Partial regularity results of solutions to the 3D Incompressible Navier–Stokes equations and other models

Wojciech S. Ożański

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The Navier–Stokes equations

\[ u_t + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty), \]
\[ \text{div} \ u = 0, \]
\[ u(0) = u_0. \]

\[ u : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}^3 \] - velocity field,
\[ p : \mathbb{R}^3 \times (0, \infty) \to \mathbb{R} \] - pressure function,
\[ \nu > 0 \] - viscosity.
The Navier–Stokes equations
Existence and uniqueness of solutions

\[ \| \nabla u_0 \|_{L^2} \]
The Navier–Stokes equations
Existence and uniqueness of solutions

\[ \parallel \nabla u_0 \parallel_{L^2} \parallel \nabla u(t) \parallel_{L^2} \]
The Navier–Stokes equations
Existence and uniqueness of solutions

\[ \| \nabla u_0 \|_{L^2} \leq \| \nabla u(t) \|_{L^2} \]
The Navier–Stokes equations
Existence and uniqueness of solutions

\[
\left\| \nabla u_0 \right\|_{L^2} \quad \left\| \nabla u(t) \right\|_{L^2}
\]
The Navier–Stokes equations
Existence and uniqueness of solutions

Leray (1934) [O. & Pooley (2017)]
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\[ \| \nabla u_0 \|_{L^2} \quad \| \nabla u(t) \|_{L^2} \]

\[ \| u_0 \|_{L^2} \quad \| u(t) \|_{L^2} \]

Leray weak solution

\[ t_0 \quad t_1 \quad t_2 \quad T \]
The Navier–Stokes equations
Existence and uniqueness of solutions
The Navier–Stokes equations
Existence and uniqueness of solutions

\[ d_H(\mathcal{T}) \leq 1/2, \quad \text{where} \quad \mathcal{T} := \{ t > 0 : \| \nabla u(t) \|_{L^2} = \infty \}. \]
The Navier–Stokes equations

Partial regularity

A pair \((u, p)\) is a suitable weak solution if

(i) (regularity of \(u\) and \(p\)) \(u \in L^\infty_t L^2_x\), \(\nabla u \in L^2\), \(u(t)\) is divergence-free for almost every \(t\), \(p \in L^{3/2}_{loc}\),

(ii) (relation between \(u\) and \(p\)) the equation

\[
-\Delta p = \sum_{i,j=1}^{3} \partial_i \partial_j (u_j u_i)
\]

holds in the sense of distribution,

(iii) (local energy inequality) the Navier-Stokes inequality,

\[
u \cdot (u_t - \Delta u + (u \cdot \nabla)u + \nabla p) \leq 0
\]

holds in the sense of distributions,

(iv) (the equation) the Navier-Stokes equations hold in the sense of distributions.

A pair \((u, p)\) is a weak solution of the Navier–Stokes inequality if conditions (i)-(iii) hold.
Theorem (Caffarelli-Kohn-Nirenberg, 1982)

Let \((u, p)\) be a weak solution of the Navier–Stokes inequality. There exist \(\varepsilon_1, \varepsilon_2\) such that

1. If

\[
\frac{1}{r^2} \int_{Q_r} |u|^3 + |p|^{3/2} \leq \varepsilon_1
\]

for any \(r > 0\), then \(u \in L^\infty(Q_{r/2})\).

2. If

\[
\limsup_{r \to 0^+} \frac{1}{r} \int_{Q_r} |\nabla u|^2 \leq \varepsilon_2
\]

then \(u \in L^\infty(Q_\rho)\) for some \(\rho > 0\).
Let

\[ S := \{(x, t): u \text{ is unbounded in any neighbourhood of } (x, t)\}. \]

**Corollary of the CKN theorem:** \( d_H(S) \leq 1, \ d_B(S) \leq 5/3. \)

**He, Wang & Zhou (2017):** \( d_B(S) \leq 2400/1903 (\approx 1.261). \)

**Scheffer (1985 & 1987):** the bound \( d_H(S) \leq 1 \) is sharp for weak solutions of the Navier–Stokes inequality,
The Navier–Stokes equations
Partial regularity

O. (2019): 1) refined constructions of Scheffer,
2) constructions satisfying the
“approximate equality”

\[-\vartheta \leq u \cdot (u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla p) \leq 0\]

for any preassigned \(\vartheta > 0\).

