ker(\eta^{(k)}) \).

These functions \( \eta_{ij} \in A^* = \text{Hom}(A,K) \) are called **representative functions** of the algebra \( A \). Another notation used for \( R(A) \) is \( A^0 \), and it is also called the **finite dual coalgebra** of \( A \). It is now easy to see that the space \( R(A) \) has a natural coalgebra structure:
for \( f \in R(A) = A^0 \) and \( f_i, g_i \) as in 1.2, the proof of the above proposition showed that \( f_j(ab) = \sum_i f_i(a)g_i(bc_j) \) so \( f_i \) (and similarly, \( g_i \)) also belong to \( R(A) \). Then one defines \( \Delta : A^0 \rightarrow A^0 \otimes A^0 \), \( \Delta(f) = \sum_i f_i \otimes g_i \) and sets \( \varepsilon(f) = f(1) \) (it is not difficult to see that \( \Delta \) is well-defined). Also, using the notations above, any finite dimensional left \( A \)-module \( V \) (i.e. representation \( \eta : A \rightarrow \text{End}(V) \)) is naturally a right \( A^0 \)-comodule, by defining \( \rho_V : V \rightarrow V \otimes A^0 \) on a basis \( \{v_i\} \) of \( V \) by \( \rho_V(v_i) = \sum_j v_j \otimes \eta_{ij} \). It is immediate that any \( A \)-module map is also an \( A^0 \)-comodule map, so we have defined a functor \( A - \text{fdmod} \rightarrow \text{fd}\mathcal{M}^{A^0} \). Conversely, any right \( A^0 \)-comodule \( (V, \rho_V) \) becomes a left \( A \)-module by the natural action \( a \ast v = \sum_{(v)} v_{[1]}(a)v_{[0]} \) and we have a functor \( \mathcal{M}^{A^0} \rightarrow A-\text{mod} \). It is obvious that the composite \( A - \text{fdmod} \rightarrow \text{fd}\mathcal{M}^{A^0} \rightarrow A - \text{fdmod} \) is the identity: for an \( A \)-module \( V \) with basis \( \{v_i\} \), we have \( a \ast v_i = \sum_j \eta_{ij}(a)v_j = a \cdot v_i \), so we reobtain the initial structure. This also works going the other way: \( \text{fd}\mathcal{M}^{A^0} \rightarrow A - \text{fdmod} \rightarrow \text{fd}\mathcal{M}^{A^0} \). Thus the categories of finite dimensional left \( A \)-modules (i.e. \( \text{Rep}(A) \)) and of finite dimensional right \( A^0 \)-comodules are equivalent.

Therefore, everytime we are looking at finite dimensional representations of an algebra, we are, in fact, looking at finite dimensional comodules over a coalgebra, and as we will show below, essentially at the full category of comodules over a coalgebra.

When \( A \) is finite dimensional, \( A^0 = A^* \) and this last equivalence of categories coincides with our earlier equivalence \( A - \text{fdmod} \approx \text{fd}\mathcal{M}^{A^*} \). There is another extension of this equivalence, this time starting with an arbitrary (possibly infinite dimensional) coalgebra \( C \) and its dual algebra \( C^* \). Specifically, the full embedding \( \mathcal{M}^C \rightarrow C^* - \text{mod} \) defined earlier has a right adjoint. First, for any algebra \( A \), a left module which is the sum of its finite dimensional submodules is called a rational module. In the case \( A = C^* \), we define the rational part of a left \( C^* \)-module \( M \) to be the sum of all its submodules whose structure come from a \( C \)-comodule as showed before, and we denote the rational part of \( M \) by \( \text{Rat}(M) \). This means \( \text{Rat}(M) \) is the largest subspace \( M' \) of \( M \) such that, for any \( m \in M' \) and \( c^* \in C^* \), there is \( \sum_i m_i \otimes c_i \in M' \otimes C \) such that \( c^* \cdot m = \sum_i c^*(c_i) m_i \). Then \( \text{Rat} : C^* - \text{mod} \rightarrow \mathcal{M}^C \) is the right adjoint of the embedding above.

We also note a very important result of comodules, the so called “Finiteness Theorem” or “Fundamental Theorem of Comodules”, which is an essential feature and is the source of many strong results which are true for comodules over coalgebras \( C \) or for modules
over the dual algebra $C^*$, but whose statements are false for modules over general algebras. This theorem states that a finitely generated submodule of a rational $C^*$-module is always finite dimensional. This shows that rational $C^*$-modules are the colimit of their finite dimensional submodules, a very special type of property.

We present several other relevant and natural mathematical contexts where coalgebras and their comodules naturally arise.

Further representation theory of algebras

A very important property of coalgebras and their comodules is the following finiteness property, which becomes apparent from the above: given a $C$-comodule $M$, the subcomodule generated by a finite subset $F$ of $M$ is finite dimensional. This is a very important feature of these objects, which makes the theory of coalgebras and their comodules a natural generalization of finite dimensional algebras and their representations. Indeed, by the above property, it is easily concluded that any finitely generated subcoalgebra of a coalgebra is finite dimensional, and therefore, we have

$$C = \lim_{\to} C_i$$

where $\{C_i\}_{i \in I}$ are all the finite dimensional subcoalgebras of $C$. Also, any $C$-comodule is the sum (limit) of its finite dimensional sub(co)modules. This fundamental finiteness property is a feature of great importance, and is the source of many special and strong results in this theory. Dualizing, we obtain that $C^* = \lim_{\to} C_i^*$, and so $C^*$ is an inverse limit of finite dimensional algebras. Such an algebra is called **profinite**. The profinite algebras are natural analogues to profinite groups, which are inverse limits of finite groups. As for profinite groups, there is an alternate characterization, which we present here. Given an algebra $A$, consider the cofinite topology to be the topology on $A$ where a basis of neighbourhoods of 0 is given by the two-sided ideals of finite codimension. An algebra $A$ is called **pseudocompact** if and only if the cofinite topology is separated and complete. Such an algebra will then have a monomorphism $A \hookrightarrow \lim_{\to} A/I$, where $I$ ranges over ideals of finite codimension (this morphism is injective since the cofinite topology is separated), and this morphism is proved to be an isomorphism by the completeness hypothesis. Also, each finite dimensional algebra $A/I$ can be dualized to a coalgebra $C_I$ (since the finite dimensional and finite dimensional coalgebras are in “perfect” duality), with corresponding
dual connecting morphisms, and one gets a coalgebra \( C = \varprojlim (A/I)^* \), which, by duality will have \( C^* \cong A \). Conversely, for any coalgebra \( C \), the algebra \( C^* \) is profinite, and therefore we have:

**Theorem 1.0.2** The following are equivalent for an algebra \( A \):

(i) \( A \) is pseudocompact.

(ii) \( A \) is profinite.

(iii) \( A = C^* \), where \( C \) is a coalgebra.

One can define a category of pseudocompact (profinite) modules over a pseudocompact algebra \( A \) (so \( A = C^* \), for a coalgebra \( C \)). The objects are again complete (Hausdorff) separated modules with respect to the cofinite topology, having a basis of neighbourhoods of 0 consisting of submodules of finite codimension. The morphisms are continuous \( A \)-linear maps. One then also proves that this category is dual to the category of comodules over the underlying coalgebra \( C \).

### 1.1 The Results

We proceed now with a general description of the results of this thesis. Each chapter is organized, in respective order, around one of the papers [I1], [I2], [I5], [IM], [I3], [I4]. Each chapter contains an introduction which describes in more detail the proper context of the problem, the history, connections, motivation and implications of the respective problem. There are two main parts: the first three chapters, which deal with a general type of question called “splitting” problems and the last three, which deal with the theory of generalized Frobenius algebras, and their applications.

The first class of problems, the splitting problems, is described as follows: consider an abelian category \( \mathcal{C} \) with a set of generators and a full subcategory \( \mathcal{A} \) of \( \mathcal{C} \), closed under colimits (for example, one could assume it is closed under subobjects, quotients and coproducts). This inclusion has a right adjoint, called the trace (or torsion) functor, which has the definition \( T(Y) = \varinjlim \{ U \mid U < Y, U \in \mathcal{A} \} \), i.e. it is the largest subobject of \( Y \) that belongs to \( \mathcal{A} \). The example most familiar to the reader - and which was a starting point of this kind of context - is that when \( \mathcal{C} \) is the category of abelian groups and \( \mathcal{A} \) is that of torsion groups. The trace (or torsion) of a group \( Y \) is the actual torsion subgroup of \( Y \).
Generalizing, one can consider modules over PID and torsion modules. In this situation, it is well known that the torsion is a direct summand in any finitely generated module (abelian group), but this does not hold for all modules. This obviously is the first step in understanding the structure of these objects, and it can be formulated as a good first step when trying to understand the general structure of objects in a category $\mathcal{C}$: looking at some “obvious or natural to consider” subcategory $\mathcal{A}$ and asking whether the torsion (trace) of every object of $\mathcal{C}$ splits off in $\mathcal{C}$. The answer to this question is negative when $\mathcal{C}$ is the category of all abelian groups and positive when it is the subcategory of finitely generated abelian groups. Many other splitting problems (such as the singular splitting problem or the “simple” torsion) have been considered (e.g. [T1, T2, T3]). A very natural context to consider for the splitting problem is the case of the inclusion $\mathcal{M}^C \to C^* - \text{mod}$ for a coalgebra $C$, or, for example $\text{Rat}(A - \text{mod}) \to A - \text{mod}$, the inclusion of the full subcategory of $A - \text{mod}$ consisting of the rational $A$-modules for an algebra $A$ (one may view this as the inclusion $\mathcal{M}^{A^0} \to A - \text{mod}$). This question will be studied in detail in Chapters 1 and 2. In Chapter 1, it is shown that if the rational part of every $C^*$-module $M$ splits off in $M$ then $C^*$ must be finite dimensional (this result was obtained by other authors too, but we give a direct short approach). In Chapter 2 we examine the splitting of the rational torsion in finitely generated $C^*$-modules. We obtain a complete result for local profinite algebras (i.e. when $C^*$ is local): if the rational torsion splits off in any finitely generated $C^*$-module, then $C^*$ must be a DVR (discrete valuation ring). In this situation, it is first shown that the rational part of any $C^*$-module coincides with the subset of all torsion elements; thus, the problem becomes a very familiar one: given a finitely generated $C^*$-module $M$, it is asking when the submodule consisting of the torsion elements (which generally does not even form a submodule!) is a direct summand in $M$. One such example is easy to think of, and that is $K[[X]]$. As mentioned, it is shown that this is about all there can be. In fact, if the profinite algebra $C^*$ is commutative (equivalently, $C$ is cocommutative) and the field is algebraically closed, then $C^*$ is a finite product of finite dimensional algebras and copies of $K[[X]]$. The result can be easily extended for non-algebraically closed fields, by replacing the $K[[X]]$ copies with $L[[X]]$ where $L$ is a finite extension of $K$. Results and examples are also obtained in the nonlocal case. The third chapter deals with the splitting of modules over profinite algebras $A$ with respect to the subclass of semiartinian modules, called the Dickson torsion. Here, it was conjectured that such a splitting cannot occur over a ring $A$ unless $A$ is semiartinian itself, meaning that the subclass of semiartinian
modules coincides with the whole category of $A$-modules (see Introduction of Chapter 3 for definition of semiartinian module). This was answered in the negative in the general case, but here we show that it holds for profinite (equivalently, pseudocompact) algebras.

Part two is dedicated to the development of generalized Frobenius and quasi-Frobenius algebras. An algebra $A$ is Frobenius if $A$ is isomorphic to $A^*$ as left (equivalently, right) $A$-modules. A Frobenius algebra is automatically finite dimensional. They are named this way in honor of G. Frobenius who first discovered this property for group algebras of finite groups. More generally, finite dimensional Hopf algebras are Frobenius algebras. They appear naturally in literature: the cohomology ring of an orientable manifold is a Frobenius algebra; it has been shown that there is an equivalence between 2 dimensional topological quantum field theories and the category of commutative Frobenius algebras. For coalgebras, the notion of co-Frobenius (the analogue of the notion of Frobenius for algebras) was introduced in [L]: a coalgebra is left co-Frobenius iff $C$ embeds in $C^*$ as left $C^*$-modules. Unlike Frobenius algebras, this is not left/right symmetric: a coalgebra can be left but not right co-Frobenius, and it is called co-Frobenius if it is co-Frobenius on both sides.

Quasi-Frobenius algebras are a generalization of Frobenius algebras, which retain only the representation theoretic properties of these algebras. Namely, for finite dimensional algebras, being quasi-Frobenius means that every projective is injective. Frobenius algebras are quasi-Frobenius, and the difference is a formula connected to multiplicities of the principal indecomposable projectives, which is equivalent to saying that each indecomposable direct summand of $AA$, has the same multiplicity in $A$ as the multiplicity of its dual in $AA$ (see 5.2.3 for the general statement in coalgebras). Quasi-co-Frobenius (QcF) coalgebras were also introduced in the literature as coalgebras which embed in an arbitrary product power of $C^*$, and they were shown to have the property that every injective is projective. In contrast to the algebra situation, they also turned out not to be a left-right symmetric concept, and this leads to naming QcF a coalgebra which is both left and right QcF. The main achievement of Chapter 4 is showing that, in fact, co-Frobenius and QcF coalgebras admit characterizations very similar to those of Frobenius algebras. It is shown that a coalgebra $C$ is co-Frobenius iff $C$ is isomorphic to its left (and then, equivalently, right) rational dual $\text{Rat}(C^*)$ - the only natural object one can take as a dual for $C$. More generally, $C$ is QcF if and only if $C$ is “weakly” isomorphic to its rational dual $\text{Rat}(C^*)$, in the sense that certain (co)product powers of these are isomorphic. These also allow the extension of many characterizations (functorial and categorical) from the finite dimensional
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1.1. THE RESULTS

case to the infinite dimensional one. Since when \( C \) is finite dimensional co-Frobenius, \( C^* \) is a Frobenius algebra, these characterizations entitle one to regard profinite algebras which are dual of co-Frobenius (respectively QcF) coalgebras as generalized Frobenius (or generalized quasi-Frobenius) algebras. These generalizations are highly non-trivial; in fact, one needs to apply Zorn’s lemma and several representation theoretic arguments to reach the conclusion. Moreover, they turn out to have applications for Hopf algebras. The concept of co-Frobenius was known to be symmetric for coalgebras which are Hopf algebras, and also equivalent to the existence of an invariant left (and right) integral. We obtain these facts and various other fundamental results of Hopf algebras, such as the uniqueness of the integral, as an immediate consequence of these results. A short proof of the bijectivity of the antipode for such Hopf algebras is also obtained as a spin-off.

The integral in Hopf algebras is an extension and a generalization of the notion of Haar measure (integral) in compact groups. More specifically, if \( G \) is a compact group, \( R(G) \) is the algebra of continuous functions on \( G \), \( \mu \) is a left invariant Haar measure on \( G \) and \( \int d\mu \) is the associated Haar integral, then \( \int d\mu \) can be regarded as a functional on the Hopf algebra \( R(G) \), and there it becomes a functional which is an integral in the Hopf algebra sense (see Chapters 4 and 5 for definition). We note with this occasion that the \( C^* \) Hopf algebra \( C(G) \) is just the continuous analogue of \( R(G) \): it is known (for example, as a consequence of Peter-Weyl theorem) that \( R(G) \) is dense in \( C(G) \). Then the structure of \( C(G) \) can be obtained from that of \( R(G) \) by continuity (and density). The uniqueness of integrals in Hopf algebras, as well as the existence results can be regarded as a natural extension of the results from compact and locally compact groups. In Chapter 5 we consider a more general definition of integrals, which was noticed before in literature. An integral in a Hopf algebra can be regarded as a morphism of right \( H \)-comodules (left \( H^* \)-modules) from \( H \) to the unit comodule \( K \). Then, for any right comodule \( M \) of a coalgebra \( H \) we can think of \( \text{hom}_{\text{comod}}(H,M) \) as the space of left invariant integrals associated to \( M \). We show that there is a very tight connection with actual integrals in the case of compact groups, which further motivates this study. In this situation (\( H = R(G) \)), these integrals can be obtained as vector integrals \( \lambda : G \to \mathbb{C}^n \) which have a “quantum” invariance of the type \( \lambda(x \cdot f) = \eta(x)\lambda(x) \), where here \( \eta(x) \in \text{Gl}_n(\mathbb{C}) \). In fact, \( \eta \) is (and can only be) a finite dimensional representation of \( G \), equivalently, a \( R(G) \)-comodule, and these quantum invariant integrals (with \( \eta \)-quantified invariance) are exactly the integral space associated to the \( R(G) \)-comodule \( \eta \). We show that all the results from Hopf algebras can be carried to the
situation with only the coalgebra structure present. Philosophically, this can be justified that for compact groups, the coalgebra structure of $R(G)$ retains the information of $G$ (both algebraic and topological), while the algebra structure is one which can be defined for any set $G$, not just for groups. We show that for left co-Frobenius coalgebras we have uniqueness of left integrals and existence of right integrals appropriately interpreted: uniqueness means \( \dim(\hom(C, M)) \leq \dim(M) \), existence means \( \dim(\hom(C, M)) \geq \dim(M) \). Also, the symmetric results extend too: a coalgebra is co-Frobenius if and only if it has existence and uniqueness of integrals for all finite dimensional comodules. These results, in turn, reobtain the symmetric characterizations of co-Frobenius coalgebras mentioned above and in fact give even more general characterizations in categorical terms, and also yield, as a consequence, new short proofs of the fundamental results on the existence and uniqueness results of classical integrals of Hopf algebras. We also note that the fact that any finite dimensional (continuous) representation of $G$ is completely reducible follows immediately as a consequence of Hopf algebra theory, and analyze the connection with integrals in depth. Finally, we give an extensive class of examples which shows that all the results are the best possible. Moreover, we use these examples to settle all the possible (previously unknown) connections between various important notions in coalgebra theory, such as co-Frobenius and QcF coalgebras (left, right, two sided) and also the (left or right or two sided) semiperfect coalgebras (coalgebras which decompose as a sum of finite dimensional comodules).

Chapter 6 is dedicated to another important aspect of QcF coalgebras, with a question which is directly suggested by the algebra case. An equivalent characterization of a finite dimensional quasi-Frobenius algebra $A$ is that $A$ is an injective cogenerator in $A\text{-mod}$, and, moreover $A$ is injective in $A\text{-mod}$ if and only if it is a cogenerator. These are the homological and categorical properties dual to the very important and obvious properties of an algebra of being a generator and projective in its category of modules. In analogy, any coalgebra is always an injective cogenerator for its comodules. It was shown that left QcF coalgebras are exactly those which are projective as left $C^*$-modules (right $C$-comodules), and that they are also generators for their left comodules. But it was unknown whether the generating condition also implies the projective one. A very large class of counterexamples is constructed as the main result of this chapter, showing that, in fact, any coalgebra can be embedded in another one which generates its comodules (or, in fact, has every comodule as a quotient). We also find exactly the necessary conditions for when this implication does take place.
Further motivation and connections to other areas of mathematics

With this brief overview of our results and the basics of the theory of coalgebras, the reader may proceed to the body of the thesis. The remainder of this introduction, while not essential for the results of the thesis, contains further motivating examples for the theory of coalgebras and comodules, as well as an outline of some future threads of research. These examples show that coalgebraic structures appear in a wide range of mathematical fields, including Algebraic Topology, Homological Algebra, Category Theory, (locally) Compact Groups, Lie Groups, Quantum Groups and Hopf Algebras, Representation Theory, and also others not mentioned here.

1.2 Category theory and Homological algebra

In what follows, we show how the (co)homology of a complex can be interpreted categorically using coalgebras and comodules, and, in fact, appears as a certain natural construction for a certain coalgebra; therefore, this construction can be extended to arbitrary coalgebras, possibly yielding new invariants.

Consider the diagram: \[ \cdots \to g_{n+1} \xrightarrow{u_{n+1}} g_n \xrightarrow{u_n} g_{n-1} \to \cdots \]. This is obviously the “skeleton” of chain complexes of $K$-vector spaces. To this diagram we can associate a $K$-category with objects the $g_n$'s and $\text{Hom}(g_n, g_{n-1}) = Ku_n$, $\text{Hom}(g_n, g_n) = \text{Id}_{g_n}$ and $u_{n-1}u_n = 0$, equivalently, $\text{Hom}(g_n, g_m) = 0$ for $m \neq n, n - 1$. Then chain complexes are just $K$-linear functors from this diagram to the category of $K$-vector spaces. Let $H$ be the coalgebra with basis $g_n, u_n, n \in \mathbb{Z}$ and comultiplication $\Delta$ and counit $\varepsilon$ given by

\[
\begin{align*}
\Delta(g_n) &= g_n \otimes g_n \\
\Delta(u_n) &= g_{n-1} \otimes u_n + u_n \otimes g_n \\
\varepsilon(g_n) &= 1 \\
\varepsilon(u_n) &= 0
\end{align*}
\]

for all integers $n$. Let us note that to any chain complex

\[ \cdots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \]

we can associate a right $H$-comodule $M$ given by $M = \bigoplus_n C_n$ and comultiplication $\rho$ given
by $\rho(\sigma_n) = \delta(\sigma_n) \otimes u_n + \sigma_n \otimes g_n = \delta_n(\sigma_n) \otimes u_n + \sigma_n \otimes g_n$, for $\sigma_n \in C_n$. Using relations (1.3)-(1.6), it is easy to see that $M$ is a $H$-comodule. We note that $H^* = \prod K g_n^* \times \prod K u_n^*$, where $g_n^*, u_n^* \in H^*$ are chosen to fulfill $u_n^*(u_m) = \delta_{nm} = g_n^*(g_m)$, $u_n^*(g_m) = g_n^*(u_m) = 0$. Conversely, starting with a $H$-comodule $M$, define $C_n = \{x \in M \mid g_n^* \cdot x = x\} (= g_n^* \cdot M)$. Then $M = \bigoplus C_n$. Indeed, if $x_n \in C_n$, then $\sum x_n = 0$ implies $\sum g_k^* \cdot x_n = \sum g_k^* g_n^* x_n = \sum \delta_{kn} g_k^* x_n = x_k$; also, since the $H$-subcomodule (= $H^*$-submodule) $H^* \cdot x$ of $M$ generated by some $x \in M$ is finite dimensional and $g_n^* \cdot x \in H^* \cdot x$, it follows that only finitely many of $g_n^* \cdot x$ are nonzero. From this, it follows without too much difficulty that $x = \varepsilon \cdot x = \sum n g_n^* \cdot x$ (the sum is finite). We can define $\delta = (\delta_n)_n$, $\delta_n : C_n \rightarrow C_{n-1}$ by $\delta_n(x) = u_n^* \cdot x$. Since $u_{n-1}^* u_n^* = 0$, we have $\delta^2 = 0$. In fact, $\delta$ can be interpreted as an element of $H^*$, $\delta = \prod u_n^*$, and the action of the morphisms $\delta_n$ on the $C_n$’s can be thought as purely multiplication with $\delta$, since multiplying $x \in C_n$ by $\delta$, one obtains an element in $C_{n-1}$. Moreover, for complexes $(C_n, \delta_n)_n$ and $(D, \alpha_n)_n$, a linear map $f : \bigoplus C_n \rightarrow \bigoplus D_n$ is a morphism of complexes if and only if $f(C_n) \subseteq D_n$ and $f \delta_n = \alpha_n f$; the first can be easily seen to be equivalent to $g_n^* \cdot f(x) = f(g_n^* \cdot x)$ and the second to $f(u_n^* \cdot x) = u_n^* \cdot f(x)$, which means that $f$ is a morphism of $H^*$-modules, or equivalently, of $H$-comodules. We have thus proved:

**Lemma 1.2.1** The category of chain complexes of modules over a field $K$ (or a ring $R$) is isomorphic to the category of right comodules over the coalgebra $H$ above. Bounded above and bounded below complexes, or chain complexes have the same feature, provided we change the coalgebra $H$ appropriately.

Thus, the “support” of homological algebra is a category of comodules over a coalgebra.

One interesting application is this: it is a well known fact that any chain complex $C$ can be decomposed as a direct sum of complexes $0 \rightarrow K_{(n)} \rightarrow K_{(n-1)} \rightarrow 0$ or $0 \rightarrow K_{(n)} \rightarrow 0$ ($n$ denotes position). This is then an immediate consequence of general corepresentation theory of coalgebras applied for the coalgebra $H$ above, which gives decompositions $H = \bigoplus K < g_{n-1}, u_n >$ as right $H$-comodules (equivalently, left $H^*$-modules) and $H = \bigoplus K < g_{n-1}, u_n >$ as left $H$-comodules. In general, any coalgebra $X$ can be decomposed in a direct product of indecomposable $C$-comodules, and here these are $K < g_{n-1}, u_n > = Kg_{n-1} + Ku_n$. One shows that given an $H$-comodule $M$, as long as 2-dimensional non-semisimple subcomodules can be found in $M$, these split off in $M$ (and these correspond to the first type of complex); whatever remains after a maximal such subobject is split off will be
semisimple and isomorphic to $K < g_n >$ (these terms correspond to the second type).

The (co)homology as a functor on comodules

We note a very interesting way of constructing the (co)homology functors, which is applicable to a general coalgebra and uses only representation theoretic notions. Thus, this might suggest that more general homological theories might exist, which might in turn offer very interesting invariants.

Let us denote $J = \prod_n K u_n^* \subseteq H^*$. It is easy to observe that this is the Jacobson radical (i.e. the intersection of all maximal left ideals) of $H^*$. Note that it is generated by $\delta = \prod_n u_n^* \times 0 \in \prod_n K u_n^* \times \prod_n K g_n^* = H^*$. Also, with notations as above, given the complex $C$ and its associated $H$-comodule $M$, we have $\bigoplus \ker(\delta_n) = \{ x \in M \mid \delta \cdot x = 0 \} = \{ x \in M \mid J \cdot x = 0 \} = J_M^1$ - the part of $M$ canceled by $J$. This is actually the semisimple part of $M$, called the socle of $M$. This is true in case of a comodule over a coalgebra (i.e. a rational $H^*$-module, but it is not necessarily true for an arbitrary module). Also, $\bigoplus \text{Im}(\delta_n) = \delta \cdot M = J \cdot M$. Since $J^2 = 0$, we have $J \cdot M \subseteq J_M^1$, and we can write

$$H_n(C) = \bigoplus_n H_n(C) = \frac{J_M^1}{J \cdot M}$$

This allows us to think of a more general situation. Let us describe it in the following. $C$ is a coalgebra, and $M$ a right $C$-comodule. We can define the following series: $M_1$ is the sum of all simple subcomodules of $C$. Because of the fundamental finiteness of comodules, any comodule contains a simple comodule and thus $M_0 \neq 0$. Then define $M_n$ inductively by $M_{n+1}/M_n$ being the socle (sum of all simple left, or equivalently, right subcomodules) of $M/M_n$. This is the Loewy series of $M$. For any comodule $M$, we have $M = \bigcup M_n$. For $M = C$ as a right comodule, this is called the coradical filtration of $C$. It can be shown that in general $J_C^1 = \{ c \in C \mid f(c) = 0, \forall f \in J \} = C_0$ and $C_0^1 = \{ f \in C^* \mid f(C_0) = 0 \} = J$. Because of this, a comodule $M$ is semisimple if and only if it is annihilated by $J$. Because of this then, for each $n$, $(J^n)_M^1 = \{ x \in M \mid x \cdot J^n = 0 \} = M_{n-1}$. We can also consider the descending chain of subcomodules of a comodule $M$: $M \supset J \cdot M \supset J^2 \cdot M \ldots$. If the series of $M$ is finite, i.e. $M = M_n$ for some $n$ assumed minimal, then we have
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\[ J^{n-k} \cdot M \subseteq M_k. \]

We can define

\[ H(M) = \bigoplus_k \frac{M_k}{J^{n-k} \cdot M} \]

and call it the homology of \( M \). For the case when \( n = 1 \) (as it happens for \( C = H \) - the coalgebra above), \( H(M) = M_0/M_1 \), and for \( C = H \) this recovers the classical homology, since as remarked, \( \bigoplus \ker \delta_n = J^n M = M_0 \). It is also the case in this general construction that \( H \) is functorial: it can be easily shown that, in general, if \( f : M \to N \) is a morphism of \( C \)-comodules, \( f(M_k) \subseteq N_k \) and obviously \( f(J^{n-k} \cdot M) = J^{n-k} \cdot f(M) \subseteq J^{n-k} \cdot N \), so this induces a morphism \( H(f) : H(M) \to H(N) \).

It is then an interesting question to explain homotopy equivalence, the long exact sequence, universal coefficient theorems and other homological properties through general constructions realizable for coalgebras. For example, the connecting morphism \( \delta \) in long exact sequence of homology would actually be induced by a certain multiplication with the element \( \delta \) (intentionally denoted by the same letter), i.e. in general by \( J \), so we would have that “\( \delta \) is the the actual \( \delta \)”.

This does show promise for further inquiry, as seen above, and might seem quite striking and surprising, but it is even further sustained by the fact that (co)homology theories have been considered by authors in literature, which start with “chain” complexes with differentials \( \delta \) which don’t have the property \( \delta^2 = 0 \), but \( \delta^3 = 0 \) or other, and this would fit into the general pattern described above as follows: given a quite general diagram \( D \), with a finitary property (e.g. that between any two vertices there are only finitely many paths), one can prove that functors from \( D \) to vector spaces (i.e. \( D \)-shaped diagrams of vector spaces, or more general of modules over a ring) form a category isomorphic to a category of comodules over a coalgebra closely related with the path coalgebra of \( D \) (but not exactly coinciding to this, but being a quotient coalgebra of the path coalgebra); in fact, these can always be thought as modules over the path algebra, and the two are in a certain duality similar to the duality between an algebra (or coalgebra) and its finite dual coalgebra (or dual algebra). Then, the general coalgebraic considerations come into place, and general “diagram” homology can be defined. For another example of such equivalence, the coalgebra \( B \) with \( K \)-basis \( g_{m,n}, u_{m,n}, w_{m,n}, z_{m,n} \) with integer \( m, n \) and comultiplication
\( \Delta \) and counit \( \varepsilon \)

\[
\begin{align*}
\Delta(g_{m,n}) & = g_{m,n} \otimes g_{m,n} \\
\Delta(u_{m,n}) & = g_{m-1,n} \otimes u_{m,n} + u_{m,n} \otimes g_{m,n} \\
\Delta(w_{m,n}) & = g_{m,n-1} \otimes w_{m,n} + w_{m,n} \otimes g_{m,n} \\
\Delta(z_{m,n}) & = g_{m-1,n-1} \otimes z_{m,n} + u_{m-1,n} \otimes u_{m,n} + u_{m,n-1} \otimes w_{m,n} + z_{m,n} \otimes g_{m,n} \\
\varepsilon(g_{m,n}) & = 1 \\
\varepsilon(u_{m,n}) & = 0 \\
\varepsilon(w_{m,n}) & = 0 \\
\varepsilon(z_{m,n}) & = 0
\end{align*}
\]

This coalgebra \( B \) is isomorphic to the tensor product of coalgebras \( H \otimes H \), where the structure given by \( \Delta_{H \otimes H}(g \otimes h) = g_1 \otimes h_1 \otimes g_2 \otimes h_2 \) and \( \varepsilon H \otimes H(g \otimes h) = \varepsilon(g)\varepsilon(h) \).

The category of double chain complexes \( (C_{m,n})_{m,n} \) (equivalently, the category of chain complexes of chain complexes of modules) is equivalent to the category of comodules over this coalgebra \( B \). Moreover, there is a certain quotient coalgebra of \( B \) by a certain coideal \( I \) such that \( B/I \) is isomorphic to the coalgebra \( H \) described before (and whose comodules are the chain complexes). This quotient map induces a functor \( \mathcal{M}^B \to \mathcal{M}^{B/I} \) by \( (M, \rho_M) \to (\text{Id}_M \otimes (B \to B/I))\rho_M \); this functor is equivalent to the total complex functor when regarded from the category of double chain complexes (equivalent to \( \mathcal{M}^B \)) to the one of chain complexes (equivalent to \( \mathcal{M}^{B/I} \)).

We note that the above considerations can be done for modules over an arbitrary ring \( R \) (chain complexes), but in that case, the notion of \( R \)-coalgebra is considered in the monoidal category of \( R \)-bimodules. In this situation, an \( R \)-coalgebra is usually called \( R \)-coring.

Thus, there are good reasons to hope that the methods and results of homological algebra (for example, long exact sequence, spectral sequences etc.) can be extended to these more general situations.

### 1.3 Algebraic Topology

It is often the case that cohomology is said to have “more” structure than homology, since there is a natural (cup) product in cohomology which is not present in homology.
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However, this is not entirely true. To fix notation, recall that if $X$ is a topological space and $R$ is a ring, we denote $C_k(X; R) = \{ \sigma \mid \sigma : \Delta_k \to X \}$ the free (left) $R$-module of chains of simplices on $X$ (where $\Delta_k = [v_0v_1 \ldots v_k]$ are the $k$-simplices $\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_i \geq 0; x_1 + \ldots + x_k \leq 1\}$) and $\{v_i\}$ is the set of vertices of $\Delta_k$. Define $\partial : C_k(X; R) \to C_{k-1}(X; R)$ by $\partial(\sigma) = \sum_i (-1)^i \sigma | [v_0 \ldots \hat{v}_i \ldots v_k]$, where $[v_0 \ldots \hat{v}_i \ldots v_k]$ is the $k-1$-simplex formed by the face opposite to $v_i$. The dual complex is then denoted $C^*(X; R)$, $\delta$ (so $\delta : C^{k-1}(X; R) \to C^k(X; R)$, $\delta(\varphi) = \varphi \circ \partial$). We then have the

- **cup product**: $\cup : C^k(X; R) \times C^l(X; R) \to C^{k+l}(X; R)$, $\varphi \cup \psi(\sigma) = \varphi(\sigma | [v_0 \ldots v_k])\psi(\sigma | [v_k \ldots v_{k+l}])$; one has $\partial(\varphi \cup \psi) = \partial(\varphi) \cup \psi + (-1)^{k}\varphi \cup \partial(\psi)$. Because of this the cup product is induced at cohomology level $H^k(X; R) \times H^l(X; R) \to H^{k+l}(X; R)$;

- **cap product**: $C_k(X; R) \times C^l(X; R) \to C_{k-l}(X; R)$, $\sigma \cap \varphi = \varphi(\sigma | [v_0 \ldots v_l])\sigma | [v_l \ldots v_k]$; one has the formula $\partial(\sigma \cap \varphi) = (-1)^l(\partial \sigma \cap \varphi - \sigma \cap \partial \varphi)$ and this allows inducing the cap product at (co)homology level too: $H_k(X; R) \times H^l(X; R) \to H_{k-l}(X; R)$.

We note that the cup product in co-homology is actually dual to a coproduct which is naturally present in homology. Define $\Delta : C_*(X; R) \to C_*(X; R) \otimes_R C_*(X; R)$ so that

$$\Delta(C_n(X; R)) \subseteq \bigoplus_{i+j=n} C_i(X; R) \otimes_R C_j(X; R)$$

Specifically, for any simplex $\sigma \in C_n(X; R)$ set

$$\Delta(\sigma) = \sum_i \sigma | [v_0 \ldots v_i] \otimes_R \sigma | [v_i \ldots v_n]$$

We can also define $\varepsilon : C_*(X; R) = \bigoplus_n C_n(X; R) \to K$ to be $\varepsilon(\sigma) = \delta_{0,n}$, where $\sigma \in C_n(X; R)$ (and $\delta_{ij}$ here is the Kroneker symbol). These form a coalgebra structure on $C_*(X; R)$, as can be easily checked. We see that the dual algebra is $A = (C_*(X; R))^* = \prod_n (C_n(X; R))^*$ with structure given for any two elements $\varphi = \prod_i \varphi_i$, $\psi = \prod_j \psi_j$ by $(\varphi \ast \psi) = \prod_n (\sum_{i+j=n} \varphi_i \cup \psi_j)$. Note that the algebra $C^*(X; R) = \bigoplus_n C^n(X; R)$ with the cup product embeds in the algebra $A$ (considered with the convolution product dual to $\Delta$) by the above formulas.

We note that we have the formula:

$$\Delta(\partial(\sigma)) = \sum_k (\partial(\sigma | [v_0 \ldots v_k]) \otimes_R \sigma | [v_k \ldots v_n] + (-1)^k \sigma | [v_0 \ldots v_k]) \otimes_R \partial(\sigma | [v_k \ldots v_n])$$
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Assume $R$ is a field. From the above equation it follows that

\[
\Delta(\text{Im } (\partial)) \subseteq \text{Im } (\partial) \otimes C_*(X; R) + C_*(X; R) \otimes \text{Im } (\partial)
\]

\[
\Delta(\ker(\partial)) \subseteq \ker(\partial) \otimes C_*(X; R) + C_*(X; R) \otimes \text{Im } (\partial)
\]

\[
\Delta(\ker(\partial)) \subseteq \text{Im } (\partial) \otimes C_*(X; R) + C_*(X; R) \otimes \ker(\partial).
\]

The first is obvious. For the second suppose that $\sigma \in \ker(\partial)$, that $\psi \in (\text{Im } (\partial))\perp$ and that $\varphi \mid \ker(\partial) = 0$. Then $\psi \circ \partial = 0$ and, for some $\alpha$, we have $\varphi = \alpha \circ \partial$. Using again the expression of $\Delta \partial(\sigma)$ it follows that

\[
(\varphi * \psi)(\sigma) = \sum_k \varphi(\sigma \mid [v_0 \ldots v_k])\psi(\sigma \mid [v_k \ldots v_n])
\]

\[
= \sum_k (\alpha(\partial(\sigma \mid [v_0 \ldots v_k]))\psi(\sigma \mid [v_k \ldots v_n])
\]

\[
+ (-1)^k\alpha(\sigma \mid [v_0 \ldots v_k])\psi(\partial(\sigma \mid [v_k \ldots v_n]))) \quad (\text{since } \psi \circ \partial = 0)
\]

\[
= (\alpha * \psi)(\partial(\sigma))
\]

\[
= 0
\]

Then, by standard linear algebra consideration, we get the desired formulas. This shows that $\Delta$ induces a morphism $\Delta : H_*(X; R) \to H_*(X; R) \otimes H_*(X; R)$, and together with the induced $\varepsilon : H_*(X; R) \to R$, it gives a coalgebra structure on $H_*(X; R)$. By the definition of $\Delta$, the cup product and the convolution product, we can easily see that $H^*(X; R)$ can be regarded as a subalgebra of the algebra dual to the coalgebra $H_*(X; R)$.

Also, with these structures in mind, note that the definition of the cap product $H_n(X; R) \times H^k(X; R) \to H_{n-k}(X; R)$, $\sigma \cap \varphi = \varphi(\sigma \mid [v_0 \ldots v_k])\sigma \mid [v_k \ldots v_n]$, or, for possibly inhomogeneous elements $\sigma$ and $\varphi$, we have $\sigma \cap \varphi = \sum_k \varphi(\sigma \mid [v_0 \ldots v_k])\sigma \mid [v_k \ldots v_n]$ means that the cup product between $\sigma$ and $\varphi$ is induced by the right action of $H^*(X; R)$ as a subalgebra of $[H_*(X; R)]^*$ on $H_*(X; R)$: recall that if $B$ is a coalgebra, then it is a $B$-comodule (left and right) and thus it has a $B^*$-module structure: $(b \cdot b^*) = b^*(b_1)b_2$ for $b \in B$, $b^* \in B^*$. In fact, in this language, the connection between the cup and the cap product $\psi(\alpha \cap \varphi) = (\varphi \cup \psi)(\alpha)$ for $\psi, \varphi \in H^*(X; R)$ and $\alpha \in H_*(X; R)$ - means nothing else but that $c^*(c \cdot b^*) = (b^* \circ c^*)(c)$, which is clear by the definition of the convolution product.

Poincaré duality of manifolds can be expressed nicely in terms of representation theory. An
algebra $A$ is called Frobenius if $A$ is isomorphic to $A^*$ as right $A$-modules. The fact that, for an orientable manifold $X$, the map $H^*(X; R) \to H_*(X; R) : \alpha \mapsto [X] \cap \alpha$ is an isomorphism means exactly that $H_*(X; R)$ is isomorphic to $H^*(X; R)$ as $H^*(X; R)$-modules; but since, in this case the (finite dimensional) $H_*(X; R)$ is the dual, as an $H^*(X; R)$-module, of $H^*(X; R)$, this means exactly that $H^*(X; R)$ is a Frobenius algebra. Generally, a Frobenius algebra is finite dimensional. One of the main research developments of this thesis is the theory of the so called infinite dimensional Frobenius algebras (with suitable definition), and these are defined by using the theory of coalgebras: a coalgebra $B$ is Frobenius if it is isomorphic to its rational dual $Rat(B^*)$ as left $B^*$-modules. Their dual algebra (which is profinite) will be called an infinite dimensional Frobenius algebra. Then, the natural question arises: is there a more general form of Poincaré duality which holds for a larger class $C$ of spaces (certain CW-complexes, including manifolds), which would be formulated as “the homology of any space in $C$ is a Frobenius coalgebra”.

We also note that the considerations of this section also work over more general rings $R$, such as quasi-Frobenius, which allows one to include $\mathbb{Z}/n$ here and thus recover torsion phenomena.

**Generalized “(co)homology” theories**

We now give an example of a “(co)homology” theory in the spirit of the previous section. Let $X$ be a topological space, and consider the chain complex $C_*$ of simplices $\sigma$ defined over a field containing a 3-rd root of unity $\omega \neq 1$. Define the differential $\partial : C_n \to C_{n-1}$ by $\partial(\sigma) = \sum_k \omega^k \sigma | [v_0 \ldots \hat{v}_k \ldots v_n]$. Then this satisfies $\partial^3 = 0$. Indeed, let us note that

$$\partial^3(\sigma) = \sum_{1 \leq i < j \leq n} (\omega^{i+j} + \omega^{i+j-1})[\sum_{1 \leq k \leq i} \omega^k \sigma | [v_0 \ldots \hat{v}_k \ldots \hat{v}_j \ldots v_n] + \sum_{i \leq k \leq j} \omega^k \sigma | [v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n] + \sum_{1 \leq k \leq i} \omega^k \sigma | [v_0 \ldots \hat{v}_i \ldots \hat{v}_j \ldots v_n]$$

$$= \sum_{1 \leq a < b < c} [(\omega^{b+c} + \omega^{b+c-1})\omega^a + (\omega^{a+c} + \omega^{a+c-1})\omega^{b-1} + (\omega^{a+b} + \omega^{a+b-1})\omega^{c-2}]$$
\[
\sigma \cdot [v_0 \ldots \hat{v}_a \ldots \hat{v}_b \ldots \hat{v}_c \ldots v_n] = \sum_{1 \leq a < b < c} \omega^{a+b+c-2} [\omega^2 + \omega + 1 + 1 + \omega^2] \sigma \cdot [v_0 \ldots \hat{v}_a \ldots \hat{v}_b \ldots \hat{v}_c \ldots v_n] = 0
\]

By the consideration of the previous section, to any “complex” \(C_*\) with such a “differential” \(\partial\) with \(\partial^3 = 0\), we can associate a generalized cohomology theory. These complexes can be viewed as comodules over the coalgebra with basis \(g_n, u_n, p_n\), where \(g_n\) will “encode” the vertices, \(u_n\) will encode the sides from \(n\) to \(n-1\) and \(p_n\) the paths obtained by composing \(n \rightarrow n-1 \rightarrow n-2\). The comultiplicative structure is given by

\[
\begin{align*}
\delta g_n &\mapsto g_n \otimes g_n \\
\delta u_n &\mapsto g_{n-1} \otimes u_n + u_n \otimes g_n \\
\delta p_n &\mapsto g_{n-2} \otimes p_n + u_{n-1} \otimes u_n + p_n \otimes p_n
\end{align*}
\]

and counit \(\varepsilon(g_n) = 1, \varepsilon(u_n) = 0, \varepsilon(p_n) = 0\). Using the definitions mentioned above for the homology of a general coalgebra, we are lead to the introduction of the homology groups given by \(\ker(\partial)/\text{Im}(\partial^2)\) and \(\ker(\partial^2)/\text{Im}(\partial)\); more precisely, these give rise to two sequences of homology groups \(H_{1,n}(X; R) = \ker(\partial_n)/\text{Im}(\partial_{n+1}\partial_{n+2})\) and \(H_{2,n}(X; R) = \ker(\partial_{n-1}\partial_n)/\text{Im}(\partial_{n+1})\).

Naturally, the same considerations work for a an \(n\)-th root of unity \(\omega \neq 1\), yielding a differential \(\partial\) such that \(\partial^n = 0\).

