
Algebraic geometry from an \mathbb{A}^1 -homotopic viewpoint

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Outline/Motivation

This course is supposed to serve as an introduction to \mathbb{A}^1 -algebraic topology, which is a term coined by Fabien Morel [Mor06b, Mor12]. Officially, the prerequisites for this class were the introductory topology sequence (homology, covering spaces, fundamental groups, manifolds, elementary differential topology, de Rham cohomology) and the algebra sequence (groups, rings, fields, linear algebra, a touch of commutative algebra and some elementary non-commutative ring theory). The true prerequisite is willingness to learn on the fly. Given that background, this course was constructed around the following conceit: it is possible and interesting to learn something about \mathbb{A}^1 -homotopy theory.

Thus, the natural question to ask is: why should one care about \mathbb{A}^1 -homotopy theory (especially if you have not yet learned homotopy theory or algebraic geometry!)? This course is essentially my answer to this question: I want to provide a “modern” approach to the study of projective modules over commutative rings.

The theory of projective modules is by now a very classical subject: the formal notion of projective module goes back to the work of Cartan–Eilenberg in the foundations of homological algebra [CE99, Chapter I.2], but examples of projective modules arose much earlier (e.g., the theory of invertible fractional ideals). The notion of projective module becomes indispensable in cohomology, e.g., group cohomology may be computed using projective resolutions. One may look at the collection of projective modules over a ring as a certain invariant of the ring itself (“representations” of the ring).

The results of Serre showed that the language of algebraic geometry might provide a good language to study projective modules over commutative unital rings [Ser55, §50 p. 242]. More precisely, Serre showed that finitely generated projective modules over commutative unital rings are precisely the same things as finite rank algebraic vector bundles over affine algebraic varieties. Serre furthermore showed that this dictionary was useful for providing a better understanding about projective modules because it allowed one to exploit an analogy between algebraic geometry and algebraic topology: projective modules over rings are analogous to vector bundles over topological spaces.

From this point of view, projective modules take on additional significance. For example, in differential topology, one may turn a non-linear problem (e.g., existence of an immersion of one manifold into another) into a linear problem by looking at associated bundles (a corresponding injection of tangent bundles in the case of immersions). In good situations, a solution to the linear problem can actually be promoted to a solution of the non-linear problem (in our parenthetical example, this is an incarnation of the Hirsch-Smale theory of immersions).

Based on this analogy, Serre observed that if R is a Noetherian ring of dimension d , one could “simplify” projective modules P of rank $r > d$ greater than the dimension: any such module could be written as a sum of a projective module of rank $r' \leq d$ and a free module of rank $r' - r$ [Ser58b]. After the work of Bass [Bas64], which furthermore amalgamated Grothendieck’s ideas regarding K-theory with Serre’s results, J.F. Adams wrote:

“This leads to the following programme: take definitions, constructions and theorems from bundle-theory; express them as particular cases of definitions, constructions and statements about finitely-generated projective modules over a general ring; and finally, try to prove the statements under suitable assumptions”.

My point of view in this class is that the Morel-Voevodsky \mathbb{A}^1 -homotopy theory provides arguably the ultimate realization of this program.

Pontryagin and Steenrod observed that one could use techniques of homotopy theory to study vector bundles on spaces having the homotopy type of CW complexes. Indeed, the basic goal of the class will be to establish the analog in algebraic geometry of this result, at least for sufficiently nice (i.e., non-singular) affine varieties. Looking beyond this, just as the Weil conjectures provides a beautiful link between the arithmetic problem of counting the number of solutions of a system of equations over a finite field and a “topologically inspired” étale cohomology theory of algebraic varieties, \mathbb{A}^1 -homotopy theory allows one to construct a link between the algebraic theory of projective modules over commutative rings, and an “algebraic-geometric” analog of the homotopy groups of spheres!

After very quickly recalling some of the topological constructions that provide sources of inspiration (and which we will attempt to mirror), I will begin a brief study of the theory of affine algebraic varieties. While this will not suffice for our eventual applications, affine varieties are, arguably, intuitively appealing, and it seemed better not to require too much algebraic-geometric sophistication at first.

Then, I will introduce a “naive” version of homotopy for algebraic varieties and, following the topological story, describe various “homotopy invariants” in algebraic geometry. Along the way, I will introduce a number of important invariants of algebraic varieties: projective modules, Picard groups, and K-theory. The ultimate goal is to prove Lindel’s theorem that shows that the functor “isomorphism classes of projective modules” is homotopy invariant, in a suitable algebraic-geometric sense, on suitably nice (i.e., non-singular, affine) algebraic varieties. Along the way, I will try to build things up in a way that motivates some of the tools used in the study of \mathbb{A}^1 -homotopy theory over a field.

There are many texts that talk about cohomology theories in algebraic geometry and these notes are not intended to be another such text. Rather, there is a hope, supported by recent results, that \mathbb{A}^1 -homotopy theory can give us information not just about cohomology of algebraic varieties, but actually about their geometry. We have attempted to illustrate this by focusing on projective modules and vector bundles on algebraic varieties.

What’s next?

To proceed from the “naive” theory to the “true” theory, requires more sophistication: one needs to know some homotopy theory of simplicial sets, Grothendieck topologies, model categories etc. The syllabus listed the following plan, which I would argue is the “next step” beyond what I now want to cover in the class. The subsequent background is written with this plan in mind.

- Week 1. Some abstract algebraic geometry: the Nisnevich topology and basic properties.
- Week 2. Simplicial sets and simplicial (pre)sheaves.
- Week 3. Model categories in brief; the simplicial and \mathbb{A}^1 -homotopy categories
- Week 4. Basic properties of the \mathbb{A}^1 -homotopy category (e.g., homotopy purity)
- Week 5. Fibrancy, cd-structures and descent
- Week 6. Classifying spaces: simplicial homotopy classification of torsors
- Week 7. \mathbb{A}^1 -homotopy classification results
- Week 8. Eilenberg-MacLane spaces and strong and strict \mathbb{A}^1 -invariance

- Week 9. Postnikov towers
- Week 10. Homotopy sheaves and \mathbb{A}^1 -connectivity
- Week 11. The unstable \mathbb{A}^1 -connectivity property and applications
- Week 12. Loop spaces and relative connectivity
- Week 13. Gersten resolutions and strong/strict \mathbb{A}^1 -invariance
- Week 14. \mathbb{A}^1 -homology and \mathbb{A}^1 -homotopy sheaves
- Week 15. \mathbb{A}^1 -quasifibrations and some computations of homotopy sheaves

Background. While there isn't a specific textbook for the class, I will use a number of different sources for some of the background material. There is formally quite a lot of background for the subject of the class and I don't expect anyone to have digested all the prerequisites in any sort of linear fashion. Instead, there will be a lot of "on-the-fly" learning and going backwards to fill in details as necessary.