O. (2019): given \(T > 0, \varepsilon > 0\) and a nonincreasing function \(e : [0, T] \rightarrow [0, \infty)\) with \(e(T) = 0\) there exist weak solutions to the Navier–Stokes inequality that blow up on a Cantor set at time \(T\), with

\[\|\|u(t)\|\|_{L^2} - e(t)\| \leq \varepsilon \quad \text{for} \ t \in [0, T].\]
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction
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$$\|u(t)\|_{L^\infty}$$
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

\[ \|u(t)\|_{L^\infty} \]

\[ 0 = t_0 \quad T = t_1 \quad t_2 \]
The “sharpness” of the bound $d_H(S) \leq 1$
Sketch of the construction

$\|u(t)\|_{L^\infty}$

$0 = t_0 \quad T = t_1 \quad t_2$
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

\[ \|u(t)\|_{L^\infty} \]

\[ 0 = t_0, T = t_1, t_2, t_3 \]
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

\[ \|u(t)\|_{L^\infty} \]

\[ 0 = t_0 \quad T = t_1 \quad t_2 \quad t_3 \]
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

$\|u(t)\|_{L^\infty} = t_0, T = t_1, t_2, t_3, T_0 = \lim_{j \to \infty} t_j$
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

$\text{supp } u(0)(\cdot, t)$
The “sharpness” of the bound $d_H(S) \leq 1$

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$\text{supp } u(0)(\cdot, t)$
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction

\[
\text{supp } u^{(0)}(\cdot, t)
\]
The “sharpness” of the bound $d_H(S) \leq 1$

Sketch of the construction
The surface growth model

\[ u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0 \quad \text{in } \mathbb{T} \times (0, \infty), \]

\[ u(0) = u_0. \]
The surface growth model

\[ u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \quad \text{in } T \times (0, \infty), \]
\[ u(0) = u_0. \]

\[ d_H(\mathcal{T}) \leq 1/4, \quad \text{where} \quad \mathcal{T} := \{ t > 0 : \|u_x(t)\|_{L^2} = \infty \}. \]
The surface growth model $u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0$

Partial regularity

Theorem (O. & Robinson, 2019)

Let $u$ be a suitable weak solution of the surface growth model.

There exist $\varepsilon_1, \varepsilon_2$ such that

1. If
\[
\frac{1}{r^2} \int_{Q_r} |u_x|^3 \leq \varepsilon_1
\]

for any $r > 0$, then $u$ is Hölder continuous in $Q_{r/2}$.

2. If
\[
\limsup_{r \to 0^+} \frac{1}{r} \int_{Q_r} |u_{xx}|^2 \leq \varepsilon_2
\]

then $u$ is Hölder continuous in $Q_\rho$ for some $\rho > 0$.

Consequently, $d_H(S) \leq 1$, $d_B(S) \leq 7/6$. 
The Navier–Stokes equations with fractional dissipation

Consider the 3D incompressible Navier–Stokes equations with $\alpha$-Laplacian,

$$u_t + (-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0, \infty),$$
$$\text{div } u = 0,$$
$$u(0) = u_0.$$

Theorem

The equations are well-posed (for any sufficiently regular $u_0$) for $\alpha \geq 5/4$. 
The Navier–Stokes equations with fractional dissipation

Theorem (Katz & Pavlović, 2002)

Let $S'$ denote the singular set in space at the first blow-up time of a local-in-time strong solution, for $\alpha \in (1, \frac{5}{4})$. Then $d_H(S') \leq 5 - 4\alpha$.

Idea:

1. A cube $Q$ is a $j$-cube if it has sidelength $2^{-j(1-\varepsilon)}$.
2. Study behaviour in time of $u_Q := \|\phi_Q P_j u\|_{L^2}$, for $j \in \mathbb{Z}$ and $j$-cubes $Q \subset \mathbb{R}^3$.
3. Deduce that $S' \subset \limsup_{k \to \infty} A_k$, where $A_k$ is a (carefully chosen) family of $k$-cubes, with cardinality $\leq c 2^k(5 - 4\alpha + \varepsilon)$.
4. It follows that $d_H(S') \leq 5 - 4\alpha + \varepsilon$. 
The Navier–Stokes equations with fractional dissipation

Theorem (Tang & Yu, 2013)
Let \((u, p)\) be a suitable weak solution, and \(S\) denote the singular set (in space-time) of \(u\), for \(\alpha \in (3/4, 1)\). Then \(d_H(S) \leq 5 - 4\alpha\).

Theorem (Colombo, De Lellis & Massaccesi, 2017)
The same claim is valid for \(\alpha \in (1, 5/4]\).
Furthermore, \(d_B(S) \leq (-8\alpha^2 - 2\alpha + 15)/3\).
(Note that the last estimate reduces to 5/3 in the case \(\alpha = 1\).)