### 1.4 Hopf algebras and related structures; compact groups

We present now another very important theory where the language of coalgebras is in full place. Hopf algebras have originally appeared in algebraic topology with the work of H. Hopf, and his results on the cohomology of a Lie group. Recall that if \(G\) is a Lie group, then \(H^*(G)\), the cohomology algebra of \(G\), can be also endowed with a comultiplication structure via the multiplication \(M : G \times G \rightarrow G\), specifically \(H^*(G) \xrightarrow{H^*(M)} H^*(G \times G) \simeq H^*(G) \times H^*(G)\) (the last isomorphism follows by the Kunneth theorem). This comultiplication turns out to be coassociative and in turn, it induces a Hopf algebra structure. Abstractly,
a Hopf algebra $H$ is a $K$-vector space which is at the same time an algebra $(H, m, u)$ and a coalgebra $(H, \Delta, \varepsilon)$ which are naturally compatible, that is, $\Delta : H \to H \otimes H$ and $\varepsilon : H \to K$ are morphisms of algebras, or, equivalently, $m$ and $u$ are morphisms of coalgebras; also, there is a linear map $S : H \to H$ called antipode (given in the Lie group case by $H^\ast(G) \xrightarrow{\alpha \mapsto \alpha^{-1}} H^\ast(G)$) which is the inverse to the identity map with respect to the convolution product on $\text{End}(H)$, $(U * V)(h) = \sum_h U(h_1)V(h_2))$. In Sweedler notation, these compatibility conditions are

$$(hg)_1 \otimes (hg)_2 = h_1g_1 \otimes h_2g_2$$

$$S(h_1)h_2 = \varepsilon(h)1 = h_1S(h_2)$$

for all $g, h \in H$. The theory of Hopf algebras has evolved as a research field in its own, and has known a very wide and rapid development over the last 30 years. Among the methods used in Hopf algebras, one of the main tools has been the representation theory. There is a very important aspect of representation theory of Hopf algebras: in contrast with the representation theory of an arbitrary algebra, the comultiplication and counit allow one to compute tensor product of representations, and the antipode allows for duals of representations. This is what one is used to having in the classical representation theory of groups. In fact, in this respect (and others) Hopf algebras are a generalization of group algebras: if $G$ is a group, then $K[G]$ becomes a Hopf algebra with comultiplication given by $g \mapsto g \otimes g$ and antipode induced by $g \mapsto g^{-1}$. More general objects, the quasi-Hopf algebras, were introduced by Drinfel’d in 1990; other types of similar structures have been introduced motivated by the quantum theory and representations of quantum groups, such as weak Hopf algebras or the weak quasi-Hopf algebras. These all support a very interesting representation theory, and are all a particular case of the general theory of Tensor Categories. These are categories which essentially behave as categories of finite dimensional representations of algebras, but moreover, have a structure similar to the one described above; that is, they have a tensor product (i.e. are monoidal) and duals for objects. Another example of this type is the representation theory of Lie algebras, since their enveloping algebras and deformed quantum enveloping algebras are Hopf algebras. As mentioned, Hopf algebras appear from different directions in mathematics, from algebraic topology, topological quantum field theory, Lie groups and Lie algebras, non-commutative algebra, non-commutative geometry, quantum groups, representation theory, compact and
locally compact groups, $C^*$-algebras, among others. Other important examples include interesting generalizations to compact and locally compact quantum groups, which are $C^*$-algebras with a suitable compatible topology and a Hopf algebra structure with the maps (comultiplication, counit, antipode) giving the structure being continuous. The motivating example of this is the $C^*$-algebra $C(G)$ of continuous functions on a compact topological group. The comultiplication is given by $\Delta : C(G) \to C(G) \otimes C(G)$, $\Delta(f) = f(xy)$ (where $\otimes$ represents the completion of the linear tensor product), the counit $\varepsilon : C(G) \to \mathbb{C}$ is given by $\varepsilon(f) = f(1)$ and the antipode $S : C(G) \to C(G)$ by $S(f) = f(x^{-1})$. Also, the locally compact quantum groups provide a new $C^*$-algebraic formalism for quantum groups, which generalizes and unifies the Kac algebra, compact quantum group and Hopf algebra approaches.

Further category theory motivation

We present another short but very important example of comodules over coalgebras (rational representations of algebras). This is given by the category of graded modules over a $G$-graded ring $R$, where $G$ is an arbitrary group. Let $G$ be a group, $R$ a $G$-graded $K$-algebra with $K$ a commutative ring, (that is, $R = \bigoplus_{g \in G} R_g$ as $K$-modules and $R_g \cdot R_h \subseteq R_{gh}$ and $1 \in R_1$) and let $R - gr$ be the category of left $G$-graded $R$-modules. These are left $R$-modules $M$ such that $M = \bigoplus_{g \in G} M_g$ as $K$-modules and $R_g \cdot M_h \subseteq M_{gh}$. Morphisms in this category are $R$-linear maps $f : M \to N$ such that $f(M_g) \subseteq N_g$. Let $C = R \otimes_K K[G]$ with the $R$-bimodule structure given by $r \cdot (s \otimes g) \cdot t_h = rst_h \otimes gh$, for $r, s \in R$ and homogeneous $t_h \in R_h, h \in G$. This becomes an $R$-coalgebra (also called $R$-coring in recent literature) with the comultiplication given by $\Delta(s \otimes g) = (s \otimes g) \otimes_R (1 \otimes g)$ and $\varepsilon(s \otimes g) = s$. Then it can be shown (see [BW]) that $R - gr$ is equivalent to $\mathcal{M}^C$ - the category of right $C$-comodules over $C$. Let us briefly explain this equivalence: any graded $R$-module $M$ becomes a $C$-comodule by the coaction $m \mapsto \sum_g m_g \otimes_R (1 \otimes g)$. Conversely, a $C$-comodule $N$ becomes an $R$-graded module by its $R$-linear structure and with the grading $N_g = \{ n \in N | n_0 \otimes n_1 \in N \otimes_R (R \otimes_K K_g) \}$. One easily checks that maps of graded modules are the same as comodule morphisms. In fact, this result follows more generally for a group $G$, a $G$-graded $K$-algebra $R$, a left $G$-set $X$ and the category of $X$-graded $R$-modules. In this respect, since it is known that the category of (co)chain complexes is equivalent to a category of $\mathbb{Z}$-graded modules, it should not be surprising that it is also
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equivalent to a category of comodules over a coalgebra.
In fact, it is the case that many types of categories are equivalent to categories of comodules
over a coalgebra. Besides the examples described before, many other examples arise from
actions and coactions of Hopf algebras, such as categories of representations of comodule
algebras, module coalgebras, Yetter-Drinfel’d modules, Doi-Koppinen modules or the so
called entwined modules over entwining structures, which encompass many such situations.
We refer to [BW] for a comprehensive covering of these notions.
As seen above, in representation theory (for example, tensor categories), it is often the case
that one is interested in an essentially small category \( C \) (a category whose isomorphisms
types of objects form a set) which has several other features. Such are:
• \( C \) is abelian (with finite colimits only);
• \( \text{Hom}(X, Y) \) is a finite dimensional vector space over a fixed field \( K \) for all objects \( X, Y \)
of \( C \);
• Each object has finite length in \( C \).
With some work along the lines of the Freyd-Mitchel theorem, it is possible to show that
such a category is equivalent to the category of finite dimensional representations of an
algebra.
Also, in many situations of an abelian category \( C \) and a subcategory \( \mathcal{A} \), with some condi-
tions, it can be obtained that \( \mathcal{A} \) is equivalent to a category of comodules over a coalgebra
\( H \) and \( C \) is equivalent to the category of modules over \( H^* \), in such a way that, through
these equivalences, the incusion functor \( \mathcal{A} \hookrightarrow C \) corresponds to the forgetful functor from
right \( H \)-comodules to left \( H^* \)-modules:

\[
\begin{array}{ccc}
\mathcal{A} & \longrightarrow & C \\
\cong & & \cong \\
\mathcal{M}^H & \subseteq & H^* - \text{mod}
\end{array}
\]

This is the type of situation which we will be considering extensively throughout this text.
Part I

Categorical Problems for Profinite Algebras
Chapter 2

When the Rational part splits off in any representation

Introduction

Let $A$ be a profinite algebra over a field $k$ and let $C$ be a coalgebra such that $A = C^*$. The category of left (resp. right) $C$-comodules is a full subcategory of the category of right (resp. left) modules over the dual algebra $A$. In [NT] it was shown that the rational part of every right $A$-module $M$ is a direct summand in $M$ if and only if $C$ is finite dimensional. In this case, the category of rational right $A$-modules is equal to the category of right $A$-modules. The aim of this chapter is to give a new and elementary proof of this result, based on general results on modules and comodules, and an old result of Levitzki, stating that a nil ideal in a right noetherian ring is nilpotent. The proof of Naăsăsescu and Torrecillas from [NT] involve several techniques of general category theory (such as localization), some facts on linearly compact modules and is based on general nontrivial and profound results of Teply regarding the general splitting problem (see [T1, T2, T3]). Another proof also based on M.L.Teply’s results is also contained in [C]. We present here a proof based on [I1]. We first prove that if $C$ has the splitting property, that is, the rational part of every right $C^*$-module is a direct summand, then $C$ has only a finite number of isomorphism types of simple (left or right) comodules. We then observe that the injective envelope of every simple right comodule contains only finite dimensional proper subcomodules. This immediately implies that $C$ is right noetherian. Then, using a quite common old idea from Abelian group theory we use the hypothesis for a direct product of modules to obtain that
every element of J, the Jacobson radical of A, is nilpotent. Finally, using a well known result in noncommutative algebra due to Levitzki, we conclude that J is nilpotent which combined with the above mentioned key observation immediately yields that C is finite dimensional.

2.1 Splitting Problem

For an \( f \in C^* \), put \( \overline{f} : C \rightarrow C, \overline{f}(x) = f(x_1)x_2 \); then \( \overline{f} \) is a morphism of right \( C \) comodules. As a key technique, we make use of the algebra isomorphism \( C^* \approx \text{Hom}(C^C, C^C) \) given by \( f \mapsto \overline{f} \) (with inverse \( \alpha \mapsto \varepsilon \circ \alpha \)), where \( \text{Hom}(C^C, C^C) \) is a ring with opposite composition.

Also if \( T \) is a simple right \( C \) subcomodule of \( C \), there exists \( E(T) \subseteq C \) an injective envelope of \( T \) and then \( C = E(T) \oplus X \) as right \( C \) comodules. As \( C^* \approx E(T)^* \oplus X^* \), we identify any element \( f \) of \( E(T)^* \) with the one of \( C^* \) equal to \( f \) on \( E(T) \) and 0 on \( X \).

**Lemma 2.1.1** If \( T \) is a simple right comodule and \( E(T) \) is the an injective envelope of \( T \), then \( E(T) \) contains only finite dimensional proper subcomodules.

**Proof.** Let \( K \varsubsetneq E(T) \) be an infinite dimensional subcomodule. Then there is a submodule \( K \varsubsetneq F \subseteq E(T) \) such that \( F/K \) is finite dimensional. We have an exact sequence of right \( C^* \) modules:

\[
0 \rightarrow (F/K)^* \rightarrow F^* \rightarrow K^* \rightarrow 0
\]

As \( F/K \) is a finite dimensional rational left \( C^* \) module, \( (F/K)^* \) is rational right module; thus \( A = \text{Rat} F^* \neq 0 \). Denote \( M = T^\perp \subseteq F^* \). Take \( u \notin M \); this corresponds to some \( v \in \text{Hom}(F, C) \) such that \( v|_T \neq 0 \). Then \( v \) is injective, because \( T \) is an essential submodule of \( F \subseteq E(T) \) and if \( \ker(v) \neq 0 \) then \( \ker(v) \cap T \neq 0 \) so \( \ker(v) \supseteq T \), which contradicts \( v|_T \neq 0 \). As \( C \) is an injective right \( C \) comodule and \( v \) is injective we have a commutative diagram:

\[
\begin{array}{ccc}
C^* & \longrightarrow & F^* \\
\| \quad & & \| \\
\text{Hom}^C(C, C) & \longrightarrow & \text{Hom}^C(F, C)
\end{array}
\]

We see that \( \text{Hom}^C(F, C) \) is generated by \( v \) as \( \text{Hom}^C(C, C) \approx C^* \) is generated by \( 1_C \). It follows that \( F^* \) is generated by any \( u \notin M \). Now if \( F^* = A \oplus B \), we see that \( A \) is finitely generated as \( F^* \) is finitely generated. So \( A \) is finite dimensional, thus \( A \neq F^* \) by the initial
CHAPTER 2. WHEN RAT SPLITS OFF IN ANY REPRESENTATION

2.1. SPLITTING PROBLEM

assumption. But now if \( a \in A \setminus M \), then \( a \) generates \( F^* \), so \( A = F^* \), and therefore \( A \subseteq M \). Also \( B \neq F^* \) as \( A \neq 0 \), so by the same argument \( B \subseteq M \), and therefore \( F^* = A + B \subseteq M \), a contradiction \( (\varepsilon \mid F \notin M) \).

Proposition 2.1.2 Let \( C \) be a coalgebra such that the rational part of every finitely generated left \( C^* \) module splits off. Then there are only a finite number of isomorphism types of simple right \( C \) comodules. Equivalently, \( C_0 \) is finite dimensional (Recall that \( C_0 \) is the sum of all simple left (equivalently, right) subcomodules of \( C \)).

Proof. Let \( (S_i)_{i \in I} \) be a set of representatives for the simple right comodules and \( \Sigma = \bigoplus_{i \in I} S_i \). Then there is an injection \( \Sigma \hookrightarrow C \) and we can consider \( E(S_i) \) an injective envelope of \( S_i \) contained in \( C \). Then the sum \( \sum_{i \in I} E(S_i) \) is direct and there is \( X < C \) such that \( \bigoplus_{i \in I} E(S_i) \oplus X = C \) as right \( C \) comodules and left \( C^* \)-modules. We have \( C^* = \prod_{i \in I} E(S_i)^* \times X^* \). If \( c^* \in E(S_i)^* \) and \( x_j \in E(S_j) \), then \( \Delta(x_j) = x_{j1} \otimes x_{j2} \in E(S_j) \otimes C \) and therefore \( c^* \cdot x_j = c^*(x_{j2})x_{j1} = 0 \) if \( j \neq i \), as \( c^* \mid_{E(S_j)} = 0 \). The same holds if \( c^* \in X^* \). Thus if \( c^* = ((c^*_i)_{i \in I}, c^*_X) \) and \( c_j \in E(S_j) \), then \( c^* \cdot c_j = c^*_j \cdot c_j \). Here \( c^*_j \) equals \( c^* \) on \( E(S_j) \) and 0 otherwise.

Now consider \( M = \prod_{i \in I} S_i \) and take \( x = (x_i)_{i \in I} \in M \), \( x_i \neq 0 \). If \( y = (y_i)_{i \in I} \in M \) then for each \( i \) we have \( S_i = C^* \cdot x_i \) as \( x_i \neq 0 \) and \( S_i \) is simple, so there is \( c^*_i \in C^* \) such that \( c^*_i \cdot x_i = y_i \). By the previous considerations, we may assume that \( c^*_i \in E(S_i)^* \) (that is, it equals zero on all the components of the direct sum decomposition of \( C \) except \( E(S_i) \)) and then there is \( c^* \in C^* \) with \( c^*_i \mid_{E(S_i)} = c^*_i \mid_{E(S_i)} \). Then one can easily see that \( c^* \cdot x_i = c^*_i \cdot x_i = y_i \), thus we may extend this to \( c^* \cdot x = y \) showing that actually \( M = C^* \cdot x \). As \( M \) is finitely generated, its rational part must split and must be finitely generated (as a direct summand in a finitely generated module), so it must be finite dimensional. But \( \bigoplus_{i \in I} S_i \subseteq (\prod_{i \in I} S_i) \), and this shows that \( I \) must be finite. As \( C \) is quasifinite (that is, \( \text{Hom}(S,C) \) is finite dimensional for every simple right \( C \)-comodule \( S \)), this is equivalent to the fact that \( C_0 \) is finite dimensional. \( \square \)

Corollary 2.1.3 \( C^* \) is a right noetherian ring.

Proof. Let \( T \) be a right simple comodule, \( E(T) \subseteq C \) an injective envelope of \( T \) and \( C = E(T) \oplus X \) as right \( C \) comodules. If \( 0 \neq I < E(T)^* \) is a right \( C^* \)-submodule, then for \( 0 \neq f \in I \) put \( K = \text{Ker} \bar{f} \). We have \( K^\perp = \{ g \in E(T)^* \mid g|_K = 0 \} = f \cdot C^* \subseteq I \). Indeed, if \( g \) is 0 on \( K \), then \( K \subseteq \text{Ker} \bar{g} \) as \( K \) is a right \( C \) subcomodule of \( E(T) \) and therefore it
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factors through $\bar{f} : \bar{g} = \alpha \bar{f} = \bar{h} \bar{f} = \bar{f} \cdot \bar{h}$ for $h = \varepsilon \circ \alpha$, so $g = f \cdot h \in f \cdot C^*$. As $K$ is finite dimensional by Lemma 2.1.1, $K^\perp = f \cdot C^*$ has finite codimension in $E(T)^*$, showing that $I \supseteq f \cdot C^*$ has finite codimension, which obviously shows that $E(T)^*$ is Noetherian.

If $C_0 = \bigoplus_{i \in F} T_i$ with $T_i$ simple right comodules then $F$ is finite by Proposition 2.1.2, so $C^* = \bigoplus_{i \in F} E(T_i)^*$ is Noetherian as each $E(T_i)^*$ are.

Put $R = C^*$. Note that $J = C_0^\perp = \{ \bar{f} | \bar{f}|_{C_0} = 0 \}$ is the Jacobson radical of $R$ and $\bigcap_{n \in \mathbb{N}} J^n = 0$. Also if $M$ is a finite dimensional right $R$-module, we have $J^nM = 0$ for some $n$, because the descending chain of submodules $(MJ^n)_n$ must stabilize and therefore $MJ^n = MJ^{n+1}MJ^n \cdot J$ implies $MJ^n = 0$ by Nakayama lemma.

**Proposition 2.1.4** Any element $f \in J$ is nilpotent.

**Proof.** As $C$ is a finite direct sum of injective envelopes of simple right comodules $E(T)$’s, it is enough to show that $f^n_{|_{E(T)}} = 0$ for some $n$ for each simple right subcomodule of $C$ and injective envelope $E(T) \subseteq C$. Assume the contrary for some fixed data $T$, $E(T)$. Let

$$M = \prod_{n \geq 1} \frac{E(T)^*}{K_n^\perp}$$

where $K_n = \text{Ker} f^n \neq E(T)$ and $K_n^\perp = \{ g \in E(T)^* | g|_{K_n} = 0 \}$. Note that $K_n \subseteq K_{n+1}$ Put $\lambda = (f^{[n/2]}|_{E(T)})$ where $[x]$ is the greatest integer less or equal to $x$. Note that if $u$ equals $f$ on $E(T)$ and $0$ on $\lambda$ then $f^n_{|_{E(T)}}$ regarded as an element of $C^*$ equals $uf^{n-1}$ (recall that we identify $E(T)^*$ as a direct summand of $C^*$).

$$\lambda = (u, uf, uf, \ldots , uf^{n-1}, uf^{n-1}, 0, \ldots ) + (0, 0, \ldots , 0, uf^n, uf^n, uf^{n+1}, \ldots ) = r_n + \mu_n \cdot f^n$$

with $r_n = (u, uf, uf, \ldots , uf^{n-1}, uf^{n-1}, 0, \ldots , 0, \ldots )$ (the morphisms are always thought restricted to $E(T)$). But then $r_n \in \prod_{p \leq n} E(T)^*/K_p^\perp \times 0$ which is a rational left $C$ comodule because $E(T)^*/K_p^\perp \simeq K_p^*$ and $K_p$ is finite dimensional. Write $M = \text{RatM} \oplus \Lambda$ as right $R$ modules and $\mu_n = q_n + \alpha_n$ with $q_n \in \text{RatM}$ and $\alpha_n \in \Lambda$. Then if $\lambda = r + \mu$ with $r \in \text{RatM}$ and $\mu \in \Lambda$ we have $r + \mu = r_n + \mu_n \cdot f^n = (r_n + q_n \cdot f^n) + \alpha_n \cdot f^n$ which shows that $\mu = \mu_n \cdot f^n$. Then if $\mu = (l_p)_{p \geq 1}$ and $\mu_n = (\mu_{n,p})_{p \geq 1}$ we get that $l_p = \mu_{n,p} \cdot f^n \in E(T)^*/K_p^\perp \cdot J^p$ for all $p$ and this shows that $l_p = 0$ by the previous remark, so $\mu = 0$. Therefore $\lambda \in \text{RatM}$, so $\lambda \cdot R$ is finite dimensional and again we get $\lambda \cdot RJ^n = 0$ for some $n$. Hence we get
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\[ f^{[p/2]+n}|_{K_p} = 0, \forall p, \text{ equivalently } \overline{f}^{[p/2]+n} = 0 \text{ on } K_p \] (because \( K_p \) is a right comodule).

For \( p = 2n + 1 \) we therefore obtain \( K_{2n+1} \subseteq K_{2n} \) so \( K_m = K_{m+1} \) for \( m = 2n \). Then if \( I = \text{Im}(\overline{f}^m) \), \( I \neq 0 \) by the assumption and there is a simple subcomodule \( T' \), \( T' \subseteq I \); then \( \overline{f}|_{T'} = 0 \) (because \( f \in J = C^1_0 \)). Take \( 0 \neq y \in T' \); then \( y = \overline{f}^m(x), x \in E(T) \) and \( 0 = \overline{f}(y) = \overline{f}^{m+1}(x) \) showing that \( x \in K_{m+1} = K_m \), therefore \( y = \overline{f}^m(x) = 0 \), a contradiction.

Theorem 2.1.5 If the rational part of every right \( C^* \) module splits off, then \( C \) is finite dimensional.

Proof. By Corollary 2.1.3 \( C^* \) is Noetherian and by the previous Proposition every element if \( J \) is nilpotent. Therefore by Levitzki’s Theorem we have that \( J \) is nilpotent. Now note that \( C_n \) is finite dimensional for all \( n \). Indeed, denoting by \( s_n(M) \) the \( n \)-th term in the Loewy series of the comodule \( M \), if \( C_0 = \bigoplus T_i \) with \( T_i \) simple right comodules, then \( C = \bigoplus_{i \in F} E(T_i) \) with \( E(T_i) \) injective envelopes of the \( T_i \)’s and \( C_n = \bigoplus_{i \in F} s_n(E(T_i)) \).

If \( C_n \) is finite dimensional, then \( s_{n+1}(E(T_i)) \) is finite dimensional as otherwise there is a decomposition \( s_{n+1}(E(T_i))/s_n(E(T_i)) = T \oplus K \) with simple \( T \) and infinite dimensional \( K \), and therefore we would find an infinite dimensional subcomodule of \( E(T_i) \) corresponding to \( K \), which is impossible. Therefore since \( J^n = 0 \) for some \( n \) and \( J^n \) has finite codimension as \( J^n = C^1_n \) and \( C_n \) is finite dimensional, we conclude that \( C \) has finite dimension. \( \square \)
Chapter 3

When does the rational torsion split off for finitely generated modules

Introduction

Let $R$ be a ring and $T$ be a torsion preradical on the category of left $R$-modules $R$–$mod$. Then $R$ is said to have splitting property provided that $T(M)$, the torsion submodule of $M$, is a direct summand of $M$ for every $M \in R$–$mod$. More generally, if $\mathcal{C}$ is a Grothendieck category and $\mathcal{A}$ is a subcategory of $\mathcal{C}$, then $\mathcal{A}$ is called closed if it is closed under subobjects, quotient objects and direct sums. To every such subcategory we can associate a preradical $t$ (also called torsion functor): for every $M \in \mathcal{C}$ we denote by $t(M)$ the sum of all subobjects of $M$ that belong to $\mathcal{A}$. We say that $\mathcal{C}$ has the splitting property with respect to $\mathcal{A}$ if $t(M)$ is a direct summand of $M$ for all $M \in \mathcal{C}$. In the case of the category of $R$-modules, the splitting property with respect to some closed subcategory is a classical problem which has been considered by many authors. In particular, when $R$ is a commutative domain, the question of when the (classical) torsion part of an $R$ module splits off is a well known problem. J. Rotman has shown in [Rot] that for a commutative domain the torsion submodule splits off in every $R$-module if and only if $R$ is a field. I. Kaplansky proved in [K1], [K2] that for a commutative integral domain $R$ the torsion part of every finitely generated $R$-module $M$ splits in $M$ if and only if $R$ is a Prüfer domain. While complete or partial results have been obtained for different cases of subcategories of $R$–$mod$ - such as the Dickson subcategory (see the next chapter for details) - or for commutative rings (see also [T1], [T2], [T3]), the general problem remains open for the
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non-commutative case and the general categorical setting.

In this chapter we investigate a special and important case of rings (algebras) $R$ arising as the dual algebra of a $K$-coalgebra $C$, $R = C^*$. We are thus situated in the realm of the theory of coalgebras and their dual algebras, a theory intensely studied over the last two decades. Then the category of the left $R$-modules naturally contains the category $\mathcal{M}^C$ of all right $C$-comodules as a full subcategory. In fact, $\mathcal{M}^C$ identifies with the subcategory $\text{Rat}(C^* \mod)$ of all rational left $C^*$-modules, which is generally a closed subcategory of $C^* \mod$. Then it is natural to study splitting properties with respect to this subcategory, and two questions regarding this splitting property with respect to $\text{Rat}(C^* \mod)$ naturally arise: first, when is the rational part of every left $C^*$-module $M$ a direct summand of $M$, and second, when does the rational part of every finitely generated $C^*$-module $M$ split in $M$. The first problem, the splitting of $C^* \mod$ with respect to the closed subcategory $\text{Rat}(C^* \mod)$ was the topic of Chapter 1. As mentioned there, it was settled originally in [NT], where it is proved that if all $C^*$-modules split with respect to $\text{Rat}$ then the coalgebra $C$ must be finite dimensional.

We consider the more general problem of when $C$ has the splitting property only for finitely generated modules, that is, the problem of when is the rational part $\text{Rat}(M)$ of $M$ a direct summand in $M$ for all finitely generated left $C^*$-modules $M$. We say that such a coalgebra has the left f.g. $\text{Rat}$-splitting property (or we say that it has the $\text{Rat}$-splitting property for finitely generated left modules). If the coalgebra $C$ is finite dimensional, then every left $C^*$-module is rational so $\mathcal{M}^C$ is equivalent to $C^* \mod$ and $\text{Rat}(M) = M$ for all $C^*$-modules $M$ and in this case $\text{Rat}(M)$ trivially splits in any $C^*$-module. Therefore we will deal with infinite dimensional coalgebras, as generally the infinite dimensional coalgebras produce examples essentially different from the ones in algebra theory.

The starting and motivating point of our research is the fact that over the ring of formal power series over a field $R = K[[X]]$ (or a division algebra), every finitely generated module splits into its torsion part and a complementary module. In this case, $R$ is the dual of the so called divided power coalgebra, and the torsion part of any module identifies with the rational submodule. Here the analogue with classical torsion splitting problems becomes obvious. In fact, what turns out to be essential in this example is the structure of ideals of $K[[X]]$, and that is, they are linearly ordered. This suggests the consideration of more
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general coalgebras, those whose left subcomodules form a chain. This turns out to be a left-right symmetric concept, and the most basic example of infinite dimensional coalgebra having the f.g. Rat-splitting property (Proposition 3.2.3 and Theorem 3.2.5). One key observation in this study is that if $C$ has the f.g. Rat-splitting property, then the indecomposable left injectives have only finite dimensional proper subcomodules, and this motivates the introduction of comodules and coalgebras $C$ having only finite dimensional proper left subcomodules, which we call almost finite (or almost finite dimensional) comodules. This proves to be the proper generalization of the phenomenon found in the case of $K[[X]]$, i.e. the set of torsion elements of a left $C^*$-module $M$ forms a submodule which coincides exactly with the rational submodule of $M$ (Proposition 3.1.5). Before turning to the study of chain comodules and coalgebras, we give several general results for coalgebras $C$ with the f.g. Rat-splitting property: they are artinian as right $C^*$-modules and injective as left $C^*$-modules, have at most countable dimension and $C^*$ is a left Noetherian ring. Moreover, such coalgebras have finite dimensional coradical and the f.g. Rat-splitting property is preserved by subcoalgebras.

The f.g. Rat-splitting property has been studied before in [C] where the last two of the above statements were proven, but with the use of very strong results of M.L. Teply; we also include alternate direct proofs. Chain coalgebras were also studied recently in [LS] and also briefly in [C] and [CGT]. However, our interest in chain coalgebras is of a different nature; it is a representation theoretic one and is directed towards our main result of this chapter, that generalizes a result previously obtained [C] in the commutative case: we characterize the coalgebras having the f.g. Rat-splitting property and that are colocal, and show that they are exactly the chain coalgebras (Section 3, Theorem 3.3.4), a result that will involve quite technical arguments. In fact, our characterizations of chain coalgebras are done as a consequence of more general discussions such as the study of chain comodules and more generally almost finite comodules and coalgebras. For example, we show that almost finite coalgebras are reflexive, and that chain coalgebras are almost finite, and thus obtain the fact that chain coalgebras are reflexive (a result also found in the recent [LS]) from our more general framework.

We provide several nontrivial examples. One will be the construction of a noncommutative chain coalgebra with coradical isomorphic to the dual of the Hamilton algebra of quaternions. However, we see that when the base field $K$ is algebraically closed or the
coalgebra is pointed, then a chain coalgebra is isomorphic to the divided power coalgebra if it is infinite dimensional or to one of its subcoalgebras otherwise. This also characterizes the divided power coalgebra over an algebraically closed field as the only local coalgebra having the above mentioned splitting property. As an application of the main result, we obtain the structure of cocommutative coalgebras having the f.g. Rat-splitting property from [C] in a more precise form: they are finite coproducts of finite dimensional coalgebras and infinite dimensional chain coalgebras. Moreover, following this model, our results allow us to generalize to the noncommutative case and show that a coalgebra that is a finite direct sum of infinite dimensional left chain comodules (serial coalgebra) has the left f.g. Rat-splitting property; moreover, this is again a left-right symmetric concept. More generally, a coproduct of such a coalgebra and a finite dimensional one again has the f.g. Rat-splitting property. We conclude by constructing a class of explicit examples of noncommutative coalgebras of this type over an arbitrary field, which will depend on a positive integer $q$ and a permutation $\sigma$ of $q$ elements.

3.1 General Considerations

Let $C$ be a coalgebra with counit $\varepsilon$ and comultiplication $\Delta$. We use the Sweedler convention $\Delta(c) = c_1 \otimes c_2$ where we omit the summation symbol. For general facts about coalgebras and comodules we refer to [A], [DNR] or [Sw1]. For a vector space $V$ and a subspace $W$ of $V$ denote by $W^\perp = \{ f \in V^* \mid f(x) = 0, \forall x \in W \}$ and for a subspace $X \subseteq V^*$ denote by $X^\perp = \{ x \in V \mid f(x) = 0, \forall f \in X \}$ (it will be understood from the context what is the space $V$ with respect to which the orthogonal is considered). Various properties of this correspondence between subspaces of $V$ and $V^*$ are well known and studied in more general settings in [DNR] (Chapter 1), [AF], [AN], [I0]. Related to that, we recall the finite topology on the dual $V^*$ of a vector space $V$: a basis of 0 for this linear topology is given by the sets $W^\perp$ with $W$ a finite dimensional subspace of $V$. Any topological consideration will refer to this topology. We often use the following: a subspace $X$ of $V^*$ is closed (in the finite topology) if and only if $(X^\perp)^\perp = X$; also, if $W$ is a subspace of $V$, then $(W^\perp)^\perp = W$. (see [DNR], Chapter 1)

For a coalgebra $C$ denote by $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ the coradical filtration of $C$, that is, $C_0$ is the coradical of $C$, and $C_{n+1} \subseteq C$ such that $C_{n+1}/C_n$ is the socle of the right (or left) $C$-comodule $C/C_n$ for all $n \in \mathbb{N}$. Then $C_n$ is a subcoalgebra of $C$ for all $n$, and the same
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C_n is obtained whether we take the socle of the left C-comodule C/C_n or of the right C-comodule C/C_n. Put C_{-1} = 0 and R = C^*. Denote J = J(C^*) the Jacobson radical of C^*. By [DNR] we have \( \bigcup_{n \in \mathbb{N}} C_n = C \), \( J = C_0^\perp \) and \( (J^{n+1})^\perp = C_n \). Then \( J^n \subseteq ((J^n)^\perp)^\perp = C_{n-1}^\perp \) and since \( \bigcup_{n \in \mathbb{N}} C_n = C \), we see that \( \bigcap_{n \in \mathbb{N}} J^n = 0 \).

For a left (right) C-comodule \( M \) with comultiplication \( \lambda : M \to C \otimes M \) (respectively \( \rho : M \to M \otimes C \)), the Sweedler notation is \( \lambda(m) = m_{-1} \otimes m_0 \) (respectively \( \rho(m) = m_0 \otimes m_1 \)). Moreover, the dual \( M^* \) of \( M \) becomes a left (right) \( C^* \)-module through the action induced by the right (left) \( C^* \)-action on \( M \) by duality: for \( m^* \in M^*, m \in M \) and \( c^* \in C^* \), \( (c^* \cdot m^*)(m) = m^*(m \cdot c^*) = c^*(m_{-1})m^*(m_0) \) (respectively \( (m^* \cdot c^*)(m) = m^*(m_0)c^*(m_1) \)).

**Lemma 3.1.1** Let \( C \) be a coalgebra over a field \( K \) and \( M \) be a left \( C \)-comodule. Then for any finitely generated left submodule \( X \) of \( M^* \), \( (X^\perp)^\perp = X \), that is, \( X \) is closed in the finite topology on \( M^* \).

**Proof.** It is enough to prove this for cyclic submodules: if \( (C^*f^\perp)^\perp = C^*f \) for all \( f \in M^* \) and \( X = C^* \cdot f_1 + \ldots + C^* \cdot f_n \) then \( (X^\perp)^\perp = \left( \bigcap_{i=1}^n (C^*f_i)^\perp \right)^\perp = \bigcap_{i=1}^n (C^*f_i)^\perp = \bigcap_{i=1}^n C^*f_i = X \) (since \( \bigcap_{i=1}^n M_i^\perp = \bigcap_{i=1}^n M_i^\perp \) for \( M_i \subseteq M \); see, for example [10], Proposition 3 or [DNR], Chapter 1; also [AN] and [AF]). Let \( X = C^*f \) and \( u : M \to C, u(m) = m_{-1}f(m_0) \), where for \( m \in M \), \( m_{-1} \otimes m_0 \in C \otimes M \) denotes the comultiplication of \( m \in M \); then \( L = (C^*f)^\perp = \{m \in M \mid (hf)(m) = 0, \forall h \in C^* \} = \{m \in M \mid f(h(m_{-1})m_0) = h(m_{-1}f(m_0)) = 0, \forall h \in C^* \} = \{m \in M \mid m_{-1}f(m_0) = 0 \} \), so \( L = \ker(u) \) (the left \( C^* \)-module structure on \( M^* \) is induced from the right \( C^* \)-module structure on \( M \) by duality). If \( g \in L^\perp \subseteq M^* \), then \( \ker(u) \subseteq \ker(g) \) and we can factor \( g \) as \( g = p \circ u \) with \( p : \text{Im}(u) \to K \). Then, defining \( h \in C^* \) as \( h = p \) on \( \text{Im}(u) \subseteq C \) and \( 0 \) on some complement of \( \text{Im}(u) \), we get \( (hf)(m) = f(m \cdot h) = f(h(m_{-1})m_0) = h(m_{-1}f(m_0)) = h(u(m)) = p \circ u(m) = g(m) \), i.e. \( g \in C^*f \). This shows that \( (C^*f^\perp)^\perp = C^*f \). \( \square \)

3.1.1 “Almost finite” coalgebras and comodules

**Definition 3.1.2** A \( C \)-comodule \( M \) will be called **almost finite** (or almost finite dimensional) if it has only finite dimensional proper subcomodules. Call a coalgebra \( C \) left **almost finite** if \( C \big/ C \) is almost finite.
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Proposition 3.1.3 Let $M$ be a left almost finite (dimensional) $C$-comodule. Then:

(i) $M$ is artinian as left $C$-comodule (equivalently, as right $C^*$-module).

(ii) Any nonzero submodule of $M^*$ has finite codimension; consequently $M^*$ is (left) Noetherian. Moreover, all submodules of $M^*$ are closed in the finite topology of $M^*$.

(iii) $M$ has at most countable dimension.

Proof. (i) Obvious.

(ii) Let $0 \neq I < M^*$ be a submodule, $0 \neq f \in I$. Then $X = (C^* \cdot f)^\perp$ is a subcomodule of $M$ which is finite dimensional and $C^* \cdot f = X^\perp$ from Lemma 3.1.1, so $C^* \cdot f$ has finite codimension, and so does $I \supseteq C^* \cdot f$. Thus $M^*$ is Noetherian and the last assertion of (ii) follows now from Lemma 3.1.1.

(iii) Assume $M$ is infinite dimensional and define inductively a sequence $(m_k)_{k \geq 0}$ such that $m_{k+1} \notin M_k = m_1 \cdot C^* + \ldots + m_k \cdot C^*$. This can be done since the $M_k$’s are finite dimensional, and then $\bigcup_{k \geq 0} M_k \subseteq M$ is infinite-countable dimensional, and thus cannot be a proper submodule of $M$. Thus $\bigcup_{k \geq 0} M_k = M$, and the proof is finished. \hfill \square

The above Proposition shows that a left almost finite coalgebra $C$ is coreflexive by [DNR], Exercise 1.5.14, since every ideal of finite codimension in $C$ is closed (Also, by a result of Radford, $C$ is coreflexive if and only every finite dimensional $C^*$-module is rational). Thus we have:

Corollary 3.1.4 Let $C$ be a left almost finite coalgebra. Then every nonzero left ideal of $C^*$ is closed in the finite topology on $C^*$ and has finite codimension, $C^*$ is Noetherian and $J^n = C^*_{n-1}$. Moreover, $C$ is coreflexive.

For a left $C^*$-module $M$ denote by $T(M)$ the set of all torsion elements of $M$, that is, $T(M) = \{ x \in M \mid \text{ann}_{C^*} x \neq 0 \}$. Since for a finite dimensional coalgebra $C$, the categories $\mathcal{M}^C$ and $C^* \mod$ are equivalent, the infinite dimensional case is the interesting one. As mentioned above, for coreflexive coalgebras, $\text{Rat}(M) = \{ x \in M \mid C^* \cdot x \text{ is finite dimensional} \} = \{ x \in M \mid \text{ann}_{C^*}(x) \text{ has finite codimension} \}$. For (infinite dimensional) almost finite coalgebras, we see that the rational submodule of a $C^*$-module has an even more special form:

Proposition 3.1.5 Let $C$ be an infinite dimensional left almost finite coalgebra. Then for every left $C^*$-module $M$ we have $\text{Rat}(M) = T(M)$; moreover, $x \in \text{Rat}(M)$ if and only if $C^* \cdot x$ is finite dimensional.
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Proof. If \( x \in \text{Rat}(M) \) then \( C^* \cdot x \) is finite dimensional and then \( \text{ann}_{C^*}(x) \) must be of finite codimension, thus nonzero, as \( C^* \) is infinite dimensional. Conversely, if \( x \in T(M) \) and \( x \neq 0 \) then \( I = \text{ann}_{C^*}(x) \) is a nonzero left ideal of \( C^* \) so it is closed by Corollary 3.1.4; thus \( I = X^\perp \) with \( X \neq C \) a finite dimensional subcomodule of \( C \). Then \( C^* \cdot x \simeq C^*/\text{ann}_{C^*}(x) = C^*/X^\perp \simeq X^* \), which is a rational left \( C^* \)-module, being the dual of a finite dimensional subcomodule of \( C \).

\[ \Box \]

3.1.2 The Splitting Property

Definition 3.1.6 We shall say that a coalgebra \( C \) has the left (right) f.g. Rat-splitting property, or that it has the left (right) Rat-splitting property for all finitely generated modules if the rational part of every finitely generated left (right) \( C^* \)-module splits off.

The following key observation, together with the succeeding study of chain coalgebras, motivates our previous introduction of almost finite comodules and coalgebras.

Proposition 3.1.7 Let \( C \) be a coalgebra such that \( \text{Rat}(M) \) splits off in every finitely generated left \( C^* \)-module \( M \). Then every indecomposable injective left \( C \)-comodule \( E \) is an almost finite comodule.

Proof. Let \( T \) be the socle of \( E \); then \( T \) is simple and \( E = E(T) \) is the injective envelope of \( T \). We show that if \( K \subseteq E(T) \) is an infinite dimensional subcomodule then \( K = E(T) \).

Suppose \( K \nsubseteq E(T) \). Then there is a left \( C \)-subcomodule (right \( C^* \)-submodule) \( K \nsubseteq L \subseteq E(T) \) such that \( L/K \) is finite dimensional. We have an exact sequence of left \( C^* \)-modules:

\[ 0 \rightarrow (L/K)^* \rightarrow L^* \rightarrow K^* \rightarrow 0 \]

As \( L/K \) is a finite dimensional left \( C \)-comodule, we have that \( (L/K)^* \) is a rational left \( C^* \)-module; thus \( \text{Rat}(L^*) \neq 0 \). Also \( L^* \) is finitely generated as it is a quotient of \( E(T)^* \) which is a direct summand of \( C^* \). We have \( L^* = \text{Rat}(L^*) \oplus X \) for some left \( C^* \)-submodule \( X \) of \( L^* \). Then \( \text{Rat}(L^*) \) is finitely generated because \( L^* \) is, so it is finite dimensional.

As \( L \) is infinite dimensional by our assumption, we have \( X \neq 0 \). This shows that \( L^* \) is decomposable and finitely generated, thus it has at least two maximal submodules, say \( M, N \). We have an epimorphism \( E(T)^* \xrightarrow{f} L^* \rightarrow 0 \) and then \( f^{-1}(M) \) and \( f^{-1}(N) \) are distinct maximal \( C^*-\)submodules of \( E(T)^* \). But by [1], Lemma 1.4, \( E(T)^* \) has only one maximal \( C^* \)-submodule which is \( T^\perp \), so we have obtained a contradiction. \[ \Box \]
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Let $C_0$ be the coradical of $C$, the sum of all simple subcomodules of $C$. By [DNR], Section 3.1, $C_0$ is a cosemisimple coalgebra that is a direct sum of simple subcoalgebras $C_0 = \bigoplus_{i \in I} C_i$ and each simple subcoalgebra $C_i$ contains only one type of simple left (or right) $C$-comodule; moreover, any simple left (or right) $C$-comodule is isomorphic to one contained in some $C_i$. A coalgebra $C$ with $C_0$ finite dimensional is called almost connected coalgebra.

The following two Propositions have also been observed in [Cu] (Lemma 3.2 and Lemma 3.3), but general powerful techniques from [T3] are used there. We provide here direct simple arguments.

**Proposition 3.1.8** Let $C$ be a coalgebra with the left f.g. Rat-splitting property. Then there is only a finite number of isomorphism types of simple left $C$-comodules, equivalently, $C_0$ is finite dimensional.

**Proof.** By the above considerations, if $S_i$ is a simple left $C$-subcomodule of $C_i$, we have that $(S_i)_{i \in I}$ forms a set of representatives for the isomorphism types of simple left $C$-comodules. Let $S$ be a set of representatives for the simple right $C$-modules. Let $E(C_i)$ be an injective envelope of the left $C$-comodule $C_i$ included in $C$; then as $C_0$ is essential in $C$ we have $C = \bigoplus_{i \in I} E(C_i)$ as left $C$-comodules or right $C^*$-modules. Then $C^* = \prod_{i \in I} E(C_i)^*$ as left $C^*$-modules. As $S_i \subseteq E(C_i)$, we have epimorphisms of left $C^*$-modules $E(C_i)^* \twoheadrightarrow S_i^* \rightarrow 0$ and therefore we have an epimorphism of left $C^*$-modules $C^* \twoheadrightarrow \prod_{i \in I} S_i^* \rightarrow 0$. But there is a one-to-one correspondence between left and right simple $C$-comodules given by $\{S_i \mid i \in I\} \ni S \mapsto S^* \in S$. Hence there is an epimorphism $C^* \twoheadrightarrow \prod_{S \in S} S \rightarrow 0$, which shows that the left $C^*$-module $P = \prod_{S \in S} S$ is finitely generated (actually generated by a single element). But then as $\text{Rat}(C^*P)$ is a direct summand in $P$, we must have that $\text{Rat}(C^*P)$ is finitely generated, so it is finite dimensional. Therefore, as $\Sigma = \bigoplus_{S \in S} S$ is a rational left $C^*$-module which is naturally included in $P$, we have $\Sigma \subseteq \text{Rat}(P)$. This shows that $\bigoplus_{S \in S} S$ is finite dimensional, so $S$ (and also $I$) must be finite. This is equivalent to the fact that $C_0$ is finite dimensional, because each $C_i$ is a simple coalgebra, thus a finite dimensional one. \hfill $\square$

**Proposition 3.1.9** If $C$ has the left f.g. Rat-splitting property then so does any subcoalgebra $D$ of $C$. 

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**3.1. GENERAL CONSIDERATIONS**

**Proof.** Let $M$ be a finitely generated left $D^*$-module. Since $C^*/D^\perp \simeq D^*$, $M$ has an induced left $C^*$-module structure and is annihilated by $D^\perp$ (that is, $D^\perp \cdot x = 0$ for all $x \in M$). Then a subspace of $M$ is a $C^*$-submodule if and only if it is a $D^*$-submodule. There is $M = T \oplus X$ a direct sum of $C^*$-modules (equivalently $D^*$-submodules, since $D^\perp$ annihilates the elements in both $T$ and $X$) with $T$ the rational $C^*$-submodule of $M$. It will suffice to show that a submodule of $M$ is rational as a $C^*$-module if and only if it is rational as a $D^*$-module. Indeed, let $m \in T = Rat_{C^*}(M)$; then there is $\sum_i m_i \otimes c_i \in T \otimes C$ such that $c^* m = \sum_i c^*(c_i) m_i$; we may assume that the $m_i$'s are linearly independent. Then for $c^* \in D^\perp \subseteq C^*$ we get $0 = c^* \cdot m = \sum_i c^*(c_i) m_i$ and so $c^*(c_i) = 0$ since the $m_i$'s are independent, showing that $c_i \in (D^\perp)^\perp = D$. Therefore $\rho(m) = \sum_i m_i \otimes c_i \in T \otimes D$, where $\rho$ is the comultiplication of $T$, and thus $m \in Rat_{D^*}(M)$.