- To get started, I will expect that people know some basic things about commutative ring theory. A good introductory textbook is [AM69], but [Mat89] is more comprehensive. For a discussion that is more algebro-geometric, you can look at [Eis95]. We will also need more detailed results about modules, for which you consult [Lam99].
- We will study affine varieties and eventually discuss sheaf cohomology on topological spaces. Beyond what I mention in the class, useful references for the theory of algebraic varieties will include [Har77, Chapters 1-2]. Useful background for the notions of sheaf cohomology we will need on topological spaces in general, and on schemes in particular, can be found in [God73] or [Har77, Chapter 3]. Implicit here is a basic understanding of some ideas from homological algebra [Wei94]. Furthermore, from the standpoint of references, I think there is now no better definitive source than Johan de Jong's Stacks Project [Sta15].
- I will also expect some familiarity with basic concepts of algebraic and differential topology, e.g., topological spaces, smooth manifolds and maps, CW complexes, singular homology, covering spaces, vector bundles, and homotopy groups as can be found in [Spa81] or [Hat02]. The point of view exposed in [AGP02] will also be useful.
- Finally, the course will, from the beginning, use category-theoretic terminology. Beyond the usual notions of categories, functors, and natural transformations, I will expect some familiarity with various kinds of universal properties, limits (and colimits) and adjoint functors and their properties, as can be picked up in [ML98] or [Kel05]. As time goes on, we will need a bit of familiarity with "size" issues in category theory, so [AR94] is also a good reference.
- One theme throughout the course will be connections with the theory of projective modules and K-theory. For the topological story, [Ati89] is a good reference, while [Hus94] is a suitable reference for the theory of fiber bundles. In the algebraic setting, [Wei13] is a good reference for K-theory, while [Lam06] will provide excellent motivation.
- As we progress, it will also be useful to know some things about the theory of quadratic forms. The theory over fields is discussed in [Lam05]; the theory over more general rings is developed in [Knu91], and [BL08] has a nice discussion from a point of view that will be closely related with ours.

Other references. The following is simply a list of references regarding topics that will appear in the class; it is by no means complete.

- Grothendieck topologies, especially the Nisnevich topology: [Nis89], [TT90], [Voe10]

- Simplicial sets: [GJ09], [Cur71] or the original sources [Kan58a, Kan58b]
- Model Categories: [Qui67], [DS95] for a survey, or [Hov99], [Hir03] for more detailed treatments.
- Sheaf theoretic homotopy theory: [Jar15] or [Bro73, BG73] for original sources.
- \mathbb{A}^1 -homotopy theory: [Voe98] for an overview, and [MV99] or [Mor06a] for (different) more detailed treatments

As is likely clear from this quick list of references, \mathbb{A}^1 -algebraic topology has a number of prerequisites and a large collection of sources of inspiration.

Notation

We use the following standard categories. All the categories under consideration are *essentially small*, i.e., equivalent to small categories (see A.1 for more details about category theory as we will need it). As a consequence we will frequently abuse notation and use the same notation for a choice of an essentially small skeletal subcategory.

- Set - objects are sets and morphisms are functions
- Grp - objects are groups and morphisms are group homomorphisms
- Ab = the full subcategory of Grp with objects consisting of abelian groups
- Mod_R - objects are (left) R -modules, and morphisms are R -module homomorphisms
- Top - objects are topological spaces and morphisms are continuous maps
- Δ - objects are non-empty finite totally ordered sets and morphisms are order-preserving functions
- sSet - objects are functors $\text{Fun}(\Delta^\circ, \text{Set})$ and morphisms are natural transformations.
- Aff_k - objects are finitely generated, commutative, unital k -algebras, morphisms are k -algebra homomorphisms.
- Cat - the category of small categories.

Warning/Disclaimer: These notes are constantly being modified (especially while the class is going on). Moreover, all the material is in very rough form, especially that which appears in later sections. I will frequently be adding/revising material in earlier sections. Thus, in the off chance that you happen to be reading along and are not taking the class, use at your own risk! Furthermore, not everything that is discussed in the notes was mentioned in class. If you do see mistakes, or find things about which you are confused (and they aren't fixed in a later version), please do not hesitate to write me for clarification!

Chapter 1

Topological preliminaries

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1.1 Lecture 1: Spaces and homotopies

A basic goal of algebraic topology is to study topological spaces by means of associated algebraic invariants. We will begin by reviewing some ideas about invariants in algebraic topology. Then we will explain how these definitions naturally give rise to a search for corresponding invariants in algebraic geometry.

1.1.1 Homotopy invariants

Write Top for the category of topological spaces, i.e., the category where objects are topological spaces and morphisms are continuous maps of topological spaces. If \mathcal{C} is some category (we would like to think of \mathcal{C} as a category of algebraic structures, e.g., sets, groups, rings, etc.), then by a \mathcal{C} -valued invariant (or just invariant for short) we will mean a functor $\mathcal{F} : \text{Top} \rightarrow \mathcal{C}$.

Write $I = [0, 1]$ for the unit interval, and for X a topological space, write $pr_X : X \times I \rightarrow X$ for the projection onto the first factor; this is a continuous function if we equip $X \times I$ with the product topology. If \mathcal{F} is an invariant, we can think of $\mathcal{F}(X \times I)$ as a continuous family of invariants parameterized by I .

Definition 1.1.1.1. A \mathcal{C} -valued invariant \mathcal{F} is homotopy invariant if, for every $X \in \text{Top}$, the map induced by the (continuous) projection map $pr_{X*} : \mathcal{F}(X \times I) \rightarrow \mathcal{F}(X)$ is an isomorphism in \mathcal{C} .

