REGULARITY RESULTS FOR STOKES TYPE SYSTEMS

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Abstract. The aim of this work is to study the regularity of solutions of Stokes type systems related to the large scale equations of the ocean and the primitive equations of the coupled system atmosphere-ocean, which have appeared in the work of J.L. Lions, R. Temam and S. Wang [LTW 1,2,3]. We prove various regularity results for strongly elliptic boundary value problems in cylinder-type domains with nonhomogeneous boundary conditions, as well as the $H^2$-regularity for the Stokes-type system built into these equations.

1. Introduction

In this work we will establish regularity results for Stokes-type systems of partial differential equations. The system of equations were introduced by J.L. Lions, R. Temam and S. Wang, for the space dimension 3, in their work on the primitive equations of the atmosphere [LTW 1], large scale equations of the ocean [LTW 2] and the coupled model atmosphere-ocean [LTW 3].

The general equations governing the state of the atmosphere, or the ocean (considered as a compressible fluid) are given by the hydrodynamic and thermodynamic equations under Coriolis forces. Lions, Temam and Wang used the approximation of the vertical momentum equations of both the atmosphere and the ocean by the hydrostatic equations in order to formulate the primitive equations of the atmosphere and the large scale equations of the ocean. They established the existence of weak solutions and their time-analyticity, as well as a maximal regularity result for the linear stationary problem associated to the primitive equations of the atmosphere [LTW 1, Lemma 2.8]; the domain in this case is $S^2 \times (0, 1)$ (i.e. no lateral boundary). Similar regularity results for the large scale equations of the ocean, which are the “raison d’être” of our study, are needed to have a better understanding of the models considered by J.L. Lions, R. Temam and S. Wang.

In their work they also considered a model for the Coupled Atmosphere and Ocean referred to as CAO (see [LTW 3,4]). The coupling is handled using, on one hand, a phenomenological formula for the momentum transfer flux near the interface (the boundary...
layer), which corresponds to the horizontal component of the wind force (the drag force) exerted on the surface of the ocean.

Horizontal wind force = $\rho C_D (v^a - v^s)|v^a - v^s|^\alpha$,

where $C_D$ is the momentum transfer coefficient or the drag coefficient, which is a nondimensional parameter, and $v^a$ and $v^s$ are the horizontal velocities of the air and the sea at the interface. On the other hand, the shear stress of the ocean (the force that the water exerts on the air) is given by:

Shear Stress of the Ocean = $\rho v^s \frac{\partial v^s}{\partial z}$.

The interface conditions are given by the well accepted physical law describing the driving mechanism as follows:

Shear Stress of the Ocean = Horizontal Wind Force.

The regularity of solutions for the linear stationary problem associated to a simplified form of the primitive equations of the coupled atmosphere-ocean (the case where the domains of the fluids present no lateral boundary i.e. the domain is $T^2 \times (-1,1)$,) is established (see [LTW4]) using the difference quotient method [ADN]. However the domain occupied by the ocean cannot be considered with no lateral boundary. Therefore, the results obtained in this paper are needed.

Due to the complexity of the operators and the domain occupied by the ocean, a simpler form of these equations in a cylinder-type domain defined in (1.1), (1.2), (1.3) will be considered in this work; the study of the regularity in the general case of operators and domain will be given in an upcoming paper.

Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^{n-1}, n \geq 2$ with a smooth boundary $\partial \mathcal{O}$; and let $h$ be a mapping of class $C^2$ on $\bar{\mathcal{O}}$ such that:

\begin{equation}
(*) \quad \exists (h_1, h_2) \in \mathbb{R}^2 \text{ such that } h_1 \leq h(x_1, \ldots, x_{n-1}) \leq h_2 < 0, \quad (x_1, \ldots, x_{n-1}) \in \mathcal{O}.
\end{equation}

We define the open set $\Omega \subset \mathbb{R}^n$ as follows:

\begin{equation}
(1.1.) \quad \Omega = \{(x', x_n) \in \mathcal{O} \times \mathbb{R}; \ h(x') < x_n < 0\},
\end{equation}

where $x' = (x_1, \ldots, x_{n-1})$.
Consider the following $n$-dimensional Stokes-type system of partial differential equations:

\begin{align}
-\Delta u(x) + \nabla_{x'} p(x') &= f(x), \quad x \in \Omega, \\
\text{div} \int_{h(x')}^0 u(x_1, \cdots, x_{n-1}, \eta) d\eta &= 0, \quad x' = (x_1, \cdots, x_{n-1}) \in \mathcal{O}.
\end{align}

Here $p(x')$ is the pressure; it can be shown that $p$ is a Lagrange multiplier of the non-local constraint (1.3). We denoted the horizontal velocity by $u = (u_1, \cdots, u_{n-1})$. The Laplacian operator in $\mathbb{R}^n$ is denoted by $\Delta$ and the gradient operator (resp. the divergence operator) in $\mathbb{R}^{n-1}$ is denoted by $\nabla_{x'}$ (resp. $\text{div}_{x'}$). For $u = (u_1(x), \cdots, u_{n-1}(x))$, $x \in \mathbb{R}^n$, we write

$$\text{div}_{x'} u = \sum_{\alpha=1}^{n-1} \frac{\partial u_\alpha}{\partial x_\alpha}(x_1, \cdots, x_n).$$

Equations (1.2), (1.3) are supplemented with one of these boundary conditions:

1.a. The Dirichlet condition: $u = 0$ on $\partial \Omega$.

1.b. The Dirichlet-Periodic condition: Here

$$\mathcal{O} = \Pi_{\alpha=1}^{n-1}(0, L_\alpha), \quad \text{and} \quad \Omega = \{(x', x_n) \in \Pi_{\alpha=1}^{n-1}(0, L_\alpha) \times \mathbb{R}; \ h(x') < x_n < 0\}, \quad \text{and}$$

$u$ and $h$ are periodic in the directions $x_1, \cdots, x_{n-1}$ with periods $L_1, \cdots, L_{n-1}$ respectively, and $u(x', 0) = u(x', h(x')) = 0$ for $x' \in \mathcal{O}$.

1.c. The mixed Dirichlet-Neumann conditions:
\[
\frac{\partial u}{\partial x_n} = g(x',0) \text{ on } \Gamma_u \text{ and } u = 0 \text{ on } \Gamma_l \cup \Gamma_b.
\]

1.d. The mixed Dirichlet-Neumann conditions:

\[
\frac{\partial u}{\partial x_n} = g(x', h(x')) \text{ on } \Gamma_u \cup \Gamma_b \text{ and } u = 0 \text{ on } \Gamma_l,
\]

where

\[
\Gamma_u = \{(x', x_n); x' \in \mathcal{O}, x_n = 0\}, \quad \text{and} \quad \Gamma_b = \{(x', x_n); x' \in \mathcal{O}, x_n = h(x')\},
\]

denote the upper and the lower boundaries of \(\Omega\) and

\[
\Gamma_l = \{(x', x_n); x' \in \partial \mathcal{O}, h(x') \leq x_n \leq 0\}
\]

denotes the lateral boundary of \(\Omega\), (see Fig. 1).

Equations (1.2), (1.3) are defined in a domain with corners, therefore the regularity of solutions is difficult to obtain. Moreover, equation (1.2) is nonlocal (with respect to the \(x_n\)-variable). The Classical methods of establishing the regularity of solutions [ADN], [Da1,2,3], [Gr1,2] fail to solve the problem. However, the problem (1.2), (1.3) presents an interesting feature; if the integration of (1.2) with respect to \(x_n\) is possible (for that we need \(\frac{\partial u}{\partial x_n} \in H^{1/2+\varepsilon}(\Omega)\), for some \(\varepsilon > 0\)), then the resulted system of equations is the \((n-1)\)-dimensional Stokes problem defined in the smooth domain \(\mathcal{O}\). According to the classical regularity result of the Stokes problem, we can conclude that \(p \in H^1(\mathcal{O})\). Now since \(p\) is independent of \(x_n\), we have \(p \in H^1(\Omega)\). The problem of regularity reduces to the regularity of an elliptic boundary value problem in a nonsmooth domain.

This paper is presented as follows: In section 2 we give some basic definitions and classical properties of Sobolev spaces, which are needed in our work. We also recall some classical results of regularity of solutions of elliptic problems in smooth domains [LM1] and in convex domains [Da1], [Gr1].

Section 3 is devoted to the regularity of nonhomogeneous elliptic boundary value problems on a cylinder. We will prove the \(H^2\)-regularity using the results obtained by M. Dauge [Da3] and P. Grisvard [Gr1].

In section 4 we prove the regularity of solutions of problem (1.2), (1.3), (1.a). Namely, we prove the following:

If \(h \in C^3(\bar{\mathcal{O}})\) and \(f \in (L^2(\Omega))^{n-1}\), then the unique solution \((u, p)\) of problem (1.2), (1.3), (1.a) belongs to

\[
(H^2(\Omega) \cap H^1_0(\Omega))^{n-1} \times H^1(\Omega).
\]
In the case of the Dirichlet-Periodic boundary condition (1.1), we prove a stronger result of regularity; we show that: If $h \in C^{k+3}(\bar{\Omega})$ and $f \in (H^k_p(\Omega))^{n-1}$, $k \geq 0$, then the unique solution $(u, p)$ of problem (1.2), (1.3), (1.1) belongs to $(H^{k+2}_p(\Omega))^{n-1} \times H^{k+1}_p(\Omega)$.

Section 5 is devoted to the mixed Dirichlet-Neumann boundary conditions (1.3) and (1.4). It also contains the regularity of solutions of the coupled system atmosphere-ocean. Finally, section 6 gives the regularity of some nonlinear problems associated to problem (1.2), (1.3).

2. Some Definitions and Classical Results

2.1 Sobolev spaces.

Let $Q$ be a Lipschitz open set in $\mathbb{R}^n$. We denote by $L^r(Q)$ the space of real functions $f$ on $Q$ such that $|f|^r$ is integrable on $Q$ for the Lebesgue measure. If $m \geq 1$ is an integer, $W^{m,r}(Q)$ is the Sobolev space of functions which belong to $L^r(Q)$ together with their derivatives of order $\leq m$; $L^r(Q)$ and $W^{m,r}(Q)$ are Banach spaces, and in the case $r = 2$, they are Hilbert spaces (we will denote $W^{m,2}(Q)$ by $H^m(Q)$).