The converse inclusion $Rat_{D^*}(M) \subseteq Rat_{C^*}(M)$ is obvious, since the $D$-comultiplication $Rat_{D^*}(M) \rightarrow Rat_{D^*}(M) \otimes D \subseteq Rat_{D^*}(M) \otimes C$ induces a $C$-comultiplication through the canonical inclusion $D \subseteq C$, compatible with the $C^*$-multiplication of $M$. \hfill $\Box$

**Proposition 3.1.10** Let $C$ be a coalgebra that has the left f.g. Rat-splitting property. Then the following assertions hold:

(i) $C$ is artinian as a left $C$-comodule (equivalently, as a right $C^*$-module).

(ii) $C^*$ is left Noetherian.

(iii) $C$ has at most countable dimension.

(iv) $C$ is injective as a left $C^*$-module.

**Proof.** (i) We have a direct sum decomposition $C = \bigoplus_{i \in F} E(S_i)$ where $C_0 = \bigoplus_{i \in F} S_i$ is the decomposition of $C_0$ into simple left $C$-comodules and $E(S_i)$ are injective envelopes of $S_i$ contained in $C$. Since $C_0$ is finite dimensional, $F$ is finite, so the result follows from Propositions 3.1.7 and 3.1.3.

(ii) Since $C^* = \bigoplus_{i \in F} E(S_i)^*$, this also follows from 3.1.3.

(iii) Similar to (i).

(iv) By [I0] Lemma 2, it is enough to prove that $E = C^C$ splits off in any left $C^*$-module $M$ in which it embeds ($E \subseteq M$) and such that $M/E$ is cyclic, generated by an element $\hat{x} \in M/E$. Let $H = \text{Rat}(C^* \cdot x) \subseteq M$; then there is $X < C^* \cdot x$ such that $H \oplus X = C^* \cdot x$. Then $E + H$ is a rational $C^*$-module so $(E + H) \cap X = 0$; also $M = C^* \cdot x + E$, so $(E + H) + X = M$, showing that $E + H$ is a direct summand in $M$. But, as $E$ is an
in injective comodule, we have that $E$ splits off in $E + H$, thus $E$ must split in $M$ and the proof is finished. □

### 3.2 Chain Coalgebras

**Definition 3.2.1** We say that a left (right) $C$-comodule $M$ is a **chain** (or **uniserial**) comodule if and only if the lattice of the left (right) subcomodules of $C$ is a chain, that is, for any two subcomodules $X, Y$ of $M$ either $X \subseteq Y$ or $Y \subseteq X$. We say a coalgebra $C$ is a **left (right) chain coalgebra** (or **uniserial coalgebra**) if $C$ is a left (right) chain $C$-comodule.

In other words, a left $C$-comodule $M$ is a chain comodule if $M$ is uniserial as a right $C^*$-module. Part of the following proposition is a somewhat different form of Lemma 2.1 from [CGT]. However, we will need to use some of the other equivalent statements bellow.

**Proposition 3.2.2** Let $M$ be a left (right) $C$-comodule. The following assertions are equivalent:

(i) $M$ is a chain comodule.

(ii) $M^*$ is a chain (uniserial) left (right) $C^*$-module.

(iii) $M$ and $M_n =$ the $n$’th Loewy term in the Loewy series of $M$ for $n \geq -1$, are the only subcomodules of $M$ ($M_{-1} = 0$).

(iv) $M^*_n = \{ u \in M^* \mid u(x) = 0, \forall x \in M_n \}$ for $n \geq -1$ and 0 are the only submodules of $M^*$.

(v) $M_n/M_{n-1}$ is either simple or 0 for all $n \geq -1$. (If $M_n/M_{n-1}$ is 0 for some $n$ then $M_k/M_{k-1}$ is 0 for all $k \geq n$.)

**Proof.** (iv)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i) If $M^*$ is uniserial, then for any two subcomodules $X, Y$ of $M$ we have $X^\perp \subseteq Y^\perp$, say. Thus we get $X = (X^\perp)^\perp \supseteq (Y^\perp)^\perp = Y$.

(i)$\Leftrightarrow$(iii) is obvious (note that (iii) this does not exclude the possibility that $M = M_n$ from some $n$ onward)

(i)$\Rightarrow$(iv) If $M$ is a chain comodule, it is enough to assume that $M$ is infinite dimensional, because of the duality of categories between finite dimensional left comodules and finite dimensional right comodules. We note that each $M^*_n$ is generated by any $u_n \in M^*_n \setminus M^*_n^{n+1}$. 

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3.2. Chain Coalgebras

Let $f \in M_n^\perp$ and write $\overline{u_n}, \overline{f} : M \to C$ for the maps $\overline{u_n}(m) = m_{-1}u_n(m_0)$ and $\overline{f}(m) = m_{-1}u_n(m_0)$. Then $u_n \in M_n^\perp \setminus M_{n+1}^\perp$ shows that $\overline{u_n}$ is a morphism of left $C$-comodules that factors to a morphism $M/M_n \to C$ which does not cancel on $M_{n+1}/M_n$ - the only simple submodule of $M/M_n$. Therefore $\text{Ker}(\overline{u_n} : M/M_n \to C) = 0$ and we have a diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & M/M_n \\
\downarrow{\overline{f}} & & \downarrow{\overline{g}} \\
C & \longrightarrow & C
\end{array}
$$

that is completed commutatively by $\overline{g}$ (as $C^n$ is injective), so that we get $g \circ \overline{u_n} = \overline{f}$. Then if $g = \varepsilon \circ \overline{g}$ we have, for $m \in M$, that $g(m_{-1})u_n(m_0) = g(m_{-1}u_n(m_1)) = \varepsilon(\overline{g}(\overline{u_n}(m))) = \varepsilon(\overline{f}(m)) = \varepsilon(m_{-1})f(m_0) = f(m)$. Thus $g \cdot u_n = f$. This shows that any cyclic submodule of $M^*$ coincides to one of the $M_n^\perp$, because for any $0 \neq f \in M^*$ there is some $n$ such that $f \in M_n^\perp \setminus M_{n+1}^\perp$, since $M = \bigcup_n M_n$. It therefore follows that for any nonzero submodule $I$ of $M^*$ there is $M_n^\perp \subseteq I$; since the $M_n$'s are (obviously) finite dimensional, $M_n^\perp$ and $I$ have finite codimension and it now easily follow from the above considerations that $I = M_k^\perp$, where $k$ is the smallest number such that $M_k^\perp \subseteq I$.

(v)\(\Rightarrow\)(iii) Let $X$ be a right submodule of $M$ and suppose $X \neq M$ and $X \neq 0$. Then there is $n \geq 0$ such that $M_n \not\subseteq X$ and let $n$ be minimal with this property. Then we must have $M_{n-1} \subseteq X$ by the minimality of $n$ and we show that $M_{n-1} = X$. Indeed, if $M_{n-1} \not\subseteq X$ we can find a simple submodule of $X/M_{n-1}$. But then $M_{n-1} \neq M$, so $M_{n-1} \neq M_n$ and as $M_n/M_{n-1}$ is the only simple submodule of $M/M_n$, we find $M_n/M_{n-1} \subseteq X/M_{n-1}$, that is $M_n \subseteq X$, a contradiction.

(i)\(\Rightarrow\)(v) If $M_{n+1}/M_n$ is nonzero and it is not simple then we can find $S_1 = X_1/M_n$ and $S_2 = X_2/M_n$ ($X_1, X_2 \subseteq M$) two distinct simple modules contained in $M/M_n$. Then $X_1 \cap X_2 = M_n$, $X_1 \neq M_n$ and $X_2 \neq M_n$. But this shows that neither $X_1 \subseteq X_2$ nor $X_2 \subseteq X_1$ which is a contradiction. $\square$

The following result shows that chain coalgebra is a left-right symmetric notion and it also characterizes chain coalgebras.

**Proposition 3.2.3** The following assertions are equivalent for a coalgebra $C$:

(i) $C$ is a right chain coalgebra.

(ii) $C_{n+1}/C_n$ is either 0 or a simple right comodule for all $n \geq -1$. 

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(iii) \( C_n \) for \( n \geq -1 \) and \( C \) are the only right subcomodules of \( C \).
(iv) \( J^n \) for \( n \geq 0 \) and 0 are the only right ideals of \( C^* \).
(v) \( C^* \) is a right (or left) uniserial ring (chain algebra).
(vi) The left comodule version of (i)-(iv).
(vii) \( C_1 \) has length less than or equal to 2.

**Proof.** The equivalence of (i)-(vi) follows from Proposition 3.2.2 and Corollary 3.1.4 (i)\( \Rightarrow \) (vii) is obvious and (vii)\( \Rightarrow \) (i) is a result from [Cu]. We note a direct argument for this case: it is enough to deal with the case when \( C_1 \) has length 2; by induction, assume \( C_k/C_{k-1} \) is simple or 0 for \( k \leq n \). Assume \( C_n \neq C_{n-1} \) and note that since \( C_n/C_{n-1} \) is the socle of \( C/C_{n-1} \), then \( C/C_{n-1} \) embeds in \( C \) and therefore \( C_{n+1}/C_{n-1} \) has length at most 2, since it embeds in \( C_1 \). Thus \( C_{n+1}/C_n \) is simple or 0. \( \square \)

**Remark 3.2.4** The above Proposition includes many of the results in [LS] sections 5.1-5.3. By Proposition 3.2.2 a chain module is almost finite and by 3.2.3 a chain coalgebra is left and right almost finite, so the results of the first section apply here. Therefore we also obtain that a chain coalgebra is coreflexive.

Next we show that a chain coalgebra is both a left and a right f.g. \( \text{Rat} \)-splitting property coalgebra. Although this follows in a more general setting as in Section 4, we also provide a direct proof that does not involve the tools used in there, but makes use of the interesting fact that for a left almost finite coalgebra \( C \) and any left \( C^* \)-comodule \( M \), \( T(M) \) is a submodule of \( M \) and is exactly the rational submodule of \( M \).

**Theorem 3.2.5** If \( C \) is a chain coalgebra, then \( C \) has the left and right f.g. \( \text{Rat} \)-splitting property.

**Proof.** Of course, we only need to consider the case when \( C \) is infinite dimensional. First notice that every torsion-free \( C^* \)-finitely generated module \( M \) is free: indeed if \( x_1, \ldots, x_n \) is a minimal system of generators, then if \( \lambda_1 x_1 + \cdots + \lambda_n x_n = 0 \) with \( \lambda_i \) not all zero, we may assume that \( \lambda_1 \neq 0 \). Without loss of generality we may also assume that \( \lambda_i C^* \supseteq \lambda_i C^* \), \( \forall i \) as any two ideals of \( C^* \) are comparable by Proposition 3.2.3. Therefore we have \( \lambda_i = \lambda_1 s_i \) for some \( s_i \in C^* \). Then \( \lambda_1 x_1 + \lambda_1 s_2 x_2 + \cdots + \lambda_1 s_n x_n = 0 \) implies \( x_1 + s_2 x_2 + \cdots + s_n x_n = 0 \) as \( M \) is torsionfree and \( \lambda_1 \neq 0 \). Hence \( x_1 \in C^* < x_2, \ldots, x_n > \), contradicting the minimality of \( n \).
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Now if $M$ is any left $C^*$-module and $T = T(M) = Rat(M)$ (by Proposition 3.1.5) then $T(M/T(M)) = 0$. Indeed take $\hat{x} \in T(M/T(M))$ and put $I = \text{ann}_{C^*}\hat{x} \neq 0$ so $I$ has finite codimension and $I$ is a two-sided ideal by Proposition 3.2.3. By Corollary 3.1.4 and Remark 3.2.4, $I$ is finitely generated and therefore $Ix$ is also finitely generated. Also, since $I = \text{ann}_{C^*}\hat{x}$, we get $Ix \subseteq T = \text{Rat}(M)$. Thus $Ix$ is finitely generated rational, so $Ix$ has finite dimension. We obviously have an epimorphism $C^*/I \to C^*/Ix$ which shows that $C^*/Ix$ is finite dimensional because $I$ has finite codimension in $C^*$. Therefore we get that $\dim(C^*/x) = \dim(C^*/Ix) + \dim(Ix) < \infty$, so then by Proposition 3.1.5 we have that $C^*/I$ is rational. Thus $x \in T$, so $\hat{x} = 0$.

Now as $M/T$ is torsion-free, there are $x_1, \ldots, x_n \in M$ whose images $\hat{x}_1, \ldots, \hat{x}_n$ in $M/T$ form a basis. Then it is easy to see that $x_1, \ldots, x_n$ are linearly independent in $M$. Then if $X = C^*x_1 + \ldots + C^*x_n$ we have $X + T = M$ and $X \cap T = 0$, because, if $a_1x_1 + \ldots + a_nx_n \in T$ then $a_1\hat{x}_1 + \ldots + a_n\hat{x}_n = 0$, which implies $a_i = 0$, $\forall i$ because $\hat{x}_1, \ldots, \hat{x}_n$ are independent in $M/T$. Thus $T(M)$ splits off in $M$ and the theorem is proved, as $T(M) = \text{Rat}_R(M)$ by 3.1.5.

We will denote by $K_n$ the coalgebra with a basis $\{c_0, c_1, \ldots, c_{n-1}\}$ and comultiplication $c_k \mapsto \sum_{i+j=k} c_i \otimes c_j$ and counit $\varepsilon(c_i) = \delta_{0,i}$. The coalgebra $\bigcup_{n \in \mathbb{N}} K_n$ has a basis $c_n, n \in \mathbb{N}$ and comultiplication and counit given by these equations. It is called the divided power coalgebra (see [DNR]). Part of the following Lemma is discussed in [CGT] Theorem 3.2; also part of it in the cocommutative case is observed in [Cu], 3.5 and 3.6. The same result appears in [LS], but with a different proof. Also Theorem 3.2.7 below can be obtained as a consequence of the general theory of serial coalgebras developed in [CGT] (Theorem 2.10 (iii) and Remark 2.12); in this respect, Lemma 3.2.6 could then be obtained as a consequence of Theorem 3.2.7. We provide here a direct argument. First, we recall that a coalgebra $C$ is called pointed if every simple $C$-comodule is 1-dimensional.

Lemma 3.2.6 Let $C$ be a finite dimensional chain coalgebra over a field $K$ and suppose that either $K$ is algebraically closed or $C$ is pointed. Then $C$ is isomorphic to $K_n$ for some $n \in \mathbb{N}$.

Proof. Let $A = C^*$; we have $\dim C_0 = 1$ because $K$ is algebraically closed (thus $\text{End}_A C_0$ is a skewfield containing $K$). Thus $\dim C_k = k$ for all $k$ for which $C_k \neq C$. As $C^*$ is finite dimensional $J^n = 0$ for some $n$ and let $n$ be minimal with this property. By Corollary 3.1.4 $J^k = C^\perp_{k-1}$. Then $J^k/J^{k+1}$ has dimension equal to the dimension of $C_k/C_{k-1}$ which is 1 for
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$k < n$, because $C_{k+1}/C_k$ is a simple comodule isomorphic to $C_0$. We then have that $J^k/J^{k+1}$ is generated by any of its nonzero elements. Choose $x \in J \setminus J^2$. We prove that $x^{n-1} \neq 0$. Suppose the contrary holds and take $y_1, \ldots, y_{n-1} \in J$. As $x$ generates $J/J^2$, there is $\lambda \in K$ such that $y_1 - \lambda x \in J$ and then $y_1x^{n-2} - \lambda x^{n-1} \in J^n$, so $y_1x^{n-2} \in J^n = 0$ because $x^{n-1} = 0$. Again, there is $\mu \in K$ such that $y_2 - \mu x \in J$ and then $y_1y_2 - \mu y_1 x \in J^3$ so $y_1y_2x^{n-3} \in J^n$ since $y_1x^{n-2} = 0$. By continuing this procedure, one gets that $y_1y_2 \ldots y_{n-2}x = 0$ and then we again find $\alpha \in K$ with $y_{n-1} - \alpha x \in J^2$, thus $y_1 \ldots y_{n-1} - \alpha y_1 \ldots y_{n-2} x \in J^n = 0$. This shows that $y_1 \ldots y_{n-1} = 0$ for all $y_1, \ldots, y_{n-1} \in J$. Thus $J^{n-1} = 0$, a contradiction.

As $x^{n-1} \neq 0$ we see that $x^k \in J^k \setminus J^{k+1}$ for all $k = 0, \ldots, n-1$, so $J^k/J^{k+1}$ is generated by the image of $x^k$. Now if $y \in A$, there is $\lambda_0 \in K$ such that $y - \lambda_0 \cdot 1_A \in J$ (either $y \in J$ or $y$ generates $A/J$). As $J/J^2$ is 1 dimensional and generated by the image of $x$, there is $\lambda_1 \in K$ such that $y - \lambda_0 - \lambda_1 x \in J^2$. Again, as $J^2/J^3$ is 1 dimensional generated by the image of $x^2$, there is $\lambda_2 \in K$ such that $y - \lambda_0 - \lambda_1 x - \lambda_2 x^2 \in J^3$. By continuing this procedure we find $\lambda_0, \ldots, \lambda_{n-1} \in K$ such that $y - \lambda_0 - \lambda_1 x - \ldots - \lambda_{n-1} x^{n-1} \in J^n = 0$, so $y = \lambda_0 + \lambda_1 x + \ldots + \lambda_{n-1} x^{n-1}$. This obviously gives an isomorphism between $A$ and $K[X]/(X^n)$. Therefore $C$ is isomorphic to $K_n$, because there is an isomorphism of $K$-algebras $K_n^* \simeq K[X]/(X^n)$.

**Theorem 3.2.7** If $K$ is an algebraically closed field and $C$ is an infinite dimensional chain coalgebra, then $C$ is isomorphic to the divided power coalgebra. The same conclusion holds for arbitrary $K$ provided the infinite dimensional chain coalgebra $C$ is pointed.

**Proof.** By the previous Lemma we have that $C_n \simeq K_n$ for all $n$. If $e \in C_0$, $\Delta(e) = \lambda e \otimes e$, $\lambda \in K$, then for $c_0 = \lambda e$ we get $\Delta(c_0) = c_0 \otimes c_0$. Suppose we constructed a basis $c_0, c_1, \ldots, c_{n-1}$ for $C_{n-1}$ with $\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j$, $\varepsilon(c_i) = \delta_{0,i}$. Denote by $A_n = C_n^*$ the dual of $C_n$; for the rest of this proof, if $V \subseteq C_n$ is a subspace of $C_n$ we write $V^\perp$ for the set of the functions on $A_n$ which are 0 on $V$. Choose $E_1 \subseteq C_0^\perp \setminus C_1^\perp$; then $E_1^n \neq 0$ and $E_1^{n+1} = 0$, as in the proof of Lemma 3.2.6 ($E_1 \in A_n$). This shows that $E_1^k \subset C_{k-1}^\perp \setminus C_k^\perp$, that $\varepsilon|_{C_n}, E_1, \ldots, E_1^n$ exhibits a basis for $A_n$ and that there is an isomorphism of algebras $A_n \simeq K[X]/(X^{n+1})$ taking $E_1$ to $X$. We can easily see that $E_1^n(c_j) = \delta_{ij}$, $\forall i, j = 0, 1, \ldots, n - 1$ and then by a standard linear algebra result we can find $c_n \in C_n$ such that $E_1^n(c_n) = 1$ and $E_1^n(c_i) = 0$ for $i < n$. Then by dualization, the relations $E_1^i(c_j) = \delta_{ij}$, for all $i, j = 0, 1, \ldots, n$ become $\Delta(c_k) = \sum_{i+j=k} c_i \otimes c_j$, $\forall k = 0, 1, \ldots, n$. Therefore we may inductively build the basis $(c_n)_{n \in \mathbb{N}}$ with $\varepsilon(c_k) = \delta_{0,k}$ and $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$, $\forall n$. \hfill \Box
A non-trivial example

In the following we construct an example of a chain coalgebra that is not cocommutative and thus is not the divided power coalgebra over $K$. Recall that if $A$ is a $k$-algebra, $\varphi : A \to A$ is a morphism and $\delta : A \to A$ is a $\varphi$-derivation (that is a linear map such that $\delta(ab) = \delta(a)b + \varphi(a)\delta(b)$ for all $a, b \in A$), we may consider the Ore extension $A[X, \varphi, \delta]$, which is $A[X]$ as a vector space and with multiplication induced by $Xa = \varphi(a)X + \delta(a)$. Let $K$ be a subfield of $\mathbb{R}$, the field of real numbers. Let $\mathbb{H}_K$ be the $K$-subalgebra of Hamilton’s quaternion algebra $\mathbb{H}$ having the set $B = \{1, i, j, k\}$ as a vector space basis over $K$. Recall that multiplication is given by the rules $i \cdot j = -j \cdot i = k; j \cdot k = -k \cdot j = i; k \cdot i = -i \cdot k = j; i^2 = j^2 = k^2 = -1$. Denote by $\sigma : \mathbb{H}_K \to \mathbb{H}_K$ the linear map defined on the basis of $\mathbb{H}_K$ by

$$\sigma = \begin{pmatrix} 1 & i & j & k \\ 1 & j & k & i \end{pmatrix}$$

It is not difficult to see then that $\sigma$ is an algebra automorphism, and that $\mathbb{H}_K$ is a division algebra (skewfield). Our example will be constructed with the aid of an Ore extension constructed with a trivial derivation: denote by $\mathbb{H}_{K,\sigma}[X] = \mathbb{H}_K[X, \sigma, 0]$ the Ore extension of $\mathbb{H}_K$ constructed by $\sigma$ with the derivation $\delta$ equal to 0 everywhere. Then a basis for $\mathbb{H}_{K,\sigma}[X]$ over $K$ consists of the elements $uX^k$, with $u \in B$ and $k \in \mathbb{N}$. Also denote by $A_n = \mathbb{H}_{K,\sigma}[X]/<X^n>$ the algebra obtained by factoring out the two-sided ideal generated by $X^n$ from $\mathbb{H}_{K,\sigma}[X]$.

**Proposition 3.2.8** The two sided ideal $<X^n>$ of $\mathbb{H}_{K,\sigma}[X]$ consists of elements of the form $f = \sum_{l=0}^{n+m} a_lX^l$. Moreover, the only (left, right, two-sided) ideals containing $<X^n>$ are the ideals $<X^l>$, for $l = 0, \ldots, n$ and consequently $A_n$ is a chain $K$-algebra.

**Proof.** It is clear by the multiplication rule $Xa = \sigma(a)X$ for $a \in B$ that elements of $H_{K,\sigma}[X]$ are of the type $\sum_{l=0}^{N} a_lX^l$ and that every element of $A_n$ is a “polynomial” of the form $f = a_0 + a_1x + \ldots + a_{n-1}x^{n-1}$, with $a_l \in \mathbb{H}_K$ and where $x$ represents the class of $X$. Such an element $f$ is invertible if and only if $a_0 \neq 0$. To see this, first note that if $a_0 = 0$ then $f$ is nilpotent, as $x$ is nilpotent and one has $f^l \in <x^l>$ by successively using the relation $xa = \sigma(a)x$. Conversely write $f = a_0 \cdot (1 + a_0^{-1}a_1x + \ldots + a_0^{-1}a_{n-1}x^{n-1})$ and note that the element $g = a_0^{-1}a_1x + \ldots + a_0^{-1}a_{n-1}x^{n-1}$ is nilpotent as before, so $1 + g$ must be invertible in $A_n$ and therefore $f$ must be invertible. Thus we may write every element
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if \( f = a_i x^i + \ldots + a_{n-1} x^{n-1} \) of \( A_n \) as the product \( f = (a_i + a_{i+1} x + \ldots + a_{n-1} x^{n-1-i}) \cdot x^l = g \cdot x^l \) with invertible \( g \). Then if \( I \) is a left ideal of \( A_n \) and \( f \in I \), we have \( f = g \cdot x^l \) for an invertible element \( g \) and some \( l \leq n \). Hence it follows that \( x^l \in I \). Taking the smallest number \( l \) with the property \( x^l \in I \), we obviously have that \( I = \langle x^l \rangle \). \( \square \)

Let \( C_n \) denote the coalgebra dual to \( A_n \). Note that \( A_n \) has a \( K \) basis \( B = \{ax^i \mid a \in B, l \in 0,1,\ldots, n-1 \} \) and we have the relations \((ax^r)(bx^s) = a\sigma^r(b)x^{r+s}\). Let \((E^a_r)_{a \in B, r \in \overline{0,n-1}}\) be the basis of \( C_n \) which is dual to \( B \), that is, \( E^a_r(bx^s) = \delta_{rs}\delta_{ab} \) for all \( a, b \in B \) and \( r, s \in \mathbb{N} \). Also, for \( r \in \mathbb{N} \) and \( a \in B \) denote by \( r \cdot a = \sigma^r(a) \) the action of \( \mathbb{N} \) on \( B \) induced by \( \sigma \).

**Proposition 3.2.9** With the above notations, denoting by \( \Delta_n \) and \( \varepsilon_n \) the comultiplication and, respectively, the counit of \( C_n \) we have

\[
\Delta_n(E^c_p) = \sum_{r+s=p; a(r-b)=\pm c} c^{-1} a(r \cdot b) E^a_r \otimes E^b_s
\]

and

\[
\varepsilon_n(E^c_p) = \delta_{p,0}\delta_{c,1}.
\]

**Proof.** For \( u, v \in B \) and \( t, l \in \mathbb{N} \) we have \( E^c_p(ux^t \cdot vx^l) = E^c_p(u(t \cdot v)x^{t+l}) \) and as \( t \cdot v \in B \) by the formulas defining \( \mathbb{H}_K \) we have that if \( d = u(t \cdot v) \) then either \( d \in B \) or \( -d \in B \). Then \( E^c_p(ux^t \cdot vx^l) = E^c_p(dx^{t+l}) = \delta_{t+l,p}\delta_{u(t \cdot v), \pm c} c^{-1} u(t \cdot v) \) as the sign of this expression must be 1 if \( d \in B \) and \( -1 \) if \( d \notin B \), and this is exactly \( c^{-1} u(t \cdot v) \) when \( u(t \cdot v) = \pm c \). We also have

\[
\sum_{r+s=p; a(r-b)=\pm c} c^{-1} a(r \cdot b) E^a_r(ux^t) E^b_s(vx^l) = \sum_{r+s=p; a(r-b)=\pm c} \delta_{t,s}\delta_{u,a}\delta_{t,a}\delta_{v,b} c^{-1} a(r \cdot b) = \delta_{t+l,p}\delta_{u(t \cdot v), \pm c} c^{-1} u(t \cdot v)
\]

and therefore we get

\[
\sum_{r+s=p; a(r-b)=\pm c} c^{-1} a(r \cdot b) E^a_r(ux^t) E^b_s(vx^l) = E^c_p(ux^t \cdot vx^l)
\]

As this is true for all \( ux^t, vx^l \in B \), by the definition of the comultiplication of the coalgebra dual to an algebra, we get the first equality in the statement of the proposition. The second one is obvious, as \( \varepsilon_n(E^c_p) = E^c_p(1 \cdot X^0) = \delta_{p,0}\delta_{c,1} \). \( \square \)
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Now notice that there is an injective map $C_n \subset C_{n+1}$ taking $E^c_n$ from $C_n$ to $E^c_n$ from $C_{n+1}$. Therefore we can regard $C_n$ as subcoalgebra of $C_{n+1}$. Denote by $C = \bigcup_{n \in \mathbb{N}} C_n$; it has a basis formed by the elements $E^c_n$, $n \in \mathbb{N}$, $c \in B$ and comultiplication $\Delta$ and counit $\varepsilon$ given by

$$\Delta(E^c_n) = \sum_{r+s=n; \ a(r \cdot b) = \pm c} c^{-1} a(r \cdot b) E^a_r \otimes E^b_s$$

and

$$\varepsilon(E^c_n) = \delta_{n,0} \delta_{c,1}.$$ 

By Proposition 3.2.8 we have that $A_n$ is a chain algebra and therefore $C_n = A^*_n$ is a chain coalgebra. Therefore, we get that the coradical filtration of $C$ is $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ and that this is a chain coalgebra which is obviously non-cocommutative.

3.3 The co-local case

Throughout this section we will assume (unless otherwise specified) that $C$ has the left f.g. Rat-splitting property and that it is a colocal coalgebra, that is, $C_0$ is a simple left (and consequently simple right) $C^*$-module. Then $C^*$ is a local algebra since $J = C^*_0$. We will also assume that $C$ is not finite dimensional. Thus, by Proposition 3.1.10, $C$ has a countable basis. We have that $C$ is the injective envelope of $C_0$ as left comodules, thus, by Proposition 3.1.7, we have that every proper left subcomodule of $C$ is finite dimensional. This implies that the $C_n$’s are finite dimensional too. Then if $I$ is a left nonzero ideal of $C^*$ different from $C^*$, Corollary 3.1.4 implies that $I$ is finitely generated and of finite codimension. Also for a (left) module $M$ over a ring $R$ denote by $J(M)$ the Jacobson radical of $M$.

Proposition 3.3.1 With the above notations, $C^*$ is a domain.

Proof. Let $S = \text{End}(C, C)$. Note that $S$ is a ring with multiplication equal to the composition of morphisms and that $S$ is isomorphic to $C^*$ by an isomorphism that takes every morphism of left $C$-comodules $f \in S$ to the element $\varepsilon \circ f \in C^*$. Then it is enough to show that $S$ is a domain. If $f : C \to C$ is a nonzero morphism of left $C$ comodules, then $\text{Ker}(f) \subsetneq C$ is a proper left subcomodule of $C$ so it must be finite dimensional. Then as $C$ is not finite dimensional we see that $\text{Im}(f) \cong C/\text{Ker}(f)$ is an infinite dimensional subcomodule of $C$. Thus $\text{Im}(f) = C$, and therefore every nonzero morphism of left comodules
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from $C$ to $C$ must be surjective. Now if $f, g \in S$ are nonzero then they are surjective so $f \circ g$ is surjective and thus $f \circ g \neq 0$. □

Proposition 3.3.2 $C^*$ satisfies ACC on principal right ideals and also on left ideals.

**Proof.** Suppose there is an ascending chain of right ideals $x_0 \cdot C^* \subsetneq x_1 \cdot C^* \subsetneq x_2 \cdot C^* \subsetneq \ldots$ that does not stabilize. Then there are $(\lambda_n)_{n \in \mathbb{N}}$ in $C^*$ such that $x_n = x_{n+1} \cdot \lambda_{n+1}$. Note that $\lambda_{n+1} \in J$, because otherwise $\lambda_{n+1}$ would be invertible in $C^*$, as $C^*$ is local, and then we would have $x_{n+1} = x_n \cdot \lambda_{n+1}^{-1}$. This would yield $x_n \cdot C^* = x_{n+1} \cdot C^*$, a contradiction. Then $x_1 = x_{n+1} \cdot \lambda_{n+1} \lambda_n \ldots \lambda_2$, so $x_1 \in J^n$ for all $n \in \mathbb{N}$, showing that $x_1 \in \bigcap_{n \in \mathbb{N}} J^n = 0$. Thus we obtain a contradiction: $x_0 \cdot C^* \subsetneq x_1 \cdot C^* = 0$. The statement is obvious for left ideals as $C \cdot C^*$ is Noetherian. □

The next proposition together with the following theorem contain the main ideas of the result.

Proposition 3.3.3 Suppose $\alpha C^*$ and $\beta C^*$ are two right ideals that are not comparable, i.e. neither one is a subset of the other. Then any two principal right ideals of $C^*$ contained in $\alpha C^* \cap \beta C^*$ are comparable.

**Proof.** Take $aC^*, bC^* \subseteq \alpha C^* \cap \beta C^*$, so $a = \alpha x = \beta y$ and $b = \alpha u = \beta v$. We may assume that $a, b \neq 0$ as otherwise the assertion is obvious. Then $\alpha, \beta, x, y, u, v$ are nonzero. Denote by $L$ the left submodule of $C^* \times C^*$ generated by $(x, u)$ and by $M$ the quotient module $C^* \times C^*/L$. We write $(s, t)$ for the image of the element $(s, t)$ through the canonical projection $\pi : C^* \times C^* \rightarrow M$. We have $(y, v) \neq (0, 0)$. Otherwise $(y, v) = \lambda (x, u)$ for some $\lambda \in C^*$, which implies $\alpha x = \beta y = \beta \lambda x$ and then $\beta \lambda = \alpha$ (because $C^*$ is a domain), a contradiction to $\alpha C^* \not\subseteq \beta C^*$. Also $\beta \cdot (y, v) = \alpha \cdot (x, u) = (0, 0)$. This shows that $(0, 0) \neq (y, v) \in T = T(M)$, so $T(M) \neq 0$. Take $X < M$ such that $M = T \oplus X$. We must have $X \neq 0$, as otherwise $(1, 0) \in T$, so there would be a nonzero $\lambda \in C^*$ and a $\mu \in C^*$ such that $\lambda \cdot (1, 0) = \mu \cdot (x, u) \in L$. But then $0 = \mu u$ implies $\mu = 0$, since $u \neq 0$, and we find $\lambda = \mu x = 0$, a contradiction.

Now note that $x$ is not invertible, since, otherwise, $\alpha x = \beta y$ implies $\alpha \in \beta C^*$ so $\alpha C^* \subseteq \beta C^*$. Likewise, $u$ is not invertible. Therefore, $x, u \in J$ since $C^*$ is local, and so $L \subseteq J \times J$. Hence $J(M) = J \times J/L$ so $M/J(M) = (C^* \times C^*/L)/(J \times J/L) \simeq (C^* \times C^*)/(J \times J)$ which has
dimension 2 as a module over the skewfield $C^*/J$. Since $M = T \oplus X$ and $M$ is finitely generated, so are $T$ and $X$ and therefore $J(X) \neq X$ and $J(T) \neq T$. Then as

$$\frac{M}{J(M)} = \frac{T}{J(T)} \oplus \frac{X}{J(X)}$$

has dimension 2 over $C^*/J$, it follows that both $T/J(T)$ and $X/J(X)$ are simple. Hence $T$ and $X$ are local, and, since they are finitely generated, it follows that each is generated by any element not belonging to its Jacobson radical. Let $T'$ (respectively $X'$) be the inverse images of $T$ (and $X$ respectively) in $C^* \times C^*$ and $t \in T'$ and $s \in X'$ be such that $C^*t + L = T'$ and $C^*s + L = X'$. We have $C^* \times C^* = T' + X' = C^*t + L + C^*s + L = (C^*t + C^*s) + L \subseteq (C^*t + C^*s) + J \subseteq C^* \times C^*$ so $(C^*t + C^*s) + J \times J = C^* \times C^*$. Therefore we obtain $C^*t + C^*s = C^* \times C^*$ because $J \times J$ is small in $C^* \times C^*$.

Write $t = (p, q) \in T'$. Then $\overline{t} = t + L \in T$ implies that there is $\lambda \neq 0$ in $C^*$ such that $\lambda \overline{t} = \overline{0} \in M$ and therefore there is $\mu \in C^*$ with $\lambda(p, q) = \mu(x, u)$. We show that either $p \notin J$ or $q \notin J$. Indeed assume otherwise: $t = (p, q) \in J \times J$. Then we get $C^*t \subseteq J \times J$. Because $C^*t + C^*s = C^* \times C^*$ we see that $(C^* \times C^*)/(J \times J)$ must be generated over $C^*$ by the image of $s$. This shows that the $C^*/J$ module $(C^* \times C^*)/(J \times J) = (C^*/J)^2$ has dimension 1 and this is obviously a contradiction.

Finally, suppose $p \notin J$, so $p$ is invertible. Then the equations $\lambda p = \mu x$ and $\lambda q = \mu u$ imply $\lambda = \mu xp^{-1}$ and $\mu xp^{-1}q = \mu u$. But $\mu \neq 0$ because $p$ is invertible and $\lambda \neq 0$. Therefore we obtain $u = xp^{-1}q$. Thus $b = \alpha u = \alpha xp^{-1}q = ap^{-1}q$, showing that $b \in aC^*$, i.e. $bC^* \subseteq aC^*$. Similarly if $q$ is invertible, we get $aC^* \subseteq bC^*$.

**Theorem 3.3.4** If $C$ is an (infinite dimensional) local coalgebra with the left f.g. Rat-splitting property, then $C$ is a chain coalgebra.

**Proof.** We first show that every pair of principal left ideals of $C^*$ are comparable. Suppose that $C^* \cdot x_0$ and $C^* \cdot y_0$ that are not comparable. Then, as they have finite codimension and $C^*$ is infinite dimensional, we have $C^* x_0 \cap C^* y_0 \neq 0$ and take $0 \neq \alpha x_0 = \beta y_0 \in C^* x_0 \cap C^* y_0$. Then the right ideals $\alpha C^*$ and $\beta C^*$ are not comparable, as otherwise, if for example $\alpha C^* \subseteq \beta C^*$, we would have a relation $\alpha = \beta \lambda$, so $\alpha x_0 = \beta \lambda x_0 = \beta y_0$. As $\beta \neq 0$ we get $\lambda x_0 = y_0$ because $C^*$ is a domain, and then $C^* y_0 \subseteq C^* x_0$, a contradiction.

By Proposition 3.3.2 the set $\{\lambda C^* \mid \lambda C^* \subseteq \alpha C^* \cap \beta C^*\}$ is Noetherian (relative to inclusion). Let $\lambda C^*$ be a maximal element. If $x \in \alpha C^* \cap \beta C^*$ then by Proposition 3.3.3
we have that \( xC^* \) and \( \lambda C^* \) are comparable and by the maximality of \( \lambda C^* \) it follows that \( xC^* \subseteq \lambda C^* \), so \( x \in \lambda C^* \). Therefore \( \alpha C^* \cap \beta C^* = \lambda C^* \). Note that \( \lambda \neq 0 \), because \( \alpha C^* \) and \( \beta C^* \) are nonzero ideals of finite codimension. Then we see that \( \lambda C^* \cong C^* \) as right \( C^* \)-modules, because \( C^* \) is a domain, and again by Proposition 3.3.3 any two principal right ideals of \( \lambda C^* = \alpha C^* \cap \beta C^* \) are comparable, so the same must hold in \( C^*_C \). But this is in contradiction with the fact that \( \alpha C^* \) and \( \beta C^* \) are not comparable, and therefore the initial assertion is proved.

Now we prove that \( J^n/J^{n+1} \) is a simple right module for all \( n \). As \( C^*/J \) is semisimple (it is a skewfield) and \( J^n/J^{n+1} \) has an \( C^*/J \) module structure, it follows that \( J^n/J^{n+1} \) is a semisimple left \( C^*/J \)-module and then \( J^n/J^{n+1} \) is semisimple also as \( C^* \)-module. If we assume that it is not simple, then there are \( f, g \in J^n \setminus J^{n+1} \) such that \( C^* \hat{f} = (C^* f + J^{n+1})/J^{n+1} \) and \( C^* \hat{g} = (C^* g + J^{n+1})/J^{n+1} \) are different simple \( C^* \)-modules, so \( C^* \hat{f} \cap C^* \hat{g} = \hat{0} \) in \( J^n/J^{n+1} \). Then \( (C^* f + J^{n+1}) \cap (C^* g + J^{n+1}) = J^{n+1} \) which shows that \( C^* f \) and \( C^* g \) cannot be comparable, a contradiction. As \( J^n = C^n_{n-1} \), we see that \( \dim(C_{n-1}) = \text{codim}(J^n) \). Then for \( n \geq 1 \), we have \( \dim(C_n/C_{n-1}) = \dim(C_n) - \dim(C_{n-1}) = \text{codim}_{C^*}(J^n) - \text{codim}_{C^*}(J^{n+1}) = \dim(J^n/J^{n+1}) = \dim(C_0) \). Because \( C_0 \) is the only type of simple right \( C \)-comodule, this last relation shows that the right \( C \)-comodule \( C_n/C_{n-1} \) must be simple. Therefore \( C \) must be a chain coalgebra.

We may now combine the results of Sections 2 and 3 and obtain

**Corollary 3.3.5** Let \( C \) be a co-local (infinite dimensional) coalgebra. Then \( C \) is a left (right) finite splitting coalgebra if and only if \( C \) is a chain coalgebra. Moreover, if the base field \( K \) is algebraically closed or the coalgebra \( C \) is pointed, then this is further equivalent to the fact that \( C \) is isomorphic to the divided power coalgebra.

**Proof.** This follows from Theorems 3.2.5, 3.2.7 and 3.3.4.

### 3.4 Serial coalgebras and General Examples

In this section we provide some nontrivial general examples of non-colocal coalgebras for which this splitting property holds.

**Lemma 3.4.1** Let \( C = D \oplus E \) be coproduct of two coalgebras \( D \) and \( E \). Then \( C \) has the left f.g. Rat-splitting property if and only if \( D \) and \( E \) have the Rat-splitting property.
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Proof. Assume $C$ has the left f.g. Rat-splitting property. It is well known that the category of modules over $C^* \simeq D^* \times E^*$ is isomorphic to the product of the category of $D^*$-modules with that of $E^*$-modules. In this respect, if $M$ is a left $C^*$-module, then $M = N \oplus P$ where $N = E^\perp \cdot M$ and $P = D^\perp \cdot M$ are $C^*$ submodules that have an induced $D^* = C^*/D^\perp$- and respectively $E^* = C^*/E^\perp$-module structure, since $D^\perp \cdot N = 0 = E^\perp \cdot P$. Also, one can check that a $D^*$-module $X$ is rational if and only if it is rational as $C^*$-module with its induced $C^*$-module structure: if $\rho : X \to X \otimes C$ is a $C$-comultiplication then we must have $\rho(X) \subseteq X \otimes D$ since $D^\perp$ cancels $X$, and $\rho$ becomes a $D$-comultiplication. Indeed, if $\rho(x) = \sum_i x_i \otimes y_i + \sum_j x'_j \otimes y'_j$ with $x_i, x'_j \in X$ assumed linearly independent, $y_i \in D$ and $y'_j \in E$, then for any $e^* \in C^*$ such that $e^*|_D = 0$, we have $0 = e^* \cdot x = \sum_j e^*(y'_j)x'_j$, so $e^*(y'_j) = 0$ by linearly independence. This shows that $x'_j \in (D^\perp)^\perp = D$ so $x'_j = 0$ for all $j$. Thus, we obtain that $Rat(D^*N) = Rat(C^*N)$ and $Rat(E^*P) = Rat(C^*P)$, and we have direct sums $N = Rat(N) \oplus N'$ and $P = Rat(P) \oplus P'$ in $D^* - \text{mod}$ and $E^* - \text{mod}$. But $N'$ and $P'$ also have an induced $C^*$-module structure with $E^* = D^\perp$ acting as 0, and we finally observe that this yields a direct sum of $C^*$ modules $M = Rat(C^*N) \oplus N' \oplus Rat(C^*P) \oplus P' = Rat(C^*M) \oplus (N' \oplus P')$.

The other implication follows from Proposition 3.1.9. \qed

We note now the following proposition which was also proved in [Cu], but with techniques involving general results of M. Teply from [T1] and [T3].

Proposition 3.4.2 Assume $C$ is a cocommutative coalgebra. Then $C$ is a f.g. Rat-splitting coalgebra if and only if it is a finite coproduct of finite dimensional coalgebras and infinite dimensional chain coalgebras. Moreover, these chain coalgebras are isomorphic to the divided power coalgebra in any of the cases:

(i) the base field is algebraically closed;

(ii) $C$ is pointed.

Proof. Since $C$ is cocommutative, $C = \bigoplus_{i=1}^n C_i$, where $C_i$ are colocal subcoalgebras of $C$. Now each of the $C_i$ must have the splitting property for finitely generated modules by Proposition 3.1.9, and therefore they must be either finite dimensional or be chain coalgebras. The converse follows from the previous Lemma and the results of Section 2. The final assertion comes from Theorem 3.2.7. \qed

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Recall, for example from [F1], 25.1.12 that a module $M$ is called a serial module if it is a direct sum of uniserial (chain) modules; a ring $R$ is said to be a left (right) serial ring if $R$ is serial when regarded as left (right) $R$-module, and a serial ring when $R$ is both left and right serial. In analogy to these definitions, for a $C$-comodule $M$ we say that $M$ is a serial comodule if it is a direct sum of uniserial -or chain- comodules. A coalgebra will be called a left (right) serial coalgebra if and only if it is a serial right (left) $C^*$-module, i.e. it is a serial left (right) $C$-comodule, and a serial coalgebra if it is both left and right serial. These definitions coincide with those in [CGT]. We note at this point that, in our definitions, a uniserial coalgebra is the same as a chain coalgebra, while a uniserial coalgebra in [CGT] is understood as a homogeneous uniserial coalgebra, that is, a coalgebra $C$ that is serial and the composition factors of each indecomposable injective comodule are isomorphic (see Definition 1.3 [CGT]). The following is a generalization of Proposition 1.6, [CGT].

Proposition 3.4.3 Let $C$ be a coalgebra. Then the following are equivalent:

(i) $C$ is a right serial coalgebra and $C_0$ is finite dimensional.

(ii) $C^*$ is a right serial algebra.

Consequently $C^*$ is serial if an only if $C$ is serial and $C_0$ is finite dimensional, equivalently, $C$ is serial and $C^*$ is semilocal.