The inclusions of the endpoints 0 and 1 into I induce continuous restriction functions $X \hookrightarrow X \times I$ and there are corresponding maps $\mathcal{F}(X) \rightarrow \mathcal{F}(X \times I)$. Since the composite of either of

these inclusion maps with the projection $X \times I \rightarrow X$ is id_X we conclude that the value of \mathcal{F} is constant in families parameterized by I . More generally, if f and g are two maps $X \rightarrow Y$ and we can find a map $H : X \times I \rightarrow Y$ such that $H(x, 0) = f$ and $H(x, 1) = g$, then we say that H is a *homotopy* between f and g . We will say that f is homotopic to g if there is a homotopy between f and g .

Exercise 1.1.1.2. 1. Show that the relation “ f is homotopic to g ” is an equivalence relation.
2. Show that composites of homotopic maps are homotopic.

If \mathcal{F} is a homotopy invariant, and f and g are homotopic maps, then the induced maps $\mathcal{F}(f)$ and $\mathcal{F}(g)$ necessarily coincide. In particular, if $f : X \rightarrow Y$ and we can find a map $g : Y \rightarrow X$ such that the composites $f \circ g$ and $g \circ f$ are homotopic to the identity, then \mathcal{F} takes the same values on X and Y . From these observations, one can conclude there is a “universal” homotopy invariant, i.e., there is a category \mathcal{H} equipped with a functor $\text{Top} \rightarrow \mathcal{H}$ that is initial among homotopy invariants; the category \mathcal{H} is called the homotopy category.

One can construct the category \mathcal{H} in a fairly explicit way: the objects are the same as in Top , but the set of morphisms in \mathcal{H} between two topological spaces X and Y is precisely the quotient of the set of continuous maps by the equivalence relation given by maps being homotopic (this is again a category, since compositions of homotopic maps are homotopic). We write $[X, Y]$ for the set of homotopy classes of maps from X to Y .

Remark 1.1.1.3. There is a variant of the discussion above that we will find useful. Namely, let Top_* be the category of based or pointed topological spaces, i.e., the objects of this category are topological spaces equipped with a distinguished point (called the base-point), and morphisms between based topological spaces are continuous maps that send the base-point to the base-point. We can then consider based analogs of the constructions above: pointed homotopies between pointed maps, pointed homotopy equivalences, and the pointed homotopy category, which will be denoted \mathcal{H}_* . We write $[X, Y]_*$ for the set of maps in the pointed homotopy category. Note that there is a forgetful functor $\text{Top}_* \rightarrow \text{Top}$ that forgets the base-point. There is also a functor $\text{Top} \rightarrow \text{Top}_*$ sending a topological space X to X_+ , which is X equipped with a disjoint base-point.

If \mathcal{F} is a \mathcal{C} -valued homotopy invariant, we will abuse notation and write \mathcal{F} for the induced functor $\mathcal{H} \rightarrow \mathcal{C}$ as well. One important source of invariants is encoded in the next definition.

Definition 1.1.1.4. A \mathcal{C} -valued homotopy invariant \mathcal{F} on Top is called *(co-)representable* if there exists $X \in \text{Top}$ and a natural isomorphism $\mathcal{F} \cong [X, -]$. In that case X is called a *representing space*. Likewise, a contravariant \mathcal{C} -valued invariant \mathcal{F} is called *representable* if it naturally isomorphic to one of the form $[-, X]$ for some space X .

Example 1.1.1.5. If S^n is the standard n -sphere, with a distinguished base-point, then homotopy sets of X are (co-)representable homotopy invariants: $\pi_n(X, x) = [S^n, X]_*$.

Remark 1.1.1.6. In order that a representable homotopy invariant $[-, X]$ take values in the category of, say, groups, the space X must have certain additional properties. Similar statements hold for co-representable homotopy invariant $[X, -]$.

Exercise 1.1.1.7. If X and Y are two pointed topological spaces, define $X \vee Y$ as the quotient of the disjoint union $X \coprod Y$ obtained by identifying the base-point in X with the base-point in Y (this space is called the *wedge sum*).

1. Show that $X \vee Y$, together with the inclusion maps $X \rightarrow X \vee Y$ and $Y \rightarrow X \vee Y$ is a coproduct in the category of pointed topological spaces, i.e., given any pointed topological space Z and pointed maps $X \rightarrow Z$ and $Y \rightarrow Z$, there is a unique map $X \vee Y \rightarrow Z$ making the resulting triangles of maps of spaces commute.
2. Show that there is a continuous map $X \vee Y \rightarrow X \times Y$ and that the two projection maps $X \times Y \rightarrow X$ and $X \times Y \rightarrow Y$ restrict to projection maps $X \vee Y \rightarrow X$ and $X \vee Y \rightarrow Y$.
3. Show that there is a map $\nabla : X \vee X \rightarrow X$ induced by the identity map $X \rightarrow X$ and the universal property of the coproduct; this map is called the fold map.

Exercise 1.1.1.8. Identify $S^1 = [0, 1]/0, 1$. Define $c : S^1 \rightarrow S^1 \vee S^1$ by sending t to $2t$ if $0 \leq t \leq \frac{1}{2}$ and $2t - 1$ if $\frac{1}{2} \leq t \leq 1$ (think of c as the map that collapses the equator in S^1 to a point). Define $j : S^1 \rightarrow S^1$ be the map on $[0, 1]$ that sends t to $1 - t$. We now show that these maps make S^1 into a co-h-space. More precisely, establish the following facts.

1. Show that the two composite maps $S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{pr} S^1$ are homotopic to the identity map.
2. There are two maps $S^1 \vee S^1 \rightarrow S^1 \vee S^1 \vee S^1$, i.e., $c \vee id$ and $id \vee c$. Show that the two composites $S^1 \rightarrow S^1 \vee S^1 \vee S^1$ obtained by composing $c \vee id$ and $id \vee c$ with c are homotopic.
3. Show that the two composites: $S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{id \vee j} S^1$ and $S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{j \vee id} S^1$ are homotopic.
4. Given a pair of pointed maps $f, g : S^1 \rightarrow X$, show that the composite

$$S^1 \xrightarrow{c} S^1 \vee S^1 \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

defines a binary operation on $[S^1, X]_*$. Conclude that $[S^1, X]_*$ has a (functorial in X) group structure.

Definition 1.1.1.9. If X and Y are pointed topological spaces, define the *smash product* $X \wedge Y$ to be the quotient $X \times Y / (X \vee Y)$.

Exercise 1.1.1.10. Generalizing the formulas above, show that $S^1 \wedge X$ has a co-h-space structure.