In the case of the Dirichlet-Periodic boundary condition (1.1), we will need the following Sobolev spaces defined on $\Omega = \{(x', x_n) \in \mathcal{O} = \Pi_{\alpha=1}^{n-1}(0, L_\alpha) \times \mathbb{R}; h(x') < x_n < 0\}$,

$$H^{m+1}_p(\Omega) = \{u \in H^{m+1}(Q); \frac{\partial^k u}{\partial x_i^k}, i = 1, \ldots, n, k = 0, \ldots, m \text{ are } \mathcal{O} - \text{periodic}\}.$$}

When $s = m + \sigma$ is not an integer $0 < \sigma < 1$, we say that the distribution $u$ defined in $Q$ is in $W^{s,r}(Q)$, if and only if

$$u \in W^{m,r}(Q) \text{ and } \int \int_{Q^2} \frac{|D^\alpha u(x) - D^\alpha u(y)|^r}{|x - y|^{n+\sigma r}} dxdy < +\infty, \text{ for } |\alpha| = m$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For $s > 0$, we define $W^{s,r}_0(Q)$ to be the closure in $W^{s,r}(Q)$ of $\mathcal{D}(Q)$, the space of infinitely differentiable functions with compact support in $Q$. We also define $W^{-s,r}(Q)$ as the dual spaces of $W^{s,r}_0(Q)$, where $\frac{1}{r} + \frac{1}{r'} = 1$. Now we give some fundamental results related to Sobolev spaces.

The Trace Theorem: If $Q$ is a bounded open set with $C^{k,1}$ boundary $\Gamma = \partial Q$ and $s - \frac{1}{2}$ is not an integer, and $s \leq k + 1$, then the mapping

$$u \to \{\frac{\partial^j u}{\partial v^j}, j = 0, 1, \cdots, \mu\}$$

of $C^{k,1}(\bar{Q}) \to \Pi_{j=0}^{\mu} C^{k-j,1}(\Gamma)$ extends uniquely by continuity to a continuous linear mapping

$$\gamma: u \to \{\frac{\partial^j u}{\partial v^j}, j = 0, 1, \cdots, \mu\}$$

(2.2)
from $H^s(Q)$ into $\prod_{j=0}^\mu H^{s-j-\frac{1}{2}}(\Gamma)$, where $\mu$ is the largest integer such that $\mu < s - \frac{1}{2}$. The mapping $\gamma$ is surjective and has a continuous inverse mapping from $\prod_{j=0}^\mu H^{s-j-\frac{1}{2}}(\Gamma)$ into $H^s(Q)$.

- If $Q$ is a bounded open set with Lipschitz boundary, then

\begin{equation}
H_0^s(Q) = H^s(Q) \text{ for } 0 < s \leq \frac{1}{2}.
\end{equation}

- If $Q$ is a bounded open set with a $C^{k,1}$ boundary $\Gamma$, $s \leq k + 1$ and $s - \frac{1}{2}$ is not an integer, then

\begin{equation}
u \in H_0^s(Q) \text{ if and only if } u \in H^s(Q) \text{ and } \frac{\partial^j u}{\partial \nu^j} = 0, \quad 0 \leq j < s - \frac{1}{2}.
\end{equation}

Moreover, the mapping

\begin{equation}u \rightarrow \tilde{u} = \text{extension of } u \text{ by 0 outside } Q\end{equation}

is continuous from $H^s(Q) \rightarrow H^s(\mathbb{R}^n)$ if and only if $0 \leq s < \frac{1}{2}$. The mapping (2.5) is continuous from $H_0^s(Q) \rightarrow H^s(\mathbb{R}^n)$ for $s > \frac{1}{2}$ if and only if $s - \frac{1}{2}$ is not an integer.

- Sobolev Imbedding Theorem [Ad]. Let $Q$ be a domain having the cone property in $\mathbb{R}^n$ and let $s > 0$, $1 < r < n$. We have the following imbeddings with continuous injections:

\begin{equation}
(i) \text{ If } n > sr, \text{ then } W^{s,r}(Q) \subset L^t(Q) \text{ for } r \leq t \leq \frac{nr}{n - sr}.
\end{equation}

\begin{equation}
(ii) \text{ If } n = sr \text{ then } W^{s,r}(Q) \subset L^t(Q) \text{ for } r \leq t < \infty.
\end{equation}

\begin{equation}
(iii) \text{ If } n < (s - j)r \text{ for some integer } j, \text{ then } W^{s,r}(Q) \subset C^j(Q).
\end{equation}

### 2.2. Some Classical Regularity Results.

The theory of regularity of solutions of elliptic boundary value problems in smooth domains is well known (see [ADN], [Ag], [LM1], [LM2], etc...). Regularity results in the Sobolev spaces $H^2$ for homogeneous elliptic problems in domains with nonsmooth boundaries can be found in [Da1], [Gr1] and the references therein. We will recall some of the results in the case of convex domains. We will also recall an important regularity result for a generalized Stokes problem in smooth domains [CA].
Let us now introduce the following framework, which will be used in the statement of the classical results and, also, in the section for the regularity of nonhomogeneous elliptic boundary value problems. Let $Q$ be a bounded open subset of $\mathbb{R}^n$ with a $C^{1,1}$ boundary and let $A$ be a second order strongly elliptic differential operator

\begin{equation}
Au = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial u}{\partial x_j}),
\end{equation}

where

\begin{equation}
\begin{aligned}
a_{ij} &\in C^{0,1}(\bar{Q}), & a_{ij} = a_{ji}, & i, j = 1, \ldots, n,
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j &\leq -\alpha|\xi|^2, & \forall \xi \in \mathbb{R}^n, x \in \bar{Q}.
\end{aligned}
\end{equation}

We have the following:

**Theorem 2.1.** Let $f \in L^2(Q)$ and $g \in H^{s-3/2}(\partial Q)$, with $0 < s \leq 2$ and $s \neq 3/2$. The unique solution of

\begin{equation}
Au = f \quad \text{in } Q, \quad \text{and} \quad u = g \quad \text{on } \partial Q
\end{equation}

belongs to

(i) $H^2(Q)$ if $Q$ is convex and $g = 0$, [Gr1].

(ii) $H^s(Q)$ if $Q$ is smooth (of class $C^{1,1}$), [LM1].

**Theorem 2.2.** Let $f \in L^2(Q)$ and $g \in H^{s-3/2}(\partial Q)$, with $1 \leq s \leq 2$. The unique solution of

\begin{equation}
Au + u = f \quad \text{in } Q \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \partial Q
\end{equation}

belongs to

(i) $H^2(Q)$, if $Q$ is convex and $g = 0$, [Gr1].

(ii) $H^s(Q)$ if $Q$ is smooth (of class $C^{1,1}$), [LM1].

We also give a classical regularity theorem for the Stokes problem in a smooth domain [CA] (see also [TE]).
Theorem 2.3. Let \( u \in (W^{1,\alpha}(Q))^n, \ p \in L^\alpha(Q), \ 1 < \alpha < +\infty, \) be solutions of the generalized Stokes problem

\[
\begin{aligned}
-\Delta u + \nabla p &= f \quad \text{in } Q, \\
\text{div } u &= g \quad \text{in } Q, \\
u &= \phi \quad \text{on } \partial Q.
\end{aligned}
\]

Let \( 1 \leq s \leq 2 \) such that \( s - \frac{1}{\alpha} \) is not an integer. If \( f \in (L^\alpha(Q))^n, \ g \in W^{s-1,\alpha}(Q) \) and \( \phi \in (W^{s-\frac{1}{s},\alpha}(\partial Q))^n \), then

\[
(2.15) \quad u \in (W^{s,\alpha}(Q))^n \quad \text{and} \quad p \in W^{s-1,\alpha}(Q).
\]


Elliptic boundary value problems in nonsmooth domains with vanishing boundary conditions have been extensively studied by many authors (see [Da 1,2,3], [Gr 1,2] and the references therein). The case of nonhomogeneous boundary conditions is less known; we are unable to find such regularity results in the literature. Hence, we give the proofs of Propositions 3.1 and 3.2 below in detail. We mention, however, the work of P. Grisvard [Gr1,2], which contains some regularity results for nonhomogeneous elliptic problems in polygonal domains; and also, the work of M. Dauge [Da3], where we can find necessary and sufficient conditions for the lifting of traces, in the case of Dirichlet and Neumann conditions when the space dimension is 3.

In the case of homogeneous boundary conditions, we should mention the results of M. Dauge [Da3] concerning the regularity of homogeneous mixed boundary problems for the Laplace operator on polyhedral curvilinear domains; these results can be used to obtain the regularity of nonhomogeneous problems for the Laplace operator on the domain \( \Omega \) defined by (1.1), (see Proposition 3.4).

In our work we assume that the domain \( Q \) is a cylinder with a smooth base, and (in the Dirichlet case) we suppose that the boundary values belong to the closure, in some appropriate Sobolev space \( H^s \), of infinitely differentiable functions with compact support on each component of \( \partial Q \).

In order to use the regularity results of homogeneous problems in convex domains [Da1], [Gr1], we will need the following localization lemma:

Lemma 3.1. Let \( B \) be a bounded open set in \( \mathbb{R}^{n-1} \) with a \( C^3 \)-boundary \( \partial B \), and let \( Q \) be the cylinder \( B \times (-1,0) \). Then, in the study of \( H^2 \)-regularity of elliptic problems on \( Q \), we can assume, without loss of generality, that \( Q \) is convex.

Proof. There exists for each \( x'_0 \in \partial B \), a ball \( B_0 = B(x'_0) \) in \( \mathbb{R}^{n-1} \) and a one-to-one mapping \( \mathcal{T} \) from \( B_0 \times (-1,0) \) to an open set \( D \times (-1,0) \subset \mathbb{R}^{n-1} \times (-1,0) \) such that:
\[ T(B \times (-1, 0) \cap B_0 \times (-1, 0)) \subset \mathbb{R}_+^{n-1} \times (-1, 0) = \]
\[ = \{(x', x_n) \in \mathbb{R}^{n-1} \times (-1, 0); x_{n-1} > 0\}, \]
\[ (3.1) \]
\[ T(\partial B \times (-1, 0) \cap B_0 \times (-1, 0)) \subset \partial \mathbb{R}_+^{n-1} \times (1, 0), \]
\[ (3.2) \]
and
\[ (3.3) \]
\[ T \in C^3(B_0 \times (-1, 0)), \ T^{-1} \in C^3(D \times (-1, 0)). \]

Let \( B_R(x'_0) \subset \subset B_0 \) and set \( B^+_0 = B_R(x'_0) \cap B, \ D' = T(B_R(x'_0)) \), and \( D^+ = T(B^+_0) \). Under the mapping \( T \), the equation \( Au = f \) in \( B^+ \times (-1, 0) \) is transformed into an equation of the same form on \( D^+ \times (-1, 0) \). We also note that the localization in the directions \( x_1, \ldots, x_{n-1} \) preserves the nature of the boundary conditions in the \( x_n \)-direction. Furthermore,

\[ (3.4) \]
\[ u \in H^s(B^+_0 \times (-1, 0)) \text{ if and only if } u \circ T^{-1} \in H^s(D^+ \times (-1, 0)). \]

Using a partition of unity of the open set \( B \), we can assume that the cylinder \( B \times (-1, 0) \) is convex.