Proof. (i)$\Rightarrow$(ii) Let $C_0 = \bigoplus_{i=1}^{k} S_i$ be a decomposition of $C_0$ into simple right comodules, and let $E(S_i)$ be an injective envelope of $S_i$ contained in $C$. Then $C = \bigoplus_{i=1}^{k} E(S_i)$ in $\mathcal{M}^C$ and $C^* - mod$. Since any other decomposition of $C$ in $\mathcal{M}^C$ is equivalent to this one, we have that $E(S_i)$ are chain comodules and then $E(S_i)^*$ are chain modules by Proposition 3.2.2. As $C^* = \bigoplus_{i=1}^{n} E(S_i)^*$ in $mod - C^*$ we get that $C^*$ is right serial.

(ii)$\Rightarrow$(i) If $C^*$ is right serial, it is a direct sum of uniserial modules $C^* = \bigoplus_{i} M_i$, each of which has to be cyclic; then we easily see that these modules have to be local (for example by [F1], 25.4.1B) and indecomposable (a finitely generated local module is indecomposable). Since there can be only a finite number of $M_i$’s in a decomposition of $C^*$, and each of the $M_i$’s are local we get that $C^*$ is semilocal, and then $C^*/J$ is semisimple ($J = C_0^\perp$). But $C^*/J = C^*/C_0^\perp = C_0^*$ and thus $C_0$ is cosemisimple finite dimensional. Then $C^* = \bigoplus_{i=1}^{k} M_i$ with $M_i$ local uniserial. Let $E_i = (\bigoplus_{j \neq i} M_j)^\perp$; since $\bigoplus_{j \neq i} M_j$ is finitely generated, it is closed in
the finite topology of $C^*$ and therefore $E_i^\perp = \bigoplus M_j$, so $E_i^\ast \simeq C^*/E_i^\perp = C^*/(\bigoplus M_j) \simeq M_i$.

Then by Proposition 3.2.2 we get that $E_i$ is a right chain $C$-comodule; also because of the anti-isomorphism of lattices between the right subcomodules of $C$ and closed right $C^*$-modules of $C^*$ (see [DNR] or [I0], Theorem 1), we get that $C = \bigoplus_{i=1}^k E_i$, with $E_i$ right chain comodules. Thus $C$ is a left serial coalgebra.

We say that a coalgebra $C$ is **purely infinite dimensional serial** if it is serial and the uniserial left (and also the uniserial right) comodules into which it decomposes are infinite dimensional. Equivalently, one can say that the injective envelope of every left (and also every right) simple $C$-comodule is infinite dimensional. It is not difficult to see that for an almost connected coalgebra it is enough to ask only that left injective envelopes are infinite dimensional: let $C = \bigoplus_{i=1}^k E(S_i)$ be a decomposition of $C$ with $S_i$ simple left comodules and $E(S_i)$ an injective envelope for each $S_i$. Assume $C$ is serial; then each $E(S_i)$ is uniserial. Then writing $L_n E(S_i)$ for the $n$-th term in the Loewy series of $E(S_i)$, we have $C_n = \bigoplus_{i=1}^k L_n E(S_i)$ and $E(S_i)$ is infinite dimensional for all $i$ if and only if $L_{n-1} E(S_i) \neq L_n E(S_i)$ for all $i$ and all $n \geq 0$ ($L_{-1} = 0$). Equivalently, $C_n/C_{n-1} \simeq \bigoplus_{i=1}^k L_n E(S_i)/L_{n-1} E(S_i)$ has length $k$ (as a module) for all $n$. Since this last condition is a left-right symmetric condition, the assertion follows. The next proposition provides the general example of this section:

**Proposition 3.4.4** Let $C$ be a purely infinite dimensional serial coalgebra which is almost connected. Then $C$ has the left (and also the right) f.g. Rat-splitting property.

**Proof.** By the previous proposition, $C^*$ is serial. Let $M$ be a finitely generated left $C^*$-module. Let $C = \bigoplus_{i=1}^k E(S_i)$ be a decomposition as above, in $C.M$, with all $E(S_i)$ chain comodules; then $C^* = \bigoplus_{i \in I} E(S_i)^*$ in $C^*-\text{mod}$. By Remark 3.2.4 and Proposition 3.1.3 each $E(S_i)^*$ is noetherian. Hence $C^*$ is Noetherian (both left and right, since $C$ is left and right serial). This shows that every finitely generated $C^*$-module is also finitely presented. Then, by [F1], Corollary 25.3.4, $M = \bigoplus_{j=1}^n M_j$ with $M_j$ cyclic uniserial left $C^*$-modules. For each $j$ there are two possibilities:

- $M_j$ is finite dimensional. Let $m_j$ be a generator of the left $C^*$-module $M_j$, and then let $I = \text{ann}_{C^*}(m_j)$. Then $I$ is a left ideal of $C^*$ and is finitely generated since $C^*$ is Noetherian, so $I = X^\perp$ for some $X \subseteq C$ by Lemma 3.1.1. Moreover, $C^*/I \simeq C^* \cdot m_j = M_j$ and so $I$
has finite codimension since $M_j$ is finite dimensional. Hence $X$ is finite dimensional and is a left submodule of $C$. Then $M_j \simeq C^*/X^\perp \simeq X^*$ and it follows that $M_j$ is rational as a dual of the rational right $C^*$-module $X$. So $\text{Rat}(M_j) = M_j$.

- $M_j$ is infinite dimensional. Let $m_j$ be a generator of $M_j$ as before, and $S = M_j/J(M_j)$. Then $S$ is a simple module because $M_j$ is local, since it is cyclic and uniserial. Let $P_i = E(S_i)^*$. Since $C^*/J = \bigoplus_{i=1}^{k} P_i/JP_i$ and all $P_i$ are local, there is some $i$ such that $P_i/JP_i \simeq S$.

Then we have a diagram

\[
\begin{array}{ccc}
P_i & \xrightarrow{p} & S \\
\downarrow{u} & & \downarrow{\pi} \\
M_j & \xrightarrow{\pi} & 0
\end{array}
\]

completed commutatively by $u$ since $P_i$ is projective, and $p, \pi$ are the canonical maps. Note that $u$ is surjective, since otherwise $\text{Im}(u) \subseteq \text{Ker}(\pi)$ because $\text{Ker}(\pi)$ is the only maximal submodule of the finitely generated module $M_i$. This cannot happen since $\pi u = p \neq 0$. By Remark 3.2.4 and Proposition 3.1.3, we see that any nonzero submodule of $P_i = E(S_i)^*$ has finite codimension. Then if $\text{Ker}(u) \neq 0$, it follows that $M_j = \text{Im}(u) \simeq P_i/\text{Ker}(u)$ would be finite dimensional, which is excluded by the hypothesis on $M_j$. This shows that $u$ is an isomorphism so $M_j \simeq E(S_i)^*$ and we now get that $M_j$ has no finite dimensional submodules besides 0, again by Remark 3.2.4 and Proposition 3.1.3. This shows that $\text{Rat}(M_j) = 0$.

Finally, if we set $\mathcal{F} = \{j \mid M_j \text{ finite dimensional}\}$, we see that $\text{Rat}(M) = \bigoplus_{j=1}^{n} \text{Rat}(M_j) = \bigoplus_{j \in \mathcal{F}} M_j$, and this shows that $\text{Rat}(M)$ is a direct summand in $M = \bigoplus_{j=1}^{n} M_j$. \qed

**Example 3.4.5** Let $K$ be a field, $q \geq 1$ and $\sigma \in S_q$ be a permutation of $\{1, 2, \ldots, q\}$. Denote by $K_q^*[X]$ the vector space with basis $x_{p,n}$ with $p \in \{1, 2, \ldots, q\}$ and $n \geq 0$. Define a comultiplication $\Delta$ and a counit $\varepsilon$ on $K_q^*[X]$ as follows:

\[
\Delta(x_{p,n}) = \sum_{i+j=n} x_{p,i} \otimes x_{\sigma^i(p),j}
\]
\[
\varepsilon(x_{p,n}) = \delta_{n,0}, \forall p \in \{1, 2, \ldots, q\}, n \geq 0
\]

It is easy to see that $\Delta$ is coassociative and $\varepsilon$ is a counit, so $K_q^*[X]$ becomes a coalgebra:

\[
(\Delta \otimes I)\Delta(x_{p,n}) = (\Delta \otimes I)(\sum_{i+j=n} x_{p,i} \otimes x_{\sigma^i(p),j})
\]
Let several steps:

\[ K \text{ is a right subcomodule of } E \]

Also, we have

\[ \sigma \]

This relation shows that the correspondence

\[ (I \otimes \Delta)(\sum_{s+u=n} x_{p,s} \otimes x_{\sigma^i(p),u}) = (I \otimes \Delta)(x_{p,n}) \]

Also, we have \[ \sum_{i+j=n} \varepsilon(x_{p,i})x_{\sigma^i(p),j} = \sum_{i=0}^{n} \delta_{i,0}x_{\sigma^i(p),n-i} = x_{p,n} \] and \[ \sum_{i+j=n} x_{p,i} \varepsilon(x_{\sigma^i(p),j}) = \sum_{j=0}^{n} x_{p,j} \delta_{j,0} = x_{p,n}, \]

showing that \( K_\sigma^q[X] \) together with these morphisms is a coalgebra. Let \( E_p \) be the vector subspace of \( K_\sigma^q[X] \) with basis \( x_{p,n}, n \geq 0 \). Note that the \( E_p \)'s are right subcomodules of \( K_\sigma^q[X] \) (obviously by the definition of \( \Delta \) and \( \varepsilon \)). We show \( E_p \) are chain comodules in several steps:

(i) Let \( E_{p,n} = \langle x_{p,0}, x_{p,1}, \ldots, x_{p,n} \rangle \) be the space with basis \( \{x_{p,0}, x_{p,1}, \ldots, x_{p,n}\} \); it is actually a right subcomodule of \( E_p \). We note that \( E_p/E_{p,n} \simeq E_{\sigma^{n+1}(p)} \). Indeed, if \( \overline{x} \) denotes the image of \( x \in E_p \) in \( E_p/E_{p,n} \), we have the following formulas for the comultiplication of \( E_p/E_{p,n} \)

\[ \overline{x_{p,m}} \mapsto \sum_{i+j=m, i \geq n+1} \overline{x_{p,i}} \otimes x_{\sigma^i(p),j} = \sum_{i+j=m-n-1} \overline{x_{p,i+n+1}} \otimes x_{\sigma^i(\sigma^{n+1}(p)),j} \]

for \( m \geq n + 1 \). The comultiplication of \( E_{\sigma^{n+1}(p)} \) is given by the formulas:

\[ x_{\sigma^{n+1}(p),s} \mapsto \sum_{i+j=s} x_{\sigma^{n+1}(p),j} \otimes x_{\sigma^i(\sigma^{n+1}(p)),j} \]

These relations show that the correspondence \( \overline{x_{p,i+n+1}} \mapsto x_{\sigma^{n+1}(p),i} \) is an isomorphism of \( K_\sigma^q[X] \)-comodules.

(ii) Let \( x = \lambda_0 x_{0,0} + \lambda_1 x_{1,0} + \ldots + \lambda_n x_{p,n} \in E_p \) and assume \( \lambda_n \neq 0 \). Let \( f \in K_\sigma^q[X]^* \) be equal to 1 on \( x_{p,n} \) and 0 on the rest of the elements of the basis \( \{x_{i,j}\} \). Then one easily sees that \( f \cdot x = \sum_{i+j \leq n} \lambda_{i+j} x_{i,j} f(x_{\sigma^i(p),j}) = \lambda_n x_{p,0} \) (the only terms remaining are the one having \( j = n, i = 0 \), and such a term occurs only once in this sum). Since \( \lambda_n \neq 0 \), we get that \( x_{p,0} \) belongs to the subcomodule generated by \( x \). This shows that \( E_{p,0} \) is contained in
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any subcomodule of $E_p$. This shows that $E_p$ is colocal and $E_{p,0}$ is its socle, which is a simple comodule.

(iii) An inductive argument now shows that $E_{p,n}$ are chain comodules for all $n$. Indeed, by the isomorphism in (i) and by (ii), we have that $E_{p,n+1}/E_{p,n} \simeq E_{\sigma^{n+1}(p),0}$. This shows that $E_p$ is a chain comodule by Proposition 3.2.2.

Since $K^q_\sigma[X] = \bigoplus_{p=1}^{q} E_p$ as right $K^q_\sigma[X]$-comodules, we see that $K^q_\sigma[X]$ is right serial, so it is serial by Proposition 3.4.3 and even purely infinite dimensional, and thus constitutes an example of a left and right f.g. Rat-splitting coalgebra by Proposition 3.4.4.

More examples can be obtained by

**Corollary 3.4.6** If $C = D \oplus E$ where $D$ is a finite dimensional coalgebra and $E$ is a purely infinite serial dimensional coalgebra, then $C$ has the both the left and the right f.g. Rat-splitting property.

**Remark 3.4.7** The fact that $K^q_\sigma[X]$ is also left serial (and then purely infinite dimensional) can also follow by noting that $K^q_\sigma[X]^{\text{op}} \simeq K^q_{\sigma^{-1}}[X]$ as coalgebras. It is also interesting to note that if $\sigma = \sigma_1 \ldots \sigma_r$ is a decomposition of $\sigma$ into disjoint cycles of respective lengths $q_1, \ldots, q_r$ (or, more generally, into mutually commuting permutations), then there is an isomorphism of coalgebras

$$K^q_\sigma[X] \simeq \bigoplus_{i=1}^{r} K^q_{\sigma_i}[X]$$

We omit the proofs here. As a final comment, we note that by the above results, some natural questions arise: is the concept of f.g. Rat-splitting left-right symmetric? That is, does the left f.g. Rat-splitting property of a coalgebra also imply the right f.g. Rat-splitting property? One should note that all the above examples have both the left and the right Rat-splitting property. Also, it would be interesting to know whether a generalization of the results in the local case hold in the general non-cocommutative case as the cocommutative case of this section and the above non-cocommutative examples seem to suggest: if $C$ has the left f.g. Rat-splitting property, can it be written as a direct sum of finite dimensional injectives and infinite dimensional chain injectives (likely in $\mathcal{C}, \mathcal{M}$), or maybe a decomposition of coalgebras as in Corollary 3.4.6. To what extent would such a decomposition characterize this property?
Chapter 4

The Dickson Subcategory Splitting for Pseudocompact Algebras

Introduction

Let $A$ be a ring and $T$ be a torsion preradical. Then $A$ is said to have splitting property provided that $T(M)$, the torsion submodule of $M$, is a direct summand of $M$ for every $A$-module $M$. More generally, if $C$ is a Grothendieck category and $A$ is a subcategory of $C$, then $A$ is called closed if it is closed under subobjects, quotient objects and direct sums.

To every such subcategory we can associate a preradical $t$ (also called torsion functor) by putting $t(M) =$ the sum of all subobjects of $M$ that belong to $A$. We say that $C$ has the splitting property with respect to $A$ if it has the splitting property with respect to $t$, that is, if $t(M)$ is a direct summand of $M$ for all $M$. The subcategory $A$ is called localizing or a Serre class if $A$ is closed and also closed under extensions. In the case of the category of left $A$ modules, the splitting property with respect to some closed subcategory is a classical problem which has been considered by many authors. In particular, the question of when the (classical) torsion part of an $A$ module splits off is a well known problem. J. Rotman has shown in [Rot] that for a commutative domain all modules split if and only if $A$ is a field. I. Kaplansky proved in [K1], [K2] that for a commutative integral domain $A$ the torsion part of every finitely generated $M$ module splits in $M$ if and only if $A$ is a Prüfer domain. While complete results have been obtained in the commutative case, the characterization of the noncommutative rings $A$ for which (a certain) torsion splits in every $A$ module (or in every finitely generated module) is still an open problem.
Another well studied problem is that of the singular splitting. Given a ring $A$ and an $A$-module $M$, denote $Z(M) = \{ x \in M \mid \text{Ann}(x) \text{ is an essential ideal of } A \}$. Then a module is called singular if $M = Z(M)$ and nonsingular if $Z(M) = 0$. Then, a ring $A$ is said to have the \textbf{(finitely generated) singular splitting property} if $Z(M)$ splits in $M$ for all (finitely generated) modules $M$. A thorough study and complete results on this problem was carried out in the work of M.L. Teply; see (also) [Gl], [FK], [FT], [T1], [T2] (for a detailed history on the singular splitting), [T3].

Given a ring $A$, the smallest closed subcategory of the category of left $A$-modules $A^{-\text{mod}}$, containing all the simple $A$-modules, is obviously the category of semisimple $A$-modules. Then one can always consider another more suitable “canonical” subcategory, namely include all simple $A$-modules and consider the smallest localizing subcategory of $A^{-\text{mod}}$ that contains all these simple modules (recall that a subcategory is called localizing if it is a closed subcategory and if it is closed under extensions). This category is called the \textbf{Dickson subcategory} of $A^{-\text{mod}}$, and it is well known that it consists of all semiarinian modules [Dk] (recall that module $M$ is called semiarinian if every non-zero quotient of $M$ contains a simple module). More generally, this construction can be done in any Grothendieck category $C$. Thus one can consider the splitting with respect to this Dickson subcategory; if a ring has this splitting property, we will say it has the \textbf{Dickson splitting property}. A remarkable conjecture in ring theory asks the question: if a ring $A$ has this splitting property, then does it necessarily follow that $A$ is semiarinian? Obviously the converse is trivially true. The answer to this question in general has turned out to be negative. In this respect, an example of J.H. Cozzens in [Cz] shows that there is a ring $R$ (a ring of differential polynomials) that is not semisimple and has the properties that every simple right $R$-module is injective (in fact it has a unique simple right module up to isomorphism) and that it is noetherian on both sides. For $A = R^{op}$, the Dickson subcategory of $A^{-\text{mod}}$ coincides with the that of semisimple $A$-modules and the (left) Dickson splitting property obviously holds since then all semisimple modules are injective ($A$ is left noetherian). However, this ring is not semisimple and thus not (left) semiarinian.

Motivated by these facts, in this chapter we consider the case when the ring $A$ is a \textbf{pseudocompact algebra}: an algebra $A$ which is a topological algebra with a basis of neighbourhoods of 0 consisting of ideals of $A$ of finite codimension and which is Hausdorff and complete. Equivalently, such an algebra is an inverse limit of finite dimensional algebras, and thus they are also called \textbf{profinite algebras} and their theory extends and general-
izes, in part, the theory of finite dimensional algebras. This class of algebras is one very intensely studied in the last 20 to 30 years. They are in fact the algebras that arise as dual (convolution) algebras of coalgebras, and the theory of the representations of such algebras is well understood through the theory of corepresentations (comodules) of coalgebras. In fact, if $A = C^*$ for a coalgebra $C$, the category of pseudocompact left $A$-modules is dual to that of the left $C$-comodules; see [DNR], Chapter 1. The main result of the chapter shows that the conjecture mentioned above holds for this class of algebras, i.e. that if $A$ is pseudocompact and has the Dickson splitting property, then $A$ is semiartinian. The particular question of whether this holds for algebras that are duals of coalgebras was also mentioned in [NT]. As a direct and easy consequence, we re-obtain the main result from Chapter 2 (and [I1] and [NT]) stating that if a coalgebra $C$ has the property that the rational submodule of every left $C^*$-module $M$ splits off in $M$, then $C$ must be finite dimensional.

We extensively use the notations and language of [DNR]; for general results on coalgebras and comodules, we also refer to the well known classical textbooks [A] and [Sw1]. We first give some general results about a coalgebra $C$ for which the Dickson splitting property for $C^*$ holds. We show that such a coalgebra $C$ must be almost connected (i.e. has finite dimensional coradical) and also that if $D \subset C$ is any subcoalgebra $C$, then $D^*$ has this Dickson splitting property. In some special cases, such as when the Jacobson radical of $C^*$ is finitely generated as a left ideal (in particular, when $C^*$ is left noetherian or when $C$ is an artinian right $C^*$-module) or when $C^*$ is a domain, then the Dickson splitting property implies that the coradical filtration of $C$ is finite, and consequently, in this case, $C^*$ is semiartinian, and moreover, it has finite Loewy length. For the general case, we first show the Dickson splitting property for $C^*$ implies $C^*$-semiartinian for colocal coalgebras (i.e. when $C^*$ is a local ring), and then treat the general case by using standard localization techniques, some general and some specific to coalgebras. The main proofs will include some extensions and generalizations of an old idea from abelian group theory and will make use of general facts from module theory but also of a number of techniques specific to coalgebra (and corepresentation) theory.
4.1 General results

For a vector space $V$ and a subspace $W \subseteq V$ denote by $W^\perp = \{ f \in V^* \mid f(x) = 0, \forall x \in W \}$ and for a subspace $X \subseteq V^*$ denote by $X^\perp = \{ x \in V \mid f(x) = 0, \forall f \in X \}$. A subspace $X$ of a coalgebra $C$, is called left coideal (or right coideal) if $X$ is a left (right) submodule of $C$; $X$ is a subcoalgebra of $C$ if $X$ is a left and right ideal of $C$. Recall from [DNR] that for any subspace $X$ of $C$, $X$ is a left coideal (or right coideal, or respectively subcoalgebra) if and only if $X^\perp$ is a left ideal (or right ideal, or respectively two-sided ideal) of $C^*$. Similarly, if $I$ is a left (or right, or two-sided) ideal of $C^*$, then $I^\perp$ is a right coideal (or left coideal, or subcoalgebra) of $C$. Moreover, if $X < C$ is a left (or right $C$) submodule (coideal) of $C$ then there is an isomorphism of left $C^*$-modules $(C/X)^* \simeq X^\perp$. We consider the finite topology on $C^*$; this is the linear topology on $C^*$ with a basis of neighbourhoods of 0 consisting of the sets $F^\perp$, for finite sets $F \subseteq C$. Recall that $(X^\perp)^\perp = X$ for any subspace of $C$ and also for a subspace $X$ of $C^*$, $(X^\perp)^\perp = \overline{X^\perp}$, the closure of $X$. Consequently, for $X \subseteq C^*$, we have $(X^\perp)^\perp = X$ if and only if $X$ is closed.

Throughout, $C$ will be a coalgebra and $\varepsilon$ will be the counit of the coalgebra. Also $J = J(A)$ will denote the Jacobson radical of $A = C^*$; then one has that $J = C_0^\perp$ and also $J^\perp = C_0$. Generally, for a left $A$-module $M$, the Jacobson radical of $M$ will be denoted $J(M)$.

Let $S$ be a system of representatives for the simple left comodules and for $S \in S$, let $C_S = \sum_{T \leq C, T\text{ simple}, T \cong S} T$. Then $C_S$ is a finite dimensional coalgebra, called the coalgebra associated to $S$, and $C_0^\perp = \bigoplus_{S \in S} C_S$ (see [DNR], proposition 2.5.3 and Chapter 3.1). Denote by $T$ the torsion preradical associated to the Dickson subcategory of $A - \text{mod}$. Note that $A/J = C^*/C_0^\perp \simeq C_0^\perp = \prod_{S \in S} A_S$ where $A_S = C_S^*$ as left $A$-modules.

**Proposition 4.1.1** With the above notations, $\Sigma = \sum_{S \in S} A_S \subseteq A/J = \prod_{S \in S} A_S$ is the socle of $A/J$ and moreover, $\Sigma = T(A/J)$.

**Proof.** For $x = (x_S)_{S \in S} \in \Pi = \prod_{S \in S} A_S$ denote $\text{supp}(x) = \{ S \in S \mid x_S \neq 0 \}$. Obviously, $\Sigma$ is a semisimple module. It is enough to see that $\Pi/\Sigma$ contains no simple submodules. Assume by contradiction that $(Ax + \Sigma)/\Sigma$ is a simple (left) submodule of $\Pi/\Sigma$; then obviously $\text{supp}(x)$ is infinite $(x \notin \Sigma)$ and write $\text{supp}(x) = I \cup J$ a disjoint union with infinite $I$ and $J$. Take $X$ such that $X \oplus C_0 = C$ and let $e_I$ be defined as $e$ on $\bigoplus_{S \in I} C_S$ and $0$ on $\bigoplus_{S \in J} C_S \oplus X$; put $x_I = e_I \cdot x$. Since $A_S \simeq C_S^*$ as $A$-bimodules, the left $C$-
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comodule structure of $C_S$ given by $\Delta : C_S \to C_S \otimes C_S$, induces a right comultiplication $A_S \to A_S \otimes C_S \subseteq A_S \otimes C$ on $A_S = C_S^*$. Then it is easy to see that $e_I \cdot x_S = 0$ if $S \notin I$ and $e_I \cdot x_S = x_S$ when $S \in I$. This shows that $\text{supp}(x_I) = I$. We have an inclusion of modules

$$\Sigma \subseteq \Sigma + Ax_I \subseteq \Sigma + Ax$$

The strict inclusions hold since $x_I \notin \Sigma$ because $\text{supp}(x_I) = I$ is infinite. Also $x \notin \Sigma + Ax_I$; otherwise, $x = \sigma + ax_I$ with $\sigma \in \Sigma$ and $a \in A$, so $\text{supp}(x) \subseteq \text{supp}(\sigma) \cup \text{supp}(x_I)$ and it follows that $J = \text{supp}(x) \setminus I \subseteq (\text{supp}(\sigma) \cup I) \setminus I \subseteq \text{supp}(\sigma)$ which is finite, a contradiction. This shows that $(\Sigma + Ax)/\Sigma$ is not simple and the proof is finished. $\square$

Corollary 4.1.2 If $C$ is a coalgebra such that $T(M)$ is a direct summand of $M$ for every cyclic $A$-module $M$, then $\mathcal{S}$ is a finite set and $C_0$ is finite dimensional.

Proof. Since $A/J$ is cyclic and $\Sigma = T(A/J)$ we see that $\Sigma = \bigoplus_{S \in \mathcal{S}} A_S$ is a direct summand of $A/J$ and thus it is itself cyclic. This shows that $\mathcal{S}$ must be finite, and therefore $C_0 = \bigoplus_{S \in \mathcal{S}} C_S$ is finite dimensional.

Proposition 4.1.3 Let $C$ be a coalgebra such that $C^*$ has the Dickson splitting property for left modules and let $D$ be a subcoalgebra of $C$. Then $D^*$ has the Dickson splitting property for left modules too.

Proof. Let $M$ be a left $D^*$-module. Let $I = D^\perp$, so we have an exact sequence

$$0 \to I \to C^* \to D^* \to 0$$

Then $M$ is a left $C^*$-module through the restriction morphism $C^* \to D^*$ and $I \subseteq \text{Ann}_{C^*}(M)$. Then there is a decomposition $M = \Sigma \oplus X$ where $\Sigma$ is the semiartinian part of $M$ as a $C^*$-module, and $X$ is a $C^*$-submodule of $M$. But $IM = 0$ and therefore $I$ also annihilates both $\Sigma$ and $X$, hence $\Sigma$ and $X$ are also $D^*$-modules. Now note that if $S$ is a $D^*$-module, then the lattice of $C^*$-submodules of $S$ coincides to that of the $D^*$-submodules since $S$ is annihilated by $I$. This shows that $S$ is semiartinian (or has no semiartinian submodule) as $D^*$-module if and only if it is semiartinian as a $C^*$-module (respectively has no semiartinian submodule). Therefore, $\Sigma$ is semiartinian as $D^*$-module (as it is a semiartinian $C^*$-module) and $X$ contains no simple $D^*$-submodule (since $X$ contains no simple
4.1. GENERAL RESULTS

C*-submodules), hence Σ is the semiartinian part of $M$ also as a $D^*$-module and it splits in $M$. □

4.1.1 Some general module facts

We dedicate a short study for a general property of modules which is obtained with a “localization procedure”, that will be used towards our main result. Although this follows in a much more general setting in Grothendieck categories ([CICN]), we leave the general case treated there aside, and present a short adapted version here. We will often use the following

Remark 4.1.4 If $A/J$ is a semisimple algebra and $M$ is a left $A$-module, then $M$ is semisimple if and only if $J \cdot M = 0$. Moreover, then for any left $A$-module $M$ we have $J(M) = JM$. Indeed, $M/J(M) \subseteq \prod_{X < M, X \text{maximal}} M/X$ which is annihilated by $J$ and thus it is semisimple; therefore $J(M/J(M)) = 0$ i.e. $JM \subseteq J(M)$. Conversely, since $M/JM$ is semisimple, $JM$ is an intersection of maximal submodules of $M$, so $J(M) \subseteq JM$.

To the end of this section, let $A$ be a ring, $e$ an idempotent of $A$. The functor $T_e = eA \otimes_A -$ : $A - \text{mod} \rightarrow eAe - \text{mod}$ is exact and has $G_e = \text{Hom}_{eAe}(eA, -)$ as a right adjoint; in fact, $eA \otimes_A M \simeq eM$ as left $eAe$-modules for any left $A$-module $M$. Recall that for $N \in eAe - \text{mod}$, the left $A$-module structure on $\text{Hom}_{eAe}(eA, N)$ is given by $(a \cdot f)(x) = f(xa)$, for $a \in A, x \in eA, f \in \text{Hom}_{eAe}(eA, N)$. Let $\psi_{e,M} : M \rightarrow \text{Hom}_{eAe}(eA, eM)$ be the canonical morphism (the unit of this adjunction); it is given by $\psi_{e,M}(m)(ea) = eam$ for $m \in M, a \in A$. The following proposition actually says that the counit of this adjunction is an isomorphism; the proof is a straightforward computation and is omitted.

Proposition 4.1.5 Let $N \in eAe - \text{mod}$ and for $n \in N$ let $\chi_n \in \text{Hom}_{eAe}(eA, N)$ be such that $\chi_n(ea) = eae \cdot n$. Then the application

$$N \rightarrow e \cdot \text{Hom}_{eAe}(eA, N) : n \mapsto e \cdot \chi_n$$

is an isomorphism of left $eAe$-modules.

Proposition 4.1.6 Let $N$ be a left $eAe$-module and $X$ an $A$-submodule of $\text{Hom}_{eAe}(eA, N)$. Denote $X(e) = \{f(e) \mid f \in X\}$. Then $X(e)$ is a submodule of $N$, and $X(e) \neq 0$ and $e \cdot X \neq 0$, provided that $X \neq 0$. 
4.1. GENERAL RESULTS

**Proof.** Let $x = f(e) \in X(e)$ for some $f \in X$ and let $a \in A$. Then $eae = eae \cdot f(e) = f(eae) = (ae \cdot f)(e) \in X(e)$ since $ae \cdot f \in X$. Moreover, if $f \neq 0$, then $f(ea) \neq 0$ for some $a$ and as above $0 \neq f(ea) = (a \cdot f)(e) \in X(e)$; also $(e \cdot (a \cdot f))(e) = (a \cdot f)(e) = f(ea) \neq 0$, so $0 \neq e \cdot (a \cdot f) \in e \cdot X$.  

**Proposition 4.1.7** If $N \in eAe - \mod$ has essential socle, then $\mathcal{G}_e(N)$ has essential socle too (as an $A$-module).

**Proof.** Let $0 \neq H < \mathcal{G}_e(N)$ be a submodule of $\mathcal{G}_e(N)$ (assume $\mathcal{G}_e(N) \neq 0$). Then $H(e) \neq 0$ by Proposition 4.1.6 and $H(e) \cap s(N) \neq 0$, where $s(N)$ is the socle of $N$. Let $\Sigma_0$ be a simple $eAe$-submodule of $H(e)$. We have an exact sequence

$$0 \to S \to \mathcal{G}_e(\Sigma_0) \to \prod_{0 \neq X < \mathcal{G}_e(\Sigma_0)} \frac{\mathcal{G}_e(\Sigma_0)}{X}$$

where $S = \bigcap_{0 \neq X < \mathcal{G}_e(\Sigma_0)} X$. Since $\mathcal{T}_e$ is exact, we have $e(\mathcal{G}_e(\Sigma_0)/X) \simeq e\mathcal{G}_e(\Sigma_0)/eX = 0$ because $e\mathcal{G}_e(\Sigma_0) \simeq \Sigma_0$ by Proposition 4.1.5, $\Sigma_0$ is simple and $eX \neq 0$ by Proposition 4.1.6. Then, it easily follows that

$$e \cdot \left( \prod_{0 \neq X < \mathcal{G}_e(\Sigma_0)} \frac{\mathcal{G}_e(\Sigma_0)}{X} \right) = \mathcal{T}_e(\prod_{0 \neq X < \mathcal{G}_e(\Sigma_0)} \frac{\mathcal{G}_e(\Sigma_0)}{X}) = 0$$

and then by the above exact sequence and the exactness of $\mathcal{T}_e$ we get $S \neq 0$. Otherwise, if $S = 0$, it follows that $\mathcal{G}_e(\Sigma_0) = 0$, so $\Sigma_0 \simeq e\mathcal{G}_e(\Sigma_0) = 0$, a contradiction. Also, $S$ is simple by construction. Let $0 \neq x \in \Sigma_0 \subseteq H(e)$ and write $x = h(e)$ for some $h \in H$. There is a monomorphism $0 \to \mathcal{G}_e(\Sigma_0) \to \mathcal{G}_e(N)$ and $0 \neq eh$ has image contained in $\Sigma_0$, since $eh(ea) = h(eae) = eae \cdot h(e) \in \Sigma_0$ and $eh(e) = h(e) \neq 0$. Therefore, we observe that $S \subseteq A \cdot eh$, by the construction of $S$. But $Aeh \subseteq H$, and thus $H$ contains the simple $A$-submodule $S$.  

**Theorem 4.1.8** Assume $A = \bigoplus_{i \in F} E_i$ as left $A$-modules, and let $E_i = Ae_i$ with orthogonal idempotents $e_i$ with $\sum_{i \in F} e_i = 1$. Let $M$ be an $A$-module such that $e_iM = \mathcal{T}_{e_i}(M)$ is a semiartinian $e_iAe_i$-module for all $i \in F$. Then $M$ is semiartinian too. Consequently, if $e_iAe_i$ is a left semiartinian ring for all $i \in F$, then $A$ is left semiartinian too.
4.2. THE DOMAIN CASE

**Proof.** Obviously there always exist such idempotents $e_i$ and $F$ is finite. It is enough to show any such $M$ contains a simple submodule. If this holds and $N$ is any submodule of $M$ where $e_iM$ is $e_iAe_i$-semiartinian, then $e_i(M/N) \simeq e_iM/e_iN$ is semiartinian over $e_iAe_i$ for all $i \in F$ and thus $M/N$ contains a simple submodule.

Let $M \rightarrow \overline{M} = \bigoplus_{i \in F} G_{e_i}(e_iM)$ be the canonical morphism, $m \mapsto (\psi_{e_i,M}(m))_{i \in F}$. This is obviously injective: $\psi_{e_i,M}(m) = 0$ for all $i \in F$ implies $e_i m = 0$, $\forall i \in F$, so $m = 1 \cdot m = \sum_{i \in F} e_i \cdot m = 0$. By Proposition 4.1.7, $\overline{M}$ has essential socle, and so $s(\overline{M}) \cap M \neq 0$ (provided $M \neq 0$) and this ends the proof. The last statement follows for $M = AA$.

**Remark 4.1.9** It not difficult to see that if $M$ is semiartinian over $A$ then $eM$ is semiartinian for every idempotent $e$; this is also a consequence of the more general results of [CICN] or can be again seen directly.

4.2 The domain case

We show that if $C$ is a coalgebra such that $C^*$ is a (local) domain and $C^*$ has the Dickson splitting property (that is, the semiartinian part of every left $C^*$-module splits off), then $C$ has finite Loewy length (in fact in this case $C^*$ will be a division algebra). We will again make use of the fact that if $X$ is a left subcomodule of $C$ then $X^\perp$ is a left ideal of $C^*$ and there is an isomorphism of left $C^*$-modules $(C/X)^* \cong X^\perp$.

**Remark 4.2.1** For an $f \in C^*$ denote by $\overline{f} : C \rightarrow C$ the morphism of left $C$-comodules defined by $\overline{f}(c) = c_1 f(c_2)$. Then the maps $C^* \rightarrow \text{End}(C^C) : f \mapsto \overline{f}$ and $\text{End}(C^C) \rightarrow C^* : \alpha \mapsto \varepsilon \circ f$ are inverse isomorphisms of $K$-algebras (for example by [DNR], Proposition 3.1.8 (i)).

**Lemma 4.2.2** Let $C$ be a coalgebra. Then $C^*$ is a domain if and only if every nonzero morphism of left (or right) $C$-comodules $\alpha : C \rightarrow C$ is surjective. Moreover, if this holds, $C^*$ is local.

**Proof.** Since $J = C_0^1$, we have $C^*/J \simeq C_0^*$ and therefore $C^*$ is local if and only if $C_0^*$ is a simple left $C^*$-module, equivalently $C_0$ is a simple comodule (left or right).

Assume first $C^*$ is a domain, so $\text{End}(C^C) \simeq C^*$ is a domain. If $C_0$ is not simple, then there is a direct sum decomposition $C_0 = S \oplus T$, where $S$ and $T$ are semisimple left $C$-comodules that are nonzero. In this case, if $E(S)$ and $E(T)$ are injective envelopes of $S$.
and $T$ respectively that are contained in $C$, we get that $C = E(S) \oplus E(T)$ because $C_0$ is essential in $C$ and $C$ is injective (as left $C$-comodules). Then defining $\alpha, \beta : C \to C$ by

$$
\alpha = \begin{cases}
\text{Id on } E(S) \\
0 \text{ on } E(T)
\end{cases}
$$

and

$$
\beta = \begin{cases}
0 \text{ on } E(S) \\
\text{Id on } E(T)
\end{cases}
$$

we obviously have $\alpha, \beta \neq 0$ and $\alpha \circ \beta = 0$, showing that $\text{End}(C,C)$ cannot be a domain in this case. Therefore $C^*$ is local. Next, if some nonzero morphism of left $C$-comodules $\alpha : C \to C$ is not surjective, then $\text{Im}(\alpha) \neq C$ so $C/\text{Im}(\alpha) \neq 0$. Therefore there is a simple left subcomodule of $C/\text{Im}(\alpha)$, that is, a monomorphism $\eta : C_0 \to C/\text{Im}(\alpha)$ (because $C_0$ is the only type of simple $C$-comodule). Then the inclusion $i : C_0 \to C$ extends to $\beta_0 : C/\text{Im}(\alpha) \to C$, such that $\beta_0 \eta = i$. If $\beta : C \to C$ is the composition of $\beta_0$ with the canonical projection $C \to C/\text{Im}(\alpha)$ then obviously $\beta|_{\text{Im}(\alpha)} = 0$, so $\beta \circ \alpha = 0$. But $\beta \neq 0$ because $\beta_0 \neq 0$ since $\beta_0$ extends a monomorphism $(i)$, and also $\alpha \neq 0$, yielding a contradiction, since $C^*$ is a domain.

The converse implication is obvious: any composition of surjective morphisms is surjective, thus nonzero. □

**Lemma 4.2.3** If $C$ is colocal (equivalently $C^*$ is local), then for any left subcomodule $Y$ of $C$, we have that $Y^*$ is a local, cyclic and indecomposable left $C^*$-module.

**Proof.** The epimorphism of left $C^*$-modules $p : C^* \to Y^* \to 0$ induces an epimorphism $C^*/J \to Y^*/J(Y^*)$ which must be an isomorphism since $C^*/J$ is simple and $Y^*/J(Y^*)$ is nonzero because $Y^*$ is finitely generated (cyclic) over $C^*$. This shows that $Y^*$ is local. If $Y^* = A \oplus B$ with $A \neq 0$, $B \neq 0$ then $A$ and $B$ would be cyclic too, and therefore we would get $J(A) \neq A$ and $J(B) \neq B$. But $A/J(A) \oplus B/J(B) = Y^*/J(Y^*)$ is simple, and therefore showing that $A/J(A) = 0$ or $B/J(B) = 0$, a contradiction. □

**Lemma 4.2.4** If $M$ is a left $C$-comodule, then $\bigcap_n (J^n \cdot M^*) = 0$.

**Proof.** Let $f \in \bigcap_n (J^n \cdot M^*)$. Pick $x \in M$, and write $\rho(x) = \sum c_s \otimes x_s \in C \otimes X$ the comultiplication of $x$. Since the Loewy series $C_0 \subseteq C_1 \subseteq \ldots$ has $\bigcup_n C_n = C$ there is some
4.2. THE DOMAIN CASE

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C_n with c_s ∈ C_n, ∀s. Also, as f ∈ J^{n+1} · M^* we can write f = ∑ j f_j m_j^* for some f_j ∈ J^{n+1} and m_j^* ∈ M^*. Since (J^{n+1})⊥ = C_n we get f_j(c_s) = 0 for all j and s. Then

\[ f(x) = (\sum_j f_j m_j^*)(x) \]
\[ = \sum_j m_j^*(x · f_j) \quad \text{(by the left } C^* \text{ – module structure on } M^*) \]
\[ = \sum_j m_j^*(\sum_s (f_j(c_s)x_s)) \]
\[ = 0 \quad \text{(because } f_j(c_s) = 0, \forall j, s) \]

This shows that f(x) = 0 and since x is arbitrary, we get f = 0. □

The Splitting property

**Proposition 4.2.5** If C is colocal and C^* has the (left) Dickson splitting property, then C^*/I is semiarithmetic for every nonzero left ideal I of C^*.

**Proof.** Let I be a nonzero left ideal of C^* and 0 ≠ f ∈ I. Let L = C^* f⊥; then since C^* f is finitely generated, it is closed in the finite topology of C^* (for example by [I2], Lemma 1.1) and we have C^* f = L⊥. Note that L ≠ C since C^* f ≠ 0. We have that L is a left coideal of C, and as L ≠ C, we can find a left coideal Y ≤ C such that Y/L is finite dimensional and nonzero (by the Fundamental Theorem of Comodules). Then there is an exact sequence of left C^*-modules

\[ 0 \rightarrow (Y/L)^* \rightarrow Y^* \rightarrow L^* \rightarrow 0 \]

Write Y^* = Σ ⊕ T, with Σ semiarithmetic and T containing no semiarithmetic (or equivalently, no simple) submodules. Then Σ ≠ 0 since 0 ≠ (Y/L)^* is a finite dimensional (thus semiarithmetic) left C^*-module contained in Y^*. But Y^* is indecomposable by Lemma 4.2.3, and therefore Y^* = Σ follows, i.e. Y^* is semiarithmetic. The above sequence shows that L^* is semiarithmetic too and since L⊥ = C^* f ⊆ I we get an epimorphism L^* ≃ C^* / L⊥ → C^* / I → 0, and therefore C^* / I is semiarithmetic. □

**Theorem 4.2.6** Let C be a coalgebra such that C^* has the Dickson splitting property for left modules. If C^* is a domain, C^* must be a finite dimensional division algebra (so it will...
have finite Loewy length).

**Proof.** Note that $C_0$ is finite dimensional by Corollary 4.1.2. We show that $C = C_0$, which will end the proof, since then $C^* = C_0^*$ is a finite dimensional semisimple algebra which is a domain, thus it must be a division algebra.

Assume $C_0 \neq C$. Then $J = J(C^*) \neq 0$ and take $f \in J$ with $f \neq 0$. Denote $M_n = (\overline{f} \circ \ldots \circ \overline{f})^{-1}(C_n) = (\overline{f^n})^{-1}(C_n)$, with $M_0 = \text{Id}^{-1}(C_0) = C_0$. Note that $M_n \subseteq M_{n+1}$ and $\overline{f^n}(M_n) = C_n$ since $\overline{f}$ is surjective by Lemma 4.2.2. Also $M_n^\perp \neq 0$, since otherwise $M_n = (M_n^\perp)^0 = 0^+ = C$ so $C_n = \overline{f^n}(M_n) = \overline{f^n}(C) = C$, while $C = C_n$ is excluded by assumption. Now

$$M^* = \frac{C^*}{M_0^\perp} \times \frac{C^*}{M_1^\perp} \times \ldots \times \frac{C^*}{M_n^\perp} \times \ldots = \prod_{n \geq 0} \frac{C^*}{M_n^\perp}$$

as a left $C^*$-module. Also put $\lambda = (\hat{\varepsilon}, \hat{\tilde{f}}, \tilde{f}^2, \ldots) \in M$ (where $\hat{h}$ denotes the image of $h \in C^*$ modulo some $M_n^\perp$). Let $M = T \oplus X$ with $T$ semiarthian and $X$ containing no semiarthian modules. If $t_n = (\hat{\varepsilon}, \hat{\tilde{f}}, \ldots, \tilde{f}^{n-1}, 0, 0, \ldots)$ then $t_n \in \prod_{0 \leq i < n} C^*/M_i^\perp \times 0$ which is semiartinian since it is a quotient of $(C^*/M_n^\perp)^n$, and $C^*/M_n^\perp$ is semiartinian by Proposition 4.2.5. Put $x_n = (0, 0, \ldots, 0, \hat{\varepsilon}, \tilde{f}, \tilde{f}^2, \ldots, \tilde{f}^n, \ldots)$. Here $\varepsilon = 1_C$ is in position “$n$”, with positions starting from “$0$”). Write $\lambda = t + x$ and $x_n = t'_n + x'_n$ with $t, t'_n \in T$ and $x, x'_n \in X$. Then

$$t + x = t_n + f^n \cdot x_n = t_n + f^n(t'_n + x'_n) = (t_n + f^n \cdot t'_n) + f^n \cdot x'_n$$

shows that $x = f^n \cdot x'_n$, since $M = T \oplus X$. Therefore if $x = (\hat{y}_p)_p \in M$ and $x'_n = (\hat{y}_{n,p})_p \in M$ for $\hat{y}_n, \hat{y}_{n,p} \in C^*/M_n^\perp$, we get $\hat{y}_p = f^n \cdot \hat{y}_{n,p}$, for all $n, p$. Since $f \in J$, we get $\hat{y}_p \in J^n \cdot (C^*/M_p^\perp)$ for all $n$. Fixing $p$ and using Lemma 4.2.4 we get that $\hat{y}_p = 0$, since $C^*/M_p^\perp \simeq M_p^\perp$. This holds for all $p$, hence $x = 0$ and then $\lambda = t \in T$. Note that $\lambda \neq 0$, since $\varepsilon \notin M_0^\perp = C_0^\perp = J \neq C$, and, as $C^* \cdot \lambda$ is semiartinian, there is some $g \in C^*$ such that $C^* g \lambda$ is a simple left $C^*$-module. Then $C^* g \hat{f}^n = (C^* g f^n + M_n^\perp)/M_n^\perp \subseteq C^*/M_n^\perp$ is either simple or $0$ for all $n$ (since it is a quotient of $C^* g \lambda$) and so it must be annihilated by $J$ (use Remark 4.1.4 for example). Thus $J \cdot g \hat{f}^n = 0$ in $C^*/M_n^\perp$ and so $J \cdot g f^n \subseteq M_n^\perp$ and then for $a \in J$ and $m \in M_n$ we have

$$0 = a \cdot g \cdot f^n(m) = a(m_1)g f^n(m_2) = a((g f^n)(m_2)) = a((g f^n)(m_2))$$

$$= a(g f^n(m)) = a(\overline{g f^n}(m))$$

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Therefore, $0 = a(\bar{g}(J^n(M_n))) = a(\bar{g}(C_n))$ for all $a \in J$, which shows that $\bar{g}(C_n) \subseteq J^\perp = C_0$. This holds for all $n$, showing that $\bar{g}(C) = g(\bigcup C_n) \subseteq C_0$. But $\bar{g} \neq 0$ since $g \neq 0$, so $\bar{g}$ has to be surjective. But this is obviously a contradiction, because $\bar{g}(C) \subseteq C_0 \neq C$, and the proof is finished.