1.1.2 The singular simplicial set

Definition 1.1.2.1. The standard n -simplex is the subset of \mathbb{R}^{n+1} with coordinate functions x_0, \dots, x_n given by

$$\Delta^n := \{(x_0, \dots, x_n) \mid \sum_{i=0}^n x_i = 1, x_i \geq 0\}$$

Example 1.1.2.2. The standard 0-simplex is a point. The standard 1-simplex is a line segment; it is homeomorphic to I . The standard 2-simplex is a triangle. Note that setting $x_i = 0$ defines an inclusion $e_i : \Delta^{n-1} \rightarrow \Delta^n$; there are $n + 1$ such maps. On the other hand, projection onto $x_i = 0$ defines a continuous map $d_i : \Delta^n \rightarrow \Delta^{n-1}$ (again, there are $n + 1$ such maps). The composites of the various maps e_i and d_i satisfy relations that we will consider in detail later.

Example 1.1.2.3. The space Δ^n is a compact subset of \mathbb{R}^{n+1} , and we can form its boundary $\partial\Delta^n$ as a topological space (i.e., the complement of the interior). Observe that $\partial\Delta^n$ is homeomorphic to S^{n-1} (by convention S^0 is 2-points).

We define $\text{Sing}_n X := C(\Delta^n, X)$, i.e., the set of continuous maps $\Delta^n \rightarrow X$. These sets are linked by the various maps we discussed above. In particular, pre-composing with the map $d_i : \Delta^n \rightarrow \Delta^{n-1}$ determines a map $d_{n,i} : \text{Sing}_{n-1} X \rightarrow \text{Sing}_n X$ called a *face map*. Likewise, pre-composing with $s_i : \Delta^{n-1} \rightarrow \Delta^n$ determines a map $s_{n,i} : \text{Sing}_n X \rightarrow \text{Sing}_{n-1} X$ called a *degeneracy map*. These maps equip the collection $\text{Sing} X$ with the structure of a simplicial set, which is a convenient way to organize both homotopic and homological information.

Example 1.1.2.4. Taking $n = 0$, observe that there are two maps $\text{Sing}_1 X \rightrightarrows \text{Sing}_0 X$; these maps can be interpreted as follows. The elements of $\text{Sing}_0 X$ are precisely the points of X . If we think of an element of $\text{Sing}_1 X$ as a map $\Delta^1 \rightarrow X$, then we can view this as a map $I \rightarrow X$ by a homeomorphism as in Example 1.1.2.2. The two maps $\text{Sing}_1 X \rightarrow \text{Sing}_0 X$ can be thought of as sending a path $I \rightarrow X$ to its endpoints. We can say two points in $X = \text{Sing}_0 X$ are equivalent if there exists an element of $\text{Sing}_1 X$ whose end-points are those given. The set $\pi_0(X)$ of path-connected components is the quotient of $\text{Sing}_0 X$ by this equivalence relation. To say things more categorically, $\pi_0(X)$ is the co-equalizer of the two maps $\text{Sing}_1 X \rightrightarrows \text{Sing}_0 X$.

Note that the assignment $X \mapsto \text{Sing} X$ is functorial, i.e., a continuous map $f : X \rightarrow Y$ induces a function $\text{Sing}_n X \rightarrow \text{Sing}_n Y$ that commute with all the maps $d_{n,i}$ and $s_{n,i}$, i.e., continuous functions induce morphisms of simplicial sets. In particular, the maps $\text{Sing} X \times Y \rightarrow \text{Sing} X$ and $\text{Sing} X \times Y$ induce a map $\text{Sing} X \times Y \rightarrow \text{Sing} X \times \text{Sing} Y$. On the other hand, given maps $\sigma : \Delta^n \rightarrow X$ and $\sigma' : \Delta^n \rightarrow Y$, there is an induced map $\sigma \times \sigma' : \Delta^n \times \Delta^n \rightarrow X \times Y$. Precomposing with the diagonal map $\Delta^n \rightarrow \Delta^n \times \Delta^n$, we thus obtain a function $\text{Sing}_n X \times \text{Sing}_n Y \rightarrow \text{Sing}_n X \times Y$. The products of the maps $d_{n,i}$ and $s_{n,i}$ on each factor induce maps $\text{Sing}_n X \times \text{Sing}_n Y \rightarrow \text{Sing}_{n-1} X \times \text{Sing}_{n-1} Y$ and $\text{Sing}_{n-1} X \times \text{Sing}_{n-1} Y \rightarrow \text{Sing}_n X \times \text{Sing}_n Y$ that make the product $\text{Sing} X \times \text{Sing} Y$ into a simplicial set itself. With respect to these structures, the map $\text{Sing} X \times Y \rightarrow \text{Sing} X \times \text{Sing} Y$ just constructed is an isomorphism of simplicial sets (i.e., the bijections at each level n commute with the face and degeneracy maps).

Definition 1.1.2.5. If X is a topological space, define a chain complex $C_*(X)$ with $C_n(X) = \mathbb{Z} \text{Sing}_n X$ and a differential $C_n(X) \rightarrow C_{n-1}(X)$ given by $\sum_{i=0}^n (-1)^i s_{n,i}$, where, by abuse of terminology, $s_{n,i} : \mathbb{Z} \text{Sing}_n X \rightarrow \mathbb{Z} \text{Sing}_{n-1} X$ is the map induced by $s_{n,i}$ as above. The singular homology $H_i(X, \mathbb{Z})$ is the homology $H_i(C_*(X))$.

Remark 1.1.2.6. With this definition, observe that singular homology is evidently functorial for continuous maps. A standard result about simplicial homotopy groups is that they are homotopy invariant, i.e., that the maps $p_{X*} : H_i(X \times I) \rightarrow H_i(X)$ induced by functoriality is a bijection for every i . This map is evidently surjective since it is split by the inclusion $X \rightarrow X \times I$ given by inclusion at any point, so it suffices to show that it is injective. The usual way to do this is to study “prism operators” to show that, more generally, given two maps $f, g : X \rightarrow Y$ that are homotopic, the induced maps $C_*(X) \rightarrow C_*(Y)$ are chain homotopic. Later, we will give a slightly different (but equivalent) approach to proving homotopy invariance that focuses more on the singular complex $\text{Sing} X$ itself.

1.1.3 Vector bundles

Suppose X is a topological space. A (finite rank, continuous) vector bundle on X is, loosely speaking, a continuous family of (finite dimensional) vector spaces parameterized by the points of X .

