On the weighted decay for solutions of the Navier-Stokes system

Igor Kukavica*

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Department of Mathematics
University of Southern California
Los Angeles, CA 90089
e-mail: kukavica@usc.edu

Abstract

We consider the question to which extend the temporal decay \(k u(\cdot, t)k_{L^2} = O(t^{-\gamma_0})\) for solutions \(u\) of the Navier-Stokes equation influences the decay of weighted norms. We prove that \(\|\varepsilon|^{a}u(\cdot, t)\|_{L^2} = O(t^{-\gamma_0+a/2})\) holds under the condition \(0 < a < n/2 + 1\). This extends the range of weights obtained in [KT2]. A wider range of exponents is also obtained for the decay of \(L^p\) norms.

Keywords: Navier-Stokes equation, time decay, space decay, strong solutions

Mathematics Subject Classification: 35Q30, 76D05, 35K55, 35K15

1 Introduction

In this paper, we address the decay rates of weighted norms for solutions of the Navier-Stokes equation with potential forces. More specifically, we are interested in to which extent the temporal decay of \(L^2\) norms influences the decay rate of the weighted norms of \(u\). The question of decay to 0 of the energy norm \(\|u(\cdot, t)\|_{L^2}\) was posed by Leray in [L]. It was answered positively by Kato in [K] for dimension \(n = 2\) and by Schonbek in space dimension 3 [S1]. Schonbek’s Fourier splitting technique was also useful in addressing the relationship between the conditions on the initial datum \(u_0\) and the rate \(\gamma_0\) in

\[\|u(\cdot, t)\|_{L^2} = O(t^{-\gamma_0})\]

where the notation (1.1) means \(\sup_{t \to -\infty} t^{\gamma_0} \|u(\cdot, t)\|_{L^2} < \infty\). In particular, by [S2, KM], solutions \(u\) of the Navier-Stokes equations decay with the rate \(\gamma_0 = (n/2)(1/p - 1/2)\) if \(u_0 \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\).

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Furthermore, Wiegner proved in [W] that if \( u_0 \in L^2 \cap B_2^{-\alpha, \infty} \) where \( 0 < \alpha \leq (n + 2)/2 \) then we have (1.1) with \( \gamma_0 = \alpha/2 \). If the initial datum is localized, i.e., \( u_0 \in \mathcal{S} \), then by the divergence-free condition it follows \( u_0 \in B_2^{-(n+2)/2, \infty} \) which then implies (1.1) with \( \gamma_0 = -n/4 - 1/2 \). This results turns out to be optimal for such data as shown by Miyakawa and Schonbek in [MS], where it was shown that \( \|u(\cdot, t)\|_{L^2} = \alpha(t^{-n/4-1/2}) \) imposes special restrictions on \( u \) which most solutions do not satisfy. For further results, cf. [B1, B2, BV, GW, KT1, Ku, Le, SW, T].

The problem we consider is the following: Given (1.1), for which \( a, p, \) and \( \beta \) can we conclude

\[
\| |x|^a u(\cdot, t) \|_{L^2} = \mathcal{O}(t^{-\beta}).
\] (1.2)

This question was raised by Amrouche et al [AGSS], and based on explicit examples for the heat equation, the conjectured rate is \( \beta = -\gamma_0 + a/2 \). It is not difficult to obtain such rates for the linear heat equation; in the case of the Navier-Stokes equation, the presence of the pressure causes difficulties with obtaining the optimal \( \beta \) as well as the range of \( a \) for which it is valid.

Amrouche et al showed in [AGSS, Theorem 4.1] that (1.2) holds with \( \beta = \gamma_0 - a\gamma_0/n \) provided \( 2 \leq n \leq 5 \) and \( a \in [0, n/2] \). In [BJ, Theorem 4.1], the obtained rate is \( \beta = n/4 - a/2 + \epsilon \) under the condition \( a \in [0, n/2] \). In [KT2], we proved the optimal rate of decay \( \beta = \gamma_0 - a/2 \) under the condition \( a \in [0, n/2] \), without any restriction on \( \gamma_0 \).