The above also shows that a profinite algebra $A$ which is a domain and has finite Loewy length (equivalently, $A = C^*$ with $C = C_n$ for some $n$) must necessarily be a division algebra. This can actually be easily proved directly by using Lemma 4.2.2, as we invite the reader to note.

### 4.3 Dickson’s Conjecture for duals of coalgebras

Denote by $T$ the torsion preradical associated to the Dickson localizing subcategory of $C^* - \text{mod}$. If $A$ is an algebra such that $A/J$ is semisimple, we again use the observation that a left $A$-module $N$ is semisimple if and only if $JN = 0$. Moreover, this implies that $N$ is semiartinian of finite Loewy length if and only if $J^nN = 0$ for some $n \geq 0$.

**Proposition 4.3.1** Let $C \neq 0$ be a colocal coalgebra such that $C^*$ has the left Dickson splitting property. Then $TC^* \neq 0$.

**Proof.** Assume otherwise. We will show that $C^*$ is a domain, which will yield a contradiction by Theorem 4.2.6, since then $C^*$ is a finite dimensional division algebra and so it is semiartinian. To see that $C^*$ is a domain, choose $0 \neq f \in C^*$ and define $\varphi_f : C^* \to C^*$ by $\varphi_f(h) = hf$; then $\varphi_f$ is a morphism of left $C^*$-modules. If $\ker(\varphi_f) \neq 0$ then by Proposition 4.2.5, $C^* \cdot f \simeq C^*/\ker(\varphi_f)$ is semiartinian. This shows that $0 \neq C^* \cdot f \subseteq T(C^*)$ which contradicts the assumption. Therefore $\ker(\varphi_f) = 0$ and it follows that $f$ is not a zero-divisor. This completes the proof. \(\square\)

**Corollary 4.3.2** Let $C$ be a colocal coalgebra. If $C^*$ has the Dickson splitting property for left $C^*$-modules, then $C^*$ is left semiartinian.

**Proof.** We have $C^* = TC^* \oplus I$, for some left ideal $I$ of $C^*$. By the previous Proposition, $TC^* \neq 0$, and since $C$ is colocal, $C^*$ is indecomposable by Lemma 4.2.3. Therefore we must have $I = 0$ and $C^* = TC^*$ is left semiartinian. \(\square\)
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4.3. THE MAIN RESULT

Theorem 4.3.3 Let $A$ be a pseudocompact algebra. If $A$ has the Dickson splitting property for left $A$-modules, then $A$ is left semiartinian.

Proof. Let $A = C^*$, for a coalgebra $C$ and let $C = \bigoplus_{i \in F} E_i$ be a decomposition of $C$ into left indecomposable injective comodules. Then $F$ is finite by Corollary 4.1.2. We have $A = C^* = \bigoplus_{i \in F} E_i^*$ with $E_i^*$ projective indecomposable left $A$-modules, so $E_i^* = Ae_i$ with $(e_i)_{i \in F}$ a complete system of indecomposable orthogonal idempotents. By [NT], Corollary 2.4 we have that each ring $e_iAe_i$ has the Dickson splitting property for left modules. By [Rad], Lemma 6 (also see [CGT2]), $eAe = eC^*e$ is also a pseudocompact algebra, dual to the coalgebra $eCe = \{e(c_1)c_2(c_3) | c \in C\}$ with counit $e$ and comultiplication $ece \mapsto ec_1e \otimes ec_2e$ (note: this comultiplication is well defined). Also, since $e_i$ are primitive, $e_iAe_i$ are local. Therefore, Corollary 4.3.2 applies, and we get that $e_iAe_i$ are semiartinian for all $i \in F$.

Now, by Theorem 4.1.8 it follows that $A$ is semiartinian too. (One can instead use the fact that $e_iAe_i - \text{mod}$ are localizations of $A - \text{mod}$, and then apply [CICN], Proposition 3.5 1,(b)). $\square$

As an immediate consequence, we obtain the following result proved first in [NT] and then independently in [Cu] and [I1]:

Corollary 4.3.4 Let $C$ be a coalgebra such that the rational submodule of every left $C^*$-module $M$ splits off in $M$. Then $C$ is finite dimensional.

Proof. Note that in this case $C^*$ has the Dickson splitting property for left modules: if $M$ is a left $C^*$-module, then $M = R \oplus X$ with $R$ rational - thus semiartinian - and $\text{Rat}(X) = 0$. Now all simple modules are rational because $C^*/J(C^*)$ is finite dimensional semisimple in this case (see, for example, [NT] or [I1, Proposition 1.2]). Thus $X$ contains no simple submodules. So $R$ is also the semiartinian part of $M$ and it is a direct summand of $M$. Thus it follows that $C^*$ is semiartinian from the previous Theorem. Now write $C^* = \text{Rat}(C^*) \oplus N$ with $\text{Rat}(N) = 0$. Then since $N$ is semiartinian we must have $N = 0$; otherwise $N$ contains simple rational submodules. Hence $\text{Rat}(C^*) = C^*$ and this module is also cyclic, so it is finite dimensional. Therefore $C$ is finite dimensional too. $\square$

We note that in several particular cases, more can be inferred when the Dickson splitting property holds (i.e. if $C^*$ is semiartinian). Recall that a coalgebra is almost connected if the coradical $C_0$ is finite dimensional.
Lemma 4.3.5 If $J = J(C^*)$ is a finitely generated left ideal for an almost connected coalgebra $C$, then a finitely generated $C^*$-module is semiartinian if and only if it has finite Loewy length.

**Proof.** It is enough to consider $M = C^*x$ for some $x \in M$. If the Loewy length of $M$ is strictly greater than $\omega$, the first infinite ordinal, then there is $f \in C^*$ such that $fx \in \mathcal{L}_{\omega+1}(M) \setminus \mathcal{L}_\omega(M)$ so $fx + \mathcal{L}_\omega(M)/\mathcal{L}_\omega(M)$ is semisimple and then it is annihilated by $J$, since $C^*/J$ is finite dimensional semisimple. Then $Jfx \subseteq \mathcal{L}_\omega(M) = \bigcup_n \mathcal{L}_n(M).$

Let $g_1, \ldots, g_s$ generate $J$ as a left ideal; then one has $g_i \cdot fx \in \mathcal{L}_\omega(M)$ and therefore $g_i \cdot fx \in \mathcal{L}_{n_i}(M)$ for some $n_i$. This shows that $Jfx \subseteq \mathcal{L}_n(M)$ with $n = \max\{n_1, \ldots, n_s\}$. Then, since $J^n$ annihilates $\mathcal{L}_n(M)$, we have that $J^{n+1}fx = J^n \cdot Jfx \subseteq J^n \mathcal{L}_n(M) = 0$. Therefore, $fx \in \mathcal{L}_{n+1}(M) \subseteq \mathcal{L}_\omega(M)$, a contradiction.

Now, note that we have $C^*x = M = \mathcal{L}_\omega(M) = \bigcup_n \mathcal{L}_n(M)$ and therefore $x \in \mathcal{L}_n(M)$ for some $n$, so $C^*x \subseteq \mathcal{L}_n(M)$, and the proof is finished. \hfill \Box

**Corollary 4.3.6** Let $C$ be a coalgebra such that $C^*$ is left semiartinian; if $J$ is finitely generated as a left ideal, then $C$ has finite coradical filtration.

**Proposition 4.3.7** If $C$ is an almost connected coalgebra with finite coradical filtration and if the Jacobson radical $J$ of $C^*$ is finitely generated as a left ideal, then $C$ is finite dimensional.

**Proof.** Let $\{f_1, \ldots, f_k\}$ be a set of generators of $J$. If $M$ is a finitely generated left $C^*$-module, say by $m_1, \ldots, m_s$, then $JM$ is also finitely generated, by $\{fm_j\}$. Indeed if $a \in J$ and $m \in M$, then $m = \sum_{j=1}^s a_jm_j$ with $a_j \in C^*$ and, since $aa_j \in J$, we get $aa_j = \sum_{i=1}^k b_{ij}f_i$.

Therefore $am = \sum_{j=1}^s aam_j = \sum_{i=1}^k \sum_{j=1}^s b_{ij}(fm_j)$. Since $JM$ is generated by the elements of the form $am$ with $a \in J$ and $m \in M$, the claim follows. Proceeding inductively, this shows that $J^n$ is finitely generated, for all $n$. Then $J^n/J^{n+1}$ is finitely generated and semisimple (because $C^*/J$ is semisimple), thus it is finite dimensional. Therefore, inductively it follows that $C^*/J^n$ is finite dimensional. Finally, some $J^n = 0$ since $C_n = C$ for some $n$. So $C^*$ is finite dimensional. \hfill \Box

**Corollary 4.3.8** If $A$ is a pseudo-compact algebra which is left semiartinian and if $J(A)$ is finitely generated as a left ideal (for example if $A$ is left noetherian), then $A$ is finite dimensional.
CHAPTER 4. THE DICKSON SUBCATEGORY SPLITTING FOR PSEUDOCOMPACT ALGEBRAS

4.3. THE MAIN RESULT

Remark 4.3.9 Naturally, the fact that $C^*$ is left semiartinian or even semiartinian of finite Loewy length (i.e. $C = C_n$ for some $n$) does not alone imply the finite dimensionality of $C$. (Thus the result is of a completely different nature of that in [NT] and [I1]). Indeed, consider the coalgebra $C$ with basis $\{g\} \cup \{x_i, i \in I\}$ for an infinite set $I$ and comultiplication given by $g \mapsto g \otimes g$ and $x_i \mapsto x_i \otimes g + g \otimes x_i$ and counit $\varepsilon(g) = 1, \varepsilon(x_i) = 0$. Then $C_0 = < g >$ and $C_1 = C$, but $C$ is infinite dimensional.

Remark 4.3.10 Thus the “Dickson Splitting conjecture” holds for the class of pseudocompact (profinite) algebras, which is the same as the class of algebras that are the duals of coalgebras. As seen above, in some situations it even follows that such an algebra $A(= C^*)$ has finite Loewy length: if the Jacobson radical is finitely generated or if the algebra is a domain (in fact, even more follows in each of these cases). Then the following question naturally arises: if $C$ is a coalgebra such that $C^*$ is left semiartinian, does it follow that $C^*$ has finite Loewy length, equivalently, does $C$ have finite coradical filtration? At the same time, one can ask the question of whether left semiartinian also implies right semiartinian for $C^*$.
Part II

The theory of Generalized Frobenius Algebras and applications to Hopf Algebras and Compact Groups
Chapter 5

Generalized Frobenius Algebras and Hopf Algebras

Introduction

An algebra $A$ over a field $K$ is called Frobenius if $A$ is isomorphic to $A^*$ as right $A$-modules. This is equivalent to there being an isomorphism of left $A$-modules between $A$ and $A^*$. This is the modern algebra language formulation for an old question posed by Frobenius. Given a finite dimensional algebra $A$ with a basis $x_1, \ldots, x_n$, left multiplication induces a representation $A \to \text{End}_K(A) = M_n(K) : a \mapsto (a_{ij})_{i,j}$ with $a_{ij} \in K$, where $a \cdot x_i = \sum_{j=1}^{n} a_{ij} x_j$. Similarly, the right multiplication produces a matrix $a'_{ij}$ by writing $x_i \cdot a = \sum_{j=1}^{n} a'_{ij} x_j$ with $a'_{ij} \in K$, and this induces another representation $A \to M_n(K) : a \mapsto (a'_{ij})_{i,j}$. Frobenius’ problem came as the natural question of when the two representations are equivalent. Frobenius algebras occur in many different fields of mathematics, such as topology (the cohomology ring of a compact manifold with coefficients in a field is a Frobenius algebra by Poincaré duality), topological quantum field theory (there is a one-to-one correspondence between 2-dimensional quantum field theories and commutative Frobenius algebras; see [Ab]) and Hopf algebras (a finite dimensional Hopf algebra is a Frobenius algebra). Frobenius algebras have consequently developed into a research subfield of algebra.

Co-Frobenius coalgebras were first introduced by Lin in [L] as a dualization of Frobenius
algebras. A coalgebra $C$ is a **left (right) co-Frobenius coalgebra** if there is a monomorphism of left (right) $C^*$-modules $C \to C^*$. However, unlike the algebra case, this concept is not left-right symmetric, as an example in [L] shows. Nevertheless, in the case of Hopf algebras, it was observed that left co-Frobenius implies right co-Frobenius. Also, a left (or right) co-Frobenius coalgebra can be infinite dimensional, while a Frobenius algebra is necessarily finite dimensional. **Co-Frobenius coalgebras** are coalgebras that are both left and right co-Frobenius. It recently turned out that this notion of co-Frobenius has a nice characterization that is analogous to a characterization of Frobenius algebras and is also left-right symmetric: a coalgebra $C$ is co-Frobenius if it is isomorphic to its left (or equivalently to its right) rational dual $\text{Rat}(C^*)$ (equivalently $C \simeq \text{Rat}(C^*)$; see [I]). This also allowed for a categorical characterization which is again analogous to a characterization of Frobenius algebras: an algebra $A$ is Frobenius iff $\text{Hom}_A(\cdot, A)$ (“the $A$-dual functor”) and $\text{Hom}_{\mathcal{K}}(\cdot, \mathcal{K})$ (“the $\mathcal{K}$-dual functor”) are naturally isomorphic functors $A \text{-mod} \to \text{mod} - A$. Similarly, a coalgebra is co-Frobenius if the $C^*$-dual $\text{Hom}_{C^*}(-, C^*)$ and the $\mathcal{K}$-dual $\text{Hom}_{\mathcal{K}}(-, \mathcal{K})$ functors are isomorphic functors $\mathcal{M}C \to \text{mod} - C^*$. If a coalgebra $C$ is finite dimensional then it is co-Frobenius if and only if $C^*$ is Frobenius, showing that the co-Frobenius coalgebras (or rather their duals) can be seen as an infinite dimensional generalization of Frobenius algebras.

Quasi-co-Frobenius (QcF) coalgebras were introduced in [GTN] and further investigated in [GMN], as a natural dualization of **quasi-Frobenius algebras** (QF algebras), which are algebras $A$ that are injective, cogenerators and artinian as left $A$-modules, equivalently, all these conditions as right modules. However, in order to allow for infinite dimensional QcF coalgebras (and thus obtain more a general notion), the definition was weakened to the following: a coalgebra is said to be **left (right) quasi-co-Frobenius** (QcF) if it embeds in $\coprod I C^*$ (a “copower” of $C^*$, i.e. a coproduct of copies of $C^*$) as left (right) $C^*$-modules. These coalgebras were shown to have many properties that were the dual analogue of the properties of QF algebras. Again, this turned out not to be a left-right symmetric concept, and **QcF coalgebras** were introduced as the coalgebras which are both left and right QcF. Our first goal is to note that the results and techniques of [I] can be extended and applied to obtain a symmetric characterization of these coalgebras. In the first main result we show that a coalgebra is QcF if and only if $C$ is “weakly” isomorphic to $\text{Rat}(C^*)$ as left $C^*$-modules, in the sense that some (co)powers of these objects are isomorphic, and
this is equivalent to asking that $C^*$ is “weakly” isomorphic to $\text{Rat}(C^*_C)$ (its right rational dual) as right $C^*$-modules. In fact, it is enough to have an isomorphism of countable (co)powers of these objects. This also allows for a nice categorical characterization, which states that $C$ is QcF if and only if the above $C^*$-dual and $K$-dual functors are (again) “weakly” isomorphic. Besides realizing QcF coalgebras as a left-right symmetric concept which is simultaneously a generalization of Frobenius algebras, co-Frobenius co-algebras and co-Frobenius Hopf algebras, we note that this also provides another characterization of finite dimensional quasi-Frobenius algebras: $A$ is QF iff $A$ and $A^*$ are weakly isomorphic in the above sense, equivalently, $\prod A \simeq \prod A^*$.

Thus these results give a nontrivial generalization of Frobenius algebras and of quasi-Frobenius algebras, and the algebras arising as duals of QcF coalgebras are entitled to be called Generalized Frobenius Algebras, or rather Generalized QF Algebras.

These turn out to have a wide range of applications to Hopf algebras. In the theory of Hopf algebras, some of the first fundamental results concerned the characterization of Hopf algebras having a nonzero integral. These are in fact generalizations of well known results from the theory of compact groups. Recall that if $G$ is a (locally) compact group, then there is a unique left invariant (Haar) measure on $G$ and an associated integral $\int$. Let $R_c(G)$ be the algebra of continuous representative functions on $G$ and an associated integral $\int$. Let $R_c(G)$ be the algebra of continuous representative functions on $G$, i.e. continuous functions $f: G \to \mathbb{R}$ such that there are $f_i, g_i : G \to K$ for $i = 1, \ldots, n$ with $f(xy) = \sum_{i=1}^n f_i(x)g_i(y)$. This is a Hopf algebra with multiplication given by the usual multiplication of functions, comultiplication given by $f \mapsto \sum_{i=1}^n f_i \otimes g_i$, counit given by $\varepsilon(f) = f(1)$ and antipode $S$ given with multiplicative inverse in the group $G$, so $S(f)(x) = f(x^{-1})$. Then, the integral $\int$ of $G$ restricted to $R_c(G)$ becomes an element of $R_c(G)^*$ that has the following property: $\alpha \cdot \int = \alpha(1) \int$, with $1$ being the constant 1 function. An element with this property in a general Hopf algebra is called a left integral, and Hopf algebras (quantum groups) having a nonzero left integral can be viewed as (“quantum”) generalizations of compact groups. That is, the Hopf Algebra can be thought of as the algebra of continuous representative functions on some abstract quantum space. Among the first of the fundamental results in Hopf algebras were the facts that the existence of a left integral is equivalent to the existence of a right integral, and the existence of these integrals is equivalent to each of several (co)representation theoretic properties of the underlying coalgebra of $H$, including
left co-Frobenius, right co-Frobenius, left QcF, right QcF, and having nonzero rational dual to the left or to the right. These were results obtained in several early papers on Hopf algebras [LS, MTW, R, Su1, Sw1]. It was then natural to ask whether the integral in a Hopf algebra is unique. Since a scalar multiple of an integral is again an integral, the natural conjecture was: the space of left integrals $\int_l$ (or that of right integrals $\int_r$) is one dimensional. This would generalize the results from compact groups. The answer to this question turned out positive, and was proved by Sullivan in [Su1]; alternate proofs followed afterwards (see [Ra, St0]). Another very important result is that of Radford, who showed that the antipode of a Hopf algebra with nonzero integral is always bijective.

We re-obtain all these foundational results as a byproduct of our co-representation theoretic results and generalizations of Frobenius algebras. They will turn out to be an easy application of these general results. We also note a very short proof of the bijectivity of the antipode by constructing a certain derived comodule structure on $H$, obtained by using the antipode and the so called distinguished grouplike element of $H$, and the properties of the comodule $H^H$. The only use we make of the full Hopf algebra structure of $H$ is through the classical Fundamental theorem of Hopf modules which gives an isomorphism of $H$-Hopf modules $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$. However, even here we will only need to use that this is a isomorphism of comodules. We thus find almost purely representation theoretic proofs of all these classical fundamental results from the theory of Hopf algebras, which become immediate easy applications of the more general results on the “generalized Frobenius algebras”. Thus, the methods and results in this chapter are also intended to emphasize the potential of these representation theoretic approaches.

5.1 Quasi-co-Frobenius Coalgebras

Let $C$ be a coalgebra over a field $K$. Let $S$ be a set of representatives for the types of isomorphism of simple left $C$-comodules and $T$ be a set of representatives for the simple right $C$-comodules. It is well known that we have an isomorphism of left $C$-comodules (equivalently, right $C^*$-modules) $C \simeq \bigoplus_{S \in S} E(S)^{n(S)}$, where $E(S)$ is an injective envelope of the simple left $C$-comodule $S$ and each $n(S)$ is a positive integer. Similarly, $C \simeq \bigoplus_{T \in T} E(T)^{p(T)}$ in $\mathcal{M}^C$, with $p(T) > 0$. We use the same notation for injective envelopes of left
comodules and for those of right comodules, as the category will always be understood from context. Also \( C^\ast \cong \prod_{S \in S} E(S)^\ast \) in \( C^\ast \text{mod} \) and \( C^\ast \cong \prod_{T \in T} E(T)^\ast \) in \( \text{mod} - C^\ast \). We refer the reader to [A], [DNR] or [Sw1] for these results and other basic facts of coalgebras.

We will again use the finite topology on duals of vector spaces: given a vector space \( V \), this is the linear topology on \( V^\ast \) that has a basis of neighbourhoods of 0 formed by the sets \( F^\perp = \{ f \in V^\ast \mid f|_F = 0 \} \) for finite dimensional subspaces \( F \) of \( V \). We also write \( W^\perp = \{ x \in V \mid f(x) = 0, \forall f \in W \} \) for subsets \( W \) of \( V^\ast \). Any topological reference will be with respect to this topology.

For a module \( M \) and a set \( I \), we convey to write \( M^{(I)} \) for the \( I \)th copower of \( M \), i.e. the coproduct of \( I \) copies of \( M \), and \( M^I \) for the \( I \)th power of \( M \), i.e. the product of \( I \) copies of \( M \). We recall the following definition from [GTN]

**Definition 5.1.1** A coalgebra \( C \) is called a right (left) quasi-co-Frobenius (QcF) coalgebra, if there is a monomorphism \( C \hookrightarrow (C^\ast)^{(I)} \) of right (left) \( C^\ast \)-modules. It is called a QcF coalgebra if it is both a left and a right QcF coalgebra.

Recall that a coalgebra \( C \) is called right semiperfect if the category \( \mathcal{M}^C \) of right \( C \)-comodules is semiperfect, that is, every right \( C \)-comodule has a projective cover. This is equivalent to the following property on left comodules: \( E(S) \) is finite dimensional for all \( S \in S \) (see [L]). In fact, this is the definition we will need to use. For convenience, we also recall the following very useful results on injective and projective comodules:

[D1, Proposition 4] Let \( Q \) be a finite dimensional right \( C \)-comodule. Then \( Q \) is injective (projective) as a left \( C^\ast \)-module if and only if it is injective (projective) as right \( C \)-comodule.

[L, Lemma 15] Let \( M \) be a finite-dimensional right \( C \)-comodule. Then \( M \) is an injective right \( C \)-comodule if and only if \( M^\ast \) is a projective left \( C \)-comodule.

Here, if \( M \) is a finite dimensional right \( C \)-comodule with coaction given by \( \rho : M \rightarrow M \otimes C : m \mapsto m_0 \otimes m_1 \), the dual space \( M^\ast \) becomes a left comodule in the following way: for each \( m^\ast \in M^\ast \), there is a unique element \( m_{-1}^\ast \otimes m_0^\ast \in C \otimes M^\ast \) such that \( m^\ast (m_0)(m_1) = m_{-1}^\ast m_0^\ast (m) \). This can easily be shown by using dual bases on \( M \) and \( M^\ast \), or the isomorphism between \( M \) and \( M^{**} \). This structure also arises in the following way: since \( M \) is a right \( C \)-comodule, it is a rational left \( C^\ast \)-module, so \( M^\ast \) is a right \( C^\ast \)-module which turns out to be rational too, and so has a compatible left \( C \)-comodule structure.

We note the following proposition that will be useful in what follows; (i)\( \Leftrightarrow\) (ii) was given
in [GTN] and our approach also gives here a different proof, along with the new characterizations.

**Proposition 5.1.2** Let $C$ be a coalgebra. Then the following assertions are equivalent:

(i) $C$ is a right QcF coalgebra.

(ii) $C$ is a right torsionless module, i.e. there is a monomorphism $C \hookrightarrow (C^*)^I$ in mod-$C^*$.

(iii) There exist a dense morphism $\psi : C(I) \rightarrow C^*$ in $C^*$-mod, that is, the image of $\psi$ is dense in $C^*$.

(iv) $\forall S \in \mathcal{S}$, $\exists T \in \mathcal{T}$ such that $E(S) \cong E(T)^*$.

**Proof.** (i)$\Rightarrow$(ii) obvious.

(ii)$\Leftrightarrow$(iii) Let $\varphi : C \rightarrow (C^*)^I \cong (C(I))^*$ be a monomorphism of right $C^*$-modules. Let $\psi : C(I) \rightarrow C^*$ be defined by $\psi(x)(c) = \varphi(c)(x)$. It is straightforward to see that the fact that $\psi$ is a morphism of left $C^*$-modules is equivalent to $\varphi$ being a morphism of right $C^*$-modules, and that $\varphi$ injective is equivalent to $(\text{Im } \psi)^\perp = 0$, which happens if and only if to $\text{Im } \psi$ is dense in $C^*$ (for example, by [DNR] Corollary 1.2.9).

(ii),(iii)$\Rightarrow$(iv) As $\text{Im } \psi \subseteq \text{Rat}(C,C^*)$, we see that $\text{Rat}(C,C^*)$ is dense in $C^*$, so $C$ is right semiperfect by Proposition 3.2.1 [DNR]. Thus $E(S)$ is finite dimensional $\forall S \in \mathcal{S}$. Also by (ii) there is a monomorphism $\iota : E(S) \hookrightarrow \prod_{j \in J} E(T_j)^*$ for some set $J$ of simple right comodules $T_j \in \mathcal{T}$. As $\dim E(S) < \infty$ there is a monomorphism to a finite direct sum $E(S) \hookrightarrow \prod_{j \in F} E(T_j)^*$ with $F$ finite and $F \subseteq J$. Indeed, if $p_j$ are the projections of $\prod_{j \in J} E(T_j)^*$, then note that $\bigcap_{j \in J} \text{ker } p_j = 0$, so there must be $\bigcap_{j \in F} \text{ker } p_j \circ \iota = 0$ for a finite $F \subseteq J$. Then $E(S)$ is injective also as a right $C^*$-module (see for example [DNR], Corollary 2.4.19), and so $E(S) \oplus X = \bigoplus_{j \in F} E(T_j)^*$ for some $X$. By [I, Lemma 1.4.1], every $E(T_j)^*$ is local indecomposable, and we claim that, as they are also cyclic projective, we will find $E(S) \cong E(T_j)^*$ for some $j \in F$. This can be easily seen by noting first that there is at least one nonzero morphism $E(S) \hookrightarrow E(S) \oplus X = \bigoplus_{j \in F} E(T_j)^* \rightarrow \bigoplus_{j \in F} T_j^* \rightarrow S_k$ (one looks at Jacobson radicals). This projection then lifts to a morphism $f : E(S) \rightarrow E(T_k)^*$ as $E(S)$ is obviously projective. Now $f$ has to be surjective since $E(T_k)^*$ is cyclic local, and then $f$ splits. Hence $E(S) \cong E(T_k)^* \oplus Y$ with $Y = 0$ as $E(S)$ is indecomposable.

(iv)$\Rightarrow$(i) Any isomorphism $E(S) \cong E(T)^*$ implies $E(S)$ finite dimensional because then $E(T)^*$ is cyclic rational; therefore $E(T) \cong E(S)^*$. Thus for each $S \in \mathcal{S}$ there is exactly
one \( T \in \mathcal{T} \) such that \( E(S) \simeq E(T)^* \). If \( \mathcal{T}' \) is the set of these \( T \)'s, then:

\[
\begin{align*}
C & \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{\pi(S)} \hookrightarrow \bigoplus_{S \in \mathcal{S}} E(S)^{\pi(S)} \simeq \bigoplus_{T \in \mathcal{T}' \subseteq \mathcal{T}} (E(T)^*)^{\pi(T)} \\
& \hookrightarrow \left( \bigoplus_{T \in \mathcal{T}} (E(T)^*)^{\pi(T)} \right)^{(N)} \subseteq \left( \prod_{T \in \mathcal{T}} (E(T)^*)^{\pi(T)} \right)^{(N)} = C^{\pi(N)}
\end{align*}
\]

\( \Box \)

From the above proof, we see that when \( C \) is right QcF, the \( E(S) \)'s are finite dimensional projective for \( S \in \mathcal{S} \), and we also conclude the following result already known from [GTN] (in fact these conditions are even equivalent); see also [DNR, Theorem 3.3.4].

**Corollary 5.1.3** If \( C \) is right QcF, then \( C \) is also right semiperfect and projective as right \( C^* \)-module.

We also immediately conclude the following

**Corollary 5.1.4** A coalgebra \( C \) is QcF if and only if the function

\[
\{E(S) \mid S \in \mathcal{S}\} \rightarrow \{E(T) \mid T \in \mathcal{T}\} : Q \mapsto Q^*
\]

is well defined and bijective.

**Definition 5.1.5** (i) Let \( \mathcal{C} \) be a category having products. We say that \( M, N \in \mathcal{C} \) are **weakly \( \pi \)-isomorphic** if and only if there are some sets \( I, J \) such that \( M^I \simeq N^J \); we write \( M \overset{\pi}{\sim} N \).

(ii) Let \( \mathcal{C} \) be a category having coproducts. We say that \( M, N \in \mathcal{C} \) are **weakly \( \sigma \)-isomorphic** if and only if there are some sets \( I, J \) such that \( M^{(I)} \simeq N^{(J)} \); we write \( M \overset{\sigma}{\sim} N \).

The study of objects of a (suitable) category \( \mathcal{C} \) up to weak \( \pi \) (\( \sigma \))-isomorphism is the study of the localization of \( \mathcal{C} \) with respect to the class of all \( \pi \) (or \( \sigma \))-isomorphisms.

Recall that in the category \( \mathcal{C} \mathcal{M} \) of left comodules, coproducts are the usual direct sums of right \( C^* \)-modules and the product \( \prod \) is given, for a family of comodules \( (M_i)_{i \in \mathcal{I}} \), by

\[
\prod_{i \in \mathcal{I}} M_i = \text{Rat}(\prod_{i \in \mathcal{I}} M_i).
\]
For easy future reference, we introduce the following conditions:
(C1) $C \simeq \text{Rat}(C'_{\ast})$ in $\text{mod-}C$.
(C2) $C \simeq \text{Rat}(C'_{\ast})$ in $\text{mod-}C$.
(C3) $\text{Rat}(C!) \simeq \text{Rat}(C'_{\ast})$ for some sets $I, J$.
(C2') $C \simeq \text{Rat}(C'_{\ast})$ in mod-\(C\).

Lemma 5.1.6 Each of the conditions (C1), (C2), (C3), (C2') implies that $C$ is QcF.

The following technical proposition is key in the proof of this lemma. This is [I, Proposition 2.4], and we include it here for completeness.

Proposition 5.1.7 Let $E(T)$ be an infinite dimensional injective indecomposable right comodule. Suppose that there is an epimorphism $E \xrightarrow{\pi} E(T) \rightarrow 0$, such that $E = \bigoplus_{\alpha \in A} E_{\lambda}$ and $E_{\lambda}$ are finite dimensional injective right comodules. Then there is an epimorphism from a direct sum of finite dimensional injective right comodules to $E(T)$ with kernel containing no non-zero injective comodules.

Proof. Denote $H = \text{Ker} \pi$ and consider the set $\mathcal{N} = \{ Q \subset H \mid Q \text{ is an injective comodule} \}$. We see that $\mathcal{N} \neq \emptyset$ as $0 \in \mathcal{N}$ and that $\mathcal{N}$ is an inductive ordered set. To see this consider a chain $(X_i)_{i \in L}$ of elements of $\mathcal{N}$ and $X = \bigcup_{i \in L} X_i$ which is a subcomodule of $H$. Let $s(X) = \bigoplus_{\lambda \in \Lambda} S_{\lambda}$ be a decomposition into simple subcomodules of the socle of $X$. Then $s(X)$ is essential in $X$ and for every $\lambda \in \Lambda$ there is an $i = i(\lambda) \in L$ such $S_{\lambda} \subset X_{i(\lambda)}$. As $X_i$ is injective, there is an injective envelope $H_\lambda$ of $S_{\lambda}$ that is contained in $X_i$.

First we prove that the sum $\sum_{\lambda \in \Lambda} H_\lambda$ is direct. To see this it is enough to prove that $H_{\lambda_0} \cap (\sum_{\lambda \in F} H_\lambda) = 0$, for every finite subset $F \subseteq \Lambda$ and $\lambda_0 \in \Lambda \setminus F$. We prove this by induction on the cardinal of $F$. If $F = \{ \lambda \}$ then $H_{\lambda_0} \cap H_\lambda = 0$ because otherwise we would have $S_\lambda = S_{\lambda_0}$, a contradiction. If the statement is proved for all sets with at most $n$ elements and $F$ is a set with $n + 1$ elements then the sum $\sum_{\lambda \in F} H_\lambda$ is direct, because $F = (F \setminus \{ \lambda \}) \cup \{ \lambda \}$ for every $\lambda \in F$ and we apply the induction hypothesis.

If $H_{\lambda_0} \cap (\sum_{\lambda \in F} H_\lambda) \neq 0$ we get that $S_{\lambda_0} \subseteq \sum_{\lambda \in F} H_\lambda$, because $S_{\lambda_0}$ is essential in $H_{\lambda_0}$. But as the sum $\sum_{\lambda \in F} H_\lambda$ is direct we have that $s(\sum_{\lambda \in F} H_\lambda) = s(\bigoplus_{\lambda \in F} H_\lambda) = \bigoplus_{\lambda \in F} s(H_\lambda) = \bigoplus_{\lambda \in F} S_{\lambda}$ so $S_{\lambda_0} \subset s(\sum_{\lambda \in F} H_\lambda) = \bigoplus_{\lambda \in F} S_{\lambda}$ which is a contradiction with $\lambda_0 \notin F$.

Now notice that $X = \bigoplus_{\lambda \in \Lambda} H_\lambda$. Since $\bigoplus_{\lambda \in \Lambda} H_\lambda$ is injective, it is a direct summand of $X$. Write
X = (\bigoplus_{\lambda \in A} H_\lambda) \oplus H'$ and suppose $H' \neq 0$. Take $S' \subseteq H'$ a simple subcomodule of $H'$. Then $S' \subseteq s(X) = \bigoplus_{\lambda \in A} S_\lambda \subseteq \bigoplus_{\lambda \in A} H_\lambda$ which is a contradiction. We conclude that $X$ is injective, thus $X \in \mathcal{N}$.

By Zorn’s Lemma we can then take $M$ a maximal element of $\mathcal{N}$. As $M$ is an injective comodule, it is a direct summand of $H$ and take $M \oplus H' = H$. It is obvious that $H$ is essential in $E = \bigoplus_{\alpha \in A} E_\alpha$, because otherwise taking $E(H)$ an injective envelope of $H$ contained in $E$, we would have $E(H) \oplus Q = E$ so $E(T) \cong E(H) \oplus Q/H \cong (E(H)/H) \oplus Q$ which is a contradiction as $Q$ is a direct sum of finite dimensional comodules and $E(T)$ is indecomposable infinite dimensional. Take $E'$ an injective envelope of $H'$ contained in $E$. If $M \oplus E' \not\subseteq E$ then there is a simple comodule $S$ contained in $E$ and such that $S \cap (M \oplus E') = 0$, because $M \oplus E'$ is a direct summand of $E$ as it is injective. Then $S \cap H = 0$, since $H \subseteq M \oplus E'$, which contradicts the fact that $H \subseteq E$ is an essential extension. Consequently, $M \oplus E' = E$ and then

$$E(T) \cong \frac{E}{H} = \frac{M \oplus E'}{M \oplus H'} = \frac{E'}{H'}$$

where $E'$ is a direct sum of finite dimensional injective indecomposable modules and $H'$ does not contain non-zero injective modules because of the maximality of $M$. \hfill \Box

Proof. of Lemma 5.1.6. Obviously (C2')\((\Rightarrow)\)(C2). In all of the above conditions (C1, C2, C3, C2') one can find a monomorphism of right $C^*$-modules $C \hookrightarrow (C^*)^J$, and thus $C$ is right QcF. Then each $E(S)$ for $S \in \mathcal{S}$ is finite dimensional and projective by Corollary 5.1.3. We first show that $C$ is also left semiperfect, along the same lines as the proofs of [I], Proposition 2.1 and [I] Proposition 2.6. For completeness, we include a short version of these arguments here. Let $T_0 \in \mathcal{T}$ and assume, by contradiction, that $E(T_0)$ is infinite dimensional. We first show that $\text{Rat}(E(T_0)^*) = 0$. Indeed, assume otherwise. Then, since $C^* = \prod_{T \in \mathcal{T}} E(T)^{\nu(T)}$ and $C = \bigoplus_{S \in \mathcal{S}} E(S)^{\nu(S)}$ as right $C^*$-modules, it is straightforward to see that each of conditions (C1-C3) implies that $\text{Rat}(E(T_0)^*)$ is injective as a left comodule, as a direct summand in an injective comodule. Thus, as $\text{Rat}(E(T_0)^*) \neq 0$, there is a monomorphism $E(S) \hookrightarrow \text{Rat}(E(T_0)^*) \subseteq E(T_0)^*$ for some indecomposable injective $E(S)$ with $S \in \mathcal{S}$. This shows that $E(S)$ is a direct summand in $E(T_0)^*$, since $E(S)$ is injective also as right $C^*$-module (by the above cited [D1, Proposition 4]). But this is a contradiction since $E(S)$ is finite dimensional and $E(T_0)^*$ is indecomposable by [I, Lemma 1.4] and
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\[ \dim E(T_0)^* = \infty. \] Thus, \( \text{Rat}(E(T_0)^*) = 0 \) as claimed.

Next, use [I, Proposition 2.3] to get an exact sequence

\[ 0 \to H \to E = \bigoplus_{\alpha \in A} E(S_{\alpha})^* \to E(T) \to 0 \]

with \( S_{\alpha} \in S \). Since each \( E(S_{\alpha})^* \) is injective in \( C^*\)-mod by [L, Lemma 15], we may assume, by the above Proposition 5.1.7, that \( H \) contains no nonzero injective right comodules. For some \( \beta \in A \neq \emptyset \), put \( E' = \bigoplus_{\alpha \in A \setminus \{\beta\}} E(S_{\alpha})^* \). Then we claim that \( H + E' = E \). Otherwise, since there is an epimorphism \( E(T) = E/H \to E/H + E' \), the finite dimensional rational right \( C^*\)-module \( (E/H + E')^* \) would be a nonzero rational submodule of \( E(T)^* \). Now, since \( H + E' = E \), we have an epimorphism \( H \to H/H \cap E' \cong H/H/E' \cong E(S_{\beta})^* \). But \( E(S_{\beta})^* \) is projective, so this epimorphism splits, and this contradicts the assumption on \( H \), since \( E(S_{\beta})^* \) is injective in \( C^*\)-mod. \( \square \)

Now, we note that if a coalgebra \( C \) is QcF, then all the conditions (C1)-(C3) are fulfilled. Indeed, we have that each \( E(S) \) \( (S \in S) \) is isomorphic to exactly one \( E(T)^* \) \( (T \in T) \) and each \( E(T)^* \) is isomorphic to some \( E(S) \). Then, (C1) follows from

\[ C^{(\mathbb{N})} = \bigoplus_{S \in S} E(S)^{(n)} = \bigoplus_{T \in T} E(T)^{(n)} = \bigoplus_{T \in T} (E(T)^*(n))^0 = (\text{Rat} C^*)^{(\mathbb{N})} \]

where we use that \( \text{Rat}(C^*) = \bigoplus_{T \in T} E(T)^* \) as right \( C^*\)-modules for left and right semiperfect coalgebras (see [DNR, Corollary 3.2.17]). For (C2),

\[ \prod_{\mathbb{N}} C = \text{Rat}(C^{(\mathbb{N})}) = \prod_{\mathbb{N}} \bigoplus_{S \in S} E(S)^{(n(S))} \]

\[ = \prod_{\mathbb{N}} \prod_{S \in S} E(S)^{(n(S))} \]

\[ = \prod_{S \in S} E(S)^{(n(S))} \]

\[ = \prod_{T \in T} E(T)^{(n(T))} = \prod_{T \in T} E(T)^{\mathbb{N}} \]
\[ \prod_{T \in \mathcal{T}} E(T)^* \times \mathbb{P}(T) = \prod_{N} C \prod_{T \in \mathcal{T}} E(T)^* \mathbb{P}(T) \]

\[ = \prod_{N} \text{Rat}(\prod_{T \in \mathcal{T}} (E(T)^{\mathbb{P}(T)})^*) = \prod_{N} \text{Rat}(\bigoplus_{T \in \mathcal{T}} E(T)^{\mathbb{P}(T)})^* \]

\[ = \prod_{N} \text{Rat}(C^*) \]

where for (*) we have used [I, Lemma 2.5] and the fact that \( E(T)^* \) are all rational since each \( E(T) \) is finite dimensional in this case and \( \prod \) is the rational part of the product in mod-\( C^* \). Finally, (3) holds because \( \text{Rat}(C^N) = \prod_{T \in \mathcal{T}} E(T)^* \mathbb{N} \) by the computations in lines 1 and 3 in the above equation and because

\[ \text{Rat}(C^N) = \text{Rat}(\prod_{N} \prod_{T \in \mathcal{T}} E(T)^* \mathbb{P}(T)) = \prod_{N} E(T)^* \mathbb{N} = \prod_{T \in \mathcal{T}} E(T)^* \mathbb{N} \]

Combining all of the above we obtain the following nice symmetric characterization which extends the one for co-Frobenius coalgebras in [I] and those for co-Frobenius Hopf algebras and Frobenius Algebras.

**Theorem 5.1.8** Let \( C \) be a coalgebra. Then the following assertions are equivalent.

(i) \( C \) is a QcF coalgebra.

(ii) \( C \simeq \text{Rat}(C_{c^*}^*) \) in \( C\mathcal{M} \).

(iii) \( C \simeq \text{Rat}(C_{c^*}^*) \) in \( C\mathcal{M} \)

(iv) \( \text{Rat}(C^I) \simeq \text{Rat}(C^J) \) in \( C\mathcal{M} \) (or mod-\( C^* \)) for some sets \( I, J \).

(v) \( C(N) \simeq \text{Rat}(C^*)^{(N)} \) as left \( C \)-comodules (right \( C^* \)-modules)

(vi) \( \prod_{N} C \simeq \prod_{N} \text{Rat}(C^*) \) as left \( C \)-comodules (right \( C^* \)-modules)

(vii) \( \text{Rat}(C^N) \simeq \text{Rat}(C^*)^N \) as left \( C \)-comodules (right \( C^* \)-modules)

(viii-xiii) The right comodules versions of (ii)-(vii).

(xiv) The association \( Q \mapsto Q^* \) determines a duality between the finite dimensional injective left comodules and finite dimensional injective right comodules.
5.1.1 Categorical characterization of QcF coalgebras

We give now a characterization similar to the functorial characterizations of co-Frobenius coalgebras and of Frobenius algebras. For a set $I$ let $\Delta_I : C \mathcal{M} \rightarrow (C \mathcal{M})^I$ be the diagonal functor and let $F_I$ be the composite functor

$$F_I : C \mathcal{M} \xrightarrow{\Delta_I} (C \mathcal{M})^I \xrightarrow{\bigoplus} C \mathcal{M}$$

that is $F_I(M) = M^{(I)}$ for any left $C$-comodule $M$.

**Theorem 5.1.9** Let $C$ be a coalgebra. Then the following assertions are equivalent:

(i) $C$ is QcF.

(ii) The functors $\text{Hom}_C(-, C^*) \circ F_J$ and $\text{Hom}_K(-, K) \circ F_I$ from $C \mathcal{M} = \text{Rat}(C^*\text{-mod})$ to $C^*\text{-mod}$ are naturally isomorphic for some sets $J, I$.

(iii) The functors $\text{Hom}_C(-, C^*) \circ F_N$ and $\text{Hom}(-, K) \circ F_N$ are naturally isomorphic.