The key feature of vector bundles is that they are *locally trivial*, i.e., for every $x \in X$, there is an open neighborhood U_x together with a homeomorphism of the restriction of the family to U_x with a product of U_x and a (finite dimensional) Euclidean space. For spaces that are not “big” an evident formalization of this notion is good. Following Vaserstein [Vas86], we make the following definition.

Definition 1.1.3.1. If X is a topological space, then a continuous real (resp. complex) vector bundle E on X is a triple (E, π, X) consisting of a surjective continuous map $\pi : E \rightarrow X$ such that (i) $\pi^{-1}(x)$ is a finite dimensional real (resp. complex) vector space, and (ii) there exist a *finite* family of functions $\{f_i\}_{i \in I}$ forming a partition of 1 such that if U_{f_i} is the locus of points where $f_i \neq 0$, then $\pi^{-1}(U_{f_i})$ is homeomorphic to $U_{f_i} \times \pi^{-1}(x)$ for some $x \in U_{f_i}$. A morphism of real (resp. complex) vector bundles is a continuous map $f : E \rightarrow E'$ covering the identity on X that is linear on each fiber. We write $\mathcal{V}^{top}(X)$ for the set of isomorphism classes of vector bundles on X .

Remark 1.1.3.2. In this generality, one can make a definition of k -vector bundle for k any topological field.

The classical Steenrod representability theorem shows that the functor isomorphism classes of topological vector bundles is both homotopy invariant and representable. However, Steenrod’s result uses a slightly different definition of vector bundle for which homotopy invariance only holds on a subcategory of all topological spaces (one version takes this subcategory to be spaces having the homotopy type of a CW complex). We use a slightly different version of this theorem that holds for arbitrary topological spaces.

Proposition 1.1.3.3 (Vaserstein). *The assignment $X \mapsto \mathcal{V}^{top}(X)$ is a (contravariant) homotopy invariant.*

Grassmannian manifolds

If V is a real or complex vector space of fixed dimension N , then we may consider the set $Gr_n(V)$ of n -dimensional subspaces of V . Equivalently, we may consider the set of $N - n$ -dimensional quotients of this vector space. The set $Gr_n(V)$ admits a natural structure of a smooth manifold. Indeed, we can cover it by open sets that are isomorphic to Euclidean space itself. We now recall several related constructions of this manifold structure as these constructions will reappear later in closely related form.

1.1.3.4 (Subspaces and graphs). Suppose $V \cong V_1 \oplus V_2$ is a direct sum decomposition of V . Any element $v \in V_1 \oplus V_2$ can be written uniquely as $v_1 + v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$. Given a linear map $L : V_1 \rightarrow V_2$, we may associate with L its graph $\Gamma_L \subset V_1 \oplus V_2$, which is the subspace consisting of vectors of the form (x, Lx) . Observe that projection that the projection $\Gamma_L \subset V_1 \oplus V_2 \rightarrow V_1$ is an isomorphism. This construction defines a bijection between $\dim V_1$ -dimensional subspaces of V complementary to V_2 and elements of $\text{Hom}(V_1, V_2)$.

Now, returning to our original situation, we can provide charts for $Gr_n V$. Every n -dimensional subspace is complementary to some $N - n$ -dimensional subspace U . If we fix a basis v_1, \dots, v_N of V , then considering those $(N - n)$ -dimensional subspaces given by $(N - n)$ -basis elements, i.e., subsets of $\{1, \dots, N\}$ consisting of $N - n$ elements, we can exhaust all possibilities. Fixing such

a subspace, those n -dimensional subspaces complementary to it form a vector space of dimension $n(N - n)$ by means of the graph construction. Thus, we obtain a cover of $Gr_n(V)$ by $\frac{N!}{(N-n)!n!}$ subsets each isomorphic to a vector space of dimension $n(N - n)$.

We now describe how to glue these open sets by describing the transition maps. Suppose we have two different decompositions $V \cong V_1 \oplus V_2$ and $V \cong V'_1 \oplus V'_2$ where $\dim V_1 = \dim V'_1$ (and thus $\dim V_2 = \dim V'_2$). Suppose we are given a linear map $L : V_1 \rightarrow V_2$ and consider the subspace of V given by Γ_L . Let us assume that Γ_L is also complementary to V'_1 , i.e., that the projection maps $\Gamma_L \rightarrow V_1$ and $\Gamma_L \rightarrow V'_1$ are both isomorphisms, i.e., the composite map $V_1 \rightarrow \Gamma_L \rightarrow V'_1$ is an isomorphism. Now, take $x \in V_1$ and consider the vector $x + Lx \in V_1 \oplus V_2$. Writing this in terms of the decomposition $V'_1 \oplus V'_2$, we see that $x + Lx = x' + L'x'$ for some unique $L' \in \text{Hom}(V'_1, V'_2)$. Therefore, $L'x' = x + Lx - x'$. Now, $x + Lx$ is the map $V_1 \rightarrow \Gamma_L \dots$

Remark 1.1.3.5. The functor $X \mapsto \mathcal{V}^{top}(X)$ is a *representable* homotopy invariant. Indeed, the representing space is the Grassmannian Gr_n parameterizing n -dimensional subspaces of an infinite dimensional real vector space.

Example 1.1.3.6. If X is any compact smooth or topological manifold, then we can always find partitions of unity as above. Therefore, the above definition of vector bundle agrees with the usual notion for such spaces. Nevertheless, there are certainly spaces that are not of this form for which the “usual” notion of vector bundle agrees with the one described above.

1.1.4 Topological spaces

Working with *all* topological spaces has pros and cons. The main simplifying feature of working with topological spaces is that definitions were short. However, the tradeoff is that theorem statements often require long lists of hypotheses on the topological spaces under consideration. For example, the usual “geometric” definition of singular homology makes sense for all topological spaces. However, singular cohomology is only a representable homotopy invariant on suitable subcategory of Top . In fact, singular cohomology can differ from the corresponding representable homotopy invariant for sufficiently pathological spaces.

More briefly, there are a number of choices regarding which category of spaces in which we could work. By and large, these choices are dictated by the problems under consideration: we would like to work in a setting where invariants are “computable” in a suitable sense. To ensure comutability, we will frequently like to make constructions on our topological spaces and be insured that the resulting constructions are suitably well-behaved.