3.1. The Dirichlet Boundary Condition Case.

**Proposition 3.1.** Let \( \varphi \in H_0^{s-1/2}(B) \) and \( \psi \in H_0^{s-1/2}(\partial B \times (-1, 0)) \) with \( \frac{1}{2} \leq s < 2, \ s \neq 1, \frac{3}{2} \). Assume that \( f \in L^2(Q) \). Then the unique solution of

\[ \begin{cases} 
Au = f & \text{in } Q, \\
u = \varphi & \text{on } B \times \{-1, 0\}, \\
u = \psi & \text{on } \partial B \times (-1, 0)
\end{cases} \]
\[ (3.5) \]
belongs to \( H^s(Q) \). Moreover,

\[ (3.6) \]
\[ \text{if } \varphi \in H_0^{1/2+\epsilon}(B) \text{ and } \psi \in H_0^{1/2+\epsilon}(\partial B \times (-1, 0)), \ \text{then } u \in H^2(Q). \]

**Proof.** The proof is based on a symmetry argument. We homogenize the boundary condition on the top and the bottom of the cylinder; the homogenization is obtained through the extension of \( \varphi \) to \( \mathbb{R}^{n-1} \). Then, we apply the reflection principle to the Laplacian and
use local regularity results to conclude the proposition in the case $A = \Delta$. Finally, we use regularity results in convex domains and interpolation arguments to conclude the proof of the proposition.

Assume that $A = \Delta$; and $\frac{1}{2} < s < 2$ and $s \neq 1, \frac{3}{2}$. We will show that $\varphi$ can be assumed to be zero. Using (2.3), (2.5), we can extend $\varphi$ by 0 in $\mathbb{R}^{n-1}$ outside $B$. Let $\tilde{\varphi}$ be the extension; we have $\tilde{\varphi} \in H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})$. Let $Q^*$ be a smooth domain containing $Q$ and contained in $\mathbb{R}^{n-1} \times (-1,0)$; we have $\tilde{\varphi} \in H^{s-1/2}(\partial Q^*)$, where $\tilde{\varphi}$ denotes the extension of $\varphi$ to $\partial Q^*$ by 0 outside $B \times \{-1,0\}$.

We note that $\bar{\Psi} \in H^{s-1/2}_0(\partial B \times (-1,0))$. Hence, if $\frac{1}{2} < s \leq 2$ and $s \neq 1, \frac{3}{2}$, we can assume, without loss of generality, that $\varphi = 0$. This completes the homogenization of the boundary condition on the top and the bottom of the cylinder.

Now we begin the reflection argument. For a function $\xi$ defined in $B \times (-1,0)$, we define its extension by reflection to $B \times (-1,1)$, denoted by $\xi^e$, as follows:
(3.8) \[ \xi^e(x',x_n) = \begin{cases} \xi(x',x_n), & \text{for } (x',x_n) \in B \times (-1,0), \\ -\xi(x',-x_n), & \text{for } (x',x_n) \in B \times (0,1). \end{cases} \]

We extend \( f, \bar{\Psi}, \) and \( u_* \) to \( B \times (-1,1) \), using (3.8), and we obtain

(3.9) \[ f^e \in L^2(B \times (-1,1)), \quad \bar{\Psi}^e \in H^{s-\frac{1}{2}}_0(\partial B \times (-1,1)). \]

Now, we consider the following boundary value problem:

\[
\begin{cases}
\Delta v^e = f^e & \text{in } B \times (-1,1), \\
v^e = 0 & \text{on } B \times \{-1,1\}, \\
v^e = \bar{\Psi}^e & \text{on } \partial B \times (-1,1).
\end{cases}
\]  

(3.10)

Using a cut-off function \( \theta(x_n) \in C_0^\infty((-1,1)) \) and Theorem 2.1, we conclude that \( v^e \in H^s(B \times (-3/4,3/4)) \) for \( s \neq 1,3/2 \).

Let \( w(x',x_n) = v^e(x',x_n) + v^e(x',-x_n) \). A straightforward computation yields \( \Delta w = 0 \) in \( B \times (-1,1) \) and \( w = 0 \) on \( \partial(B \times (-1,1)) \). Hence \( w = 0 \) in \( B \times (-1,1) \) and \( v^e(x',0) = 0 \). Therefore,

(3.11) \[ v^e = u_* \text{ in } B \times (-1,0) \text{ and } u_* \in H^s(B \times (-3/4,0)). \]

Thus, \( u = u_* + \Phi \in H^s(B \times (-3/4,0)). \)

The reflection argument, used above, can also be applied to the hyperplane \( x_n = -1 \) and, thanks to Theorem 2.1, we conclude (as in (3.11)) that \( u \in H^s(B \times (-1,-1/2)) \). Hence \( u \in H^s(Q) \). We have proved

**Lemma 3.2.** Let \( u_p \) be the unique solution of the following boundary value problem (with \( \varphi \) and \( \psi \) as in Proposition 2.1)

(3.12) \[
\begin{cases}
\Delta u_p = 0 & \text{in } Q, \\
u_p = \varphi & \text{on } B \times \{-1,0\}, \\
u_p = \psi & \text{on } \partial B \times (-1,0).
\end{cases}
\]

Then, \( u_p \in H^s(Q) \) for \( 1/2 < s < 2 \), \( s \neq 1,3/2 \). Moreover,
if \( \varphi \in H^{1/2+\epsilon}_0(B) \) and \( \psi \in H^{1/2+\epsilon}_0(\partial B \times (-1,0)) \), then \( u_p \in H^2(Q) \).

We will continue the proof of Proposition 3.1 when \( s = 2 \) using Grisvard’s results [Gr1] and Lemma 3.1 and then conclude by an interpolation argument. Assume that \( s = 2 \), then it is clear that \( Au_p \in L^2(Q) \). We write the equation satisfied by \( u - u_p \) as follows:

\[
(3.13) \quad \begin{cases} 
A(u - u_p) = f - Au_p \in L^2(Q), \\
u - u_p = 0 \quad \text{on} \quad \partial Q.
\end{cases}
\]

Thanks to Lemma 3.1, we can assume that \( Q \) is convex and, according to Theorem 2.1, we conclude that \( u - u_p \in H^2(Q) \cap H^1_0(Q) \). Hence, \( u \in H^2(Q) \).

For \( s < 2 \), the proof is completed using the interpolation theory (see [LM1], [Tr]). We have shown that the operator \( A \), in (3.13), is invertible and \( A^{-1} \) is a well defined continuous operator from \( L^2(Q) \) into \( H^2(Q) \cap H^1_0(Q) \subset H^2(Q) \). Thanks to the Lax-Milgram Theorem, \( A^{-1} \) maps \( H^{-1}(Q) \) into \( H^1(Q) \). It is clear that the domain \( Q \) admits a \( C^2 \)– extension operator (see[Ad]). Hence interpolating between \( H^{-1}(Q) \) and \( L^2(Q) \), on one hand, and \( H^1(Q) \) and \( H^2(Q) \) on the other hand, we conclude that \( A^{-1} \) is a continuous linear operator from \( H^{s-2}(Q) \) into \( H^s(Q) \) for \( s \neq 3/2 \) (see [LM1] for more details). Therefore, if \( 1/2 < s < 2, s \neq 3/2, s \neq 1 \), and \( u_p \in H^s(Q) \), then \( Au_p \in H^{s-2}(Q) \) and the solution of

\[
(3.14) \quad \begin{cases} 
A(u - u_p) = f - Au_p \in H^{s-2}(Q), \\
u - u_p = 0 \quad \text{on} \quad \partial Q
\end{cases}
\]

belongs to \( H^s(Q) \). Hence, \( u \in H^s(Q) \). The proof of (3.6) follows the same lines, since the extension of \( \phi \) by zero is possible. The proof is complete.

**Remark 3.1.** When the domain \( Q \) is a polyhedral tridimensional domain and the operator \( A \) is the Laplacian (which is the case of our domain \( \Omega \) defined in (1.1)). The regularity, when the boundary conditions are homogeneous, has been studied by M. Dauge [Da2,3] and she obtained results similar to ours, if we assume that the boundary values are zero.

### 3.2. The Neumann Boundary Condition.

This subsection is devoted to the regularity of solutions of elliptic boundary value problems with nonhomogeneous Neumann boundary conditions. The main result is the following:

**Proposition 3.2.** Let \( \varphi \in H^{s-3/2}_0(B) \) and \( \psi \in H^{s-3/2}_0(\partial B \times (-1,0)) \) with \( 3/2 \leq s < 2 \). Assume that \( f \in L^2(Q) \). Then the unique solution of
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\begin{equation}
\begin{aligned}
Au + u &= f \quad \text{in } Q, \\
\frac{\partial u}{\partial \nu} &= \varphi \quad \text{on } B \times \{-1,0\}, \\
\frac{\partial u}{\partial \nu} &= \psi \quad \text{on } \partial B \times (-1,0)
\end{aligned}
\end{equation}

belongs to $H^s(Q)$. Moreover,

\begin{equation}
\text{if } \varphi \in H^{1/2+\varepsilon}_0(B \times \{-1,0\}) \text{ and } \psi \in H^{1/2+\varepsilon}_0(\partial B \times (-1,0)) \text{ then } u \in H^2(Q).
\end{equation}

**Proof.** We proceed as in the proof of Proposition 3.1. First, we assume that $3/2 \leq s < 2$ and show that $\varphi$ can be taken equal to zero. Then, we use a reflection argument when $A$ is the Laplacian and $3/2 \leq s \leq 2$. The conclusion of the proof is done with the interpolation argument which is similar to the one given in the proof of Proposition 3.