**Proof.** Since for any left comodule $M$, there is a natural isomorphism of left $C^*$-modules $\text{Hom}_C(M, C) \simeq \text{Hom}_K(M, K)$, then for any sets $I, J$ and any left $C$-comodule $M$ we have the following natural isomorphisms:

$$\text{Hom}_K(M^{(I)}, K) \simeq \text{Hom}_C(M^{(I)}, C) \simeq \text{Hom}_C(M, C^I) \simeq \text{Hom}_C(M, \text{Rat}(C^I))$$

$$\text{Hom}_C(M^{(I)}, C^*) \simeq \text{Hom}_C(M, (C^*)^I) \simeq \text{Hom}_C(M, \text{Rat}((C^*)^I))$$

Therefore, by the Yoneda Lemma, the functors of (ii) are naturally isomorphic if and only if $\text{Rat}(C^I) \simeq \text{Rat}(C^*J)$. Thus, by Theorem 5.1.8 (ii), these functors are isomorphic if and only if $C$ is QcF. Moreover, in this case, by the same theorem the sets $I, J$ can be chosen countable. \hfill \Box

**Remark 5.1.10** The above theorem states that $C$ is QcF if and only if the $C^*$-dual functor $\text{Hom}_{C^*}(-, C^*)$ and $K$-dual $\text{Hom}(-, K)$ functor from $C \mathcal{M}$ to $C^*\text{-mod}$ are isomorphic in a “weak” sense, namely they are isomorphic only on the objects of the form $M^{(N)}$ in a way that is natural in $M$, i.e. they are isomorphic on the subcategory of $C \mathcal{M}$ consisting of objects $M^{(N)}$ with morphisms $f^{(N)}$ induced by any $f : M \rightarrow N$. If we consider the category $C$ of functors from $C \mathcal{M}$ to $C^*\text{-mod}$ with morphisms the classes (which are not necessarily sets) of natural transformations between functors, then the isomorphism in (ii) can be restated
5.2. co-Frobenius coalgebras and Applications to Hopf Algebras

We now present an interesting characterization of left co-Frobenius coalgebras which shows what is the difference between them and left QcF coalgebras. As we will see, roughly speaking, co-Frobenius is QcF plus a certain condition connecting the multiplicities of left and right injective indecomposable comodules. First, we present a Lemma which is very useful in computations and helpful also in the next chapter for understanding the spaces of algebraic integrals introduced there. Therefore, although the proof is standard and also follows from the Hom-Tensor relations, we include it here.

Lemma 5.2.1 If $M \in \mathcal{M}_C$ and $N \in \mathcal{M}_C$ then $\text{Hom}_C^{\ast}(M, N^\ast) \cong \text{Hom}_C^{\ast}(N, M^\ast)$: $\varphi \mapsto \psi$, where $\psi(n)(m) = \varphi(m)(n)$ naturally in $M$ and $N$; more precisely $\text{Hom}(C^\ast, M^\ast C^\ast) \cong \text{Hom}(N C^\ast, M^\ast C^\ast)$.

Proof. The same formula defines the inverse function and gives a bijection of sets provided we show that $\varphi$ is a morphism of left $C^\ast$-modules if and only if $\psi$ is a morphism of right $C^\ast$-modules. For this, let $c^\ast \in C^\ast$, $m \in M$ and $n \in N$ and note that

$$(c^\ast \cdot \varphi(m))(n) = \varphi(c^\ast \cdot m)(n) \iff c^\ast(n_{-1})\varphi(m)(n_0) = \varphi(c^\ast(m_1)m_0)(n)$$

$$(\psi(c^\ast(n_{-1})n_0)(m) = \psi(n)(m_0)c^\ast(m_1)$$

$$(\psi(n \cdot c^\ast)(m) = (\psi(n) \cdot c^\ast)(m)$$

and this proves the statement. Finally, the naturality in $M$ and $N$ is also a straightforward computation which we omit here. \hfill \square

Proposition 5.2.2 Let $C$ be a left QcF coalgebra and $\sigma : T \rightarrow S$ be defined such that $\sigma(T) = S$ if and only if $E(T) \cong E(S)^\ast$ (this is well defined by the above Proposition). Then $C$ is left co-Frobenius if and only if $n(\sigma(T)) \geq p(T)$, for all $T \in T$.

Proof. If $C \hookrightarrow C^\ast$ in $C^\ast$-mod then for each $T \in T$ there is a monomorphism $\varphi : E(T)^{p(T)} \hookrightarrow \prod_{S \in S} E(S)^{n(S)}$. Now, $C$ is semiperfect since it is left QcF (see [GTN]) and so
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Each $E(T)$ is finite dimensional. Therefore, as before, we may again find a finite subset $F$ of $S$ and a monomorphism $E(T)^{p(T)} \hookrightarrow \bigoplus_{S \in F} E(S)^*$ and, as in the proof of Proposition 5.1.2 we get that for some finite $F \subseteq S$ and $Y$:

$$E(T)^{p(T)} \oplus Y = \bigoplus_{S \in F} E(S)^{\ast n(S)}$$

Again, since all the $E(S)^*$ are local cyclic indecomposable, we get that $E(T) \simeq E(S)^*$ for some $S \in F$; moreover, there have to be at least $p(T)$ indecomposable components isomorphic to $E(T)$ on the right hand side of the above equation. But since $E(S)^*$ and $E(S')^*$ are not isomorphic when $S$ and $S'$ are not, we conclude that we must have $n(S) \geq p(T)$ for the $S$ with $E(S)^* \simeq E(T)$, i.e. $S = \sigma(T)$ and $n(\sigma(T)) \geq p(T)$.

Conversely, if $p(T) \leq n(\sigma(T))$ we have monomorphisms of left $C^*$-modules

$$C = \bigoplus_{T \in T} E(T)^{p(T)} \hookrightarrow \bigoplus_{T \in T} E(T)^{n(\sigma(T))} \hookrightarrow \bigoplus_{S \in S} E(S)^{\ast n(S)} \subseteq \prod_{S \in S} E(S)^{\ast n(S)} = C^*$$

Let $C_S = \sum_{S' \simeq S, S' \leq C} S'$ be the simple subcoalgebra of $C$ associated to $S$. Then $C_S$ is a simple coalgebra which is finite dimensional, and $C_S \simeq S^{n(S)}$. The dual algebra $C_S^*$ of $C_S$ is a simple finite dimensional algebra, $C_S^* = (S^*)^{n(S)}$ as left $C_S^*$-modules (or $C^*$-modules) and thus $C_S^* \simeq M_{n(S)}(\Delta_S)$, where $\Delta_S = \text{End}_{C^*}(S^*)$ is a division algebra. By Lemma 5.2.1 we also have $\Delta_S \simeq \text{End}(S_{C^*})$, and it is easy to see that the isomorphism in Lemma 5.2.1 also preserves the multiplicative structure thus giving an isomorphism of algebras. Let $d(S) = \dim(\Delta_S)$. Then, as $C_S^* \simeq M_{n(S)}(\Delta_S) = (S^*)^{n(S)}$, we have $\dim(C_S) = \dim(C_S^*) = d(S) \cdot n(S)^2 = n(S) \cdot \dim S$ and therefore $\dim(S) = \dim(S^*) = n(S)d(S)$. For a right simple comodule $T$ denote $d'(T) = \dim(\text{End}(C^*T))$; note that $d'(T) = d(T^*)$ since $\text{End}(C^*T) \simeq \text{End}(T^*_{C^*})$ by the same Lemma 5.2.1. Similarly for right simple comodules $T$, $\dim(T) = d'(T)p(T)$. Then we also have $p(T) = n(T^*)$. Denote by $C_0$ the coradical of $C$; then we have that $C_0 = \bigoplus_{S \in S} C_S$.

Remark 5.2.3 Let $C$ be a left $QcF$ coalgebra, and assume that $\text{End}(S) = K$ for all simple left (equivalently, right) comodules $S$ (for example, this is true if $C$ is pointed or the base-field $K$ is algebraically closed). Then $C$ is left co-Frobenius if and only if $\dim(\text{soc}(E)) \leq \dim(\text{cosoc}(E))$ for every finite dimensional indecomposable injective right comodule $E$, ...
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where cosoc($E$) represents the cosocle of $E$. Indeed, in this case, $d(S) = 1 = d'(T)$ and if $E(T) \simeq E(S)^*$, then $S^* = \text{cosoc}(E(T))$, so $n(\sigma(T)) = n(S) = \dim(S) = \dim(\text{cosoc}(E(T)))$ and $p(T) = \dim(T) = \dim(\text{soc}(E(T)))$.

Before giving the main applications to Hopf algebras, we start with two easy propositions that will contain the main ideas of the applications. First, for a QcF coalgebra $C$

(i) Let Proposition 5.2.4 $C$ be a QcF coalgebra such that

\[ \bigoplus_{S \in S} E(S)^n(S) \simeq \bigoplus_{S \in S} E(T)^{n(S) \times I} \simeq \bigoplus_{S \in S} E(S)^{p(\varphi(S))} \]

for all $S \in \mathcal{S}$; similarly, as $C$ is also left co-Frobenius we get $n(S) \geq p(\varphi(S))$ for all $S \in \mathcal{S}$. Hence $n(S) = p(\varphi(S))$ for all $S \in \mathcal{S}$ and this implies $C = \bigoplus_{S \in S} E(S)^n(S) \simeq \bigoplus_{T \in T} E(T)^{p(T)} = \text{Rat}(C_{C*})$. Conversely, if $C \simeq \text{Rat}(C_{C*})$ as right $C^*$-modules then, by the proof of (i), when $I$ and $J$ have one element, we get that $n(S) = p(\varphi(S))$ for all $S \in \mathcal{S}$, which implies that we also have $C = \bigoplus_{T \in T} E(T)^{p(T)} \simeq \bigoplus_{S \in S} E(S)^{n(S)} = \text{Rat}(C_{C*})$ as left $C^*$-modules, so $C$ is co-Frobenius.

The above Proposition 5.2.4 (ii) shows that the results of this chapter are a generalization of the results in [1].

Proposition 5.2.5 Let $C$ be a QcF coalgebra such that $C^k \simeq \text{Rat}(C_{C*})$ in $\text{mod-}C^*$ and $C^l \simeq \text{Rat}(C_{C*})$ in $C^*\text{-mod}$ for some $k, l \in \mathbb{N}$. Then $C$ is co-Frobenius and $k = l = 1$. 

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Proof. As in the proof of Proposition 5.2.4 we get $k \cdot n(S) = p(\varphi(S))$ for all $S \in \mathcal{S}$. Similarly, using $C^l \simeq (\text{Rat}(C^\cdot C^*))$ in $C^*-\text{mod}$ we get $l \cdot p(T) = n(\varphi^{-1}(T))$ for $T \in \mathcal{T}$ so $n(S) = l \cdot p(\varphi(S))$. These two equations give $k = l = 1$ and the conclusion follows as in Proposition 5.2.4 (ii). □

Let $H$ be a Hopf algebra over a basefield $K$. Recall that a left integral for $H$ is an element $\lambda \in H^*$ such that $\alpha \cdot \lambda = \alpha(1) \lambda$, for all $\alpha \in H^*$. The space of left integrals for $H$ is denoted by $\int_l$. The right integrals and the space of right integrals $\int_r$ are defined analogously. As mentioned in the introduction to this chapter, we shall need the fundamental theorem of Hopf modules which provides an isomorphism of right $H$-Hopf modules

$$\int_l \otimes H \simeq \text{Rat}(H^*H^*) : t \otimes h \mapsto t \leftarrow h = S(h) \mapsto t$$

where for $x \in H$ and $\alpha \in H^*$ we define $x \rightarrow \alpha$ by $(x \rightarrow \alpha)(y) = \alpha(yx)$ and $\alpha \leftarrow x = S(x) \rightarrow \alpha$. In fact, we will only need that this is an isomorphism of right $H$-comodules (left $H^*$-modules). Similarly, $H \otimes \int_r \simeq \text{Rat}(H_H^*)$.

**Theorem 5.2.6** (Lin, Larson, Sweedler, Sullivan)

If $H$ is a Hopf algebra, then the following assertions are equivalent.

(i) $H$ is a right co-Frobenius coalgebra.

(ii) $H$ is a right QcF coalgebra.

(iii) $H$ is a right semiperfect coalgebra.

(iv) $\text{Rat}(H^*H^*) \neq 0$.

(v) $\int_l \neq 0$.

(vi) $\dim \int_l = 1$.

(vii-xii) The left-right symmetric version of the above (i-vi), respectively.

**Proof.** (i)⇒(ii)⇒(iii) is clear and (iii)⇒(iv) is a property of semiperfect coalgebras (see [DNR, Section 3.2]).

(iv)⇒(v) follows by the isomorphism $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$ and (vi)⇒(v) is trivial.

(v)⇒(i), (vi) and (vii-xii). Since $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$ in $\mathcal{M}^H$, we have $H^{(j)} \simeq \text{Rat}(H^*H^*)$ so, by Theorem 5.1.8, $H$ is both left and right QcF and it then follows that $\int_r \neq 0$ (by the left hand side version of (ii)⇒(v)) and $H^{(j)} \simeq \text{Rat}(H^*_H^*)$. We can now apply Propositions 5.2.4 and 5.2.5 to get first that $\dim \int_l < \infty$, $\dim \int_r < \infty$ and then that $H$ is both left and right co-Frobenius, so (i) and (vii) hold. Again by Proposition 5.2.5 we get that, more
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precisely, \( \dim I_l = \dim I_r = 1 \).

By symmetry, we also have (vii) \( \Rightarrow \) (viii) \( \Rightarrow \) (x) \( \Rightarrow \) (xi) \( \Rightarrow \) (xii) \( \Rightarrow \) (vii) \( \Rightarrow \) (i), so the proof is finished. \( \square \)

The following corollary was the initial form of the result proved by Sweedler [Sw2]

**Corollary 5.2.7** The following are equivalent for a Hopf algebra \( H \):

(i) \( H^* \) contains a finite dimensional left ideal.
(ii) \( H \) contains a left coideal of finite codimension.
(iii) \( \int_l \neq 0 \).
(iv) \( \text{Rat}(H^*H^*) \neq 0 \).

**Proof.** (i) \( \Leftrightarrow \) (ii) It can be seen by a straightforward computation that there is a bijective correspondence between finite dimensional left ideals \( I \) of \( H^* \) and coideals \( L \) of finite codimension in \( H \), given by \( I \mapsto L = I^\perp \). Moreover, it follows that any such finite dimensional ideal \( I \) of \( H^* \) is of the form \( I = L^\perp \) with \( \dim(H/L) < \infty \), so \( I = L^\perp \simeq (H/L)^* \) is then a rational left \( H^* \)-module, thus \( I \subseteq \text{Rat}(H^*H^*) \). This shows that (ii) \( \Rightarrow \) (iv) also holds, while (iii) \( \Rightarrow \) (ii) is trivial, and (vi) \( \Rightarrow \) (iii) is contained in the last theorem. \( \square \)

The bijectivity of the antipode

Let \( t \) be a nonzero left integral for \( H \). Then it is easy to see that the one dimensional vector space \( Kt \) is a two sided ideal of \( H^* \). Also, by the definition of integrals, \( Kt \subseteq \text{Rat}(H^*H^*) = \text{Rat}(H^*_{H^*}) \), since \( H \) is semiperfect as a coalgebra (see [DNR, chapter 3]).

Thus \( Kt \) also has a left comultiplication \( Kt \to H \otimes Kt : t \mapsto a \otimes t \), for some \( a \in H \) and then the coassociativity and counit property for \( H^*Kt \) imply that \( a \) has to be a grouplike element. This element is called the distinguished grouplike element of \( H \). In particular \( t \cdot \alpha = \alpha(a)t \), for all \( \alpha \in H^* \). See [DNR, Chapter 5] for some more details.

For any right \( H \)-comodule \( M \) denote by \( ^aM \) the left \( H \)-comodule structure on \( M \) with comultiplication

\[
M \to H \otimes M : m \mapsto m_{-1}^a \otimes m_0^a = aS(m_1) \otimes m_0
\]

(\( S \) denotes the antipode). It is straightforward to see that this defines an \( H \)-comodule structure.
Proposition 5.2.8 The map $p : {}^aH \to \text{Rat}(H^*)$, $p(x) = x \mapsto t$ is a surjective morphism of left $H$-comodules (right $H^*$-modules).

Proof. Since the injectivity of $H \simeq \int_l \otimes H \xrightarrow{\sim} \text{Rat}(H^*)$ is given by $h \mapsto t \leftarrow h = S(h) \mapsto t$, we get the surjectivity of $p$. We need to show that $p(x)_{−1} \otimes p(x)_0 = x^a_{−1} \otimes p(x^a_0)$. For this, having the left $H$-comodule structure of $\text{Rat}(H^*)$ in mind, it is enough to show that for all $\alpha \in H^*$, we have $p(x)_0\alpha(p(x)_{−1}) = p(x) \cdot \alpha = \alpha(x^a_{−1})p(x^a_0)$. Indeed, for $g \in H$ we have:

$$(x \mapsto t) \cdot \alpha)(g) = t(g_1x)\alpha(g_2) = t(g_1x_1\varepsilon(x_2))\alpha(g_3) = t((g_1x_1)(\alpha \leftarrow x_3))(g_2x_2) = t((g_1x_1)(\alpha \leftarrow x_2)((g_1x_2)) = (t \cdot (\alpha \leftarrow x_2))(g_1) = (\alpha \leftarrow x_2)(a)t(g_2x_1) \quad (a \text{ is the distinguished grouplike of } H) = \alpha(aS(x_2))(x_1 \mapsto t)(g)$$

and this ends the proof. \hfill \Box

Let $\pi$ be the composite map $^aH \xrightarrow{p} \text{Rat}(H^*_H) \xrightarrow{\sim} H \otimes \int_r \simeq H$, where the isomorphism $H \otimes \int_r \simeq \text{Rat}(H^*_H)$ is defined analoguously to $\int_l \otimes H \simeq \text{Rat}(H^*H^*)$. Since $^H^H$ is projective in $^H_M$, this surjective map splits by a morphism of left $H$-comodules $\varphi : H \leftarrow ^aH$, so $\pi\varphi = \text{Id}_H$. This leads to another proof of:

Theorem 5.2.9 The antipode of a co-Frobenius Hopf algebra is bijective.

Proof. Since the injectivity of $S$ is immediate from the injectivity of the map $H \to H^* : x \mapsto t \leftarrow x$, as noticed by Sweedler [Sw2], we only observe the surjectivity. The fact that $\varphi$ is a morphism of comodules means $\varphi(x)_{−1} \otimes \varphi(x)_0 = x_1 \otimes \varphi(x_2)$, i.e. $aS(\varphi(x)_2) \otimes \varphi(x)_1 = x_1 \otimes \varphi(x_2)$. Since $a = S(a^{-1}) = S^2(a)$, by applying $\text{Id} \otimes \varepsilon\pi$ we get $S(a^{-1})S(\varphi(x)_2)\varepsilon\pi(\varphi(x)_1) = x_1\varepsilon\pi\varphi(x_2) = x_1\varepsilon(x_2) = x$, so $x = S(\varepsilon\pi(\varphi(x)_1)\varphi(x)_2a^{-1})$. \hfill \Box
Chapter 6

Abstract integrals in algebra

Introduction

Let $G$ be a compact group. It is well known that there is a unique (up to multiplication) left invariant Haar measure $\mu$ on $G$, and a unique left invariant Haar integral. If $H$ is a Hopf algebra over a field $K$, an element $\lambda \in H^*$ is called a left integral for $H$ if $\alpha \lambda = \alpha(1)\lambda$ for all $\alpha \in H^*$. For a compact group $G$, let $R_c(G)$ be the $\mathbb{C}$-coalgebra (Hopf algebra) of representative functions on $G$, consisting of all $f : G \to \mathbb{C}$ such that there are continuous $u_i, v_i : G \to \mathbb{C}$ with $i = 1, \ldots, n$ such that $f(xy) = \sum_{i=1}^{n} u_i(x)v_i(y)$ for all $x, y \in G$. Then the Haar integral, restricted to $R_c(G)$, is an integral in the Hopf algebra (coalgebra) sense (see for example [DNR, Chapter 5]). The uniqueness of integrals for compact groups has a generalization for Hopf algebras: if a nonzero (left) integral exists in $H$, then it was shown by Radford [R] that it is unique, in the sense that the dimension of the space of left integrals equals 1.

For a Hopf algebra $H$, it is easy to see that a left integral $\lambda$ is the same as a morphism of right $H$-comodules (left $H^*$-modules) from $H$ to the right $H$-comodule $K$ with coaction $K \to K \otimes H : a \mapsto a \otimes 1_H$. Then it is natural to generalize this definition to arbitrary finite dimensional right $H$-comodule $M$ by putting $\int_M = \text{Hom}_{H^*}(H, M)$. The advantage of this definition is that it can be considered for arbitrary coalgebras, where in contrast to the Hopf algebra case, there is no canonical comodule structure on $K$. We give an explicit description of the space of these generalized integrals for the case of the representative coalgebra (Hopf algebra) of a (locally) compact group and also give an interpretation at
the group level. More precisely, we will consider vector-valued integrals, \( \int : C(G) \to \mathbb{C}^n = V \) (or \( \int : L^1(G) \to \mathbb{C}^n \)) with the “quantum-invariance” property: there is a function \( \eta : G \to \text{End}(V) = \text{End}(\mathbb{C}^n) \) such that \( x \cdot f = \eta(x) \cdot (\int f) \) for every \( f \in R_c(G) \) and \( x \in G \).

It turns out that \( \eta \) must actually be a representation of \( G \). Then \( V \) with the left \( G \)-action is turned naturally into a right \( R_c(G) \)-comodule and the integral restricted to \( R_c(G) \) turns out to be an algebraic integral in the above sense, that is, \( \int \in \text{Hom}^{R_c(G)}(R_c(G), V) \).

We note that in the case of a locally compact group \( G \), the coalgebra structure of \( R_c(G) \) is the one encoding the information of representative functions, and so of the group itself: the comultiplication of \( R_c(G) \) is defined by \( \Delta(f) = \sum_{i=1}^n u_i \otimes v_i \), where \( f(xy) = \sum_{i=1}^n u_i(x)v_i(y) \) for all \( x, y \in G \). The algebra structure is given by \( (f \ast g)(x) = f(x)g(x) \), and this comes by “dualizing” the comultiplication of the coalgebra structure of \( \mathbb{C}[G] \), which is defined as \( x \mapsto x \otimes x \) for \( x \in G \). Since this coalgebra structure does not involve the group structure of \( G \) in any way (\( G \) might as well be a set), it is to be expected that only the coalgebra structure of \( R_c[G] \) will encapsulate information on \( G \). This suggests that a generalization of the results on existence and uniqueness of integrals should be possible for the case of coalgebras.

With this in mind, we generalize the existence and uniqueness results from Hopf algebras to the pure coalgebraic setting. For a coalgebra \( C \) and a finite dimensional right \( C \)-comodule \( M \) we define the space of left integrals \( \int_{l,M} = \text{Hom}_{C^*}(C, M) \) and similarly for left \( C \)-comodules \( N \) let \( \int_{r,N} = \text{Hom}_{C^*}(C, N) \) be the space of right integrals. We note that this definition has been considered before in literature; see [DNR, Chapter 5.4]. It is noted there that if \( C \) is a co-Frobenius coalgebra, then \( \dim(\text{Hom}_{C^*}(C, M)) \leq \dim(\text{Hom}_{C^*}(C, M)) \); this result was proved in [St0] for certain classes of co-Frobenius coalgebras (finite dimensional, or cosemisimple, or which are Hopf algebras). Our goal is to prove here far more general results, and to generalize the well known theorem that a Hopf algebra is co-Frobenius if and only if it has nonzero left integrals (equivalently, it has right integrals), and in this case, the integral is unique up to scalar multiplication.

Based on the case of Hopf algebras and \( M = \mathbb{K} \), it is natural to think of the relation \( \dim(\int_{l,M}) \leq \dim(M) \) as a “uniqueness” of integrals for \( M \) and of the relation \( \dim(\int_{l,M}) \geq \dim(M) \) as “existence of integrals”. We first show that for a coalgebra which is (just) left co-Frobenius, the “uniqueness” of left integrals holds for all right \( C \)-comodules \( M \) and
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the “existence” of right integrals holds as well for all left \( C \)-comodules \( N \). Examples are provided later on to show that the converse statements do not hold (even if both left and right existence - or both left and right uniqueness - of integrals are assumed). On the way, we produce some interesting characterizations of the more general quasi-co-Frobenius coalgebras. These will show that the co-Frobenius and quasi-co-Frobenius properties are fundamentally about a certain duality between the left and right indecomposable components of \( C \), and the multiplicities of these in \( C \) (Propositions 5.1.2 and 5.2.2).

One of the main results of the chapter is Theorem 6.2.1, which extends the results from Hopf algebras. It states that a coalgebra is co-Frobenius if and only if existence and uniqueness of left integrals hold for all right \( C \)-comodules \( M \) or, equivalently, for all left comodules \( M \). This adds to the previously known symmetric characterization of co-Frobenius coalgebras from the previous chapter and [I], where it is shown that \( C \) is co-Frobenius if and only if \( C \) is isomorphic to its left (or, equivalently, right) rational dual \( \text{Rat}(C^* \cdot C) \). Moreover, it is shown there that this is further equivalent to the \( C^* \)-dual functor \( \text{Hom}_{C^*}(\cdot, C^*) \) and the \( K \)-dual functor \( \text{Hom}_K(\cdot, K) \) from \( C^*\)-mod \( \text{mod-C}^* \) being isomorphic when restricted to the category of left (equivalently, right) rational \( C^* \)-modules, which is the same as that of right \( C \)-comodules. This has an interesting comparison to the algebra case: if the two functors were isomorphic on the whole category of left \( C^* \)-modules, one would have that \( C^* \) is a Frobenius algebra (by well known facts of Frobenius algebras, see [CR]), so \( C^* \) (and \( C \)) would be finite dimensional. This again illustrates that the co-Frobenius coalgebra concept is a generalization of Frobenius algebras to the infinite dimensional case. Here, the Theorem 6.2.1 also allows us to extend this view by giving a new interesting characterization of co-Frobenius coalgebras: \( C \) is co-Frobenius if and only if the functors \( \text{Hom}_{C^*}(\cdot, C^*) \) and \( \text{Hom}_K(\cdot, K) \) are isomorphic (only) on the subcategory of \( C^*\)-mod consisting of finite dimensional rational left \( C^* \)-modules. This is then further equivalent to the corresponding statement for \( \text{mod-C}^* \). In fact, quite interestingly, we note that for \( C \) to be co-Frobenius, it is enough for these two functors to be isomorphic when evaluated in vector spaces, but by an isomorphism which is not necessarily natural; the existence of a natural isomorphism of functors with values in \( C^* \)-modules follows thereafter. As further applications, we obtain the well known equivalent characterizations of Hopf algebras with nonzero integrals of Lin, Larson, Sweedler, Sullivan as well as the uniqueness of integrals.

We also give an extensive class of examples which will show that all the results in the
paper are the best possible. On the side, we obtain interesting examples (of one sided and two sided) semiperfect, QcF and co-Frobenius coalgebras showing that all possible inclusions between these classes are strict. For example, we note that there are left and right QcF coalgebras which are left co-Frobenius but not right co-Frobenius, or which are neither left nor right co-Frobenius. These coalgebras are associated to graphs and are usually subcoalgebras of the path coalgebra. We look at the abstract spaces integrals in the case of the representative Hopf algebra (coalgebra) of compact groups $G$, and note that the abstract integrals are in fact restrictions of unique vector integrals $\int$ on $C(G)$, the algebra of complex continuous functions on $G$, which have a certain “quantum”-invariance: $\int (x \cdot f) dh = \eta(x) \int f dh$, where $\eta$ is a finite dimensional representation of $G$. In particular, we note a nice short Hopf algebra proof of the well known fact (due to Peter and Weyl) that every finite dimensional representation of a compact group is completely reducible, and we also obtain the statements on the existence and uniqueness of “quantum” integrals for compact groups.

### 6.1 The General results

Recall that if $C$ is a coalgebra, then $C = \bigoplus_{S \in S} E(S)^{n(S)}$ as left $C$-comodules, where $S$ is a set of representatives of simple left $C$-comodules, $E(S)$ is an injective envelope of the left comodule $S$ contained in $C$ and $n(S) > 0$ are natural numbers. Similarly, $C = \bigoplus_{T \in T} E(T)^{p(T)}$ as right comodules, with $p(T) > 0$ and $T$ a set of representatives for the simple right $C$-comodules. We always use the letter $S$ to mean a simple left comodule and $T$ for a simple right $C$-comodule. If $M$ is a right $C$-comodule then, as usual, it is a left $C^*$-module, so $M^*$ has a natural structure of a right $C^*$-module: $(m^* \cdot c^*)(m) = m^*(c^* \cdot m) = m^*(m_0)c^*(m_1)$. Moreover, if $M$ is a finite dimensional comodule, then $M^*$ is a rational (finite dimensional) right $C^*$-module and so it has a compatible left $C$-comodule structure, i.e. $M^* \in \mathcal{C_M}$. The left $C^*$-module and (in the finite dimensional case) right $C$-comodule structures on $N^*$ for a left $C$-comodule $N$ are defined similarly.

**Definition 6.1.1** Let $M$ be a right $C$-comodule. The space of the **left integrals** of $M$ will be $\int_{l,M} = \text{Hom}_{\mathcal{CM}}(C^C, M^C)$, the set of morphisms of right $C$-comodules, regarded as a left $C^*$-module by the action $(c^* \cdot \lambda)(c) = \lambda(c \cdot c^*) = \lambda(c^*(c_1)c_2)$. Similarly, if $N \in \mathcal{CM}$ then the space of **right integrals** is $\int_{r,N} = \text{Hom}(C^C, C^N)$, regarded as a right $C^*$-module.
Note that, using Lemma 5.2.1, since for a finite dimensional comodule $M$ we have $M \simeq M^{**}$, we have

$$\int_{l,M} = \text{Hom}_{\mathcal{M}}(C^C, M^C) \simeq \text{Hom}_{\mathcal{M}}(C, M^{**}) \simeq \text{Hom}_{C^*}(M^*, C^*) = \text{Hom}_{C^*}(M^*, \text{Rat}(C_{C^*}^*)).$$

We will sometimes write just $\int_{l,M}$ if there is no danger of confusion, that is, if the comodule $M$ or $N$ has only one comodule structure (for example, it is not a bimodule).

**Proposition 6.1.2** The following assertions are equivalent:

(i) $\dim(\int_{l,M}) \leq \dim(M)$ for all finite dimensional $M \in \mathcal{M}^C$.

(ii) $\dim(\int_{l,T}) \leq \dim(T)$ for all simple comodules $T \in \mathcal{M}^C$.

If $C$ is a left QcF coalgebra, then these are further equivalent to

(iii) $C$ is left co-Frobenius.

Moreover, if $C$ is left QcF then:

(a) $\int_{l,T} \neq 0$ for all $T \in \mathcal{T}$ if and only if $C$ is also right QcF.

(b) $\dim(\int_{l,T}) \geq \dim(T)$ if and only if $C$ is also right co-Frobenius.

**Proof.** (ii)$\Rightarrow$(i) We prove (i) by induction on the length of $M$ (or on $\dim(M)$). For simple modules it holds by assumption (ii). Assume the statement holds for comodules of length less than length($M$). Let $M'$ be a proper subcomodule of $M$ and $M'' = M/M'$; we have an exact sequence $0 \to \text{Hom}_{\mathcal{M}}(C, M') \to \text{Hom}_{\mathcal{M}}(C, M) \to \text{Hom}_{\mathcal{M}}(C, M'')$ and therefore $\dim(\int_{l,M}) = \dim(\text{Hom}_{\mathcal{M}}(C^C, M)) \leq \dim(\int_{l,M'}) + \dim(\int_{l,M''}) \leq \dim(M') + \dim(M'') = \dim(M)$ by the induction hypothesis. (i)$\Rightarrow$(ii) is obvious.

Assume $C$ is left QcF and let $\sigma : \mathcal{T} \to S$ be such that $E(T) \simeq E(\sigma(T))^*$ as given by Proposition 5.1.2.

(i)$\Leftrightarrow$(iii) Let $T_0 \in \mathcal{M}^C$ be simple. Then there exists at most one $T \in \mathcal{T}$ such that $\text{Hom}_{\mathcal{M}}(E(T), T_0) \neq 0$. Indeed, for any $T \in \mathcal{T}$ we have $E(T) \simeq E(S)^*$ for $S = \sigma(T)$. Since $T_0^*$ is a rational $C^*$-module, applying Lemma 5.2.1 we get $\text{Hom}_{\mathcal{M}}(E(T), T_0) = \text{Hom}_{\mathcal{M}}(E(T), T_0^*) = \text{Hom}_{C^*}(T_0^*, E(T)^*) = \text{Hom}_{C^*}(T_0^*, E(S)) = \text{Hom}_{C^*}(T_0^*, S)$ which is nonzero if and only if $T_0^* \simeq S = \sigma(T)$. This can only happen for at most one $T$. Thus we get that $\int_{l,T_0} = \text{Hom}_{\mathcal{M}}(C^C, T_0) = \text{Hom}_{\mathcal{M}}(\bigoplus_{T \in \mathcal{T}} E(T)^{p(T)}, T_0) = \prod_{T \in \mathcal{T}} \text{Hom}_{\mathcal{M}}(E(T), T_0)^{p(T)}$ is 0 if $T_0 \notin \text{Im} \sigma$, or $\int_{l,T_0} = \text{Hom}_{\mathcal{M}}(E(T), T_0)^{p(T)} = \text{Hom}_{C^*}(T_0^*, S)^{p(T)}$ with $S = \sigma(T) = \cdots$
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Let $T_0^*$ as above. In this latter case, we have

$$\dim(\int_{l,T_0}) = p(T) \dim(\text{Hom}_{\mathcal{M}}(T_0^*, T_0^*)) = p(T) d(T_0^*)$$

while $\dim(T_0) = \dim(T_0^*) = n(T_0^*) d(T_0^*) = n(\sigma(T)) d(T_0^*)$. Since $\dim(\int_{l,T_0}) = 0 \leq \dim(T_0)$ for $T_0^* \notin \text{Im}(\sigma)$, we get that $\dim(\int_{l,T_0}) \leq \dim(T_0)$ holds for all $T_0$ if and only if this is so for $T_0$ in the image of $\sigma$. By the above equalities, this is further equivalent to $p(T) \leq n(\sigma(T))$, for all $T \in T$. By Proposition 5.2.2 this is equivalent to $C$ being left co-Frobenius. This finishes (i)⇔(iii) under the supplementary hypothesis of $C$ being left QcF.

For (a) if $C$ is left QcF then, since $\sigma$ is automatically injective and $\int_{l,T_0} \neq 0$ if and only if $T_0^* \in \text{Im}(\sigma)$, we see that $\sigma$ is bijective if and only if $\int_{l,T} \neq 0$, for all $T \in T$. The surjectivity of $\sigma$ means that for all $S \in \mathcal{S}$, there is some $T$ such that $E(S)^* \simeq E(T)$, or $E(S) \simeq E(T)^*$, which is equivalent to $C$ being right QcF by Proposition 5.1.2.

(b) Using (a) and the above facts, $\sigma$ is bijective. Note that the condition $\dim(\int_{l,T_0}) = p(T) d(T_0^*) \geq \dim(T_0) = n(\sigma(T)) d(T_0^*)$, with $T_0^* = \sigma(T)$, for all $T_0$ is equivalent to $n(\sigma(T)) \leq p(T)$ for all $T$, which can be rewritten as $p(\sigma^{-1}(S)) \geq n(S)$ for all $S \in \mathcal{S}$. This means that $C$ is right co-Frobenius by the right QcF version of Proposition 5.2.2.

**Corollary 6.1.3** If $C$ is a left co-Frobenius coalgebra, then $\dim(\int_{l,M}) \leq \dim(M)$ for all finite dimensional $M \in \mathcal{M}^C$.

**Remark 6.1.4** We see by the above characterization of left QcF coalgebras, that if $C$ is left QcF, then $\int_{r,S} \neq 0$ for all $S \in \mathcal{S}$. Indeed, let $T = S^*$ and $S_0 \in \mathcal{S}$ be such that $E(T) \simeq E(S_0)^*$. Then the monomorphism $T \hookrightarrow E(S_0)^*$ produces a nonzero epimorphism $E(S_0) \twoheadrightarrow T^* = S \rightarrow 0$ so $\text{Hom}_{\mathcal{M}}(C,S) \neq 0$. Therefore, $\int_{r,N} \neq 0$ for all $N$, because every comodule $N$ contains some simple comodule $S \in \mathcal{S}$. We thus observe the following interesting

**Corollary 6.1.5** The following are equivalent for a coalgebra $C$:

(i) $C$ is left QcF and $\int_{l,T} \neq 0$ for all simple left rational $C^*$-modules $T$.

(ii) $C$ is right QcF and $\int_{r,S} \neq 0$ for all simple right rational $C^*$-modules $S$.

**Proposition 6.1.6** Let $C$ be a left co-Frobenius coalgebra. Then $\dim(\int_{r,N}) \geq \dim(N)$ for all finite dimensional $N \in C\mathcal{M}$.
6.2 Co-Frobenius coalgebras and Hopf algebras

The next theorem generalizes the existence and uniqueness of left and right integrals from co-Frobenius Hopf algebras to the general case of co-Frobenius coalgebras, showing that, as in the Hopf algebra case, these are actually equivalent to the coalgebra being co-Frobenius. It is noted in [DNR, Remark 5.4.3] that for co-Frobenius coalgebras $\dim(\text{Hom}_{C^*}(C, M)) \leq \dim(M)$. This was shown above in Proposition 6.1.2 to hold in the more general case of
left co-Frobenius coalgebras (with actual equivalent conditions) and the following gives the mentioned generalization:

**Theorem 6.2.1** A coalgebra \( C \) is co-Frobenius if and only if \( \dim(\int_{l,M}) = \dim(M) \) for all finite dimensional right \( C \)-comodules \( M \) or, equivalently, \( \dim(\int_{r,N}) = \dim(N) \) for all finite dimensional left \( C \)-comodules \( N \).

**Proof.** \( \Rightarrow \) Since \( C \) is left co-Frobenius, Proposition 6.1.2 shows that \( \dim(\int_{l,M}) \leq \dim(M) \) for finite dimensional right comodules \( M \) and as \( C \) is also right co-Frobenius, the right co-Frobenius version of Proposition 6.1.6 shows that \( \dim(\int_{r,M}) \geq \dim(M) \) for such \( M \).

\( \Leftarrow \) Let \( T \) be a simple right \( C \)-comodule and \( S = T^* \). Let \( X \) be the socle of \( \text{Rat}(C^*_S) \) and \( X_S = \sum_{S' < C^*, S' \simeq S} S' \) be the sum of all simple sub(co)modules of \( C^* \) isomorphic to \( S \). It is easy to see that \( X = \bigoplus_{S \in S} X_S \) and \( X_S \) is semisimple isomorphic to a direct sum of comodules isomorphic to \( S \), that is \( X_S \simeq S^{(i)} = \bigsqcup_I S \). Then \( \text{Hom}_{C^*}(C, T) = \text{Hom}_{C^*}(C, T^{**}) = \text{Hom}_{C^*}(T^*, C^*) = \text{Hom}_{C^*}(S, C^*) = \text{Hom}_{C^*}(S, X_S) \) so we obtain \( \dim(\text{Hom}_{C^*}(C, T)) = \dim(\text{Hom}_{C^*}(S, X_S)) \). If \( I \) has cardinality greater than \( n(S) \) then \( \dim(\text{Hom}_{C^*}(S, X_S)) > \dim(\text{Hom}_{C^*}(S, S^{n(S)})) = d(S)n(S) = \dim(S) = \dim(T) \) so \( \dim(\text{Hom}_{C^*}(C, T)) > \dim(T) \) and this contradicts the hypothesis. Then we get that \( I \) is finite and \( \dim(\text{Hom}_{C^*}(C, T)) = |I| \cdot \dim(\text{Hom}_{C^*}(S, S)) = d(S) \cdot |I| = \dim(T) = \dim(S) = d(S) \cdot n(S) \) and thus \( |I| = n(S) \). This shows that \( X_S \simeq S^{n(S)} \simeq C_S \). Hence \( X = \bigoplus_{S \in S} X_S \simeq \bigoplus_{S \in S} C_S \simeq C_0 \) as left \( C \)-comodules (right \( C^* \)-modules).

Next, we show that \( \text{Rat}(C^*_C) \) is injective: let \( 0 \rightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \rightarrow 0 \) be an exact sequence of finite dimensional left \( C \)-comodules. It yields the exact sequence of vector spaces \( 0 \rightarrow \text{Hom}_{C^*}(N'', \text{Rat}(C^*_C)) \xrightarrow{g^*} \text{Hom}_{C^*}(N, \text{Rat}(C^*_C)) \xrightarrow{f^*} \text{Hom}_{C^*}(N', \text{Rat}(C^*_C)) \). Evaluating dimensions we get

\[
\dim(\text{Hom}_{C^*}(N', \text{Rat}(C^*_C))) = \dim(\text{Hom}_{C^*}(N', C^*)) = \dim(\int_{l,(N')^*}) = \dim(N')^* = \dim(N') = \dim(N) - \dim(N'') = \dim(\int_{l,N^*}) - \dim(\int_{l,(N'')^*}) = \dim \text{Hom}_{C^*}(N, C^*) - \dim \text{Hom}_{C^*}(N'', C^*) = \dim \text{Hom}_{C^*}(N, \text{Rat}(C^*_C)) - \dim \text{Hom}_{C^*}(N'', \text{Rat}(C^*_C)) = \dim(\text{Im} f^*)
\]
and this shows that \( f^* \) is surjective. Then, by [DNR, Theorem 2.4.17] we get that \( \text{Rat}(C^*_{C^*}) \) is injective.

Then, since \( X \) is the socle of the injective left comodule \( \text{Rat}(C^*_{C^*}) \), we get \( \text{Rat}(C^*_{C^*}) \simeq E(X) \) because \( X \) is essential in \( \text{Rat}(C^*_{C^*}) \). But \( X \simeq C_0 \) in \( C^* \mathcal{M} \), so \( E(X) \simeq E(C_0) \simeq C \), i.e. \( C \simeq \text{Rat}(C^*_{C^*}) \). By [I, Theorem 2.8] we get that \( C \) is left and right co-Frobenius.

We now give the applications of these general results to the equivalent characterizations of co-Frobenius Hopf algebras and the existence and uniqueness of integrals for Hopf algebras. Recall that if \( H \) is a Hopf algebra, \( \lambda \in H^* \) is called a left integral for \( H \) if \( h^* \cdot \lambda = h^*(1)\lambda \) for all \( h^* \in H^* \). This is equivalent to saying that the 1-dimensional vector space \( K\lambda \) is a left ideal of \( H^* \) which is rational, and its right comultiplication \( \rho : K\lambda \to K\lambda \otimes H \) is just \( \rho(\lambda) = \lambda \otimes 1 \). Let \( \int_l \) denote the space of all left integrals of \( H \), and defined similarly, let \( \int_r \) be the space of all right integrals. Note that \( \int_l = \text{Hom}(H^*K \cdot 1, H^*H^*) = \int_{l,H}^* \) where \( K \cdot 1 \) is the right \( H \)-comodule with comultiplication given by \( 1 \mapsto 1 \otimes 1_H \); indeed \( \phi : K \cdot 1 \to H^* \) with \( \phi(1) = \lambda \in H^* \), is a morphism of left \( H^* \)-modules if and only if \( \lambda \) is an integral: \( \phi(h^* \cdot 1) = h^* \cdot \phi(1) \iff h^*(1)\phi(1) = h^* \cdot \phi(1) \).

We will need to use the isomorphism of right \( H \)-comodules \( \int_l \otimes H \simeq \text{Rat}(H^*H^*) \) from [Sw2], pp.330-331 (see also [DNR, Chapter 5]). This is in fact an isomorphism of \( H \)-Hopf modules, but we only need the comodule isomorphism (we will not use the right \( H \)-module structure of \( \text{Rat}(H^*H^*) \)). The above mentioned isomorphism is a direct easy consequence of the fundamental theorem of Hopf modules.

We note that only one of the results of the previous section (Proposition 6.1.2 or Corollary 6.1.3) is needed to derive the well known uniqueness of integrals for Hopf algebras.

**Corollary 6.2.2 (Uniqueness of Integrals of Hopf algebras)** Let \( H \) be a Hopf algebra. Then \( \dim(\int_l) \leq 1 \).

**Proof.** If \( \int_l \neq 0 \), then there is a monomorphism of left \( H^* \)-modules \( H \hookrightarrow \int_l \otimes H \simeq \text{Rat}(H^*H^*) \hookrightarrow H^* \). Therefore \( H \) is left co-Frobenius and Corollary 6.1.3 (or Proposition 6.1.2) shows that \( \dim(\int_l) = \dim(\int_{l,K}) \leq \dim(K) = 1 \). \( \square \)

We can however derive the following more general results due to Lin, Larson, Sweedler, Sullivan [L, LS, Su1].