In this paper, we improve the result of [KT2] by extending the range to \( a \in [0, n/2 + 1] \). Also, we prove that (1.1) implies

\[
\| |x|^a u(\cdot, t) \|_{L^p} = \mathcal{O}(t^{-\gamma_0 + a - (n/2)(1/2 - 1/p)})
\] (1.3)

under the condition \( a \in [0, n/p' + 1] \) (in [KT2] the same was obtained with \( a \in [0, n/p'] \)). There is an argument suggesting that the range \( a < n/p' + 1 \) might indeed be optimal at least for the case \( \gamma_0 = n/4 + 1/2 \). Namely, by [M, BM, DS], there exist solutions with spatial decay \( n + 1 \) which is exactly \( n/p' + 1 \) if \( p = \infty \). It is not clear whether our result holds for the endpoint \( a = n/p' + 1 \).

### 2 Notation and the main theorem

Let \( u \in C([0, \infty), H^1(\mathbb{R}^n)) \) be a solution of the Navier-Stokes equation

\[
\begin{align*}
  u_t - \Delta u + (u \cdot \nabla)u + \nabla \pi &= 0 \\
  \nabla \cdot u &= 0
\end{align*}
\] (2.1)

with an initial condition \( u(\cdot, 0) = u_0 \). We have set the viscosity to 1, which can be achieved by rescaling. We assume that \( u_0 \) is divergence-free and well localized, i.e., \( u_0 \in \mathcal{S} \). We also assume that the solution
satisfies
\[ \|u(\cdot, t)\|_{L^2} = O(t^{-\gamma_0}) \]
and
\[ \|u(\cdot, t)\|_{L^\infty} = O(1) \]
as \( t \to \infty \). Throughout this paper, the notation \( \phi(t) = O(t^a) \) means \( \sup_{t \geq 1} t^{-a} |\phi(t)| < \infty \). The above assumption implies that the solution is strong and thus infinitely smooth in both space and time variables. By [W], we may assume \( \gamma_0 \geq n/4 + 1/2 \). The following is our main result.

**Theorem 2.1.** Let \( u \) be as above. Then
\[ \|x|^a u(\cdot, t)\|_{L^p} = O(t^{-\gamma_0 + a/2 - (n/2)(1/2 - 1/p)}) \] (2.2)
for all \( p \in [2, \infty) \) and \( a \in [0, n/p' + 1) \), where \( p' = p/(p - 1) \).

In [KT2], we proved the same statement but with a weaker assertion \( a \in [0, n/p') \). In the proof, we need the following simple fact: If \( \eta \in \mathcal{C}_0^\infty(\mathbb{R}^n) \) and \( \alpha > 2 - n/p \). Then
\[ \frac{x^\alpha}{|x|^2} \eta(x) \in L^p_x(\mathbb{R}^n), \quad p \in (1, \infty), \quad a \in \left[0, \alpha - 2 + \frac{n}{p}\right] \] (2.3)
where \( L^p_x(\mathbb{R}^n) \) denotes the standard Lebesgue-Sobolev space. The proof consists of partitioning the function smoothly into dyadic shells and using Sobolev type inequalities on every shell (cf. [KT2]).

**Proof of Theorem 2.1.** As the first step in the proof, we claim that
\[ \|x|^a D_x u(\cdot, t)\|_{L^p} = O(t^{-\gamma_0 + a/2 - (n/2)(1/2 - 1/p)}) \] (2.4)
for all \( a \in [0, 2 + n - n/p) \), where \( D_x \) denotes the gradient in \( x \). The vorticity \( \omega_{ij} = \partial_i u_j - \partial_j u_i \) satisfies
\[ \partial_t \omega_{ij} - \Delta \omega_{ij} + \partial_i (u_k \omega_{kj}) - \partial_j (u_k \omega_{ki}) = 0, \quad i, j = 1, \ldots, n. \] (2.5)
From [KT2], we recall
\[ \|x|^a D^b_x \omega(\cdot, t)\|_{L^p} = O(t^{-\gamma_0 - 1/2 + a/2 - b/2 - (n/2)(1/2 - 1/p)}) \] as \( t \to \infty \) (2.6)
for all \( p \in [2, \infty) \), \( a \geq 0 \), and \( b \in \mathbb{N}_0 \), where \( D^b_x \) denotes the totality of all partial derivatives of order \( b \). As in [KT2], we have \( \hat{\omega}_{kij}(0, t) = 0 \), where \( \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx \) is the Fourier transform (in the \( x \) variable). Moreover, we assert
\[ \nabla \hat{\omega}_{kij}(0, t) = 0 \] (2.7)
for all \( t \in \mathbb{R}^n \), which is equivalent to
\[ \int x_m \omega_{ij}(x, t) dx = 0, \quad i, j, m \in \{1, \ldots, n\}. \] (2.8)
Note that we can not conclude that \( \int x_m \partial_i u_j \) equals \( -\delta_{mi} \int u_j \) using integration by parts since we do not know whether \( \int |x||u| < \infty \) (here, \( \delta_{im} = 0 \) if \( i \neq m \) and \( \delta_{im} = 1 \) if \( i = m \)). Indeed, this property is not preserved by the evolution [BM]. We may conclude this at time \( t = 0 \):

