A note on characteristic functions.

By definition, the characteristic function of a (real-valued) random variable $\xi$ is a (complex-valued) function $\phi$ of the real variable $t$:

$$\phi(t) = \mathbb{E}e^{\sqrt{-1}\xi t}.$$

The definition immediately implies the following properties of $\phi$:

(1) $\phi(0) = 1$
(2) $|\phi(t)| \leq 1$ for all $t$;
(3) $\phi$ is (uniformly) continuous [check out the uniform part].

The main necessary and sufficient result is known as the Bochner-Khinchin theorem: a complex-valued function $\phi$ of a real variable $t$ is a characteristic function of some random variable if and only if all the following three properties hold

(1) $\phi(0) = 1$
(2) $\phi$ is continuous for all $t$
(3) for every collection $t_1, \ldots, t_n$ of real numbers the matrix $(\phi(t_i - t_j), i, j = 1, \ldots, n)$ is Hermitian and non-negative definite.

The third property is not so easy to verify. One famous sufficient condition is due to Polya: If $\phi(t)$ is even, $\phi(0) = 1$, $\phi$ is convex for $t > 0$, and $\lim_{t \to +\infty} \phi(t) = 0$, then $\phi$ is a characteristic function of an absolutely continuous random variable. For more, see [1,3].

Here is a necessary condition [1, Theorem 4.1.1]: if $\phi$ is a characteristic function and $\phi(t) = 1 + o(t^2), t \to 0$, then $\phi(t) = 1$ for all $t$ [indeed, the random variable with such a characteristic function must have zero mean and zero variance]. In particular, if $r > 2$, then $e^{-|t|^r}$ is not a characteristic function.

Another necessary condition is due to Marcinkiewicz (see [2], no proof...): if $\phi(t) = e^{p(t)}$ is a characteristic function and $p = p(t)$ is a polynomial, then the degree of $p$ is at most 2. For example, $e^{-|t|^2 - t^4}$ is not a characteristic function.

Some other facts:

(1) If $\xi$ is absolutely continuous, then $\lim_{|t| \to \infty} |\phi(t)| = 0$ [Riemann-Lebesque];
(2) If $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$, then $\xi$ is absolutely continuous with pdf

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\sqrt{-1}tx} \phi(t)dt.$$

Question 1. Is $e^{-|t|^t - t^4}$ a characteristic function? [Technically, $|t| - t^4$ is not a polynomial...]

Question 2. Let $X_1, X_2, \ldots$ be iid random variables with support on the standard mid-third Cantor set; the cdf of $X_1$ is the Cantor ladder (or Devil’s staircase...); the characteristic function of $X_1$ is $\phi(t) = e^{\sqrt{-1}t/2} \prod_{k=1}^{\infty} \cos(t/3^k)$. Does there exist an $n > 1$ such that the sum $X_1 + \ldots + X_n$ is absolutely continuous?

References.