Steenrod summarized the situation as follows [Ste67]: “For many years, algebraic topologists have been laboring under the handicap of not knowing in which category of spaces they should work. Our need is to be able to make a variety of constructions and to know that the results have good properties without the tedious spelling out at each step of lengthy hypotheses...” He then goes on to spell out the properties he expects a “convenient category” of topological spaces to have: it must be large enough to consider all spaces of “interest”, it must be closed under various standard operations including formation of subspaces, product spaces, function spaces, unions of expanding sequences of spaces and compositions of these operations and it must be small enough that certain reasonable propositions about the standard operations are true.

All of these properties can be formulated more categorically. Existence of products and function spaces satisfying natural properties can be phrased as existence of a suitable closed symmetric monoidal structure on the category of spaces and the other properties can be phrased in terms of existence of limits and colimits. The colimits and limits in \mathbf{Top} are inherited from colimits and limits in the category of sets; it is for essentially this reason that the following result holds.

Lemma 1.1.4.1. *The category of topological spaces has all small limits and colimits.*

Proof. Indeed, by the existence theorem for limits and colimits, it suffices to show that \mathbf{Top} has products and equalizers and coproducts and coequalizers. The product in \mathbf{Top} is the usual set-theoretic product equipped with the product topology. The equalizer of two morphisms $f, g : A \rightarrow B$ is the set-theoretic equalizer of these two maps given the subspace topology. Likewise, the coproduct is given by the disjoint union, and the co-equalizer is the usual set-theoretic co-equalizer equipped with the quotient topology. \square

The following examples give ways in which the category \mathbf{Top} is not “reasonable.”

Example 1.1.4.2. The function space $(-)^X$ is not right adjoint to $\times X$ in \mathbf{Top} .

Example 1.1.4.3. For example, given a pair of continuous maps $f_1 : A \rightarrow B_1$ and $f_2 : A \rightarrow B_2$, we can form the space $B_1 \cup_A B_2$; this is the quotient of the disjoint union $B_1 \coprod B_2$ where the points $f_1(a)$ is identified with $f_2(a)$. One special case of this construction is the usual gluing construction of topological spaces: if U and V are topological spaces, and $U \cap V$ is an open subset of U and V , then we can form the coproduct $X := U \cup_{U \cap V} V$, which is the topological space obtained by gluing U and V along $U \cap V$. The way we have defined the glued space X , X need not be a Hausdorff space in general (e.g., take two copies of the real line and glue via the identity map along $\mathbb{R} \setminus 0$). Indeed, to guarantee that the glued space is Hausdorff, it is necessary and sufficient to assume that the map $U \cap V \rightarrow U \times V$ is a closed map. Thus from this point of view it is important that \mathbf{Top} is really the category of *all* topological spaces, not just the Hausdorff spaces.

Steenrod used the following definition for his “convenient category” of smooth spaces; it includes all locally countable spaces and all first countable spaces (i.e., metrizable topological spaces).

Definition 1.1.4.4. The category \mathbf{CGHaus} has objects consisting of Hausdorff topological spaces X that are *compactly generated*, i.e., any subset that intersects every compact set in a closed set is itself closed.

The complication that arises in considering \mathbf{CGHaus} is that limits and colimits are not computed using their corresponding set-theoretic counterparts. Nevertheless, it is Cartesian closed and both complete and cocomplete. In any case, we make no attempt to give a complete discussion of the technical aspects. The point is simply to show that, even classically, it is necessary to make choices. We will be confronted with analogous problems in the algebro-geometric situation under consideration.

Chapter 2

Affine varieties, \mathbb{A}^1 -invariance and naive \mathbb{A}^1 -homotopies

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A basic goal of \mathbb{A}^1 -algebraic topology is to study algebraic varieties over a base k (sometimes, but not always, a field) using associated algebraic objects that are homotopy invariant in a suitably algebraic sense. In this section, we introduce the basic properties of algebraic varieties in suitable generality that it can be used when we want to “do abstract homotopy theory.” Along the way, we introduce algebro-geometric analogs of homotopy invariants and produce the first elementary examples of such invariants.

2.1 Lecture 2: Affine varieties

We begin by analyzing a special class of varieties: affine varieties. Loosely speaking, these are the varieties defined by finitely many polynomial equations in a polynomial ring of some number of variables. When the base k is an algebraically closed field, one can use intuition from usual calculus/analytic geometry to study such objects. We will augment this intuition by studying what happens when the base k is not an algebraically closed field. More strongly: every commutative unital ring is an algebra over the ring of integers \mathbb{Z} , and it will sometimes be convenient for us to take the base $k = \mathbb{Z}$ sometimes.

2.1.1 Affine varieties

We begin by studying affine varieties over a base k . Intuitively speaking affine varieties are very familiar objects: they are simultaneous vanishing loci of a finite collection of polynomials in finitely many variables. While this should always serve as important inspiration, this definition is only correct when one works over an algebraically closed base field. The basic premise of affine algebraic geometry is that an affine variety is equivalent to its ring of functions. We begin with a definition of affine schemes in general that takes this point of view seriously. You can think of our definition as adding several layers of complexity to the intuitive idea of affine variety above:

- the topological space underlying an affine variety can have points that are not closed;
- the ring of coordinate functions can have nilpotent elements; and
- the base k may not be a field.

The Zariski topology

If R is any commutative ring, we can associate with R a topological space called its spectrum as follows.

Definition 2.1.1.1. Suppose R is a commutative unital ring,

1. $\text{Spec } R :=$ the set of prime ideals in R ;
2. for a subset T of R (not necessarily an ideal!) $V_T :=$ prime ideals containing T ;
3. given an element $f \in R$, $D(f) :=$ prime ideals not containing f .

Exercise 2.1.1.2. Suppose R is a commutative unital ring. Show that

1. Every non-zero ring has a maximal ideal.
2. The set $\text{Spec } R$ is empty if and only if R is the zero ring.

Exercise 2.1.1.3. If I and J are ideals in a commutative unital ring R , then show that

1. if T is a subset of R , and (T) is the ideal generated by T , then $V_T = V_{(T)}$;
2. V_I is empty if and only if I is the unit ideal;
3. $V_I \cup V_J = V_{I \cap J}$;
4. if I is an ideal and \sqrt{I} is its radical, then $V_I = V_{\sqrt{I}}$;
5. for any set of ideals $\{I_\alpha\}_{\alpha \in A}$, $\bigcap_{\alpha \in A} V(I_\alpha) = V(\bigcup_{\alpha \in A} I_\alpha)$;
6. if $f \in R$, then $D(f) = \bigcap V_f = \text{Spec } R$;
7. if $f, g \in R$, then $D(fg) = D(f) \cap D(g)$;
8. if $\{f_i\}_{i \in I}$ is a set of elements in R , then $\bigcup_{i \in I} D(f_i)$ is the complement in $\text{Spec } R$ of $V_{\{f_i\}_{i \in I}}$;
9. if $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$;
10. if $f \in R$ and $D(f) = \text{Spec } R$, then f is a unit.