\[
\int x_m \omega_{ij}(x, 0) dx = \int x_m (\partial_i u_{0j} - \partial_j u_{0i}) = -\delta_{im} \int u_{0j} + \delta_{jm} \int u_{0i} = 0
\]

since \( u_0 \in \mathcal{S} \) is divergence-free. Now, fix an arbitrary \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( \eta \equiv 1 \) on \( B_1(0) \) and \( \eta \equiv 0 \) on \( B_2(0)^c \). Then, using (2.5) and integration by parts,

\[
\int x_m \omega_{ij}(x, t) \eta \left( \frac{x}{R} \right) dx - \int x_m \omega_{ij}(x, 0) \eta \left( \frac{x}{R} \right) dx
\]

\[
= \int_0^t \int \left( \frac{2}{R} \delta_{km} \omega_{ij}(x, s) \partial_k \eta \left( \frac{x}{R} \right) + \frac{1}{R^2} x_m \omega_{ij}(x, s) \Delta \eta \left( \frac{x}{R} \right) + \delta_{im} u_k(x, s) \omega_{kj}(x, s) \eta \left( \frac{x}{R} \right) \right. \]

\[
\left. + \delta_{jm} u_k(x, s) \omega_{ik}(x, s) \partial_j \eta \left( \frac{x}{R} \right) - \frac{1}{R} x_m u_k(x, s) \omega_{ij}(x, s) \partial_j \eta \left( \frac{x}{R} \right) \right) dx \; ds
\]

and thus, writing \( \omega_{ij} = \partial_i u_j - \partial_j u_i \) and integrating by parts, we get

\[
\int x_m \omega_{ij}(x, t) \eta \left( \frac{x}{R} \right) dx - \int x_m \omega_{ij}(x, 0) \eta \left( \frac{x}{R} \right) dx
\]

\[
= \int_0^t \int \left( \frac{2}{R} \delta_{km} \omega_{ij}(x, s) \partial_k \eta \left( \frac{x}{R} \right) + \frac{1}{R^2} x_m \omega_{ij}(x, s) \Delta \eta \left( \frac{x}{R} \right) \right. \]

\[
- \frac{1}{R} \delta_{im} u_k(x, s) \omega_{kj}(x, s) \partial_k \eta \left( \frac{x}{R} \right) + \frac{1}{2R} \delta_{jm} u_k(x, s) \omega_{ik}(x, s) \partial_j \eta \left( \frac{x}{R} \right) \]

\[
+ \frac{1}{2R} \delta_{km} u_k(x, s) \omega_{ij}(x, s) \partial_j \eta \left( \frac{x}{R} \right) - \frac{1}{R} x_m u_k(x, s) \omega_{ij}(x, s) \partial_j \eta \left( \frac{x}{R} \right) \right) dx \; ds.
\]

The equation (2.8) follows for all \( t \geq 0 \) by sending \( R \to \infty \). (Note that \( x_m u_k \omega_{kj} \in L^1(\mathbb{R}^n \times (0, t)) \) for all \( t \geq 0 \).