Assume that $3/2 \leq s < 2$ and let $\beta = s - 3/2$. We have $0 \leq \beta < 1/2$, so, by (2.3), (2.5), we can extend $\varphi$ to $\mathbb{R}^{n-1}$ by 0 outside $B$. We denote the extension by $\tilde{\varphi}$; we have $\tilde{\varphi} \in H^3(\mathbb{R}^{n-1})$. Let $Q^*$ be a smooth domain containing $Q$ and contained in $\mathbb{R}^{n-1} \times (-1,0)$. We still denote the extension of $\varphi$ to $\partial Q^*$, by 0 outside $B \times \{-1,0\}$, by $\tilde{\varphi}$. We have, obviously, $\tilde{\varphi} \in H^3(\partial Q^*)$.

Now, we consider the following boundary value problem:

\begin{equation}
\begin{aligned}
-\Delta \Phi + \Phi &= 0 \quad \text{in } Q^*, \\
\frac{\partial \Phi}{\partial \nu} &= \varphi \quad \text{on } \partial Q^*.
\end{aligned}
\end{equation}

By Theorem 2.2, we have $\Phi \in H^2(Q^*)$. Let $u_* = u - \Phi$ and $\Psi^* = \text{the trace of } \frac{\partial \Phi}{\partial \nu}$ on $\partial B \times (-1,0)$. We have, by the Trace theorem, $\Psi^* \in H^3(\partial B \times (-1,0))$. Now $u_*$ solves the following problem:

\begin{equation}
\begin{aligned}
-\Delta u_* + u_* &= f \quad \text{in } Q, \\
\frac{\partial u_*}{\partial \nu} &= 0 \quad \text{on } B \times \{-1,0\}, \\
\frac{\partial u_*}{\partial \nu} &= \Psi - \Psi^* = \bar{\Psi} \quad \text{on } \partial B \times (-1,0).
\end{aligned}
\end{equation}
From now on, we will assume that \( \frac{3}{2} \leq s \leq 2 \) and start the reflection argument. For a function \( \xi \) defined in \( B \times (-1, 0) \), we define its extension \( \xi^e \), by reflection, to \( B \times (-1, 1) \) as follows:

\[
\xi^e(x', x_n) = \begin{cases} 
\xi(x', x_n) & \text{for } (x', x_n) \in B \times (-1, 0), \\
\xi(x', -x_n) & \text{for } (x', x_n) \in B \times (0, 1).
\end{cases}
\]

We define the extension of \( f \) and \( \Phi \) using (3.19), and obtain

\[ f^e \in L^2(B \times (-1, 1)) \quad \text{and} \quad \Phi^e \in H^3(\partial B \times (-1, 1)). \]

Now we consider the following boundary value problem:

\[
\begin{aligned}
-\Delta v^e + v^e &= f^e & \text{in } B \times (-1, 1), \\
\frac{\partial v^e}{\partial \nu} &= 0 & \text{on } B \times \{-1, 1\}, \\
\frac{\partial v^e}{\partial \nu} &= \Phi^e & \text{on } \partial B \times (-1, 1).
\end{aligned}
\]

Using a cut-off function \( \theta(x_n) \) and Theorem 2.2, we conclude that \( v^e \in H^s(B \times (-3/4, 3/4)) \).

Let \( w(x', x_n) = v^e(x', x_n) - v^e(x', -x_n) \). A straightforward computation yields

\[
-\Delta w + w = 0 \quad \text{in } B \times (-1, 1) \quad \text{and} \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial(B \times (-1, 1)).
\]

Hence, \( w = 0 \) in \( B \times (-1, 1) \) and \( \frac{\partial v^e}{\partial \nu}(x', 0) = 0 \), which implies that \( v^e = u_* \) in \( B \times (-1, 0) \) and \( u_* \in H^s(B \times (-3/4, 0)) \). Thus, \( u = u_* + \Phi \) belongs to \( H^s(B \times (-3/4, 0)) \). The reflection about \( x_n = -1 \) concludes the proof when \( A = -\Delta \). The remainder of the proof is completed with Lemma 3.1 and Theorem 2.2 (when the domain is convex and \( s = 2 \)). For \( s < 2 \), we use the interpolation argument as in the proof of Proposition 3.1. We omit the details.

Results similar to those of Proposition 3.1 and 3.2 cannot be obtained, if we mix the boundary conditions (see [Da2,3]). However, in the case of the Laplacian operator defined in a polyhedral domain, such as the domain of the ocean \( \Omega \) defined by (1.1), we can obtain \( H^2 \)-regularity under a supplementary assumption on the dihedral angles of \( \Omega \).

Combining Propositions 3.1 and 3.2, via localization in the \( x_n \)-direction, we have the following:

**Proposition 3.3.** Let \( \varphi \in H^{s-3/2}_0(\mathcal{O}) \), \( \psi_\ell \in H^{s-1/2}_0(\Gamma_\ell) \), and \( \psi_b \in H^{s-1/2}_0(\Gamma_b) \) with \( 3/2 \leq s < 2 \); and assume that \( f \in L^2(\mathcal{O}) \). Then, the unique solution of
\[
\begin{cases}
-\Delta u + u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \varphi & \text{on } \partial \Omega, \\
u = \psi_\ell (\text{resp. } \psi_b) & \text{on } \Gamma_\ell (\text{resp. } \Gamma_b)
\end{cases}
\]

belongs to \(H^s(\Omega)\). Moreover,

\[
\text{if } \psi_b = H_0^{3/2+\varepsilon}(\Gamma_b), \varphi \in H_0^{1/2+\varepsilon}(\partial \Omega), \text{ and } \psi_\ell \in H_0^{3/2+\varepsilon}(\Gamma_\ell), 0 < \varepsilon < \frac{1}{2}, \text{ then } u \in H^2(\Omega).
\]

Note that, in Proposition 3.3, the boundary conditions are mixed at corners with the angle \(\frac{\pi}{2}\) (at the top boundary of \(\Omega\)). We do not need to make any supplementary assumption on \(h\), in this case. However, mixing the type of boundary conditions at the bottom boundary of \(\Omega\), requires the following assumption on \(\Omega\):

\[(H) \quad \nabla_{x^i} h(z) = 0 \text{ for every } z \text{ in a neighborhood of } \partial O.
\]

The assumption \((H)\) implies that the dihedral angles of \(\Omega\) are equal to \(\pi/2\). We should mention that the assumption \((H)\) is needed even for zero boundary conditions (see [Da3]). We state the following:

**Proposition 3.4.** Let \(\varphi_b \in H_0^{s-3/2}(\Gamma_b), \varphi_u \in H_0^{s-3/2}(\partial \Omega), \text{ and } \psi \in H_0^{s-1/2}(\Gamma_\ell)\) with \(3/2 \leq s < 2\), and assume that \(\Omega\) satisfies \((H)\). Then, for all \(f \in L^2(\Omega)\), the unique solution of

\[
\begin{cases}
-\Delta u + u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \varphi_b (\text{resp. } \varphi_u) & \text{on } \Gamma_b (\text{resp. } \partial \Omega), \\
u = \psi & \text{on } \Gamma_\ell
\end{cases}
\]

belongs to \(H^s(\Omega)\). Moreover,

\[
\text{if } \varphi_b \in H_0^{1/2+\varepsilon}(\Gamma_b), \varphi_u \in H_0^{1/2+\varepsilon}(\partial \Omega), \text{ and } \psi \in H_0^{3/2+\varepsilon}(\Gamma_\ell), \text{ then } u \in H^2(\Omega).
\]

**Proof.** Thanks to the assumption \((H)\), and by localization in the \(x_1, \cdots, x_{n-1}\) directions, we can assume that \(\Omega\) is a right cylinder. Using the symmetry argument, as in the proof of Proposition 2.2, and [Da3, Theorem 2] we conclude the proof of Proposition 3.4. We omit the details.
4. Regularity of the Stokes-type system

The proof of the regularity of solutions of the Stokes-type system is based on the integration of equation (1.2) with respect to $x_n$; this integration is done via a partial regularity result for the solution; i.e. $u \in H^{3/2+\varepsilon}(\Omega)$ for $0, \varepsilon < 1/2$. This yields a Stokes problem on the smooth domain $\mathcal{O}$. Therefore, thanks to Theorem 2.3, $p \in H^1(\mathcal{O})$. We transform problem (1.1), (1.2) into an elliptic boundary value problem, and use Lemma 3.1 and Theorem 2.1 to obtain the regularity.

In order to prove the partial regularity result stated above, we will transform the domain $\Omega$ into a cylinder $Q = \mathcal{O} \times (-1,0)$ and use Proposition 3.1.

4.1. Transformation of the domain.

Lemma 4.1. Problem (1.2), (1.3) with one of the boundary conditions (1.a), (1.b), (1.c) or (1.d) is equivalent to the following:

$$
\begin{align*}
- [\Delta_{n-1} \tilde{u}_i + D_n(y_n^2 h^2 |\nabla H|^2 + H^2)D_n \tilde{u}_i] + 2 \sum_{\alpha=1}^{n-1} D_n(\frac{\partial H}{\partial x_{\alpha}} H y_n D_{\alpha} \tilde{u}_i) \\
- h y_n \Delta H D_n \tilde{u}_i + 2 y_n^2 h |\nabla H|^2 D_n \tilde{u}_i + 2 H \sum_{\alpha=1}^{n-1} \frac{\partial H}{\partial x_{\alpha}} D_{\alpha} \tilde{u}_i + \frac{\partial p}{\partial x_i} = \tilde{f}_i, \\
i = 1, \ldots, n-1
\end{align*}
$$

(4.1)

$$
\text{div}_{y'} \int_0^0 \tilde{u}(y', \eta) d\eta = -\nabla H \cdot \int_{-1}^0 \tilde{u}(y', \eta) d\eta
$$

(4.2)

where

$$
\tilde{u}(y) = u(x), \ \tilde{f}(y) = f(x) \ \text{and} \ H(y') = \frac{1}{h(y')}.
$$

Proof. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n, \psi(x_1, \cdots, x_n) = (\psi_i(x))_{1 \leq i \leq n}$ be such that

$$
y_i = \psi_i(x) = x_i, \ 1 \leq i \leq n-1 \ \text{and} \ y_n = \psi_n(x) = -H(x_1, \cdots, x_{n-1})x_n
$$

(4.3)

Let $\tilde{u}(y) = u(x)$ and $\tilde{f}(y) = f(x)$, we have

$$
\Delta u(x) = \tilde{L}\tilde{u}(y) = \sum_{i,j=1}^n \tilde{a}_{ij} \frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j} + \sum_{i=1}^n \tilde{b}_i \frac{\partial \tilde{u}}{\partial x_i}
$$