**Theorem 6.2.3** Let \( H \) be a Hopf algebra. Then the following assertions are equivalent:

(i) \( H \) is a left co-Frobenius coalgebra.
(ii) $H$ is a left $QcF$ coalgebra.
(iii) $H$ is a left semiperfect coalgebra.
(iv) $\text{Rat}(H,H^*) \neq 0$.
(v) $\int_l \neq 0$.
(vi) $\int_{l,M} \neq 0$ for some finite dimensional right $H$-comodule $M$.
(vii) $\dim\int_l = 1$.
(viii-xiv) The right hand side versions of (i)-(vii)

**Proof.** (i)⇒(ii)⇒(iii)⇒(iv) are properties of coalgebras ([L], [GTN], [DNR, Chapter 3]),
(vii)⇒(v) is trivial and (iv)⇔(v) follows by the isomorphism $\int_l \otimes H \simeq \text{Rat}(H,H^*)$; also
(v)⇒(vi) and (vi) implies $\int_{l,M} \simeq \text{Hom}_{H^*}(M^*,H^*) \neq 0$, so $\text{Rat}(H,H^*) \neq 0$ ($M^*$ is rational),
and thus (iv) and (v) follow. Now assume (v) holds; then (i) follows since the isomorphism
of right $H$-comodules $\int_l \otimes H \simeq \text{Rat}(H,H^*)$ shows that $H \hookrightarrow H^*$ in $H^*$-mod, i.e. $H$ is
left co-Frobenius. Moreover, in this case, since $\int_r = \int_{r,K1}$, Proposition 6.1.6 shows that
$\dim(\int_r) \geq \dim(K1) = 1$. In turn, by the equivalences (viii)-(xii), i.e. the right hand side of equivalences of (i)-(v), $H$ is also right co-Frobenius and Proposition 6.1.2 shows that
$\dim(\int_l) \leq 1$ so $\dim(\int_l) = 1$ and similarly $\dim(\int_r) = 1$. Hence, (v)⇒(i), (vii) & (viii), and
this ends the proof. □

**Further applications**

We use the above results to give a new characterization of co-Frobenius coalgebras. We have shown that a $K$-coalgebra $C$ is co-Frobenius if and only if the functors $\text{Hom}_{C^*}(-,C^*)$ and $\text{Hom}_K(-,K)$: $C^*$-mod→mod-$C^*$ are isomorphic when restricted to the category of right $C$-comodules $M^C$ (rational left $C^*$-modules). We will show that it is enough for the two functors to be isomorphic only on the finite dimensional rational $C^*$-modules, or even more generally, that for every finite dimensional rational comodule, the $C^*$-dual and the
$K$-dual have the same dimension.

**Theorem 6.2.4** The following are equivalent for a coalgebra $C$:
(i) $C$ is co-Frobenius.
(ii) The functors $\text{Hom}_{C^*}(-,C^*)$ and $\text{Hom}_K(-,K)$ are isomorphic when restricted to the
category of finite dimensional right $C$-comodules.
(iii) For every rational finite dimensional left $C^*$-module $M$, the $C^*$-dual and the $K$-dual
are isomorphic as right $C^*$-modules. Equivalently, they have the same dimension, that is,
dim(Hom_{C^*}(M, C^*)) = dim(M).

(iv-v) The right $C^*$-module version of (ii)-(iii).

**Proof.** (i)$\Rightarrow$(ii) is shown in chapter 5 and (ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i) If $N$ is a left $C$-comodule, then $M = N^*$ is a right $C$-comodule and $dim(M^*) = dim(Hom_{C^*}(M, C^*))$ and therefore

$$\dim\left(\int_{r \cdot N} \right) = dim(Hom_{C^*}(C^r C, C N)) = dim(Hom_{C^*}(C, M^*))$$

$$= dim(Hom(C^r M, C^* C^*)) \quad \text{(by Lemma 5.2.1)}$$

$$= dim(M) \quad \text{(by hypothesis)}$$

$$= dim(N)$$

and the result follows now as an application of Theorem 6.2.1. \hfill \Box

### 6.3 Examples and Applications

We provide some examples to show that most of the general results given above are in some sense the best possible for coalgebras.

**Example 6.3.1** Let $C$ be the $K$-coalgebra having $g, c_n$, $n \geq 1, n \in \mathbb{N}$ as a basis and comultiplication $\Delta$ and counit $\varepsilon$ given by

$$\Delta(g) = g \otimes g \quad , \quad \Delta(c_n) = g \otimes c_n + c_n \otimes g \quad \forall n$$

$$\varepsilon(g) = 1 \quad , \quad \varepsilon(c_n) = 0 \quad \forall n$$

i.e. $g$ is a grouplike element and each $c_n$ is a $(g, g)$-primitive element. Then $S = Kg$ is essential in $C$ and $S$ is the only type of simple $C$-comodule. Also $C/S \simeq \bigoplus_{n}(KC_n + S)/S \simeq \bigoplus_{n} S$. Then $Hom_{C^*}(C/S, S) \simeq Hom_{C^*}(\bigoplus_{n} S, S) \simeq \prod_{n} Hom_{C^*}(S, S)$ and since there is a monomorphism $Hom_{C^*}(C/S, S) \to Hom_{C^*}(C, S) = \int_{S}$, it follows that $\int_{S}$ is infinite dimensional. In fact, it can be seen that $\int_{S} \simeq Hom_{C^*}(C/S, S)$: we have an exact sequence $0 \to Hom_{C^*}(C/S, S) \to Hom_{C^*}(C, S) \to Hom_{C^*}(S, S)$. The last morphism in this sequence is 0, because otherwise it would be surjective, since $dim(Hom_{C^*}(S, S)) = 1$, and this would imply that the inclusion $S \subseteq C$ splits, which is not the case. Thus we have $dim(S) \leq dim\int_{S}$.
for every simple comodule $S$. Then, for any $C$-comodule $N$, there exists a monomorphism $S \hookrightarrow N$ which produces a monomorphism $\int_S \hookrightarrow \int_N$ and therefore $\int_N$ is always infinite dimensional and so $\dim(\int_N) \geq \dim(N)$, for all finite dimensional $C$-comodules $N$ (and since $C$ is cocommutative, this holds on both sides). Nevertheless, $C$ is not co-Frobenius, since it is not even semiperfect: $C = E(S)$. This shows that the converse of Proposition 6.1.6 does not hold.

**Example 6.3.2** Let $C$ be the divided power series $K$-coalgebra with basis $c_n$, $n \geq 0$ and comultiplication $\Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j$ and counit $\varepsilon(c_n) = \delta_{0n}$. Then $C^* \simeq K[[X]]$ - the ring of formal power series, and the only proper subcomodules of $C$ are $C_n = \bigoplus_{i=0}^n Kc_n$. Since all these are finite dimensional, $C$ has no proper subcomodules of finite codimension, and we have $\int_N = \Hom(C, N) = 0$ for any finite dimensional $C$-comodule $N$. (Again this holds both on left and on the right.) Thus “uniqueness” $\dim(\int_N) \leq \dim(N)$ holds for all $N$, but $C$ is not co-Frobenius since it is not even semiperfect ($C = E(Kc_0)$).

We give a construction which will be used in a series of examples, and will be used to show that the Propositions in the first sections are the best possible results. Let $\Gamma$ be a directed graph, with the set of vertices $V$ and the set of edges $E$. For each vertex $v \in V$, let us denote by $L(v)$ the set of edges coming into $v$ and by $R(v)$ the set of edges going out of $v$. For each edge $m$ we denote by $l(m)$ its starting vertex and $r(m)$ its end vertex: $l(v) \bullet \overset{m}{\longrightarrow} \bullet r(v)$. We define the coalgebra structure $K[\Gamma]$ by defining $K[\Gamma]$ to be the vector space with basis $V \sqcup E$ and comultiplication $\Delta$ and counit $\varepsilon$ defined by

$$\Delta(v) = v \otimes v, \quad \varepsilon(v) = 1 \text{ for } v \in V;$$

$$\Delta(m) = l(m) \otimes m + m \otimes r(m)$$

Denote by $\langle x, y, \ldots \rangle$ the $K$-vector space with spanned by $\{x, y, \ldots\}$ We note that this is the second term in the coradical filtration in the path coalgebra associated to $\Gamma$, and it is not difficult to see that this actually defines a coalgebra structure. Notice that the socle of $K[\Gamma]$ is $\bigoplus_{v \in V} \langle v \rangle$ and the types of simple (left, and also right) comodules are $\{ \langle v \rangle | v \in V \}$. We also note that there is a direct sum decomposition of $K[\Gamma]$ into
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indecomposable injective left $K[\Gamma]$-comodules

$$K[\Gamma]K[\Gamma] = \bigoplus_{v \in V} K[\Gamma] < v; m | m \in L(v) >$$

and also a direct sum decomposition into indecomposable right $K[\Gamma]$-comodules

$$K[\Gamma]K[\Gamma] = \bigoplus_{v \in V} < v; m | m \in R(v) >^{K[\Gamma]}$$

To see this, note that each of the components in the above decompositions is a left (respectively right) subcomodule of $K[\Gamma]$ and that it has essential socle given by the simple (left and right) $K[\Gamma]$-comodule $< v >$. For $v \in V$ let $E_r(v) = < v; m | m \in R(v) >^{K[\Gamma]}$ and $E_l(v) = K[\Gamma] < v; m | m \in L(v) >$. We have an exact sequence of right $K[\Gamma]$-comodules

$$0 \rightarrow < v >^{K[\Gamma]} \rightarrow E_r(v)^{K[\Gamma]} \rightarrow \bigoplus_{m \in R(v)} < r(m) >^{K[\Gamma]} \rightarrow 0$$

Since $< v >$ is the socle of $E_r(v)$, this shows that a simple right comodule $< w >$, with $w \in V$, is a quotient of an injective indecomposable component $E_r(v)$ whenever $w = r(m)$ for some $m \in R(v)$. This can happen exactly when $E_r(v)$ contains some $m \in L(w)$. Therefore we have $\text{Hom}^{K[\Gamma]}(E_r(v), < w >) = 0$ whenever $m \notin L(w)$ for every $m \in R(v)$, and $\text{Hom}^{K[\Gamma]}(E_r(v), < w >) = \prod_{m \in L(w) \mid l(m) = v} K$. Thus

$$\text{Hom}^{K[\Gamma]}(K[\Gamma], < w >) = \text{Hom}^{K[\Gamma]}(\bigoplus_{v \in V} E_r(v), < w >) = \prod_{v \in V} \text{Hom}^{K[\Gamma]}(E_r(v), < w >)$$

$$= \prod_{v \in V} \prod_{m \in L(w) \mid l(m) = v} K = \prod_{m \in L(w)} < w >$$

i.e. $\dim(\int_{l,<w>}) = \dim K^{L(w)}$. Similarly, we can see that $\dim(\int_{r,<w>}) = \dim K^{R(w)}$.

We will use this to study different existence and uniqueness of integrals properties for such coalgebras. Also, we note a fact that will be easy to use in regards to “the existence of integrals” for a coalgebra $C$: suppose $\dim(\int_{r,S}) = \infty$ for all simple left $C$-comodules $S$, and for an arbitrary finite dimensional left $C$-comodule $N$, let $S$ be a simple subcomodule of $N$. Then the exact sequence $0 \rightarrow \text{Hom}^C(C, S) \rightarrow \text{Hom}^C(C, N)$ shows that $\dim(\int_{r,N}) = \infty$, so existence of right integrals holds trivially in this case.
We also note that the above coalgebra has the following:

• uniqueness of left (right) integrals if $|L(w)| \leq 1$ (respectively $|R(w)| \leq 1$) for all $w \in V$, since in this case, $\dim(\int_{L,w}) \leq 1$ for all simple right comodules $T = \langle w \rangle$, and this follows by Proposition 6.1.2

• existence of left (right) integrals if $|L(w)| = \infty$ (respectively $|R(w)| = \infty$) for all $w \in V$, since then $\dim(\int_{L,w}) = \dim(KL(w)) = \infty$ and it follows from above.

• $K[\Gamma]$ is left (right) semiperfect if and only if $R(w)$ (respectively $L(w)$) is finite for all $w \in V$, since if $R(w)$ (respectively $L(w)$) is infinite for some $w \in V$ then $K[\Gamma]$ cannot be left (right) QcF nor left (right) co-Frobenius.

• If $|R(w)| \geq 2$ for some $w \in V$, then $K[\Gamma]$ is not left QcF. Otherwise, $E_r(w) \cong E_l(v)^*$, with $E_l(v) = \langle v; m \mid m \in L(v) \rangle$ with both $E_r(w), E_l(v)$ finite dimensional; but $E_l(v)$ has socle $\langle v \rangle$ of dimension 1, so $E_r(w) \cong E_l(v)^*$ is local by duality. But $\dim(E_r(w)/ \langle w \rangle) = |R(w)| \geq 2$ and $E_r(w)/ \langle w \rangle$ is semisimple, so it has more than one maximal submodule, which is a contradiction. Similarly, if $|L(w)| \geq 2$ then $K[\Gamma]$ is not right QcF (nor co-Frobenius).

**Example 6.3.3** Let $\Gamma$ be the graph

\[ \cdots \longrightarrow \bullet^{x-1} \longrightarrow \bullet^{x_0} \longrightarrow \bullet^{x_1} \longrightarrow \cdots \longrightarrow \bullet^{x_n} \longrightarrow \cdots \]

and $C = K[\Gamma]$. By the above considerations, we see that $C$ has the existence and uniqueness property of left integrals of simple modules: $\dim(\int_{L,T}) = \dim(T) = 1$ for all right simple $C$-comodules $T$. But this coalgebra is not left QcF (nor co-Frobenius) because $|R(x_0)| = 2$ and it is also not right QcF, because $E_l(y_0)$ is not isomorphic to a dual of a right injective $E_r(v)$, as it can be seen directly by formulas, or by noting that $E_r(x_0)^* = \langle x_0, [x_0x_1], [x_0y_0] \rangle^*$ and $E_r(y_0) = \langle y_0 \rangle^*$ are the only duals of right injective indecomposables containing the simple left comodule $\langle y_0 \rangle$, and they have dimensions 3 and 1 respectively.

This shows that the characterization of co-Frobenius coalgebras from Theorem 6.2.1 cannot be extended further to requiring existence and uniqueness only for simple comodules, as in the case of Hopf algebras, where existence for the simple comodule $K1$ is enough to infer
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Example 6.3.4 Let $\Gamma$ be the directed graph (tree) obtained in the following way: start with the tree below $W$ (without a designated root):

This has infinitely many arrows going into the center-point $c$ and infinitely many going out. Then for each “free” vertex $x \neq c$ of this graph, glue (attach) another copy $W$ such that the vertex $x$ becomes the center of $W$, and one of the arrows of this copy of $W$ will be the original arrow $xc$ (or $cx$) with orientation. We continue this process for “free” vertices indefinitely to obtain the directed graph $\Gamma$ which has the property that each vertex $a$ has an infinite number of (direct) successors and an infinite number of predecessors. Thus $|R(a)| = \infty$ and $|L(a)| = \infty$ for all the vertices $a$ of $\Gamma$, so we get $\dim(\int_{l,M}) = \infty$ and $\dim(\int_{r,N}) = \infty$ for all $M \in \mathcal{M}$ and $N \in \mathcal{C} \mathcal{M}$. Just as example 6.3.1 this shows that the converse of Proposition 6.1.2 does not hold even if we assume “existence” of left and right integrals. The example here is non-cocommutative and has many types of isomorphism of simple comodules, and all spaces of integrals are infinite dimensional.

Example 6.3.5 Consider the poset $V = \bigsqcup_{n \geq 0} \mathbb{N}^n$ with the order given by the “levels” diagram

$0 \to \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \ldots$

and for elements in consecutive levels we have that two elements are comparable only in the situation $(x_0, x_1, \ldots, x_n) < (x_0, x_1, \ldots, x_n, x)$ with $x_0 = 0$ and $x_1, \ldots, x_n, x \in \mathbb{N}$. This makes $V$ into a poset which is actually a tree with root $v_0 = (0)$. Visually, we can see this
as in the diagram (the arrows indicate ascension):

(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 0) \rightarrow (0, 0, n_2) \rightarrow (0, 1, 0) \rightarrow (0, 1) \rightarrow (0, 1, n_2) \rightarrow (0, 1, n_1, 0) \rightarrow (0, n_1, 0) \rightarrow (0, n_1, 1) \rightarrow (0, n_1) \rightarrow (0, n_1, n_2) \rightarrow (0, n_1, n_2, 0) \rightarrow \ldots
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Let \( \Gamma \) be the above tree, i.e. having vertices \( V \) and sides (with orientation) given by two consecutive elements of \( V \). For each pair of consecutive vertices \( a, b \) we have exactly one side \([ab]\) and the comultiplication is

- \( \Delta(a) = a \otimes a \) and \( \varepsilon(a) = 1 \) for \( a \in V \) (i.e. \( a \) is a grouplike element)
- \( \Delta([ab]) = a \otimes [ab] + [ab] \otimes b \) and \( \varepsilon([ab]) = 0 \) for \( [ab] \in E \), that is, \( b \in S(a) \) (i.e. \( [ab] \) is \((a, b)\)-primitive)

We see that here we have \( |L(v)| \leq 1 \) for all \( v \in V \). In fact \( |L(v)| = 1 \) for \( v \neq v_0 \) and \( |L(v_0)| = 0 \). So uniqueness of left integrals holds: \( \dim \int_{L,M} \leq \dim M \), for all finite dimensional rational left \( K[\Gamma]^* \)-modules \( M \) by the considerations on general construction of the \( K[\Gamma] \). Since \( |R(v)| = \infty \) for all \( v \in V \), the same construction shows that \( \dim(\int_{r,N}) \geq \dim N \) for \( N \in K[\Gamma]^*M \) (existence of right integrals). However, this coalgebra is not left co-Frobenius (nor QcF) because \( |R(v)| = \infty \) for all \( v \in V \). This shows that the converse of Proposition 6.1.6 and Corollary 6.1.3 combined does not hold. More generally, for this purpose, we could consider an infinite rooted tree (that is, a tree with a pre-chosen root) with the property that each vertex has infinite degree.

We also note that this coalgebra is not right co-Frobenius (nor QcF) either, because the dual of a left injective indecomposable comodule cannot be isomorphic to a right injective indecomposable comodule, since the latter are all infinite dimensional.

Example 6.3.6 As seen in the previous example, it is also not the case that “left uniqueness” and “right existence” of integrals imply the fact that \( C \) is right co-Frobenius; this can also be seen because there are coalgebras \( C \) that are left co-Frobenius and not right co-Frobenius (see [L] or [DNR, Chapter 3.3]). Then the left existence and right uniqueness hold by the results in Section 1 (Corollary 6.1.3 and Proposition 6.1.6) but the coalgebra is not right co-Frobenius. Also, this shows that left co-Frobenius does not imply either uniqueness of right integrals or existence of left integrals, since in this case, any combination of existence and uniqueness of integrals would imply the fact that \( C \) is co-Frobenius by Theorem 6.2.1.

Since integrals are tightly connected to the notions of co-Frobenius and QcF coalgebras, we also give some examples which show the fine non-symmetry of these notions; namely, we note that there are coalgebras which are QcF (both left and right), co-Frobenius on one side but not co-Frobenius. Also, it is possible for a coalgebra to be semiperfect (left and right) and QcF only on one side.
First, we note that the above general construction for graphs can be “enhanced” to produce non-pointed coalgebras. Namely, using the same notations as above, if $\Gamma$ is a labeled graph, i.e. a graph such that there is a positive natural number $n_v = n(v)$ attached to each vertex $v \in V$, then consider $K[\Gamma]$ to be the coalgebra with a basis $\langle (v_{ij})_{i,j=1,\ldots,n(v)}; (m_{ij})_{i=1,\ldots,n(m),j=1,\ldots,n(r(m))} \mid v \in V, m \in \mathcal{E} \rangle$ and comultiplication and counit given by

$$\Delta(v_{ij}) = \sum_{k=1}^{n_v} v_{ik} \otimes v_{kj}$$
$$\Delta(m_{ij}) = \sum_{k=1}^{n(m)} l(m)_{ik} \otimes m_{kj} + \sum_{k=1}^{n(r(m))} m_{ik} \otimes r(m)_{kj}$$
$$\varepsilon(v_{ij}) = \delta_{ij}$$
$$\varepsilon(m_{ij}) = 0$$

Again, we can denote by $S_l(v,i) = K < v_{ki} \mid k = 1,\ldots,n_v >$ and $S_r(v,i) = K < v_{ik} \mid k = 1,\ldots,n_v >$; these will be simple left and right $K[\Gamma]$-comodules, respectively. Also, let $E_l(v,i) = K < v_{ki}, k = 1,\ldots,n_v; m_{qi}, q = 1,\ldots,n_l(m), m \in L(v) >$ and put $E_r(v,i) = K < v_{ik}, k = 1,\ldots,n_v; m_{iq}, q = 1,\ldots,n_r(m), m \in R(v) >$; these are the injective envelopes of $S_l(v,i)$ and $S_r(v,i)$ respectively. Let $S_{l/r}(v) = S_{l/r}(v,1)$ and $E_{l/r}(v) = E_{l/r}(v,1)$; these are representatives for the simple left/right $K[\Gamma]$-comodules, and for the indecomposable injective left/right $K[\Gamma]$-comodules.

**Example 6.3.7** Consider the labeled graph $\Gamma$ in the diagram below

The vertices $a^n$ have labels positive natural numbers $p_n$. (They will represent the simple subcoalgebras of the coalgebra $C = K[\Gamma]$, which are comatrix coalgebras of the respective size.) Between each two vertices $a^{n-1}, a^n$ there is a side $x^n$. The above coalgebra $C = K[\Gamma]$ then has a basis $\{a^n_{ij} \mid i,j = 1,\ldots,p_n; n \in \mathbb{Z}\} \cup \{x^n_{ij} \mid i = 1,\ldots,p_{n-1}; j = 1,\ldots,p_n; n \in \mathbb{Z}\}$ and structure

$$\Delta(a^n_{ij}) = \sum_{k=1}^{p_n} a^n_{ik} \otimes a^n_{kj}$$
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\[ \Delta(x_{ij}^n) = \sum_{k=1}^{p_n-1} a_{ik}^{n-1} \otimes x_{kj}^n + \sum_{k=1}^{p_n} x_{ik}^n \otimes a_{kj}^n \]
\[ \varepsilon(a_{ij}^n) = \delta_{ij} \]
\[ \varepsilon(x_{ij}^n) = 0 \]

With the above notations, let \( E_r(n) = E_r(a^n) = E_r(a^n, 1) \) and \( E_l(n) = E_l(a^n) = E_l(a^n, 1) \). We note that \( E_l(n)^* \simeq E_r(n-1) \) for all \( n \). First, note that if \( M \) is a finite dimensional left \( C \)-comodule with comultiplication \( \rho(m) = m_{-1} \otimes m_0 \), then \( M^* \) is a right \( C \)-comodule with comultiplication \( R \) such that
\[ R(m^*) = m_0^* \otimes m_1^* \] if and only if
\[ m_0^*(m)m_1^*(m_0) = \]

This follows immediately by the definition of the left \( C^* \)-action on \( M^* \). We then have the following formulas giving the comultiplication of \( E_r(n-1) =< a_{1k}^{n-1} | 1 \leq k \leq p_{n-1}; x_{1k}^n | 1 \leq k \leq p_n > \)
\[ a_{1k}^{n-1} \mapsto \sum_j a_{1j}^{n-1} \otimes a_{jk}^{n-1} \]
\[ x_{1k}^{n-1} \mapsto \sum_j a_{1j}^{n-1} \otimes x_{jk}^n + \sum_j x_{1j}^n \otimes a_{jk}^n \]

and for \( E_l(n) =< a_{k1}^n | 1 \leq k \leq p_n; x_{k1}^n | 1 \leq k \leq p_{n-1} > \) we have
\[ a_{k1}^n \mapsto \sum_j a_{kj}^n \otimes a_{j1}^n \]
\[ x_{k1}^n \mapsto \sum_j a_{kj}^{n-1} \otimes x_{j1}^n + \sum_j x_{kj}^n \otimes a_{j1}^n \]

Let \( \{ A_{k1}^n | 1 \leq k \leq p_n; X_{k1}^n | 1 \leq k \leq p_{n-1} \} \) be a dual basis for \( E_l(n)^* \). Then, on this basis, the right comultiplication of \( E_l(n)^* \) reads:
\[ X_{k1}^n \mapsto \sum_i X_{1i}^n \otimes a_{ik}^{n-1} \]
\[ A_{k1}^n \mapsto \sum_i X_{1i}^n \otimes x_{ik}^n + \sum_i A_{1i}^n \otimes a_{ik}^n \]

Indeed, this can be easily observed by testing equation (6.1) for the dual bases \( \{ a_{k1}^n; x_{k1}^n \} \)
and \( \{A^p_{k_1}; X^p_{k_1}\} \). This shows that the 1-1 correspondence \( a_{1k}^{n-1} \leftrightarrow X_{k_1}^n; x_{1k}^p \leftrightarrow A^p_{k_1} \) is an isomorphism of right \( C \)-comodules. Therefore, \( E_l(n)^* \simeq E_r(n-1) \), and then also \( E_r(n) \simeq E_l(n+1)^* \) for all \( n \); thus we get that \( C \) is \( \text{QcF} \) (left and right). One can also show this by first proving this coalgebra is Morita equivalent to the one obtained with the constant sequence \( p_n = 1 \), which is \( \text{QcF} \) in a more obvious way, and using that \( \text{QcF} \) is a Morita invariant property. Note that even the hypothesis of \( \text{QcF} \) is not co-Frobenius on either side.

Remark 6.3.8 It is stated in [Wm] (see also review MR1851217) that a coalgebra \( C \) which is \( \text{QcF} \) on both sides must have left uniqueness of integrals (\( \dim(\text{Hom}^C(C,M)) \leq \dim(M) \) for \( M \in \mathcal{M}^C \)). By Proposition 6.1.2, this is equivalent to \( C \) being also left co-Frobenius. Nevertheless, by the above example we see that there are coalgebras which are both left and right \( \text{QcF} \), but not co-Frobenius on either side. Note that even the hypothesis of \( \text{QcF} \) being left \( \text{QcF} \) and right co-Frobenius would not be enough to imply the fact that left uniqueness of integrals holds.

In fact, in the above example, denoting \( S_{l/r}(n) = S_{l/r}(a_n,1) \), we have an exact sequence of right comodules \( 0 \rightarrow S_r(n) \rightarrow E_r(n) \rightarrow S_r(n+1) \rightarrow 0 \). Also \( K[\Gamma]K[\Gamma] = \bigoplus E_r(n)^{p_n} \) as right comodules. Therefore \( \dim \text{Hom}^{K[\Gamma]}(K[\Gamma], S_r(m)) = \dim \prod_n \text{Hom}^{K[\Gamma]}(E_r(n), S_r(m))^{p_n} = p_{m-1} \) since \( \text{Hom}^{K[\Gamma]}(E_r(n), S_r(m)) = \text{Hom}^{K[\Gamma]}(S_r(n+1), S_r(m)) = \delta_{n+1,m} \) (the \( E_r(n)'s \) are also local). Comparing this to \( \dim(S_r(m)) = p_m \), we see that any inequality is possible (unless some monotony property of \( p_n \) is assumed, as above).

Example 6.3.9 Let \( C = K[\Gamma] \) where \( \Gamma = (V, E) \) is the graph:

\[
\bullet^0 \xrightarrow{x_0} \bullet^1 \xrightarrow{x_1} \bullet^2 \xrightarrow{x_2} \bullet^3 \xrightarrow{x_3} \ldots
\]

By the above considerations, \( \dim(E_r(v_n)) = 2 \) for all \( n \) and \( \dim(E_l(v_n)) = 2 \) if \( n > 0 \), \( \dim(E_l(v_0)) = 1 \). Also, \( E_r(v_n) \simeq E_l(v_{n+1})^* \) for all \( n \). These show that this coalgebra is semiperfect (left and right), it is left \( \text{QcF} \) (and even left co-Frobenius) but it is not right
QcF since $E_t(v_0)$ is not isomorphic to any of the $E_t(v_n)^*$’s for any $n$ (by dimensions).

The above examples also give a lot of information about the (left or right) semiperfect, QcF and co-Frobenius algebras: they show that any inclusion between such classes (such as, for example, left QcF and right semiperfect coalgebras into left and right QcF coalgebras) is a strict inclusion. Inclusions are in order since left (right) co-Frobenius coalgebras are left (right) QcF, and left (right) QcF coalgebras are left (right) semiperfect.

**Example 6.3.10** Consider the coalgebras associated to the following graphs:

(a) The labeled graph

```
  ▼
 ▼
...→ 1 → 1 → 2 → 1 → ...
 ▼
      ▼
  ▼
```

With considerations as above, this coalgebra can be easily seen to be left QcF, not right semiperfect and not left co-Frobenius.

(b) (Same as above, but without the labels)

```
...→ 1 → 1 → 2 → 1 → ...
  ▼
      ▼
  ▼
```

This coalgebra will be left co-Frobenius (since it is left QcF and there are no labels, so multiplicities are all 1) but not right semiperfect (infinitely many arrows go into one of the vertices).
(c) The labeled graph

This coalgebra is left and right semiperfect, left QcF, not right QcF (more than two sides go into a vertex) and not left nor right co-Frobenius (as in the above Example 6.3.7).

The following table sums up all these examples with respect to the left or right semiperfect, QcF or co-Frobenius coalgebras. In the column to the left, the coalgebras in the various examples are cited (Example 6.3.7 contains examples for more than one situation); the other columns record the properties of these coalgebras. Moreover, all these coalgebras are non-cocommutative. Further minor details here are left to the reader.

<table>
<thead>
<tr>
<th>Example</th>
<th>left semiperfect</th>
<th>right semiperfect</th>
<th>left QcF</th>
<th>right QcF</th>
<th>left co-Frob</th>
<th>right co-Frob</th>
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<td>Ex 6.3.3</td>
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<td>Ex 6.3.4</td>
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<tr>
<td>Ex 6.3.7(1)</td>
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<td>Ex 6.3.7(2)</td>
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<tr>
<td>Ex 6.3.7(3)</td>
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<tr>
<td>Ex 6.3.9</td>
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<td>Ex 6.3.10(a)</td>
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<tr>
<td>Ex 6.3.10(b)</td>
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<tr>
<td>Ex 6.3.10(c)</td>
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6.4 Locally Compact Groups

We examine what the generalized integrals represent for the case of locally compact groups. First we look at a very simple case. Consider the measure $d\mu_t(x) = e^{itx}dx$ on the group $(\mathbb{R}, +)$ for some $t \in \mathbb{R}$, that is, $\int_{\mathbb{R}} f(x)d\mu(x) = \int_{\mathbb{R}} f(x)e^{itx}dx$ for $f \in L^1(\mathbb{R})$. Then this
measure $\mu_t$ has a special type of “invariance”, since $\int_{\mathbb{R}} f(x+a)d\mu_t(x) = \int_{\mathbb{R}} f(x+a)e^{ita}dx = \int_{\mathbb{R}} (f(x)e^{it(x-a)})dx = e^{-ita} \int_{\mathbb{R}} f(x)d\mu_t(x)$. Equivalently, this means that for any Borel set $U$ we have $\mu_t(U + a) = \int_{\mathbb{R}} \chi_U(x-a)d\mu_t(x) = e^{ita}\mu_t(U)$, that is, translation by $a$ of a set has the effect of “scaling” its measure by $e^{ita}$. Note that here $t$ could be any complex number.

We generalize this for a locally compact group $G$ with left invariant Haar measure $\lambda$.

Let $\mu$ be a complex vector measure on $G$, that is, $\mu = (\mu_1,\ldots,\mu_n)$ and so $\mu(U) = (\mu_1(U),\ldots,\mu_n(U))$ $\in \mathbb{C}^n$ for each Borel subset $U$ of $G$. We will be looking at the above type of invariance for such a measure $\mu$. That is, we study measures such that right translation of $U$ by $g \in G$ will have the effect of scaling $\mu(U)$ by $\eta(g)$, where $\eta(g)$ must be an $n \times n$ matrix, i.e. $\mu(U \cdot g) = \alpha(g) \cdot \mu(U)$. With the natural left action of $G$ on the set of all functions $f : G \to \mathbb{C}$ defined by $(g \cdot f)(x) = f(xy)$, this becomes

$$\int_G g \cdot \chi_U d\mu = \int_G \chi_U(x \cdot g)d\mu(x) = \int_G \chi_{Ug^{-1}}d\mu$$

$$= \mu(U \cdot g^{-1}) = \alpha(g^{-1})\mu(U)$$

$$= \alpha(g^{-1}) \int_G \chi_U d\mu$$

This is extended to all $L^1$ functions $f$ by $\int_G x \cdot f d\mu = \eta(x) \int_G f d\mu$, where $\eta(x) = \alpha(x^{-1})$.

Note that we have $\eta(xy) \int_G f d\mu = \int_G xy \cdot f d\mu = \eta(x) \int_G y \cdot f d\mu = \eta(x)\eta(y) \int_G f d\mu$. This leads to the following definition:

**Definition 6.4.1** Let $G$ be a topological group and $\int : C_c(G) \to V = \mathbb{C}^n$ be a linear map, where $C_c(G)$ is the space of continuous functions of compact support on $G$. We say $\int$ is a quantum $\eta$-invariant integral (quantified by $\eta$) if $\int(x \cdot f) = \eta(x) \int(f)$ for all $x \in G$, where $\eta : G \to \text{End}(V)$.

We note that the “quantum” factor $\eta$ is a representation which is continuous if the linear map $\int$ is itself continuous, where the topology on $C_c(G)$ is that of uniform convergence, i.e. that given by the sup norm $\|f\| = \sup_{x \in G} |f(x)|$ for $f \in C_c(G)$. For example, by general facts of measure theory, if $G$ is locally compact $\int = (\lambda_1,\ldots,\lambda_n)$ is continuous whenever $\int = \int d\mu$, where $\mu = (\mu_1,\ldots,\mu_n)$ and $\mu_i$ are positive measures, i.e. $\lambda_i = \int (-)d\mu_i$ is positive in the sense that $\lambda_i(f) \geq 0$ whenever $f \geq 0$. More generally, this occurs when $\mu_i = \mu_{i_1} - \mu_{i_2} + i(\mu_{i_3} - \mu_{i_4})$, where $\mu_{ij}$ are all positive measures, since any $\sigma$-additive complex measure $\mu_i$ can be written this way. Similarly, it can be seen that any
continuous \( \lambda_i : C_c(G) \to \mathbb{C} \) can be written as
\( \lambda_i = (\lambda_{i1} - \lambda_{i2}) + i(\lambda_{i3} - \lambda_{i4}) \) with \( \lambda_{ij} \) positive linear functionals which can be represented as \( \lambda_{ij} = \int (-)d\mu_{ij} \) by the Riesz Representation Theorem. We refer to [Ru] for the facts of basic measure theory.

**Proposition 6.4.2** Let \( G \) be a locally compact group, \( \int : C_c(G) \to V = \mathbb{C}^n \) be a quantum \( \eta \)-invariant integral and \( W = \text{Im} (\int) \). Then:

(i) \( W \) is an \( \eta \)-invariant subspace of \( V \), that is, \( \eta(x)W \subseteq W \) for all \( x \in G \).

Consider the map \( \eta : G \to \text{End}(W) \) induced by (i), and denote it also by \( \eta \). Then:

(ii) \( \eta \) is a representation of \( G \), so \( \eta : G \to GL(W) \).

(iii) \( \eta \) is a continuous representation if \( \int \) is continuous.

**Proof.** (i) For \( w = \int (f) \in W \) we have \( \eta(x)w = \eta(x)\int (f) = \int (x \cdot f) \in W \).

(ii) For any \( w = \int (f) \in W \) and \( x, y \in G \) we have \( \eta(xy)w = \eta(xy)\int (f) = \int (xy \cdot f) = \eta(x)\int (y \cdot f) = \eta(x)\eta(y)f = \eta(x)\eta(y)w \) and so \( \eta(xy) = \eta(x)\eta(y) \), since here \( \eta \) is considered with values in \( \text{End}(W) \). Since \( 1 \cdot f = f \), we get \( \eta(1) = \text{Id}_W \). Hence \( \text{Id}_W = \eta(1) = \eta(x^{-1}) = \eta(x)\eta(x^{-1}) = \eta(x^{-1}\eta(x)) \), so \( \eta : G \to GL(W) \) is a representation.

We note that this result holds also for the case when \( V \) is an infinite dimensional complex vector space.

(iii) Let \( w_1, \ldots, w_k \) be an orthonormal basis of \( W \) and let \( \varepsilon \) be fixed. For each \( i \), let \( f_i \in C_c(G) \) be such that \( w_i = \int f_i \). Since \( \int \) is continuous, we can choose \( \delta_i \) such that \( |\int (g - f_i)| < \varepsilon \) whenever \( ||g - f_i|| < \delta_i \), and let \( \delta = \min \{ \delta_i : i = 1, \ldots, n \} / 2 \). Since \( f_i \) is compactly supported and continuous, it is also uniformly continuous, and therefore there is a neighbourhood \( U_i \) of 1, - which may be assumed symmetric, such that if \( y^{-1}z \in V_i \) then \( |f_i(z) - f_i(y)| < \delta \). Therefore, if \( x \in U_i \) and \( y \in G \) we see that
\[
|\int (x \cdot f_i - f_i)(y)| = |f_i(xy) - f_i(y)| < \delta, \text{ so } ||x \cdot f_i - f_i|| \leq \delta < \delta_i.
\]

Hence,
\[
|(\eta(x) - \text{Id})(w_i)| = |\eta(x)\int (f_i) - \int (f_i)| = |\int (xf_i - f_i)| < \varepsilon
\]

Then this holds for all \( x \in U = \bigcap_{i=1}^k U_i \). For any \( w = \sum_{i=1}^k a_i w_i \in W \) and for all \( x \in U \) we have
\[
|\eta(x) - \text{Id})(w)| = \left| \sum_{i=1}^k a_i(\eta(x) - \text{Id})(w_i) \right| \leq \sum_{i=1}^k |a_i| \cdot |(\eta(x) - \text{Id})(w_i)|
\]

\[ 114 \]
\[
\leq \varepsilon \sum_{i=1}^{k} |a_i| \leq \varepsilon \cdot \sqrt{k \sum_{i=1}^{k} a_i^2} \leq \varepsilon \sqrt{k} \|w\|
\]

This shows that the norm of \(\eta(x) - \text{Id}\) (as a continuous linear operator on \(W\)) is small enough for \(x \in U\): \(|\|\eta(x) - \eta(1)\| = |\|\eta(x) - \text{Id}\| \leq \varepsilon \sqrt{k}\), so \(\eta\) is continuous at 1. Thus it is continuous everywhere since it is a morphism of groups \(\eta : G \to GL(W)\). \(\square\)

Now consider \(R_c(G)\) be the coalgebra (and actually Hopf algebra) of continuous representative functions on \(G\). It is well known that any continuous (not necessary unitary) finite dimensional representation of \(\eta : G \to GL(V) \subseteq \text{End}(V)\) becomes a right \(R_c(G)\)-comodule in the following way: if \((v_i)_{i=1}^{n}\) is a basis of \(V\), one writes

\[
g \cdot v_i = \sum_{j} \eta_{ji}(g)v_j
\]

and then it is straightforward to see that \(\eta_{ij}(gh) = \sum_{k} \eta_{ik}(g)\eta_{kj}(h)\) so \(\eta_{ij} \in R_c(G)\) and \(\rho : V \to V \otimes R_c(G)\),

\[
v_i \mapsto \sum_{j} v_j \otimes \eta_{ij}
\]

is a comultiplication. Conversely, the action of (6.2) defines a representation of \(G\) whenever \(V\) is a finite dimensional \(R_c(G)\)-comodule defined by (6.3). Also, the formula in (6.2) defines a continuous representation, since the linear operations on \(V\) - a complex vector space with the usual topology - are continuous, and the maps \(\eta_{ij}\) are continuous too. Moreover, \(\varphi : V \to W\) is a (continuous) morphism of left \(G\)-modules if and only if it is a (continuous) morphism of right \(R_c(G)\)-comodules. That is, the categories of finite dimensional right \(R_c(G)\)-comodules and of finite dimensional \(G\)-representations are equivalent. We can now give the interpretation of generalized algebraic integrals for locally compact groups:

**Proposition 6.4.3** Let \(\eta : G \to \text{End}(V)\) be a (continuous) finite dimensional representation of \(G\) and \(\int : C_c(G) \to V = \mathbb{C}^n\) be an \(\eta\)-invariant integral as above, i.e.

\[
\int x \cdot f = \eta(x) \int f
\]

and let \(\lambda : R_c(G) \to V\) be the restriction of \(\int\) to \(R_c(G) \subseteq C_c(G)\): \(\lambda(f) = \int f\). Then \(\lambda \in \int_V = \text{Hom}^{R_c(G)}(R_c(G), V)\), in the sense of Definition 6.1.1.
6.4. LOCALLY COMPACT GROUPS

CHAPTER 6. ABSTRACT INTEGRALS IN ALGEBRA

Proof. It is enough to show that $\lambda$ is a morphism of left $G$-modules. But this is true, since $x \cdot \lambda(f) = \eta(x) \int_G f d\mu = \int_G x \cdot f d\mu = \lambda(x \cdot f)$. □

We finish with a theorem for uniqueness and existence of $\eta$-invariant integrals. First we note that, as an application of a purely algebraic result, we can get the following nice and well known fact in the theory of compact groups:

Proposition 6.4.4 Let $G$ be a compact group. Then every finite dimensional continuous (not necessary unitary) representation $\eta : G \to GL(V)$ of $G$ is completely reducible.

Proof. By the above comments, the statement is equivalent to showing that $V$ is cosemisimple as a $R_c(G)$-comodule. But $R_c(G)$ is a Hopf algebra $H$ whose antipode $S$ has $S^2 = \text{Id}$ (since $S(f)(x) = f(x^{-1})$) and it has integrals (in the Hopf algebra sense) - the left Haar integral, as it also follows by the above Proposition. This integral is nonzero and defined on all $f \in R_c(G)$, since $G$ is compact. Then a result of [Su1] (with a very short proof also in [DNT2]) applies, which says that an involutory Hopf algebra with non-zero integrals is cosemisimple. Therefore, $R_c(G)$ is cosemisimple so $V$ is cosemisimple. □

Remark 6.4.5 It is well known that any continuous representation $V$ of $G$ is completely reducible, but for infinite dimensional representations, the decomposition is in the sense of Hilbert direct sums of Hilbert spaces.

For a continuous finite dimensional representation $\eta : G \to GL(V)$, let $\text{Int}_\eta$ denote the space of all continuous quantum $\eta$-invariant integrals on $C(G)$. Then we have:

Theorem 6.4.6 (Uniqueness of quantum invariant integrals)
Let $G$ be a compact group, and $\eta : G \to GL(V)$ a (continuous) representation of $G$. Then

$$\dim(\text{Int}_\eta) \leq \dim V$$

Proof. By the Peter-Weyl theorem, it is known that the continuous representative functions $R_c(G)$ are dense in the space of all continuous functions $C(G)$ in the uniform norm. Therefore, the morphism of vector spaces $\text{Int}_\eta \to \int_{l,V}$ given by the restriction is injective. Since $\dim(\int_{l,V}) = \dim(V)$ by Theorem 6.2.1, we get the conclusion. □
Remark 6.4.7 In particular, we can conclude the uniqueness of the Haar integral in this way. However, the existence cannot be deduced from this, since, while the uniqueness of the Haar measure is not an essential feature of this part of the Peter-Weyl theorem, the existence of the left invariant Haar measure on $G$ is.

The existence of quantum integrals can be easily obtained constructively from the existence of the Haar measure as follows. We note that for any $v \in V$, we can define $H_\eta(v) \in \text{Int}_\eta$ by $H_\eta(v)(f) = \int_G f(x^{-1})\eta(x) \cdot v$, where $\int$ is the (a) left translation invariant Haar measure on $G$. Let $e_1, \ldots, e_n$ be a $\mathbb{C}$-basis of $V$ and $\eta(x) = (\eta_{i,j}(x))_{i,j}$ be the coordinates formula for $\eta$. Then $H_\eta(v)(f) = (\sum_{j=1}^n \int_G f(x^{-1})\eta_{i,j}(x)v_j)_{i=1}^n$. We note that this is well defined. That is, $H_\eta(v)$ is a quantum $\eta$-invariant integral: let $\Lambda = H_\eta(v) = (\Lambda_i)_{i=1}^n$; then using the fact that $\int$ is the left translation invariant Haar integral, we get

$$
\lambda_i(x \cdot f) = \sum_{j=1}^n \int_G f(y^{-1}x)\eta_{kj}(y)v_j dy \quad \text{(substitute } y = xz)
$$

$$
= \sum_{j=1}^n \int_G f(z^{-1})\eta_{kj}(xz)v_j dz \quad \text{(since } \int \text{ is the left Haar integral)}
$$

$$
= \sum_{k=1}^n \sum_{j=1}^n f(x^{-1})\eta_{ik}(x)\eta_{kj}(z)v_j dz \quad \text{(since } \eta \text{ is a morphism)}
$$

$$
= \sum_{k=1}^n \eta_{ik}(x) \left( \sum_{j=1}^n \int_G f(z^{-1})\eta_{kj}(z)v_j dz \right)
$$

$$
= \sum_{k=1}^n \eta_{ik}(x) \Lambda_k(f)
$$

So $\Lambda(x \cdot f) = \eta(x) \cdot \Lambda(f)$, i.e. $\Lambda \in \text{Int}_\eta$. Define $\theta_\eta : \text{Int}_\eta \to V$ by $\theta_\eta(\Lambda) = (\sum_{i=1}^n \Lambda_i(\eta_{ij}))_{i=1}^n$.