Theorem (O., 2020)
Let \(\alpha \in (1, 5/4)\) and let \((u, p)\) be any Leray-Hopf weak solution (i.e. not necessarily suitable), and \(S'\) denote the singular set in space of \(u\). Then
(i) \(d_H(S') \leq 5 - 4\alpha\),
(ii) \(d_B(S') \leq (-16\alpha^2 + 16\alpha + 5)/3\).
The Navier–Stokes equations with fractional dissipation

Idea: (inspired by the approach of Katz & Pavlović (2002))

1. A cube $Q$ is a $j$-cube if it has sidelength $2^{-j(1-\varepsilon)}$.

2. Study behaviour in time of $u_Q := \| \phi_Q P_j u \|_{L^2}$ (for $j \in \mathbb{Z}$ and $j$-cubes $Q \subset \mathbb{R}^3$), inside every interval of regularity.

3. Use weak $L^2$ continuity in time of $u$ to obtain an estimate uniform in time $\Rightarrow$ (i).

4. Observe that, for every $j$, $S' \subset \bigcup_{k=C(\alpha)j}^j A_k$ (rather than $\limsup_{k \to \infty}$), where $A_k$ is a (carefully chosen) family of $k$-cubes, with cardinality $\leq c2^k(5-4\alpha+\varepsilon)$.

5. Cover every $k$-cube from $A_k$ with $k < j$ by $j$-cubes $\Rightarrow$ (ii).
Thank you for your attention.
The surface growth model $u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0$

Nonlinear parabolic Poincaré inequality

Theorem (O. & Robinson, 2017)

Any distributional solution $u$ to SGM,

$$u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0,$$

on a cylinder $Q_{2r}$ satisfies

$$\|u - [u]_{Q_r}\|_{L^3(Q_r)} \leq C \left( r \|u_x\|_{L^3(Q_{2r})} + r^{1/3} \|u_x\|_{L^3(Q_{2r})}^2 \right),$$

where

$$[u]_{Q_r} := \frac{1}{|Q_r|} \int_{Q_r} u.$$
The surface growth model $u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0$

Partial regularity; main iteration

Lemma (The main iteration)

Given $\theta \in (0, 1/4)$ there exist $\varepsilon_0 = \varepsilon_0(\theta)$ and $R = R(\theta)$ such that for weak solution $u$ of the SGM and any cylinder $Q_r$ with $r < R$

$$\frac{1}{r^2} \int_{Q_r} |u_x|^3 < \varepsilon_0$$

implies

$$\frac{1}{(\theta r)^2} \int_{Q_{\theta r}} |u_x|^3 < c_* \theta^3 \varepsilon_0,$$

where $c_*$ is a universal constant.
The surface growth model \( u_t + uu_{xxxx} + \partial_{xx}u_x^2 = 0 \)

Partial regularity; main iteration

Suppose that \( \varepsilon_k \to 0, r_k \to 0 \) and cylinders \( Q_{r_k} \) are such that

\[
\frac{1}{r_k^2} \int_{Q_{r_k}} |\partial_x u|^3 = \varepsilon_k, \quad \frac{1}{(\theta r_k)^2} \int_{Q_{\theta r_k}} |\partial_x u|^3 \geq c_\star \theta^3 \varepsilon_k.
\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \)

Partial regularity; main iteration

Suppose that \( \varepsilon_k \to 0, r_k \to 0 \) and cylinders \( Q_{r_k} \) are such that

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\]

Step 1. Consider the rescalings

\[
u_k(x, t) := \frac{u(x_k + x r_k, t_k + t r_k^4) - [u]_{Q_{r_k}/2}}{\varepsilon_k^{1/3}}.\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0 \)

Partial regularity; main iteration

Suppose that \( \varepsilon_k \to 0, r_k \to 0 \) and cylinders \( Q_{r_k} \) are such that

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\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0 \)

Partial regularity; main iteration

Then

\[
\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_* \theta^5,
\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0 \)

Partial regularity; main iteration

Then

\[
\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_* \theta^5, \quad \int_{Q_{1/2}} u_k = 0,
\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \)

Partial regularity; main iteration

Then
\[
\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_\star \theta^5, \quad \int_{Q_{1/2}} u_k = 0,
\]

\[
\int_{Q_1} u_k \phi_t = \int_{Q_1} u_k \phi_{xxxx} + \varepsilon_k^{1/3} \int_{Q_1} (\partial_x u_k)^2 \phi_{xx}, \quad \phi \in C_0^\infty(Q_1).
\]
The surface growth model $u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0$

Partial regularity; main iteration

Then

$$\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_* \theta^5, \quad \int_{Q_{1/2}} u_k = 0,$$

$$\int_{Q_1} u_k \phi_t = \int_{Q_1} u_k \phi_{xxxx} + \varepsilon_{k}^{1/3} \int_{Q_1} (\partial_x u_k)^2 \phi_{xx}, \quad \phi \in C_0^\infty(Q_1).$$

Step 2. Take a limit in $k$.