(4.4)
where

$$\begin{align*}
\tilde{a}_{ij} &= \sum_{r,s=1}^{n} \frac{\partial \psi_i}{\partial x_r} \frac{\partial \psi_j}{\partial x_s} \delta_{rs} = \sum_{r=1}^{n} \frac{\partial \psi_i}{\partial x_r} \frac{\partial \psi_i}{\partial x_r} = \tilde{a}_{ji} \\
\tilde{b}_i &= \sum_{r,s=1}^{n} \frac{\partial^2 \psi_i}{\partial x_r \partial x_s} \delta_{rs} = \sum_{r=1}^{n} \frac{\partial^2 \psi_i}{\partial x_r^2},
\end{align*}$$

(4.5)

which implies that

$$\begin{align*}
\tilde{a}_{ii} &= (\frac{\partial \psi_i}{\partial x_i})^2 = 1, \quad 1 \leq i \leq n-1, \quad \tilde{a}_{nn} = h^2 y_n^2 |\nabla H|^2 + H^2, \\
\tilde{a}_{ij} &= 0, \quad \text{for } 1 \leq i, j \leq n-1, \quad i \neq j, \quad \tilde{a}_{nj} = -\frac{\partial H}{\partial x_j} x_n = +\frac{\partial H}{\partial x_j} y_n, \\
\tilde{b}_i &= 0 \quad \text{for } 1 \leq i \leq n-1, \quad \tilde{b}_n = \sum_{r=1}^{n} \frac{\partial^2 \psi_n}{\partial x_r^2} = -x_n \Delta H.
\end{align*}$$

(4.6)

Thus

$$\begin{align*}
\bar{L}\tilde{u}_i &= \sum_{\alpha=1}^{n-1} \frac{\partial^2 \tilde{u}_i}{\partial y_\alpha^2} + (y_n^2 h^2 |\nabla H|^2 + H^2) \frac{\partial^2 \tilde{u}_i}{\partial y_n^2} + h y_n \Delta H \frac{\partial \tilde{u}_i}{\partial y_n} + \\
&\quad + 2 \sum_{\alpha=1}^{n-1} \frac{\partial H}{\partial y_\alpha} y_n \frac{\partial^2 \tilde{u}_i}{\partial y_\alpha \partial y_n}.
\end{align*}$$

(4.7)

$$\bar{L}\tilde{u}_i = \sum_{\alpha=1}^{n-1} \frac{\partial^2 \tilde{u}_i}{\partial y_\alpha^2} + (y_n^2 h^2 |\nabla H|^2 + H^2) \frac{\partial^2 \tilde{u}_i}{\partial y_n^2} + 2 \sum_{\alpha=1}^{n-1} \frac{\partial H}{\partial y_\alpha} y_n \frac{\partial \tilde{u}_i}{\partial y_\alpha} + \\
&\quad + h y_n \Delta H \frac{\partial \tilde{u}_i}{\partial y_n} - 2 y_n h^2 |\nabla H|^2 \frac{\partial \tilde{u}_i}{\partial y_n} + 2 H \sum_{\alpha=1}^{n-1} \frac{\partial H}{\partial y_\alpha} \frac{\partial \tilde{u}_i}{\partial y_\alpha} + \frac{\partial p}{\partial y_i} = \tilde{f}_i(y),
$$

for $i = 1, \cdots, n-1$, and (4.1) is proved. For later use, we rewrite (4.7) as follows:

$$\bar{L}\tilde{u}_i = \Delta_{n-1} \tilde{u}_i + \tilde{A}_n \tilde{u}_i + \tilde{B}_n \tilde{u}_i,$$

(4.8)
where $\tilde{B}$ is the linear differential operator of order 1 involving only derivatives with respect to $y_n$, and $\tilde{A}_n$ is a second order differential operator vanishing when applied to a function independent of $y_n$. We have

\[
\begin{align*}
\tilde{A}_n \tilde{u}_i &= (y_n^2 h^2(y')|\nabla H|^2 + H^2) \frac{\partial^2 \tilde{u}_i}{\partial y_n^2} + 2 \sum_{\alpha=1}^{n-1} \frac{\partial H}{\partial y_\alpha} H y_n \frac{\partial^2 \tilde{u}_i}{\partial y_\alpha \partial y_n}, \\
\tilde{B}_n \tilde{u}_i &= h y_n \nabla H \frac{\partial \tilde{u}_i}{\partial y_n}.
\end{align*}
\]

Since $h$ is at least of class $C^3$, we have

\[
(4.10) \quad u \in H^s(\Omega) \iff \tilde{u} \in H^s(Q), \quad 0 \leq s \leq 2.
\]

where $Q = \mathcal{O} \times (-1,0)$.

For the proof of (4.2), note that

\[
(4.11) \quad \frac{\partial \tilde{u}_i}{\partial y_i} = \frac{\partial u_i}{\partial x_i} + \frac{\partial H}{\partial y_i} x_n \frac{\partial u_i}{\partial x_n}.
\]

**Remark 4.1.** Using the Lax-Milgram theorem, we can prove the existence and the uniqueness of the solution $(u,p)$ of problem (1.2), (1.3) with one of the boundary conditions (1.a), (1.b), (1.c) or (1.d). The proof is similar to the one for the Stokes problem [TE], (we refer to [LTW 1,2] for more details).

**Remark 4.2.** The operator $\tilde{L} - \tilde{B}_n$, obviously, satisfies conditions (2.9), (2.10) and (2.11). Therefore, we can apply the results of section 3 to $\tilde{L} - \tilde{B}_n$ and also to $\tilde{L}$.

### 4.2. The Regularity in the Dirichlet Boundary Condition Case.

The proof of the regularity is based on the decomposition of $u$ into the sum of a function satisfying a homogeneous elliptic boundary value problem on $\mathcal{O}$, and another function satisfying a nonhomogeneous boundary value problem of the type studied in section 3.

From Remark 4.1, we have $\nabla_{n-1} p \in (H^{-1}(\mathcal{O}))^{n-1}$. Let $v$ be the unique solution of

\[
(4.12) \quad \begin{cases}
\Delta_{n-1} v = \nabla_{n-1} p & \text{in } \mathcal{O}, \\
v = 0 & \text{on } \mathcal{O}.
\end{cases}
\]
The Lax-Milgram theorem implies that $(v \in H^1_0(\mathcal{O}))^{n-1}$. Let $w = \tilde{u} - v$; we have, obviously, $\tilde{A}_n v + \tilde{B}_n v = 0$. Let $w = \tilde{u} - v$; the function $w$ satisfies the following nonhomogeneous boundary value problem:

\begin{equation}
-\tilde{L}w = -\tilde{L}\tilde{u} + \Delta_{n-1} v = -\tilde{L}\tilde{u} + \nabla_{n-1} p = \tilde{f}.
\end{equation}

Consider the nonhomogeneous boundary value problem

\begin{equation}
\begin{cases}
-\tilde{L}w = \tilde{f} & \text{in } \mathcal{O} \times (-1,0), \\
w = 0 & \text{on } \partial\mathcal{O} \times (-1,0), \\
w = -v & \text{on } \mathcal{O} \times \{-1,0\}.
\end{cases}
\end{equation}

We have $v \in (H^{1-\varepsilon}_0(\mathcal{O}))^{n-1}$ for $0 < \varepsilon < 1/2$. Then, Proposition 3.1 implies that $w \in (H^{3/2-\varepsilon}(\mathcal{O}))^{n-1}$. Moreover, equation (4.2) implies that

\begin{equation}
\text{div}_{n-1} \int_{-1}^{0} \tilde{u}(y', \eta) d\eta \in H^1(\mathcal{O}).
\end{equation}

Hence

\begin{equation}
\text{div}_{n-1} v(y') = \text{div}_{n-1} \int_{-1}^{0} \tilde{u}(y', \eta) d\eta - \text{div}_{n-1} \int_{-1}^{0} w(y', \eta) d\eta \in H^{1/2-\varepsilon}(\mathcal{O}).
\end{equation}

Now, we consider the following Stokes problem defined in $\mathcal{O}$:

\begin{equation}
\begin{cases}
-\Delta_{n-1} v + \nabla_{n-1} p = 0 & \text{in } \mathcal{O}, \\
\text{div } v \in H^{1/2-\varepsilon}(\mathcal{O}) & \text{is given,} \\
v = 0 & \text{on } \mathcal{O}.
\end{cases}
\end{equation}

Thanks to Theorem 2.3, we have $v \in (H^{3/2-\varepsilon}_0(\mathcal{O}))^{n-1}$ and $p \in H^{1/2-\varepsilon}(\mathcal{O})$, for all $\varepsilon$, $0 < \varepsilon < \frac{1}{2}$. Moreover, by (2.4), we have $v \in (H^{3/2-\varepsilon}_0(\mathcal{O}))^{n-1}$.

We return to problem (4.14) and conclude, by Proposition 3.1, that $w \in (H^{2-\varepsilon}(Q))^{n-1}$. Now, since,

\[
\frac{\partial w}{\partial y_n} = \frac{\partial \tilde{u}}{\partial y_n} - \frac{\partial v}{\partial y_n} = \frac{\partial \tilde{u}}{\partial y_n},
\]

we have obtained our partial regularity result; i.e.
\[
\frac{\partial \tilde{u}}{\partial y_n} \in (H^{1-\varepsilon}(Q))^{n-1} \text{ for all } 0 < \varepsilon < 1/2.
\]

**Proposition 4.1.** Let \((\tilde{u}, p) \in (H^1_0(Q))^n \times L^2(\mathcal{O})\) be a solution of (4.1), (4.2), (1.a) with \(\tilde{f} \in (L^2(Q))^n\). Then we have

\[
(4.18) \quad \forall \varepsilon, \ 0 < \varepsilon < \frac{1}{2}, \ \frac{\partial \tilde{u}}{\partial y_n} \in (H^{1-\varepsilon}(Q))^{n-1}.
\]

Now we are ready to prove the \(H^2\)–regularity of solutions of problem (1.2), (1.3) with the Dirichlet boundary condition. Thanks to Proposition 4.1, we are able to integrate (4.1) with respect to \(y_n\). The resulted system of equations is the \((n-1)\)-dimensional Stokes problem on the smooth domain \(\mathcal{O}\).