Then we have $\theta_\eta \circ H_\eta = \text{Id}_V$. Indeed, we have

$$
\theta_\eta(H_\eta(v)) = \sum_{j=1}^n H_\eta(v)_j(\eta_{ij})
$$

$$
= \sum_{j=1}^n \sum_{k=1}^n \int_G \eta_{ij}(x^{-1})\eta_{jk}(x)v_k dx
$$
\[= \sum_{k=1}^{n} \int_{G} \eta_{ik}(1_G) v_k dx \quad (\text{since } \eta(x^{-1})\eta(x) = \eta(1_G) = \text{Id}_n \in \mathcal{M}_n(\mathbb{C}))\]

\[= \sum_{k=1}^{n} \int_{G} \delta_{ik} v_k dx = \int_{G} v_i dx = v_i\]

where we have assumed, without loss of generality, that \(\int_{G}\) is a normalized Haar integral.
Chapter 7

The Generating Condition for Coalgebras

Introduction

Let $R$ be a ring or an algebra. There are two basic categorical properties of rings, which are very important for homological algebra and the theory of rings and modules: in its category of left (right) modules, $R$ is both projective and a generator. The dual properties, namely $R$ being injective or a cogenerator in its category of left (right) modules, have been the subject of much study in ring theory (see for example [F2], 4.20-4.23, 3.5 and references therein). The rings (algebras) that satisfy both conditions are the same as the pseudo-Frobenius (PF) rings, which are defined equivalently as rings such that every faithful right module is a generator. There are many known equivalent characterizations of these rings as well as many connections of these rings with other notions, such as the quasi-Frobenius (QF) rings, semiperfect rings, perfect rings, FPF rings or Frobenius algebras. They have been introduced as generalizations of Frobenius algebras, and they retain much of the module (representation) theoretic properties of these algebras. The following theorem recalls some equivalent characterizations of PF-rings (see [F2, 4.20]) and QF-rings (see also [CR]):

Theorem 7.0.8 (1) $R$ is right PF if and only if it satisfies either one of the following conditions:

(i) $R$ is an injective cogenerator in $\text{mod-}R$.

(ii) $R = \bigoplus_{i=1}^{n} e_iR$ with $e_i^2 = e_i$ and $e_iR$ is indecomposable injective with simple socle for all
(2) $R$ is a QF-ring if and only if every injective right $R$-module is projective and if and only if every injective left $R$-module is projective.

Dually, analogous questions have been raised in the case of coalgebras and comodules over coalgebra. A coalgebra $C$ over a field $K$ is always a cogenerator for its comodules and is also injective as a comodule over itself. The dual properties in the coalgebra situation, corresponding to the selfinjectivity and cogenerator properties of a ring (or an algebra), are that of a coalgebra being projective as a right (or left) comodule or being a generator for the right (or left) comodules. These conditions were studied for coalgebras in [GTN] and [GMN], where quasi-co-Frobenius (QcF) coalgebras were introduced as the dualization of QF-algebras and in some respects of PF rings. It is proved there that

**Theorem 7.0.9** The following assertions are equivalent for a coalgebra $C$.

(i) $C$ is left QcF, i.e. $C$ embeds in a direct sum of copies of $C^*$ as left $C^*$-modules.

(ii) $C$ is a torsionless left $C^*$-module, i.e. $C$ embeds in a direct product of copies of $C^*$.

(iii) Every injective right $C$-comodule is projective.

(iv) $C$ is a projective right $C$-comodule.

(v) $C$ is a projective left $C^*$-module.

(The equivalence of (i) and (ii) was included in proposition 5.1.2). Moreover, if these hold, then $C$ is a generator in $^C\mathcal{M}$, the category of left $C$-comodules. As mentioned earlier, this concept is not left-right symmetric, unlike the algebra counterpart, the QF-algebras (see [DNR, Example 3.3.7] and [GTN, Example 1.6]). It is shown in [GMN] (see also [DNR, Theorem 3.3.11]) that a coalgebra is QcF if and only if $C^C$ is a generator and projective in $\mathcal{M}^C$, equivalently, $^C\mathcal{C}$ is a projective generator in $^C\mathcal{M}$. These are further equivalent to $C$ being a generator for both $\mathcal{M}^C$ and $^C\mathcal{M}$, characterizations that dualize known characterizations of finite dimensional QF algebras. It remains open whether the fact that $C$ is a generator for $^C\mathcal{M}$ is actually enough to imply the fact that $C$ is left QcF, i.e. if it implies that $C$ is projective as left $C^*$-module. In fact, the question has been has been studied very recently in [NTvO], where some partial results are given. Among these, it is shown that the answer to this question is positive when $C$ has finite coradical filtration. The general question however is left there as an open question.

In the case of a ring $R$, there is no implication between the property of being left self-injective and that of $R$ being left cogenerator. An example of a ring $R$ which is a non-
injected cogenerator in mod-$R$ is the $K$-algebra with basis $1 \cup \{ e_i \mid i = 0, 1, 2 \ldots \} \cup \{ x_i \mid i = 0, 1, 2 \ldots \}$ with identity 1 and with $e_i x_j = \delta_{i,j} x_j$, $x_j e_i = \delta_{i,j-1} x_j$, $e_i e_j = \delta_{i,j} e_i$ and $x_i x_j = 0$ for all $i, j$ - see [F1, 24.34.2, p. 215]. Conversely, if a ring $R$ is a right cogenerator, then it is right selfinjective if and only if it is semilocal (see again [F1, 24.10-24.11]), and there are selfinjective rings which are not semilocal, and thus they are not right cogenerators. Such an example can even be obtained as a profinite algebra, that is, an algebra which is the dual of a coalgebra - see Example 7.3.5.

We will say that a coalgebra has the right generating condition if it is a generator in $\mathcal{M}^C$. There are two main results in this chapter. First, we examine some conditions under which the right generating condition of a coalgebra implies that $C$ is right QcF (projective as right $C^*$-module). Among these, we consider three important conditions in the theory of coalgebras: semiperfect coalgebras, coalgebras of finite coradical filtration, and coalgebras of finite dimensional coradical (almost connected). We show that

\begin{itemize}
  \item [\textbf{(*)}] a coalgebra with the right generating condition and whose indecomposable injective left components are of finite Loewy length is necessarily right QcF. The converse is known to hold.
\end{itemize}

Therefore, for a coalgebra $C$ with the right generating condition, the above is an equivalence, and the coalgebra $C$ being QcF is further equivalent to $C$ being right semiperfect (see [L]). As a consequence, we see that implication (\textbf{*}) holds whenever the coalgebra has finite coradical filtration, and this allows us to reobtain the main result of [NTvO] in a direct short way. Secondly, we show that every coalgebra $C$ embeds in a coalgebra $C_\infty$ that has the right generating condition. In fact, $C_\infty$ will even have all of its finite dimensional right comodules as quotients. Thus, starting with a coalgebra $C$ which is not right semiperfect, we will get a coalgebra $C_\infty$ which is not right semiperfect (see [L]) and thus, by well known properties of coalgebras, $C_\infty$ will not be right QcF. Moreover, if we start with a connected coalgebra (coalgebra whose coradical has dimension 1) over an algebraically closed field, we show that the coalgebra $C_\infty$ can be constructed to be local as well, therefore showing that the third mentioned condition for coalgebras - the coalgebra having finite dimensional coradical - is not enough for the right generating condition to imply the QcF property.
CHAPTER 7. THE GENERATING CONDITION FOR COALGEBRAS
7.1. LOEWY SERIES AND THE LOEWY LENGTH OF MODULES

7.1 Loewy series and the Loewy length of modules

We first recall a few well known facts on the Loewy series of modules. Let $M$ be a module over a ring $R$. We denote $L_0(M) = 0$ and $L_1(M) = s(M)$ - the socle of $M$, the sum of all the simple submodules of $M$. The Loewy series of $M$ is defined inductively as follows: if $L_n(M)$ is defined, $L_{n+1}(M)$ is such that $L_{n+1}(M) = s(M/L_n(M))$. More generally, if $\alpha$ is an ordinal, and $(L_\beta(M))_{\beta<\alpha}$ have been defined, then

- when $\alpha = \beta + 1$ is a successor, one defines $L_{\beta+1}(M)$ such that $L_{\beta+1}(M) = s(M/L_\beta(M))$;
- when $\alpha$ is a limit (i.e. not a successor), one defines $L_\alpha(M) = \bigcup_{\beta<\alpha} L_\beta(M)$.

We also write $M_\alpha = L_\alpha(M)$. If $M = M_\alpha$ for some $\alpha$, we say that $M$ has its Loewy length defined and the least ordinal $\alpha$ with this property will be called the Loewy length of $M$; we will write $lw(M) = \alpha$. It is known that the modules having the Loewy length defined are exactly the semiartinian modules, that is, the modules $M$ such that $s(M/N) \neq 0$ for any submodule $N$ of $M$ with $N \neq M$. We refer to [N] as a good source for these facts. We also recall a few well known facts on the Loewy length of modules. Throughout this chapter, only modules of finite Loewy length will be used; however these properties hold in general for all modules. In the following, whenever we write $lw(M)$ we understand that this implicitly also means the Loewy length of $M$ is defined (and for our purposes, it will also be enough to assume that $lw(M)$ is finite).

**Proposition 7.1.1** For any ordinal $\alpha$ (or $\alpha$ non-negative integer) we have:

(i) If $N$ is a submodule of $M$ then $L_\alpha(N) \subseteq L_\alpha(M)$ and in fact $L_\alpha(N) = N \cap L_\alpha(M)$.
(ii) If $f : N \to M$ is a morphism of modules, then $f(L_\alpha(N)) \subseteq L_\alpha(M)$.
(iii) If $N$ is a submodule of $M$ then $lw(N) \leq lw(M)$, $lw(M/N) \leq lw(M)$ and $lw(M) \leq lw(N) + lw(M/N)$.
(iv) $L_\alpha(\bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} L_\alpha(M_i)$ and $lw(\bigoplus_{i \in I} M_i) = \sup_{i \in I} lw(M_i)$.

Let $(C, \Delta, \varepsilon)$ be a coalgebra over an arbitrary field $K$ and let $M$ be a right $C$-comodule with comultiplication $\rho : M \to M \otimes C$. It is well known that, when viewed as a left $C^*$-module, $M$ has its Loewy length defined and in fact $lw(M) \leq \omega_0$, the first infinite ordinal. The coradical filtration of $C$ is defined by $C_0 = L_1(C)$, ..., $C_n = L_{n+1}(C)$. Let $J = J(C^*)$, the Jacobson radical of $C^*$. By [DNR, Proposition 2.5.3] we have $(J^n)\perp = C_{n-1}$, where, as before, for $I < C^*$ we set $I^\perp = \{c \in C | f(c) = 0, \forall f \in I\}$ and for $X \subseteq C$
we set \( X^\perp = \{ c^* \in C^* | c^*(x) = 0, \forall x \in X \} \). Also, if \( M \) is a right \( C \)-comodule (so a left \( C^* \)-module), \( M^* \) becomes a right \( C^* \)-module in the usual way by the “dual action”: \( (m^* \cdot a)(m) = m^*(am) \) for \( m^* \in M^*, m \in M, a \in C^* \). The following Lemma gives the connection between the Loewy length of \( M \) and \( M^* \) and also provides a way to compute it for comodules of finite Loewy length.

**Lemma 7.1.2** Let \((M, \rho)\) be a right \( C \)-comodule. Then the following are equivalent:

(i) \( J^n \cdot M = 0 \).

(ii) \( M^* \cdot J^n = 0 \).

(iii) \( \text{Im} \rho \subseteq M \otimes C_{n-1} \).

(iv) \( \text{lw}(M) \leq n \).

**Proof.** (i)\( \Rightarrow \)(ii) is straightforward.

(ii)\( \Rightarrow \)(i) If \( f \in J^n \) and \( m \in M \), then for all \( m^* \in M^* \) we have \( 0 = (m^* \cdot f)(m) = m^*(f \cdot m) \). Since this is true for all \( m^* \in M^* \), we get \( f \cdot m = 0 \).

(iii)\( \Leftrightarrow \)(iv) The map \( \rho : M \to M \otimes C \) is a morphism of \( C \)-comodules by the coassociativity property. Moreover, \( M \otimes C \cong \bigoplus_{i \in I} I \), where \( I \) is a \( K \)-basis of \( M \). Since \( C_{n-1} = L_n(C) \), using these isomorphisms, we have that \( L_n(\bigoplus_{i \in I} I) = \bigoplus_{i \in I} C_{n-1} \) and so \( L_n(M \otimes C) = M \otimes C_{n-1} \).

Therefore, if (iii) holds then since \( \rho \) is also injective (by the counit property) we get \( \text{lw}(M) = \text{lw}(\rho(M)) \leq \text{lw}(M \otimes C_{n-1}) = n \). Conversely, if (iv) holds, then \( M = L_k(M) \) for some \( k \leq n \), so \( \rho(M) \subseteq L_k(M \otimes C) \subseteq L_n(M \otimes C) = M \otimes C_{n-1} \).

(i)\( \Rightarrow \)(iii) For \( m \in M \), let \( \rho(m) = \sum_{i=1}^k m_i \otimes c_i \in M \otimes C \) where, by a standard linear algebra observation, we choose the \( m_i \)'s to be linearly independent. For all \( f \in J^n \) we have \( 0 = f \cdot m = \sum_{i=1}^k f(c_i)m_i \) and thus \( f(c_i) = 0 \) for all \( i \). That is, \( c_i \in (J^n)^\perp = C_{n-1} \) for all \( i = 1, \ldots, k \).

(iii)\( \Rightarrow \)(i) is true, since \( J^n \subseteq (J^n)^\perp = C_{n-1} \).

Since the dual of a finite dimensional right \( C \)-comodule is a finite dimensional left \( C \)-comodule, we have

**Corollary 7.1.3** If \( M \) is a finite dimensional right \( C \)-comodule (rational left \( C^* \)-module), then \( M^* \in \mathcal{C} \mathcal{M} \) and \( \text{lw}(M) = \text{lw}(M^*) \).
7.2 The generating condition

Let $S$ (respectively $T$) denote a system of representatives of simple left (respectively right) $C$-comodules. Then $C \simeq \bigoplus_{S \in S} E(S)^{n(S)}$ as left $C$-comodules, with $n(S)$ positive integers and $E(S)$ the injective envelopes of the comodule $S$. Similarly, $C = \bigoplus_{T \in T} E(T)^{p(T)}$ as right $C$-comodules. Then we obviously have that $C$ generates $M_C$ if and only if $(E(T))_{T \in T}$ is a system of generators in $M_C$. Recall that a coalgebra $C$ is right (left) semiperfect if and only if $E(S)$ is finite dimensional for all $S \in S$ (resp. every $E(T)$ with $T \in T$ is finite dimensional; see [L] or [DNR, Chapter 3]). We first give a simple proposition that explains what is the property that coalgebras with the generating condition are missing to be QcF.

**Proposition 7.2.1** Let $C$ be a coalgebra. Then $C$ is left QcF if and only if $C$ is left semiperfect and is a generator in $C M$.

**Proof.** $\Rightarrow$ is already known (see [DNR, Chapter 3]).

$\Leftarrow$ It is well known that we have $C = \bigoplus_{i \in I} E(T_i)$ a direct sum of right comodules (left $C^*$-modules), where $T_i$ are simple right comodules, $C_0 = \bigoplus_{i \in I} T_i$ is the coradical of $C$ and $E(T_i)$ are injective envelopes of $T_i$ contained in $C$. For each $T_i$, the comodule $E(T_i)$ is finite dimensional, so $E(T_i)^*$ is a finite dimensional right $C^*$-modules which is rational, that is, it has a left $C$-comodule structure. Then there is an epimorphism of right $C^*$-modules $\phi_i : C^{n_i} \to E(T_i)^* \to 0$, where $n_i$ can be taken to be a (finite) number since $E(T_i)$ is finite dimensional. By duality, this gives rise to a morphism $\psi_i : E(T_i) \simeq (E(T_i)^*)^* \to (C^*)^{n_i}$, given by $\psi_i(x)(c) = \phi_i(c)(x)$. Since $\phi_i$ is a surjective morphism of right $C^*$modules, it is easy to see that $\psi_i$ is an injective morphism of left $C^*$-modules. We then get a monomorphism of left $C^*$-modules $\bigoplus_{i \in I} \psi_i : \bigoplus_{i \in I} E(T_i) \hookrightarrow \bigoplus_{i \in I} (C^*)^{n_i}$, a copower of $C^*$, so $C$ is left QcF. $\Box$

The next proposition will be the key step in proving the main results of this section.

**Proposition 7.2.2** Suppose $C$ generates $M_C$. If $S \in S$ is such that $lw(E(S)) = n$, then for each finite dimensional subcomodule $N$ of $E(S)$ with $lw(N) = n$, there is $T \in T$ such that $N \simeq E(T)^*$.

**Proof.** Note that since $N$ has simple socle, $N^*$ is a right $C$-comodule which is local, say with a unique maximal subcomodule $X$. This is due to the duality $X \leftrightarrow X^*$ between finite
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dimensional left and finite dimensional right $C$-comodules. Let $\bigoplus_{i \in I} E(T_i) \xrightarrow{\varphi} N^* \to 0$ be an epimorphism in $\mathcal{M}^C$; then $\exists i \in I$ such that $\varphi(E(T_i)) \subseteq X$, and then (for example by Nakayama’s lemma) we have $\varphi(E(T_i)) = N^*$. Put $T = T_i$. We have a diagram of left $C^*$-modules

\[
\begin{array}{ccc}
E(T) & \xrightarrow{\varphi} & N^* \\
\downarrow^r & & \downarrow^p \\
0 & & 0
\end{array}
\]

which is completed commutatively by a morphism $p$, since $E(S)^*$ is a direct summand in $C^*$ (the vertical map is the natural one). Let $P = \text{Im}(p)$; by (the left comodule version of) Lemma 7.1.2, $J^n \cdot E(S)^* = 0$ and so $J^n \cdot P = p(J^n \cdot E(S)^*) = 0$. But $P$ is finitely generated (even cyclic, since $E(S)^*$ is so), and $P$ is also a right $C$-comodule (rational left $C^*$-module), and therefore it is finite dimensional. Thus its Loewy length is defined and $lw(P) \leq n$ by the same Lemma. Also, $\varphi|_P$ is injective. Indeed, otherwise $T \subseteq \ker \varphi \cap P = \ker(\varphi|_P) \neq 0$, since $T$ is essential in $E(T)$. Then $T = L_1(E(T)) = L_1(P)$ and so $lw(P/T) = lw(P/L_1(P)) < lw(P) \leq n$ (by the definition of Loewy length). But $\varphi$ factors through $\varphi : P/T \to N^*$ and therefore, using also Corollary 7.1.3, $lw(P/T) \geq lw(N^*) = n - a contradiction.

Since $\varphi \circ p = r$ is surjective, $\varphi|_P$ is an isomorphism with inverse $\theta$. This shows that the inclusion $\iota : P \hookrightarrow E(T)$ splits off: $\theta \circ \varphi \circ \iota = \theta \circ \varphi|_P = \text{id}_P$. Since $E(T)$ is indecomposable, $P = E(T)$. Hence $\varphi$ is an isomorphism and $E(T) \cong N^*$, so $N \cong E(T)^*$ since they are finite dimensional.

**Proposition 7.2.3** Suppose $C$ satisfies the right generating condition. Then for each $S \in \mathcal{S}$ such that $E(S)$ has finite Loewy length, there exists $T \in \mathcal{T}$ such that $E(S) \cong E(T)^*$ and $E(S)$ is finite dimensional.

**Proof.** Let $n = lw(E(S))$. First note that there exists at least one finite dimensional subcomodule $N$ of $E(S)$ such that $lw(N) = n$: take $x \in L_n(E(S)) \setminus L_{n-1}(E(S))$ and put $N = x \cdot C^*$ the left subcomodule (equivalently, right $C^*$-submodule) generated by $x$. Then $L_{n-1}(N) \neq N$ since otherwise $N \subseteq L_{n-1}(E(S))$, and therefore $n \leq lw(N) \leq lw(E(S)) = n$.

Let $N_0 = N$. Assuming $E(S)$ is not finite dimensional, we can inductively build a sequence $(N_k)_{k \geq 0}$ of finite dimensional subcomodules of $E(S)$ such that $N_k/N_{k-1}$ is simple for all $k \geq 1$, since simple comodules are finite dimensional. Applying Proposition 7.2.2 we
see that each \( N_k \) is local, since each is the dual of a comodule with simple socle (same argument as above; this also follows from the more general \([I, \text{Lemma 1.4}]\)). Then \( N_k/N_0 \) has a composition series

\[
0 = N_0/N_0 \subseteq N_1/N_0 \subseteq N_2/N_0 \subseteq \ldots \subseteq N_{k-1}/N_0 \subseteq N_k/N_0
\]

with each term of the series being local. Thus, by duality, \( M_k = (N_k/N_0)^* \) has a composition series

\[
0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots \subseteq X_{k-1} \subseteq X_k = (N_k/N_0)^*
\]

with \( X_i \simeq (N_k/N_i)^* \), because of the short exact sequences of left \( C^* \)-modules and right \( C \)-comodules

\[
0 \rightarrow (N_k/N_i)^* \rightarrow (N_k/N_0)^* \rightarrow (N_i/N_0)^* \rightarrow 0.
\]

Therefore, \( M_k/X_i \simeq (N_i/N_0)^* \) has simple socle (by duality), since \( N_i/N_0 \) are all local.

Then, by definition, the above series of \( M_k \) is the Loewy series and so \( lw(N_k/N_0) = lw(M_k) = k \). But then \( k = lw(N_k/N_0) \leq lw(N_k) \leq lw(E(S)) = n \) for all \( k \), which is absurd. Therefore \( E(S) \) is finite dimensional. This also shows that the sequence \((N_k)_{k \geq 0}\) must terminate with some \( N_k = E(S) \), because it can be continued whenever \( N_k \neq E(S) \). Since \( N_k \simeq E(T)^* \) for some \( T \in T \) by Proposition 7.2.2, this ends the proof. \( \square \)

**Theorem 7.2.4** Let \( C \) be a coalgebra satisfying the right generating condition. Then the following conditions are equivalent:

(i) The injective envelope (as comodules) of every simple left comodule has finite Loewy length.

(ii) \( C \) is right semiperfect.

(iii) \( C \) is right \( QcF \).

These conditions hold in particular if \( C = C_n \) for some \( n \), i.e. \( C \) has finite coradical filtration.

**Proof.** We note that (iii)\( \Rightarrow \) (ii)\( \Rightarrow \) (i) are obvious so we only need to prove (i)\( \Rightarrow \) (iii). By Proposition 7.2.3, for all \( S \in \mathcal{S} \) there is \( T \in \mathcal{T} \) such that \( E(S) \simeq E(T)^* \), so each \( E(S) \) is projective as right \( C^* \)-module and it also embeds in \( C^* \). Therefore, \( C \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \) is projective as right \( C^* \)-module (and then also as left \( C \)-comodule). It also follows that since each \( E(S) \) embeds in \( C^* \) (as direct summand), we have an embedding \( C \simeq \bigoplus_{S \in \mathcal{S}} E(S)^{n(S)} \to \)
\[ \bigoplus_{S \in \mathcal{S}} C^{* \mathcal{H}(S)}. \]

Note that the argument above provides another proof for Proposition 7.2.1. In particular, it provides a direct proof for [NTvO, Theorem 4.1]. We also note that the property used in the above proofs, that the dual of every indecomposable injective left \( C \)-comodule is an indecomposable injective right comodule, is proved to be equivalent to the coalgebra \( C \) being QeF in [IM], [I3] and in chapter 5. The above considerations constitute another (direct) argument.

### 7.3 A general class of examples

In this section we construct the general examples of this chapter. The first goal is to start with an arbitrary coalgebra \( C \) and build a coalgebra \( D \) such that \( C \subseteq D \) and \( D \) satisfies the right generating condition.

Let \((C, \Delta, \varepsilon)\) be a coalgebra and \((M, \rho_M)\) a finite dimensional right \( C \)-comodule. Then \( \text{End}(M^C) \) - the set of comodule endomorphisms of \( M \) (equivalently, endomorphisms of \( M \) as left \( C^* \)-module) is a finite dimensional algebra considered with the opposite composition as multiplication. Considering \( \text{End}(M^C) \) as acting on \( M \) on the right, \( M \) becomes a \( C^* \)-\( \text{End}(M^C) \) bimodule. Denote \((A_M, \delta_M, e_M)\) the finite dimensional coalgebra dual to \( \text{End}(M^C) \); then it is easy to see that \( M \) is an \( A_M \)-\( C \) bicomodule, with the induced left \( A_M \)-comodule structure coming from the structure of a right \( \text{End}(M^C) \)-module. This holds since there is an equivalence of categories \( \mathcal{M}_{\text{End}(M^C)} \simeq A_M \mathcal{M} \), since \( A_M \) is finite dimensional. Let \( r_M : M \to A_M \otimes M \) be the left \( A_M \)-comultiplication of \( M \). As usual we use the Sweedler notation:

\[
\begin{align*}
\rho_M(m) &= m_0 \otimes m_1 \in M \otimes C \text{ for } m \in M \\
r_M(m) &= m_{(-1)} \otimes m_{(0)} \in A_M \otimes M \text{ for } m \in M \\
\Delta(c) &= c_1 \otimes c_2 \in C \otimes C \text{ for } c \in C \\
\delta_M(a) &= a_{(1)} \otimes a_{(2)} \in A_M \otimes A_M \text{ for } a \in A_M
\end{align*}
\]
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The compatibility relation between the left $A_M$-comodule and the right $C$-comodule structures of $M$ is then

\[(\ast)\quad m_{(-1)} \otimes m_{(0)0} \otimes m_{(0)1} = m_{0(-1)} \otimes m_{0(0)} \otimes m_1\]

We now proceed with the first step of our construction. Let $\mathcal{R}(C)$ be a set of representatives for the isomorphism types of finite dimensional right $C$-comodules. With the above notations, let

\[C' = (\bigoplus_{M \in \mathcal{R}(C)} A_M) \oplus (\bigoplus_{M \in \mathcal{R}(C)} M) \oplus C\]

and define $\delta : C' \to C' \otimes C'$ and $e : C' \to K$ by

\[
\begin{align*}
\delta(a) &= \delta_M(a) = a(1) \otimes a(2) \in A_M \otimes A_M \subseteq C' \otimes C' \text{ for } a \in A_M \text{ and } M \in \mathcal{R}(C) \\
\delta(m) &= r_M(m) + \rho_M(m) = m_{(-1)} \otimes m_{(0)} + m_0 \otimes m_1 \in A_M \otimes M + M \otimes C \subseteq C' \otimes C' \\
&\quad \text{for } m \in M \text{ and } M \in \mathcal{R}(C) \\
\delta(c) &= \Delta(c) = c_1 \otimes c_2 \in C \otimes C \subseteq C' \otimes C' \text{ for } c \in C
\end{align*}
\]

(E1) Here, everything is understood as belonging to the appropriate - corresponding component of the tensor product $C' \otimes C'$.

\[
\begin{align*}
e(a) &= e_M(a), \text{ for } a \in A_M \text{ and } M \in \mathcal{R}(C) \\
e(m) &= 0, \text{ for } m \in M \text{ and } M \in \mathcal{R}(C) \\
e(c) &= \varepsilon(c) \text{ for } c \in C
\end{align*}
\]

(E2) It is not difficult to see that $(C', \delta, e)$ is a coalgebra. For example, for $m \in M$ and $M \in \mathcal{R}(C)$

\[
\begin{align*}
(\delta \otimes \text{Id})\delta(m) &= (\delta \otimes \text{Id})(m_{(-1)} \otimes m_{(0)} + m_0 \otimes m_1) \\
&= m_{(-1)(1)} \otimes m_{(-1)2} \otimes m_{(0)} + m_{0(-1)} \otimes m_{0(0)} \otimes m_0 + m_{00} \otimes m_{01} \otimes m_1
\end{align*}
\]

and

\[
\begin{align*}
(\text{Id} \otimes \delta)\delta(m) &= (\text{Id} \otimes \delta)(m_{(-1)} \otimes m_{(0)} + m_0 \otimes m_1) \\
&= m_{(-1)} \otimes m_{(0)(-1)} \otimes m_{(0)(0)} + m_{(-1)} \otimes m_{(0)0} \otimes m_{(0)1} + m_0 \otimes m_{11} \otimes m_{12}
\end{align*}
\]
and here the first, second and third terms are equal respectively because of the coassociativity property of $M$ as left $A_M$-comodule, the compatibility from (*) and the coassociativity property of $M$ as right $C$-comodule. Also, we have $(e \otimes \text{Id})\delta(m) = (e \otimes \text{Id})(m_{(-1)} \otimes m_{(0)} + m_0 \otimes m_1) = e(m_{(-1)}) \otimes m_{(0)} + e(m_0) \otimes m_1 = 1 \otimes e_M(m_{(-1)})m_{(0)} = 1 \otimes m \text{ etc.}$

For $(M, \rho_M) \in \mathcal{R}(C)$, since $C \subseteq C'$ is an inclusion of coalgebras, $M$ has an induced right $C'$-comodule structure by $\rho : M \rightarrow M \otimes C \subseteq M \otimes C'$ (the “co-restriction of scalars”).

**Proposition 7.3.1** (i) Let $X(C) = (\bigoplus_{M \in \mathcal{R}(C)} A_M) \oplus (\bigoplus_{M \in \mathcal{R}(C)} M)$. Then $X(C)$ is a right $C'$-subcomodule of $C'$ and $C \otimes X(C) = C'$ as right $C'$-comodules.

(ii) If $M \in \mathcal{R}(C)$ and $Z_M = (\bigoplus_{N \in \mathcal{R}(C)} A_N) \oplus (\bigoplus_{N \in \mathcal{R}(C) \setminus \{M\}} N) \oplus C = A_M \oplus (\bigoplus_{N \in \mathcal{R}(C) \setminus \{M\}} A_N \oplus N) \oplus C$, then $Z_M$ is a right $C'$-subcomodule of $C'$ and $C'/Z_M \simeq M$ as right $C'$-comodules.

**Proof.** Using the relations defining $\delta$, we have $\delta(X(C)) \subseteq X(C) \otimes C'$. Thus (i) follows.

For (ii), let $p : C' = M \oplus Z_M \rightarrow M$ be the projection. We have $\delta(Z_M) \subseteq \bigoplus_{N \in \mathcal{R}(C)} (A_N \otimes A_N) \oplus \bigoplus_{N \in \mathcal{R}(C) \setminus \{M\}} (A_N \otimes N + N \otimes C) \oplus C \subseteq Z_M \otimes C'$. Then for $c' = m + z \in C'$, $m \in M$ and $z \in Z_M$, we have $(p \otimes \text{Id}_{C'})\delta(z) = 0$ and so

$$(p \otimes \text{Id}_{C'})\delta(m + z) = (p \otimes \text{Id}_{C'})(m_{(-1)} \otimes m_{(0)} + m_0 \otimes m_1)$$

$$= p(m_0) \otimes m_1 = m_0 \otimes m_1$$

$$= p(m + z)_0 \otimes p(m + z)_1 = (\rho_M \circ p)(m)$$

So $p$ is a morphism of right $C'$-comodules. Since $p = \text{Ker}(p) = Z_M$, (ii) follows. \hfill \square

We now proceed with the last steps of our construction. Build the coalgebras $C^{(n)}$ inductively by setting $C^{(0)} = C$ and $C^{(n+1)} = (C^{(n)})'$ for all $n$. Let $\delta_n, \varepsilon_n$ be the comultiplication and counit of $C^{(n)}$. We have $C^{(n+1)} = C^{(n)} \oplus X(C^{(n)})$ as $C^{(n+1)}$-comodules by Proposition 7.3.1(i). Let

$$C_\infty = \bigcup_{n \geq 1} C^{(n)}$$

as a coalgebra with $\delta_\infty$ and $\varepsilon_\infty$ defined as $\delta_\infty|_{C^{(n)}} = \delta_n$ and $\varepsilon_\infty|_{C^{(n)}} = \varepsilon_n$. We also note that $\delta_\infty(X(C^{(n)})) = \delta_n(X(C^{(n)})) \subseteq X(C^{(n)}) \otimes C^{(n)} \subseteq X(C^{(n)}) \otimes C_\infty$, so each $X(C^{(n)})$ is a
right $C_\infty$-subcomodule in $C_\infty$ and we therefore actually have

$$C_\infty = C \oplus \bigoplus_{n \geq 1} X(C^{(n)}) = C^{(n)} \oplus \bigoplus_{k \geq n+1} X(C^{(k)})$$

(7.1)

as right $C_\infty$-comodules. We can now conclude our

**Theorem 7.3.2** The coalgebra $C_\infty$ has the property that every finite dimensional right $C_\infty$-comodule is a quotient of $C_\infty$. Consequently, $C_\infty$ satisfies the right generating condition.

**Proof.** If $(N, \rho_N) \in \mathcal{M}^{C_\infty}$ is finite dimensional, then the coalgebra $D$ associated to $N$ is finite dimensional (see [DNR, Proposition 2.5.3]. This follows since the image of $\rho_N$ is finite dimensional and then the second tensor components in $N \otimes C_\infty$ from a basis of $\rho_N(N)$ span a finite dimensional coalgebra. If $d_1, \ldots, d_k$ is a basis of $D$, then there is an $n$ such that $d_1, \ldots, d_k \in C^{(n)}$, i.e. $D \subseteq C^{(n)}$. So $\rho_N : N \to N \otimes D \subseteq N \otimes C^{(n)} \subseteq N \otimes C_\infty$. Thus $N$ has an induced right $C^{(n)}$-comodule structure and so $\exists M \in \mathcal{R}(C^{(n)})$ such that $N \simeq M$ as $C^{(n)}$-comodules. Thus, by proposition 7.3.1(ii), there is an epimorphism $C^{(n)} \to N \to 0$ of right $C^{(n)}$-comodules. Then this is also an epimorphism in $\mathcal{M}^{C}\infty$. By equation (7.1) $C^{(n)}$ is a quotient of $C_\infty$ (in $\mathcal{M}^{C}\infty$) and consequently $N$ must be a quotient of $C_\infty$ as right $C_\infty$-comodules. Since any right $C_\infty$-comodule is the sum of its finite dimensional subcomodules, the statement follows.

**Example 7.3.3** Let $C$ be a coalgebra which is not right semiperfect. Then $C_\infty$ is not right semiperfect either, since a subcoalgebra of a semiperfect coalgebra is semiperfect (see [DNR, Corollary 3.2.11]). Then $C_\infty$ cannot be right QcF, since right QcF coalgebras are right semiperfect (see [DNR, Corollary 3.3.6]; see also [GTN]). So $C_\infty$ is not left projective by Theorem 7.0.9. But still, $C_\infty$ is a generator for the category of right $C_\infty$-comodules.

**Remark 7.3.4** It is also possible for a coalgebra to be generator and not projective in $\mathcal{M}^C$. Indeed, just take a coalgebra $C$ which is right QcF but not left QcF. Then $C$ generates $\mathcal{M}^C$ but $C^C$ is not projective since it is not left QcF. (Such a coalgebra exists, e.g. see [DNR, Example 3.3.7].)

**Example 7.3.5** Let $A$ be the algebra dual to the coalgebra $C$ of [DNR, Example 3.3.7 and Example 3.2.8]. This is left QcF and not right QcF, and $C_0$ is not finite dimensional. Thus $C^*$ is not semilocal ($C^*/J \simeq C_0^*$). By [DNR, Corollary 3.3.9], $C^*$ is right selfinjective, and it cannot be a right cogenerator since it is not semilocal.
Another construction

In the following we build another example of a coalgebra that has the right generating condition but is not right QcF. This will be a colocal coalgebra, that is, a coalgebra whose coradical is a simple (even 1-dimensional) coalgebra. Thus, this will show that another important condition in the theory of coalgebras, the condition that the coradical is finite dimensional, is not enough to have that the right generating condition implies that the coalgebra is right QcF.

Let $K$ be an algebraically closed field and $(C, \Delta, \varepsilon)$ be a colocal pointed $K$-coalgebra, so the coradical $C_0$ of $C$ is $C_0 = Kg$, with $g$ a grouplike element: $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The coalgebra $C$ is also called connected in this case. Let $\mathcal{L}(C)$ be a set of representatives for the indecomposable finite dimensional right $C$-comodules. Keeping the same notations as above, we note that $\text{End}(M^C)^{op}$ is a local $K$-algebra, since $M \in \mathcal{L}(C)$ is indecomposable. Moreover, its residue field is canonically isomorphic to $K$ since it is a finite dimensional division $K$-algebra over the algebraically closed field $K$. Thus, each $A_M$ is a colocal coalgebra and there exists a unique morphism of coalgebras $\sigma_M : K \to A_M$, with $g_M = \sigma_M(1)$ being the unique grouplike of $A_M$.

Let $(C^\sim, \delta, e)$ be the coalgebra defined by the same relations (E1) and (E2) as $C'$ above. Let $I$ be generated by the elements $\{g-g_M \mid M \in \mathcal{L}(C)\}$ as a vector space. They will even form a $K$-basis. Then $I$ is a coideal since $\delta(g-g_M) = g \otimes (g-g_M) + (g-g_M) \otimes g$ and $\varepsilon(g-g_M) = 0$. Let $\Sigma^\sim = (\bigoplus_{M \in \mathcal{L}(C)} Kg_M) \oplus Kg$, $\Sigma = Kg \subset C$ and $\Sigma^\perp = \Sigma^\sim / I$. Let $C^\vee = C^\sim / I$. With these notations we have

**Proposition 7.3.6** $C^\vee$ is a colocal pointed coalgebra.

**Proof.** Let $\sigma : K \to C$ be the canonical “inclusion” morphism $\sigma(1) = g$. The dual algebra of $C^\vee$ is $(C^\vee)^* = (C^\sim / I)^* \simeq I^\perp \subseteq (C^\sim)^* = C^* \times (\prod_{M \in \mathcal{L}(C)} (M^* \times A_M^*))$. Let $B = I^\perp$ which is a subalgebra of $(C^\sim)^*$ and let $J_M$ and $J$ denote the Jacobson radicals of $A_M^*$ and $C^*$ respectively. Note that $B$ consists of all families $(a; (m^*, a_M)_{M \in \mathcal{L}(C)}) \in (C^\sim)^*$ with $a_M(g_M) = a(g)$, equivalently, $\sigma_M(a_M) = \sigma(a)$. If two such families add up to the identity element $1_B$ of $B$, say $(a; (m^*, a_M)_{M \in \mathcal{L}(C)}) + (b; (n^*, b_M)_{M \in \mathcal{L}(C)}) = (1; (0, 1)_{M \in \mathcal{L}(C)})$, then $a_M + b_M = 1 \in A_M^*$ and $a + b = 1 \in A$. So $a \notin J$ or $b \notin J$, since $A$ is local. Say $a \notin J = Kg^+$, that is $a(g) \neq 0$. Then $a_M(g_M) = a(g) \neq 0$ and so all $a_M$ and $a$ are invertible. Thus $(a; (m^*, a_M)_{M \in \mathcal{L}(C)})$ is invertible with inverse $(a^{-1}; -a^{-1}m^*a_M^{-1}, a_M^{-1})_{M \in \mathcal{L}(C)})$. This
shows that $B = I^\perp$ is local with Jacobson radical $J \times (M^* \times J_M)$ and therefore, by duality, it is not difficult to see that $C^\vee$ is colocal with coradical $\Sigma^\vee$ respectively.

**Remark 7.3.7** We can easily see that we have a morphism of coalgebras $C \hookrightarrow A_M \oplus M \oplus C \twoheadrightarrow (A_M \oplus M \oplus C)/K \cdot (g - g_M)$ which is injective. Then, it is also easy to see that $C^\vee$ is the direct limit of the family of coalgebras $\{C\} \cup \{(A_M \oplus M \oplus C)/K \cdot (g - g_M)\}_{M \in \mathcal{L}(C)}$ with the above morphisms. In fact, the algebra $A \times M^* \times A_M^*$ dual to $A_M \oplus M \oplus C$ is the upper triangular “matrix” algebra with obvious multiplication:

$$
\begin{pmatrix}
C^* & M^* \\
0 & A_M^*
\end{pmatrix}
$$

We note that $C$ embeds in $C^\vee$ canonically as a coalgebra following the composition of morphisms $C \hookrightarrow C^\sim \twoheadrightarrow C^\sim/I = C^\vee$, since $g \notin I$ so $C \cap I = 0$. This allows us to view each right $C$-comodule $M$ as a comodule over $C^\vee$, by the “corestriction” of scalars $M \to M \otimes C \to M \otimes C^\sim \to M \otimes C^\vee$. Let $p_M : C^\sim \to C^\sim/I \to M$ be the projection.

**Proposition 7.3.8** (i) $p_M$ is a morphism of right $C^\vee$-comodules.

(ii) Each $M \in \mathcal{L}(C)$ is a quotient of $C^\vee/\Sigma^\vee$.

(iii) $C/\Sigma$ is a direct summand in $C^\vee/\Sigma^\vee$ as right $C^\vee$-comodules; in fact, if we denote $X^\sim(C) = \bigoplus_{N \in \mathcal{L}(C)} (A_N \oplus N)$ and $X^\vee(C) = (X^\sim(C) + I)/I$ we have an isomorphism of right $C^\vee$-comodules:

$$
\frac{C^\vee}{\Sigma^\vee} \simeq \frac{C}{\Sigma} \oplus X^\vee(C)
$$

**Proof.** (i) By Proposition 7.3.1 $p_M$ is a morphism of right $C^\sim$-comodules, and then it is also a morphism of $C^\vee$-comodules via corestriction of scalars. Since the projection $C^\sim \to C^\sim/I = C^\vee$ is a morphism of coalgebras, it is also a morphism of right $C^\vee$-comodules, and since it also factors through $I$, we get that $p_M$ is a morphism of $C^\vee$-comodules.

(ii) follows since $p_M$ is a morphism of right $C^\vee$-comodules which is 0 on $\Sigma^\vee$.

(iii) Note that the coradical $\Sigma$ of $C$ is identified with $\Sigma^\vee$ by the inclusion $C \hookrightarrow C^\vee$. Also both $C$ and $X^\sim(C)$ are right $C^\sim$-subcomodules of $C^\sim$ (just as above for $X(C)$ in $C'$), and then $C$ and $X^\vee(C)$ are also $C^\vee$-subcomodules in $C^\vee$. Since we also have an isomorphism

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of vector spaces

\[ \frac{C^\vee}{\Sigma^\vee} \cong \frac{C}{\Sigma} = \bigoplus_{M \in \mathcal{C}(C)} \left( \frac{A_M}{Kg_M} \oplus M \right) \oplus \frac{C}{Kg} = \frac{C}{\Sigma} \oplus X^\vee(C) = \frac{C}{\Sigma^\vee} \oplus X^\vee(C), \]

the proof is finished.

To end the second construction, start with an arbitrary pointed colocal coalgebra over an algebraically closed field \( K \). Denote \( C^{[0]} = C \) and \( C^{[n+1]} = (C^{[n]})^\vee \) for all \( n \geq 0 \). Put \( C^\vee = \bigcup C^{[n]} \). Then we have

**Theorem 7.3.9** The coalgebra \( C^\vee \) is colocal and has the property that any indecomposable finite dimensional right \( C^\vee \)-comodule is a quotient of \( C^\vee \). Consequently, \( C^\vee \) has the right generating condition.

**Proof.** Since all the coalgebras \( C^{[n]} \) are colocal, say with common coradical \( \Sigma \), so will be \( C^\vee \). Let \( (M, \rho) \) be a finite dimensional indecomposable \( C^\vee \)-comodule. Then, as before, \( \rho(M) \subseteq M \otimes C^{[n]} \) for some \( n \), since \( \dim M < \infty \). So \( M \) has an induced structure of a right \( C^{[n]} \)-comodule, and by Proposition 7.3.8(ii), \( M \) is a quotient of \( C^{[n+1]}/\Sigma \). But Proposition 7.3.8 together with the construction of \( C^\vee \), ensure that

\[ \frac{C^\vee}{\Sigma} = \frac{C^{[n+1]}}{\Sigma} \oplus \bigoplus_{k \geq n+1} X^\vee(C^{[k]}). \]

Moreover, since each \( X^\vee(C^{[k]}) \) is a right \( C^{[k]} \)-subcomodule in \( C^{[k]}/\Sigma \), which is in turn a \( C^\vee \)-subcomodule of \( C^\vee/\Sigma \), it follows that the \( X^\vee(C^{[k]}) \) are actually \( C^\vee \)-subcomodules in \( C^\vee/\Sigma \). Therefore, \( C^{[n+1]}/\Sigma \) splits off in \( C^\vee \), and so \( C^\vee \) has \( M \) as a quotient. The final conclusion follows since any finite dimensional comodule is a coproduct of finite dimensional indecomposable ones. \( \square \)

**Example 7.3.10** Let \( C \) be a connected (i.e. pointed colocal) coalgebra which is not right semiperfect. Then \( C^\vee \) is not right semiperfect but has the right generating condition. Then, as in example 7.3.3, \( C^\vee \) is not right QcF. More specifically, we can take \( C = \mathbb{C}[[X]]^\circ \), the divided power coalgebra over the field of complex numbers, which has a basis \( c_n \) for \( n \geq 0 \) with comultiplication \( \Delta(c_n) = \sum_{i+j=n} c_i \otimes c_j \) and counit \( \varepsilon(c_n) = \delta_{0,n} \)- the Kroneker symbol.
Remark 7.3.11 We could have arranged for $C_\infty$ also to have all its finite dimensional comodules as quotients. Indeed, for this, it is enough that at each step of the construction - in passing from $C$ to $C^\vee$ - to consider the direct sum constructing $C^\vee$ to contain countably many copies of each right $C$-comodule $M$, that is, $C^\vee = \left( \bigoplus_{N \in \mathcal{L}(C)} \bigoplus_{M \in L(C)} (A_M \oplus M) \right) \oplus C / I$.

Then every finite dimensional comodule will be decomposed in a direct sum of finitely many indecomposable comodules, which we will be able to generate as a quotient of only one $C^{[n]} / \Sigma$ for some $n$, since enough of these indecomposable components can be found in $C^{[n]} / \Sigma$. In fact, it is easy to see that every finite or countable sum of $M \in \mathcal{L}(C^{[n-1]})$ can be obtained as a quotient of $X^\vee(C^{[n]}) / \Sigma = \bigoplus_{N \in \mathcal{L}(C^{[n-1]})} A_M \oplus M$. 
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