Remark 2.1.1.4. Given a ring R and an element $f \in R$, the sets $D(f)$ are called the basic (or principal) open sets of R .

Exercise 2.1.1.5. If R is a commutative unital ring, defining closed sets to be sets of the form V_T equips $\text{Spec } R$ with the structure of a topological space. The sets $D(f)$ for a basis for this topology.

If $\varphi : R \rightarrow S$ is any ring homomorphism, then for any prime ideal $\mathfrak{p} \subset S$, $\varphi^{-1}(\mathfrak{p})$ is prime, so there is an induced function $\text{Spec } S \rightarrow \text{Spec } R$.

Exercise 2.1.1.6. If $\varphi : R \rightarrow S$ is a ring homomorphism, then the induced function $\text{Spec } S \rightarrow \text{Spec } R$ is continuous. Moreover, Spec is a contravariant functor from the category of commutative unital rings to Top .

Remark 2.1.1.7. We think of R as the ring of “regular functions” on the topological space $\text{Spec } R$.

Motivating examples of spectra

Example 2.1.1.8. If k is a field, then $\text{Spec } k$ is, as a topological space, a single point with the discrete topology.

Example 2.1.1.9. Suppose R is a domain that is not a field. In this case (0) is a prime ideal and therefore is a point of $\text{Spec } R$. On the other hand, since (0) is contained in every ideal, it follows that this point is not closed and, in fact, contains every point of $\text{Spec } R$ in its closure. The point (0) is the generic point of $\text{Spec } R$.

Example 2.1.1.10. Take $R = k[\epsilon]/\epsilon^2$. In this case, R has ideals (ϵ) and (0) . The ideal (ϵ) is prime and determines a closed point of $\text{Spec } R$. The point corresponding to (0) is contained in (ϵ) . One evocative image for $\text{Spec } R$ in this case is a closed point together with “nilpotent fuzz”.

Exercise 2.1.1.11. Draw a picture of $\text{Spec } k[x]$. In particular, observe that $\text{Spec } k[x]$ is not a Hausdorff topological space.

Example 2.1.1.12. More generally, a polynomial ring in n -variables $k[x_1, \dots, x_n]$ is a reduced, integral k -algebra and $\text{Spec } k[x_1, \dots, x_n]$ is denoted \mathbb{A}_k^n (affine k -space).

Exercise 2.1.1.13. Let R be a commutative unital ring.

1. Suppose $S \subset R$ a multiplicative set. Show that the ring homomorphism $R \rightarrow R[S^{-1}]$ induces a homeomorphism

$$\text{Spec } R[S^{-1}] \longrightarrow \{\mathfrak{p} \in \text{Spec } R \mid S \cap \mathfrak{p} = \emptyset\},$$

where the topology on the right hand side is the subspace topology induced from the Zariski topology on $\text{Spec } R$.

2. If $f \in R$, then the map $R \mapsto R_f$ induces a homeomorphism $\text{Spec } R_f \rightarrow D(f) \subset \text{Spec } R$.

Exercise 2.1.1.14. If R is a commutative unital ring and $I \subset R$ is an ideal, then the map $R \rightarrow R/I$ induces a homeomorphism

$$\text{Spec } R/I \rightarrow V_I \subset \text{Spec } R.$$

Affine schemes

We now proceed to give the general definition of an affine k -scheme.

Definition 2.1.1.15. Fix a base k (e.g., \mathbb{Z} or a field). The category of (finite type) *affine k -schemes* is the opposite of the category of (finitely generated) commutative, unital k -algebras. If k is a field, a finite type k -algebra will be called an *affine k -algebra*. We write Aff_k for the category of affine k -schemes and ring homomorphisms.

Remark 2.1.1.16. Typically we will abuse terminology and say $X = \text{Spec } R$ is an affine scheme.

In this definition, we have made no assumptions about zero-divisors or nilpotent elements in R .

Definition 2.1.1.17. If R is a commutative unital ring, then say that

1. R is *reduced* if R has no nilpotent elements;
2. R is *integral* if R is an integral domain.

Example 2.1.1.18. If R is a commutative unital ring, then R has a nilradical $N(R)$, which is equal to the intersection of all prime ideals in R . In that case, there is always a map $R \rightarrow R/N(R)$. The induced map $\text{Spec } R/N(R) \rightarrow \text{Spec } R$ identifies $\text{Spec } R/N(R)$ as a closed subset of $\text{Spec } R$. You can show that, as topological spaces, the map $\text{Spec } R/N(R) \rightarrow \text{Spec } R$ is a homeomorphism.

Definition 2.1.1.19. A topological space X is *reducible* if it can be written as the union of two non-empty proper closed subsets (and *irreducible* if it is not reducible).

Proposition 2.1.1.20. Suppose R is a commutative unital ring.

1. For a prime $\mathfrak{p} \subset R$, the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$.
2. The irreducible closed subsets of $\text{Spec } R$ are precisely those of the form $V(\mathfrak{p})$ for \mathfrak{p} a prime ideal.
3. Under the correspondence described in Point (2), the irreducible components of $\text{Spec } R$ correspond precisely with the minimal prime ideals.

Proof. Exercise. □

Example 2.1.1.21. If R is an integral domain, then $\text{Spec } R$ is irreducible.

Exercise 2.1.1.22. Show that $\text{Spec } R$ is irreducible if and only if $\sqrt{(0)}$ is a prime ideal.

Definition 2.1.1.23. If k is a field, by an *affine k -algebra* we will mean a finitely generated reduced k -algebra. The category of affine k -varieties is the opposite of the category of reduced, affine k -algebras; we write $\text{Var}_k^{\text{aff}}$ for the category of affine k -varieties.

Remark 2.1.1.24. According to our definition, affine varieties can be *reducible*.

Dimension

Definition 2.1.1.25. If R is a commutative ring, recall that a chain of prime ideals of length n in R is a sequence $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$, where each inclusion is proper. The *Krull dimension* of R is the supremum of the lengths of chains of prime ideals. We will say that an affine scheme $X = \text{Spec } R$ has dimension d if R has Krull dimension d .