Let

\[
(4.19) \quad \tilde{U}(y') = \int_{-1}^{0} \tilde{u}(y', \eta) d\eta, \quad \tilde{F}(y') = \int_{-1}^{0} \tilde{f}(y', \eta) d\eta.
\]

We have

\[
(4.20) \quad \Delta_{n-1} \tilde{U}(y') = \int_{-1}^{0} \tilde{L}\tilde{u}(y', \eta) d\eta + \int_{-1}^{0} \tilde{f}_1(y', \eta) d\eta
\]

where

\[
(4.21) \quad \tilde{f}_1(y', \eta) = -\tilde{A}_n \tilde{u} - \tilde{B}_n \tilde{u},
\]

and

\[
(4.22) \quad \frac{\partial \tilde{U}}{\partial y_n} = \int_{-1}^{0} \tilde{f}_1(y', \eta) d\eta = - \int_{-1}^{0} (\eta^2 h^2(y')) |\nabla H|^2 + H^2 \frac{\partial^2 \tilde{u}}{\partial y_n^2} d\eta - h^2 \int_{-1}^{0} \eta \frac{\partial^2 \tilde{u}}{\partial y_n \partial y_\alpha} d\eta - h \Delta H \int_{-1}^{0} \eta \frac{\partial \tilde{u}}{\partial y_n} d\eta.
\]

Integrating by parts the first and second integrals in the right hand side of (4.22), we obtain, for the first one,
\begin{align}
(4.23) \quad & -H^2 \left[ \frac{\partial \bar{u}}{\partial y_n}(y', 0) - \frac{\partial \bar{u}}{\partial y_n}(y', -1) \right] + h^2(y')|\nabla H|^2 \frac{\partial \bar{u}}{\partial y_n}(y', -1) + 2 \int_{-1}^0 \eta h^2(y') \frac{\partial \bar{u}}{\partial y_n} d\eta \\
& \quad \text{which belongs, by Proposition 4.1, to } L^2(\mathcal{O}). \text{ For the second integral, we have}
\end{align}

\begin{align}
(4.24) \quad & \int_{-1}^0 \eta \frac{\partial^2 \bar{u}}{\partial y_n \partial y_\alpha} d\eta = \frac{\partial \bar{u}}{\partial y_\alpha}(y', -1) - \int_{-1}^0 \frac{\partial \bar{u}}{\partial y_\alpha}(y', \eta) d\eta, \quad \alpha = 1, \ldots, n - 1,
\end{align}

and since \( \bar{u} = 0 \) on \( \mathcal{O} \times \{-1\} \), we conclude that

\begin{align}
(4.25) \quad & \int_{-1}^0 \eta \frac{\partial^2 \bar{u}}{\partial y_n \partial y_\alpha} d\eta = -\int_{-1}^0 \frac{\partial \bar{u}}{\partial y_\alpha}(y', \eta) d\eta \in L^2(\mathcal{O}).
\end{align}

We have proved the following:

**Lemma 4.2.** Let \( \bar{u} \in (H^1(\mathcal{Q}))^n \) be such that \( \bar{u} = 0 \) on \( \mathcal{O} \times \{-1\} \) and \( \frac{\partial \bar{u}}{\partial y_n} \in H^{1/2+\varepsilon}(\mathcal{Q}) \), for some \( \varepsilon > 0 \). Then

\begin{align}
\int_{-1}^0 \tilde{L} \bar{u}(y', \eta) d\eta = \Delta_{n-1} \left( \int_{-1}^0 \bar{u}(y', \eta) d\eta \right) - \tilde{F}_1(y')
\end{align}

with \( \tilde{F}_1(y') \in L^2(\mathcal{O}) \).

By Lemma 4.2, we can write the equations satisfied by \((\tilde{U}, p)\)

\begin{align}
(4.26) \quad & \begin{cases}
- \Delta_{n-1} \tilde{U} + \nabla_{n-1} p = \tilde{F} - \tilde{F}_1 \in L^2(\mathcal{O}), \\
\text{div}_{n-1} \tilde{U} = -\nabla H \cdot \tilde{U} \in H^1(\mathcal{O}), \\
\tilde{U} = 0 \quad \text{on } \partial \mathcal{O}.
\end{cases}
\end{align}

Theorem 2.3 allows us to affirm that \( p \in H^1(\mathcal{O}) \). Now we return to (4.1) and write it in the form

\begin{align}
(4.27) \quad & \begin{cases}
- \tilde{L} \bar{u} = \tilde{f} - \nabla_{n-1} p \quad \text{in } Q, \\
\bar{u} = 0 \quad \text{on } \partial Q.
\end{cases}
\end{align}

Proposition 3.1 concludes that \( \bar{u} \in (H^2(\mathcal{Q}))^{n-1} \). We have proved the following:
Theorem 4.1. If $h \in C^3(\mathcal{O})$ and $f \in (L^2(\Omega))^n$, then the unique solution $(u, p)$ of problem (1.2), (1.3), (1.a) satisfies

\[(4.28) \quad u \in (H^2(\Omega) \cap H^1_0(\Omega))^n \quad \text{and} \quad p \in H^1(\Omega).\]

Theorem 4.1 cannot be improved to obtain higher regularity for the solution of problem (1.2), (1.3), (1.a); i.e. if $f \in (H^1(\Omega))^n$, $u$ needs not to be in $H^3(\Omega)$. Hence, we cannot get a better regularity result. The following example confirms our claim:

Example 2.1. Let $\Omega = B \times (-1, 1)$, where $B$ is the unit ball in $\mathbb{R}^2$, and let $f$ be an odd function with respect to $x_3$. The uniqueness of the solution of the problem $-\Delta \omega = f$ in $B \times (-1, 1)$ with zero boundary condition implies that $\omega$ is an odd function. Hence,

\[(4.29) \quad \text{div}_{x'} \int_{-1}^1 \omega(x_1, x_2, \eta) d\eta = 0, \quad (x_1, x_2) \in B.\]

Therefore, $(\omega, 0)$ is the unique solution of (1.2), (1.3), (1.a) when $\Omega = B \times (-1, 1)$ and, if $f \in (H^1(\Omega))^2$, $u$ needs not to be in $(H^3(\Omega))^2$ (see [Gr1] for more details).

4.3. The Regularity in the Periodic-Dirichlet Boundary Condition Case.

In this subsection we prove the regularity of the solution of the Stokes-type system with the Dirichlet–periodic condition (1.b). This case is much easier than the previous one, and it can be studied using the classical method of difference quotients [ADN]. We will give a shorter proof using Theorem 4.1 and a localization argument. We will also prove a stronger result, which does not hold in the case of the Dirichlet boundary condition. The maximal regularity result is given in Theorem 4.3 below. Now we prove

Theorem 4.2. Assume that $u$ and $h$ are periodic with respect to $x_1, \ldots, x_{n-1}$. Then, if $f \in (L^2(\Omega))^n$ and $h \in C^3(\mathcal{O})$, the unique solution $(u, p)$ of problem (1.2), (1.3), (1.b) belongs to $(H^2_p(\Omega))^{n-1} \times H^1_p(\Omega)$.

Proof. We extend $u$, $p$, $f$, $h$ and the first derivatives of $u$ by periodicity in the directions $x_1, \ldots, x_{n-1}$. Let $\theta \in C_0^\infty(\mathbb{R}^{n-1})$ be such that

\[0 \leq \theta \leq 1 \quad \text{and} \quad \theta(x') = 1 \quad \text{for} \quad x' \in \mathcal{O}.\]

Consider $\mathcal{O}^*$ to be a smooth domain in $\mathbb{R}^{n-1}$ containing the support of $\theta$ and let

\[(4.30) \quad Q^* = \{(x', x_n) \in \mathbb{R}^n; \quad x' \in \mathcal{O}^*, \quad h(x') < x_n < 0\}.\]

We set $\overline{\pi} = \theta u$ and $\overline{\rho} = \theta p$. A straightforward computation yields
\[\begin{aligned}
- \Delta_n \mathbf{p}(x) + \nabla_{h-1} \mathbf{p}(x') &= \mathbf{f}(x) = \theta f + \nabla \theta \nabla u + u \Delta_{n-1} \theta, \quad x \in Q^*, \\
\text{div}_{n-1} \int_{h(x')}^0 \mathbf{p}(x_1, \cdots, x_{n-1}, \eta) d\eta &= \nabla \theta \int_{h(x')}^0 u(x', \eta) d\eta = \mathbf{f}(x'), \quad x' \in \mathcal{O}^*, \\
\mathbf{p} &= 0 \quad \text{on } \partial Q^*.
\end{aligned}\]

(4.31)

Thanks to Theorem 4.1, we have

\[\begin{aligned}
\mathbf{p} &\in (H^2(Q^*))^{n-1}, \quad p \in H^1(Q^*), \\
u &\in (H^2_p(\Omega))^{n-1}, \quad p \in H^1_p(\Omega).
\end{aligned}\]

(4.32)

therefore

\[\begin{aligned}
u &\in (H^2(Q^*))^{n-1}, \quad p \in H^1(Q^*), \\
u &\in (H^2_p(\Omega))^{n-1}, \quad p \in H^1_p(\Omega). 
\end{aligned}\]

Now we prove the following maximal regularity result:

**Theorem 4.3.** Assume that \( h \in C^{k+3}(\bar{\mathcal{O}}) \), and \( u \) and \( h \) are periodic with respect to \( x_1, \cdots, x_{n-1} \). Then, if \( f \in (H^k_p(\Omega))^{n-1}, k \geq 0 \), the unique solution \((u, p)\) of problem (1.2), (1.3), (1.b) belongs to \((H^{k+2}_p(\Omega))^{n-1} \times H^{k+1}_p(\Omega)\).

**Proof.**

(i) The case \( k = 0 \) is Theorem 4.2.

(ii) Assume that \( h \in C^{k+4}(\bar{\mathcal{O}}) \) and \( f \in (H^{k+1}_p(\Omega))^{n-1} \). We know already that \( u = \theta u \in (H^{k+2}(\Omega))^{n-1} \) and \( \mathbf{p} = \theta p \in H^{k+1}(\Omega) \). The change of variables

\[\eta = \frac{-1}{h(x')} \zeta = -H(x') \zeta\]

(4.34)

gives

\[\begin{aligned}
\text{div}_{n-1} \int_{h(x')}^0 \frac{\partial \mathbf{p}}{\partial x_{\alpha}}(x', \zeta) d\zeta &= \text{div}_{n-1} \int_{-1}^0 \frac{\partial \mathbf{p}}{\partial x_{\alpha}}(x', \eta) h(x') d\eta \\
&= \nabla h \cdot \int_{-1}^0 \frac{\partial \mathbf{p}}{\partial x_{\alpha}}(x', \eta) d\eta.
\end{aligned}\]

(4.35)
which belongs to $H^{k+1}(O^*)$. Now we take the derivative of the first equation in (4.31) with respect to $x_\alpha$, $1 \leq \alpha \leq n-1$, and obtain

$$
-\Delta_n \frac{\partial \mathbf{u}}{\partial x_\alpha} + \nabla_{n-1} \frac{\partial \mathbf{p}}{\partial x_\alpha} = \frac{\partial \mathbf{f}}{\partial x_\alpha} \in (H^k(Q^*))^{n-1},
$$

and, obviously, $\frac{\partial \mathbf{u}}{\partial x_\alpha} = 0$ on $\partial Q^*, 1 \leq \alpha \leq n-1$. Hence, Theorem 4.1 yields

$$
\frac{\partial \mathbf{u}}{\partial x_\alpha} \in (H^{k+2}(Q^*))^{n-1}, \frac{\partial \mathbf{p}}{\partial x_\alpha} \in H^k(Q^*), 1 \leq \alpha \leq n-1,
$$

we also have

$$
\frac{\partial^2 \mathbf{u}}{\partial x_n^2} = -\Delta_{n-1} \mathbf{u} + \nabla_{n-1} \mathbf{p} - \mathbf{f}
$$

which belongs to $(H^{k+1}(Q^*))^{n-1}$. Therefore,

$$
\mathbf{u} \in (H^{k+3}(Q^*))^{n-1} \text{ and } \mathbf{p} \in H^{k+2}(Q^*).$$

The proof is complete.