There exists $v \in L^3(Q_{1/2})$ and a subsequence $k_n \to \infty$ such that

$$u_{k_n} \rightharpoonup v, \quad \partial_x u_{k_n} \rightharpoonup \partial_x v \quad \text{in} \quad L^3(Q_{1/2}).$$

By interior regularity of solutions to $v_t + v_{xxxx} = 0$

$$\|v\|_{L^\infty(Q_{1/2})} \leq (c_*/8)^{1/3}. $$
The surface growth model \( u_t + u_{xxxx} + \partial_x u_x^2 = 0 \)

Partial regularity; main iteration

Then
\[
\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_* \theta^5, \quad \int_{Q_{1/2}} u_k = 0,
\]

\[
\int_{Q_1} u_k \phi_t = \int_{Q_1} u_k \phi_{xxxx} + \varepsilon_1^{1/3} \int_{Q_1} (\partial_x u_k)^2 \phi_{xx}, \quad \phi \in C_0^\infty(Q_1).
\]

Step 2. Take a limit in \( k \).

There exists \( v \in L^3(Q_{1/2}) \) and a subsequence \( k_n \to \infty \) such that

\[
u_{k_n} \rightharpoonup v, \quad \partial_x u_{k_n} \rightharpoonup \partial_x v \quad \text{in} \ L^3(Q_{1/2}).
\]

By interior regularity of solutions to \( v_t + v_{xxxx} = 0 \)

\[
\| v \|_{L^\infty(Q_{1/2})} \leq (c_*/8)^{1/3}.
\]

Step 3. Use the A-L lemma to get \( \partial_x u_{k_n} \to \partial_x v \) in \( L^3(Q_{1/4}) \).
The surface growth model \( u_t + u_{xxxx} + \partial_{xx}u_x^2 = 0 \)

Partial regularity; main iteration

Then

\[
\int_{Q_1} |\partial_x u_k|^3 = 1, \quad \int_{Q_\theta} |\partial_x u_k|^3 \geq c_* \theta^5, \quad \int_{Q_{1/2}} u_k = 0,
\]

\[
\int_{Q_1} u_k \phi_t = \int_{Q_1} u_k \phi_{xxxx} + \varepsilon_k^{1/3} \int_{Q_1} (\partial_x u_k)^2 \phi_{xx}, \quad \phi \in C_0^\infty(Q_1).
\]

Step 2. Take a limit in \( k \).

There exists \( v \in L^3(Q_{1/2}) \) and a subsequence \( k_n \to \infty \) such that

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u_{k_n} \rightharpoonup v, \quad \partial_x u_{k_n} \rightharpoonup \partial_x v \quad \text{in} \ L^3(Q_{1/2}).
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By interior regularity of solutions to \( v_t + v_{xxxx} = 0 \)

\[
\|v\|_{L^\infty(Q_{1/2})} \leq (c_*/8)^{1/3}.
\]

Step 3. Use the A-L lemma to get \( \partial_x u_{k_n} \rightharpoonup \partial_x v \) in \( L^3(Q_{1/4}) \).

Step 4. Obtain a contradiction:

\[
1 \leq \frac{1}{c_* \theta^5} \int_{Q_\theta} |v_x|^3 \leq \frac{1}{8 \theta^5 |Q_\theta|} = \frac{1}{2}. \quad \square
\]
The surface growth model \( u_t + u_{xxxx} + \partial_{xx} u^2_x = 0 \)

The local Serrin condition

Theorem (O., 2018)

*If \( u \) is a weak solution of the SGM in a cylinder \( Q = I \times B \) and \( u_x \in L^{q'}(I; L^q(B)) \), where the exponents \( q', q \geq 2 \) are such that \( \frac{4}{q'} + \frac{1}{q} = 1 \) and \( q' < \infty \), then \( u \in C^\infty(Q) \).*
The surface growth model \( u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \)

The local Serrin condition

**Theorem (O., 2018)**

If \( u \) is a weak solution of the SGM in a cylinder \( Q = I \times B \) and \( u_x \in L^{q'}(I; L^q(B)) \), where the exponents \( q', q \geq 2 \) are such that

\[
\frac{4}{q'} + \frac{1}{q} = 1 \quad \text{and} \quad q' < \infty,
\]

then \( u \in C^\infty(Q) \).

Q. How about \( q' = \infty \)?
The surface growth model \( u_t + u_{xxxx} + \partial_{xx} u_x^2 = 0 \)

The local Serrin condition

**Theorem (O., 2018)**

*If \( u \) is a weak solution of the SGM in a cylinder \( Q = I \times B \) and

\[
 u_x \in L^{q'}(I; L^q(B)),
\]

where the exponents \( q', q \geq 2 \) are such that

\[
 \frac{4}{q'} + \frac{1}{q} = 1 \quad \text{and} \quad q' < \infty,
\]

then \( u \in C^\infty(Q) \).*

Q. How about \( q' = \infty \)?

→ \( L_{3,\infty} \) condition in the NSE (Escauriaza, Seregin & Šverák, 2003)