Using \( \dot{\omega}_{kj}(0, t) = 0 \) and (2.7), we have by the Taylor formula

\[
\dot{\omega}_{kj}(\xi, t) = \frac{1}{2} \xi_i \xi_m \partial_m \dot{\omega}_{kj}(0, t) + \frac{1}{2} \xi_i \xi_m \xi_n \int_0^1 (1 - s)^2 \partial_{lmq} \dot{\omega}_{kj}(s \xi, t) \; ds
\]

\[
= f^{(1)}_{kj}(\xi, t) + f^{(2)}_{kj}(\xi, t).
\]

Denote \( L = (-\Delta)^{1/2} \). By

\[
\partial_i u_j = \frac{\xi_i \xi_k}{|\xi|^2} \dot{\omega}_{kj}, \quad i, j = 1, \ldots, n
\]
and the Hausdorff-Young inequality, we obtain
\[
\| x^n \partial x_j \|_{L^p} \leq C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) \dot{\omega}_{kj}(\xi, t)}{|\xi|^2} \right) \right\|_{L^{p'}} + C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) (1 - \eta(\sqrt{t} \xi)) \dot{\omega}_{kj}(\xi, t)}{|\xi|^2} \right) \right\|_{L^{p'}}
\]
\[
\leq C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) f_{kj}(1)}{|\xi|^2} \right) \right\|_{L^{p'}} + C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) f_{kj}(2)}{|\xi|^2} \right) \right\|_{L^{p'}}
\]
\[
+ C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) \dot{\omega}_{kj}(\xi, t)}{|\xi|^2} \right) \right\|_{L^{p'}} = I_1 + I_2 + I_3.
\]

For the first term, we have
\[
I_1 = C \left\| L^a \left( \frac{\xi \xi_k \eta(\sqrt{t} \xi) \dot{\omega}_{kj}(0, t)}{|\xi|^2} \right) \right\|_{L^{p'}}
\]
\[
= \frac{C}{2} \left\| L^a \left( \frac{\xi \xi_k \xi \xi_m \eta(\sqrt{t} \xi)}{|\xi|^2} \right) \right\|_{L^{p'}} |\dot{\omega}_{kj}(0, t)|. \tag{2.9}
\]

The third factor in the far right expression of (2.9) is estimated as
\[
|\dot{\omega}_{kj}(0, t)| \leq C\|x|^2 \omega\|_{L^1} \leq C\|x|^2 \omega\|_{L^2}^{1/2} \|x^n \omega\|_{L^2}^{1/2} = O(t^{-\gamma_0 + n/4 + 1/2}).
\]

(Here we used the inequality \(\|f\|_{L^1} \leq \|f\|_{L^2}^{1/2} \|x^n \|_{L^2}^{1/2}\) which is, for nonzero \(f\), obtained by substituting \(b = (\|x^n \|_{L^2}/\|f\|_{L^2})^{1/2}\) in the inequality \(\|f\|_{L^1} \leq \|(x + b)^n \|_{L^2} \|x + b\|_{L^2}^n \leq Cb^{-n/2}(\|x^n \|_{L^2} + b^n \|f\|_{L^2})\).) For the second factor in the far right expression of (2.9), making a substitution and using a chain rule
\[
L^a(f(b\xi)) = b^a(L^a f)(b\xi) \tag{2.10}
\]
we get
\[
\left\| L^a \left( \frac{\xi \xi_k \xi \xi_m \eta(\sqrt{t} \xi)}{|\xi|^2} \right) \right\|_{L^{p'}} = t^{-(2-a)/2-n/2p'} \left\| L^a \left( \frac{\xi \xi_k \xi \xi_m \eta(\xi)}{|\xi|^2} \right) \right\|_{L^{p'}} = O(t^{-(2-a)/2-n/2p'})
\]
provided \(p' (2 - a) > -n\), i.e., \(2 + n - n/p > a\). We conclude
\[
I_1 = O(t^{-\gamma_0 - 1/2 + a/2-n/4+n/2})
\]
derived under the condition \(a < 2 + n - n/p\). Now, we find \(q, r \in (p', \infty)\) such that \(1/q + 1/r = 1/p'\) with \(q(a - 3) < n\) and \(r > n\). (For this, simply take \(r = 2n\) and \(q = (1/p' - 1/r)^{-1}\).) Then, by [KPV, p. 334],
\[
I_2 = \left\| \frac{\xi \xi_k \eta(\sqrt{t} \xi) f_{kj}(2)}{|\xi|^2} \right\|_{L^{p'}} \leq C \left\| L^a \left( \frac{\xi \xi_k \xi \xi_m \xi \xi_g \eta(\sqrt{t} \xi)}{|\xi|^2} \right) \right\|_{L^{p'}} \left\| \int_0^1 (1 - s)^2 \partial_{\xi \xi m q} \dot{\omega}_{kj} (s\xi, t) \right\|_{L^r} \leq C \left\| L^a \left( \frac{\xi \xi_k \xi \xi_m \xi \xi_g \eta(\sqrt{t} \xi)}{|\xi|^2} \right) \right\|_{L^{p'}} \left\| \frac{\xi \xi_k \xi \xi_m \xi \xi_g \eta(\sqrt{t} \xi)}{|\xi|^2} \right\|_{L^{p'}} \left\| \int_0^1 (1 - s)^2 \partial_{\xi \xi m q} \dot{\omega}_{kj} (s\xi, t) \right\|_{L^r}. \tag{2.11}
\]
By the change of variable, by (2.10), and by (2.3), we have