5. regularity of solutions of the mixed Dirichlet-Neumann Problem

5.1. The Ocean equations with nonhomogeneous boundary conditions. In this section we deal with the mixed boundary conditions. We will be concerned with two types of conditions:

\[ (M_1) \begin{cases} 
 u = 0 & \text{on } \Gamma_b \cup \Gamma_\ell, \\
 \frac{\partial u}{\partial \nu} = \varphi_1 & \text{on } O
\end{cases} \]

and

\[ (M_2) \begin{cases} 
 u = 0 & \text{on } \Gamma_\ell, \\
 \frac{\partial u}{\partial \nu} = \varphi_2 & \text{on } \Gamma_b, \\
 \frac{\partial u}{\partial \nu} = \varphi_1 & \text{on } O.
\end{cases} \]
The boundary data \( \varphi_1 \) and \( \varphi_2 \) satisfy

\[
\varphi_1 \in H^{3/2+\varepsilon}_0(\mathcal{O}) \quad \text{and} \quad \varphi_2 \in H^{3/2+\varepsilon}_0(\Gamma_b).
\]

We note that, for the boundary condition \((M_1)\), the boundary conditions are mixed at a corner with the angle \( \pi \). However, for the boundary condition \((M_2)\), they are also mixed at the corner between \( \Gamma_b \) and \( \Gamma_\ell \); and in view of Proposition 3.4, we will make the supplementary assumption \((H)\):

\[
\nabla_{x'} h(z) = 0, \quad \text{for every} \ z \in \text{a neighborhood of} \ \partial \mathcal{O}.
\]

By the Lax-Milgram theorem, we have, for the two types of boundary conditions \((M_1)\) and \((M_2)\), the existence and the uniqueness of solutions to (1.2), (1.3), \((M_1)\) and to (1.2), (1.3), \((M_2)\), i.e. the solution \((u, p)\) belongs to \((H^1(\Omega))^n \times L^2(\Omega)\). We will prove the following:

**Theorem 5.1.** Assume that \( h \in C^3(\bar{\Omega}) \) and \( \partial \mathcal{O} \in C^3 \). In the case of the boundary condition \((M_2)\), we assume, also, that \((H)\) holds. Let \( f \in (L^2(\Omega))^{n-1} \), \( \varphi_1 \in (H^{3/2+\varepsilon}_0(\mathcal{O}))^{n-1} \) and \( \varphi_2 \in (H^{3/2+\varepsilon}_0(\Gamma_b))^{n-1}, 0 < \varepsilon < 1/2 \). Then, the unique solution \((u, p)\) to (1.2), (1.3), \((M_1)\) or \((M_2)\) belongs to \((H^2(\Omega))^{n-1} \times H^1(\Omega)\).

**Proof.** We proceed as in the proof of the Dirichlet boundary condition case. First, we solve (4.22) and obtain \( v \in H^1_0(\Omega) \). Then, we consider the following elliptic boundary problems:

\[
(5.2_{M_1}) \quad \begin{cases}
- \Delta w = f & \text{in } \Omega, \\
  w = 0 & \text{on } \Gamma_\ell, \\
  w = -v & \text{on } \Gamma_b, \\
  \partial w / \partial \nu = \varphi_1 & \text{on } \mathcal{O},
\end{cases}
\]

and

\[
(5.2_{M_2}) \quad \begin{cases}
- \Delta w = f & \text{in } \Omega, \\
  w = 0 & \text{on } \Gamma_\ell, \\
  \partial w / \partial \nu = -\left( \frac{\nabla v_1 \cdot \nabla h}{\sqrt{1 + |\nabla h|^2}}, \ldots, \frac{\nabla v_{n-1} \cdot \nabla h}{\sqrt{1 + |\nabla h|^2}} \right) + \varphi_2 & \text{on } \Gamma_b, \\
  \partial w / \partial \nu = \varphi_1 & \text{on } \mathcal{O}.
\end{cases}
\]

Thanks to Proposition 3.3 (resp. Proposition 3.4), the solution \( w \) of \((5.2_{M_1})\) (resp. \((5.2_{M_2})\)) belongs to \((H^{3/2-\varepsilon}(\Omega))^{n-1} \), for any \( \varepsilon > 0 \). Therefore, the distribution \( \int_{h(x')}^0 w(x', \eta) d\eta \) belongs to \( H^{3/2-\varepsilon}(\Omega) \). Moreover, since \( v \) is independent of \( x_n \), we have (with \( u = v + w \))
\[- \text{div}_{x'}(hv) = \text{div}_{x'} \int_{h(x')}^{0} v(x')d\zeta = - \text{div}_{x'} \int_{h(x')}^{0} u(x', \zeta)d\zeta \]
\[+ \text{div}_{x'} \int_{h(x')}^{0} w(x', \zeta)d\zeta + \text{div}_{x'} \int_{h(x')}^{0} w(x', \zeta)d\zeta,\]

the right hand side above belongs to $H^{1/2-\varepsilon}(\Omega)$, and since
\[\text{div}_{x'}v = \frac{1}{h(x')} \text{div}_{x'}(hv) - \frac{\nabla h \cdot v}{h},\]
we have $\text{div}_{x'}v \in H^{1/2-\varepsilon}(\Omega)$.

Now we consider the Stokes problem satisfied by $(v, p)$; i.e. problem (4.27). The regularity of this problem is given by Theorem 2.3, and we have $v \in (H_0^{3/2-\varepsilon}(\Omega))^{n-1}$. We return to (5.2$_{M_1}$) (resp. (5.2$_{M_2}$)) and conclude, with Proposition 3.3 (resp. Proposition 3.4), that $w \in H^{2-\varepsilon}(\Omega)$, in the two cases of boundary conditions (5.2$_{M_1}$) and (5.2$_{M_2}$). We have
\[\frac{\partial u}{\partial x_n} = \frac{\partial w}{\partial x_n} \in (H^{1-\varepsilon}(\Omega))^{n-1}.\]
Hence, we can integrate equation (1.2) with respect to $x_n$ and obtain, thanks to Theorem 2.3, $p \in H^1(\Omega)$.

Finally, we use Proposition 3.3, in the case of the boundary condition (M$_1$) and Proposition 3.4, in the case of boundary condition (M$_2$), to conclude that the solution $u$ belongs to $H^2(\Omega)$.

5.2. Regularity of the coupled system.

In this subsection we prove the $H^2$–regularity of the solution of the following coupled system of equations (the Transmission Problem):

\[
- \Delta u^a(x) + \nabla_{x'} p^a(x^1) = f^a(x), \quad x \in \Omega^a, \\
\text{div}_{x'} \int_{h(x')}^{L} u^a(x', \eta)d\eta = 0 \quad x' \in \mathbb{R}^{n-1},
\]

and

\[
- \Delta u^s(x) + \nabla_{x'} p^s(x') = f^s(x) \quad x \in \Omega^s \\
\text{div}_{x'} \int_{h(x')}^{0} u^s(x', \eta)d\eta = 0 \quad x' \in \Gamma_i.
\]
Here the domain $\Omega^s$ is defined by (1.1) and $\Omega^a = \mathbb{T}^{n-1} \times (0, L)$. The boundary of $\Omega^s$ is the union of

\begin{equation}
\Gamma_i = \mathcal{O} \times \{0\}, \quad \Gamma_b = \{(x', x_n); \ x' \in \mathcal{O}, \ x_n = h(x')\},
\end{equation}

and

\begin{equation}
\Gamma_\ell = \{(x', x_n); \ x' \in \partial \mathcal{O}, \ h(x') \leq x_n \leq 0\}.
\end{equation}

The boundary of $\Omega^a$ is the union of $\Gamma_i, \ \mathbb{R}^{n-1} - \Gamma_i = \Gamma_e$, and $\Gamma_u = \mathbb{R}^{n-1} \times \{L\}$ (see Figure 5.2).

Equations (5.5), (5.6) are supplemented with the following boundary conditions:

\begin{equation}
\begin{cases}
    u^a = 0 & \text{on } \Gamma_u, \\
    \frac{\partial u^a}{\partial x_n} = g(x')|u^a|^\alpha u^a & \text{on } \Gamma_e, \\
    u^s = 0 & \text{on } \Gamma_b \cup \Gamma_\ell
\end{cases}
\end{equation}

and

\begin{equation}
\frac{\partial u^a}{\partial x_n} = -\frac{\partial u^s}{\partial x_n} = g(x')|u^a - u^s|^\alpha (u^a - u^s) & \text{on } \Gamma_i,
\end{equation}

where $\alpha \geq 0$, and $g$ is a smooth positive function defined on $\mathbb{R}^{n-1}$.

---

**Figure 5.2**
Problem (5.5), (5.6), (5.9), and (5.10) is a simplified form of the linearized stationary primitive equations of the coupled system atmosphere-ocean (see [LTW3]). Using the Lax-Milgram Theorem, we can show the existence and uniqueness of solutions \((u^a, p^a)\) and \((u^s, p^s)\) in \((H^1(\Omega^a))^n \times L^2(\Omega^a)\) and \((H^1(\Omega^s))^n \times L^2(\Omega^s)\) respectively, whenever \(f^a \in (L^2(\Omega^a))^n\) and \(f^s \in L^2(\Omega^s)^n\). Therefore, by the Trace Theorem, we have

\[
(5.11) \quad u^a|_{\Gamma_i} \in (H^{1/2}(\Gamma_i))^n, \quad u^s|_{\Gamma_i} \in (H^{1/2}(\Gamma_i))^n.
\]

First, we assume that \(\alpha = 0\) and \(n \geq 2\) (linear boundary conditions), and prove

**Theorem 5.2.** The unique solution \((u^a, p^a), (u^s, p^s)\) of problem (5.5), (5.6), (5.9), and (5.10) with \(\alpha = 0\), and \(f^a \in (L^2(\Omega^a))^n\), \(f^s \in (L^2(\Omega^s))^n\), and \(h \in C^3(\bar{\Gamma}_i)\) satisfies

\[
(5.12) \quad (u^a, p^a) \in (H^2(\Omega^a))^n \times H^1(\Omega^a), \quad (u^s, p^s) \in (H^2(\Omega^s))^n \times H^1(\Omega^s).
\]

**Proof.** Using the boundary conditions (5.10) and (5.11), we have

\[
(5.13) \quad \frac{\partial u^s}{\partial x_n}|_{\Gamma_i} = -\frac{\partial u^a}{\partial x_n} = g(u^s - u^a) \in (H^{1/2}(\Gamma_i))^n \subset (H_0^{1/2-\varepsilon}(\Gamma_i))^n, \forall \varepsilon, 0 < \varepsilon < \frac{1}{2}.
\]

Thanks to Theorem 5.1 (with an interpolation argument), we can conclude that

\[
(5.14) \quad u^s \in (H^{2-\varepsilon}(\Omega^a))^n \quad \text{and} \quad p^s \in H^{1-\varepsilon}(\Omega^s), \forall \varepsilon, 0 < \varepsilon < 1/2.
\]

The same argument applied to \(u^a\), defined in \(\mathbb{T}^n \times (0, L)\), implies, with Theorem 4.2, that

\[
(5.15) \quad u^a \in (H^{2-\varepsilon}(\Omega^a))^n \quad \text{and} \quad p^a \in H^{1-\varepsilon}(\Omega^a), \forall \varepsilon, 0 < \varepsilon < 1/2.
\]

Hence, by the Trace theorem, we have

\[
(5.16) \quad \frac{\partial u^s}{\partial x_n}|_{\Gamma_i} \in (H^{3/2-\varepsilon}(\Gamma_i))^n \quad \text{and} \quad \frac{\partial u^a}{\partial x_n}|_{\Gamma_i} \in (H^{3/2-\varepsilon}(\Gamma_i))^n, \forall \varepsilon, 0 < \varepsilon < 1/2.
\]

The integration the first equations in (5.5) and (5.6) is possible, thanks to (5.15). We conclude, as in the proof of Theorem 4.1, that

\[
(5.17) \quad p^a \in H^1(\Omega^a) \quad \text{and} \quad p^s \in H^1(\Omega^s).
\]
Moving the gradient of the pressure $p^a$ (resp. $p^s$), in the first equation of (5.5) (resp. (5.6)), to the right hand side, we obtain nonhomogeneous elliptic boundary value problems on $\Omega^a$ (resp. $\Omega^s$). Thanks to (5.16) and Proposition 3.3, we conclude that

(5.18) $u^a \in (H^2(\Omega^a))^{n-1}$ and $u^s \in (H^2(\Omega^s))^{n-1}$.

The proof is complete.

**Remark 5.1.** Under the supplementary assumption $(H)$, one can prove $H^2$–regularity of the solutions to problem (5.5), (5.6) with the following boundary conditions:

(5.19) $\begin{cases}
\frac{\partial u^a}{\partial x_n} = 0 & \text{on } \Gamma_u, \\
\frac{\partial u^a}{\partial x_n} = g(x') |u^a|^\alpha u^a & \text{on } \Gamma_e, \\
u^s = 0 & \text{on } \Gamma_\ell,
\end{cases}$

and

(5.20) $\begin{cases}
\frac{\partial u^s}{\partial \nu} = 0 & \text{on } \Gamma_b, \\
\frac{\partial u^a}{\partial x_n} = -\frac{\partial u^s}{\partial x_n} = g(x') |u^a - u^s|^\alpha (u^a - u^s) & \text{on } \Gamma_i.
\end{cases}$

These boundary conditions have been introduced in [LTW3] and seem to be more natural than the boundary conditions (5.9), (5.10).

The proof of the regularity in this case of boundary conditions is identical to the one for Theorem 5.2, we only need to use Proposition 3.4 instead of Proposition 3.3. We omit the details.

Now we give an example to show that Theorem 5.2 is optimal, in the sense that no maximal regularity result can be obtained for the coupled system.

**Example 5.1.** Let $\Omega^s = B \times (-1, 0)$, where $B$ is the unit ball in $\mathbb{R}^2$. Let $f^s$ be an even function with respect to $x_3$ and define $f^a$ in $B \times (0, 1)$ by:

$f^a(x', x_3) = f^s(x', -x_3)$.

Consider the system (5.5), (5.6) defined in $\Omega^s \cup \Omega^a$, as given above, and consider the following boundary conditions:
\[
\begin{aligned}
&\begin{cases}
  u^a = 0 \quad \text{on } B \times \{1\}, \\
  u^s = 0 \quad \text{on } B \times \{-1\}, \\
  u^s = 0 \quad \text{on } \partial B \times (-1,0), \\
  u^a = 0 \quad \text{on } \partial B \times (0,1), \\
  \frac{\partial u^s}{\partial x_3} = -\frac{\partial u^a}{\partial x_3} = (u^s - u^a).
\end{cases}
\end{aligned}
\]

A change of variable \( x_3 \to -x_3 \) gives the same system of equations with \( u^a \) and \( u^s \) interchanged. Hence

\[
u^a = u^s = 0 \quad \text{on } B \times \{0\}.
\]

The system is no longer coupled and example 4.1 implies that there is no maximal regularity.

### 6. Regularity of Some Nonlinear Problems Related to the Stokes Type System

We consider the coupled system studied in section 5 and generalize the \( H^2 \)-regularity result obtained, in the case \( \alpha = 0 \) (Theorem 5.2), to the case when \( 0 < \alpha \leq 1 \) and \( n = 3 \). We will need the following imbedding Theorem [Gr1], [Zo]:

**Theorem 6.1.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \). Then for \( s_1 \leq s_2, \beta_1 \geq \beta_2 \) such that \( s_2 - \frac{n}{\beta_2} = s_1 - \frac{n}{\beta_1} \), we have the following imbedding with continuous injection:

\[
(6.1) \quad W^{s_2,\beta_2}(\Omega) \subset W^{s_1,\beta_1}(\Omega).
\]

We will assume in this section that \( n = 3 \). We recall that

\[
u^a \big|_{\Gamma_i} \in (H^{1/2}(\Gamma_i))^2 \quad \text{and} \quad u^s \big|_{\Gamma_i} \in (H^{1/2}(\Gamma_i))^2.
\]

Hence, by the Sobolev imbedding theorem (2.6), we have

\[
(6.2) \quad u^a \big|_{\Gamma_i}, u^s \big|_{\Gamma_i} \in (L^4(\Gamma_i))^2.
\]

Therefore,

\[
(6.3) \quad \int_{\Gamma_i} (|u^a - u^s|^{\alpha + 1})^{\frac{4}{\alpha + 1}} d\sigma_i < \infty,
\]

which implies that
\[(6.4) \quad \frac{\partial u^a}{\partial x_n}|_{\Gamma_i} \in (L^{\frac{4}{\alpha+1}}(\Gamma_i))^2, \quad \frac{\partial u^s}{\partial x_n}|_{\Gamma_i} \in (L^{2}(\Gamma_i))^2.\]

Since \(0 < \alpha \leq 1\), we have \(\frac{4}{\alpha + 1} \geq 2\). The integration of the first equation in (5.6) with respect to \(x_3\) yields

\[(6.5) \quad \begin{cases} - \Delta u^s + \nabla p^s \in (L^{\frac{4}{\alpha+1}}(\Gamma_i))^2 \subset (L^2(\Gamma_i))^2, \\ \text{div} \, U^s \in H^1(\Gamma_i), \\ U^s = 0 \quad \text{on } \partial \Gamma_i, \end{cases}\]

where \(U^s = \int_{-1}^0 u^s(x', \eta)d\eta\). Thanks to Theorem 2.3, we have \(p^s \in H^1(\Omega^s)\). Hence \(-\Delta u^s \in (L^2(\Omega^s))^2\) and \(\frac{\partial u^s}{\partial x_n}|_{\Gamma_i} \in (L^2(\Gamma_i))^2, \quad u^s = 0 \quad \text{on } \Gamma_\ell \cup \Gamma_b\).

According to Proposition 3.3, the following is true:

\[(6.6) \quad u^s \in (H^{3/2}(\Omega^s))^2 \quad \text{and} \quad u^s|_{\Gamma_i} \in (H^1(\Gamma_i))^2.\]

The same argument applied to \(u^a\) gives

\[u^a \in (H^{3/2}(\Omega^a))^2 \quad \text{and} \quad u^a|_{\Gamma_i} \in (H^1(\Gamma_i))^2.\]

Hence

\[(6.7) \quad (u^a - u^s)|_{\Gamma_i} \in (W^{1,\beta_2}(\Gamma_i))^2, \quad \text{for any } 1 < \beta_2 < 2,\]

we will choose \(\beta_2 = 4/3\).

We use Theorem 6.1 with \(\beta_1 = 2\) and \(s_2 = 1\) to obtain

\[(6.8) \quad W^{1,\beta_2}(\Gamma_i) \subset H^2-\frac{\beta_2}{2}(\Gamma_i).\]

Hence, \(\frac{\partial u^s}{\partial x_n}|_{\Gamma_i} \in (H^{2-\frac{\beta_2}{2}}(\Gamma_i))^2\) and Proposition 3.3. implies that

\[u^s \in (H^{3/2+2-\frac{\beta_2}{2}}(\Omega^s))^2.\]

Now, for \(\beta_2 = 4/3\), we have \(u^s \in (H^2(\Omega^s))^2\). A similar argument yields \(u^a \in (H^2(\Omega^a))^2\). We proved

**Theorem 6.2.** The unique solution of problem (5.5), (5.6) with the boundary condition (5.9) and (5.10), in the case \(0 \leq \alpha \leq 1\) and \(n = 3\), satisfies:

\[(6.9) \quad (u^s, p^s) \in (H^2(\Omega^s))^2 \times H^1(\Omega^s), \quad \text{and} \quad (u^a, p^a) \in (H^2(\Omega^a))^2 \times H^1(\Omega^a).\]
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REFERENCES