\[ \left\| L^a \left( \frac{\xi_1 \xi_2 \xi_3}{|\xi|^4} \eta(\sqrt{T_\xi}) \right) \right\|_{L^q} \leq C t^{(a-3-n/q)/2}. \]

On the other hand, using the Hausdorff-Young inequality, we get

\[ \left\| \int_0^1 (1-s)^2 \partial_t m q \omega_{kj}(s \xi, t) \, ds \right\|_{L^r} \leq \int_0^1 (1-s)^2 \| \partial_t m q \omega_{kj}(s \xi, t) \|_{L^q} \, ds. \]

Note that

\[ \| \partial_t m q \omega_{kj}(s \xi, t) \|_{L^r} = \frac{1}{s^{n/r}} \| \partial_t m q \omega_{kj}(s \xi, t) \|_{L^r} \leq \frac{C}{s^{n/r}} \| x^3 \omega_{kj}(s \xi, t) \|_{L^r} = \frac{1}{s^{n/r}} \mathcal{O}(t^{-\gamma_0+1-(n/r)(1-2/r')}), \]

and thus since \( n/r < 1 \),

\[ \left\| \int_0^1 (1-s)^2 \partial_t m q \omega_{kj}(s \xi, t) \, ds \right\|_{L^r} = \mathcal{O}(t^{-\gamma_0+1-(n/2)(1/2-1/r')}). \]

Therefore, the first term on the far right of (2.11) is less than or equal to \( C t^{-\gamma_0-1/2+a/2-(n/2)(1/2-1/p)} \).

The same estimate holds for the second term in (2.11), and we get

\[ I_2 = \mathcal{O}(t^{-\gamma_0-1/2+a/2-(n/2)(1/2-1/p)}). \]

For the term \( I_3 \), we have

\[ I_3 \leq C \left\| L^a \left( \frac{\xi_2}{|\xi|^4} (1-\eta(\sqrt{T_\xi})) \right) \right\|_{L^r} \| \xi^2 \omega_{kj} \|_{L^q} + C \left\| \frac{\xi_2}{|\xi|^4} (1-\eta(\sqrt{T_\xi})) \right\|_{L^r} \| \xi^2 L^a \omega_{kj} \|_{L^q} \]

\[ = \mathcal{O}(t^{-\gamma_0-1/2+a/2-(n/2)(1/2-1/p)}) \]

using \( \| f \|_{L^s} \leq C \| f \|_{L^2}^{3/2-1/s} \| x^n f \|_{L^2}^{1/s-1/2} \) from [GK], valid for \( s \in [1, 2] \), with \( f = |\xi|^2 \omega_{kj} \) and \( f = |\xi|^2 L^a \omega_{kj} \). We conclude that (2.4) holds subject to \( a \in [0, 2 + n - n/p) \). Using the weighted inequality from [CKN] on (2.4), we get

\[ \| |x|^a u \|_{L^p} \leq C \| x^{a+1} \nabla u \|_{L^p} \]

provided \( a < 1 + n - n/p \), and the theorem is proven.

\[ \square \]

References