Remark 2.1.1.26. For an arbitrary ring, this number need not be finite. The Krull dimension of $\text{Spec } R$ coincides with the dimension of $\text{Spec } R$ as a topological space (the topological definition involves the lengths of chains of irreducible subspaces).

Example 2.1.1.27 (Hyperbolic quadrics). Fix a base field k . Consider the subvariety of \mathbb{A}^{2n} , with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ defined by the equation $\sum_i x_i y_i = 1$. The expression $\sum_i x_i y_i$ is called the hyperbolic quadratic form and $Q_{2n-1} := \text{Spec } k[x_1, \dots, x_n, y_1, \dots, y_n] / (\sum_i x_i y_i - 1)$; the subscript labels the Krull dimension $2n - 1$ of the coordinate ring. Likewise, in \mathbb{A}^{2n+1} (with additional coordinate z), we set $Q_{2n} := \text{Spec } k[x_1, \dots, x_n, y_1, \dots, y_n, z] / (\sum_{i=1}^n x_i y_i - z(z+1))$. One obtains an isomorphic variety by replacing $z(1+z)$ by $z(1-z)$. Once again, the subscript $2n$ is the Krull dimension of coordinate ring.

Exercise 2.1.1.28. Show that if 2 is a unit in k , then the ring $k[x_1, \dots, x_n, y_1, \dots, y_n, z]/(\sum_{i=1}^n x_i y_i - z^2 - 1)$ is isomorphic to $k[x_1, \dots, x_n, y_1, \dots, y_n, z]/(\sum_{i=1}^n x_i y_i - z(z+1))$.

Exercise 2.1.1.29. Show that if -1 is a square in k and 2 is invertible in k , then Q_{2n-1} is isomorphic to the “usual” sphere defined by $\sum_{i=1}^{2n-1} w_i^2 - 1$ and Q_{2n} is isomorphic to the variety defined by the equation $\sum_{i=1}^{2n} w_i^2 - 1$.

Abstract vs. embedded varieties

To connect the above definitions more closely with geometric intuition, fix an affine k -algebra A . Just as in topology, there are an “abstract” and “embedded” point of view on affine k -varieties. By assumption A is finitely generated, so we can choose a surjection $k[x_1, \dots, x_n] \rightarrow A$. Such a surjection corresponds to a map $\text{Spec } A \rightarrow \mathbb{A}_k^n$; this map identifies $\text{Spec } A$ as a closed subset of \mathbb{A}_k^n via the conclusion of Exercise 2.1.1.14. Now, since $k[x_1, \dots, x_n]$ is a Noetherian k -algebra, any ideal $I \subset k[x_1, \dots, x_n]$ is finitely generated. Thus the kernel of $k[x_1, \dots, x_n] \rightarrow A$ is a finitely generated ideal $I \subset k[x_1, \dots, x_n]$. By picking generators f_1, \dots, f_r of I , we see that $\text{Spec } A$ can be identified as the closed subset of \mathbb{A}^n defined by the equations f_1, \dots, f_r . Thus, we conclude that every affine k -variety is a closed subset of affine space.

Remark 2.1.1.30. Fix a field k . A number of natural questions arise: what is the minimal dimension affine space into which a given affine k -variety embeds? In other words, for a given affine k -algebra A , what is the minimal n for which there exists a surjection $k[x_1, \dots, x_n] \rightarrow A$. Given two embeddings of an affine k -variety in an affine space, is there an isomorphism of affine space with itself that maps one embedding into the other? In other words, given two surjections $\varphi_1 : k[x_1, \dots, x_n] \rightarrow A$ and $\varphi_2 : k[x_1, \dots, x_n] \rightarrow A$, does there exist an isomorphism $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ such that $\varphi_1 = \varphi_2 \circ \varphi$?

There are analogs of the (weak) Whitney embedding theorem in the algebro-geometric setting [Sri91].

General topology of spectra

It is possible to characterize those topological spaces that are homeomorphic to prime ideal spectra of rings, but to do so requires a bit of general topology. We mention this here for the sake of curiosity, but also to explain how far the topological spaces that are spectra of rings are from the “standard” topological spaces one studies in algebraic topology (e.g., Hausdorff).

Definition 2.1.1.31. A topological space X is called:

1. T_0 if given any two points $x, x' \in X$, there exists an open neighborhood U of x not containing x' ;
2. *quasi-compact* if every open cover of X admits a finite open subcover;
3. *quasi-separated* if the intersection of two quasi-compact subsets is again quasi-compact;

Exercise 2.1.1.32. If R is a commutative unital ring, characterize the quasi-compact open subsets of $\text{Spec } R$ as finite unions of basic open sets. Conclude that $\text{Spec } R$ is both quasi-compact and quasi-separated. Show that $\text{Spec } R$ is T_0 .

An irreducible component of a topological space X is a maximal irreducible subset of X . A point $x \in X$ is a generic point if the closure $\bar{x} = X$. We saw above that integral affine k -schemes have unique generic points (corresponding to the zero ideal (0)) and are therefore irreducible. In fact, Hochster characterized topological spaces that can appear as $\text{Spec } R$ for some ring R ; for more details, we refer the reader to [Sta15, Tag 08YF].

Theorem 2.1.1.33 ([Hoc69, p. 43]). *A topological space X is $\text{Spec } R$ for a commutative ring R if and only if X is quasi-compact, T_0 , the quasi-compact open subsets of X form a basis for the open subsets of X , are closed under finite intersections, and every non-empty irreducible component of X has a unique generic point.*

2.1.2 The functor of points

We would like to think of the variety $\text{Spec } A = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ as the simultaneous “vanishing locus” of f_1, \dots, f_r , but we have to take care in doing this. Indeed, if we look at the ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$, then the “vanishing locus” of $x^2 + y^2 - 1$ over \mathbb{R} is simply a circle. However, there are other maximal ideals besides those corresponding to points on the graph. Indeed, there are maximal ideals corresponding to complex solutions of the equations.

To explain this more clearly, suppose we are given another k -algebra T (for test). A homomorphism $A \rightarrow T$ corresponds, using the description above, to specifying elements x_1, \dots, x_n in T such that $f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n) = 0$ in T . In other words, a map $\text{Spec } T \rightarrow \text{Spec } A$ corresponds to a “solution of the equations defining A with coefficients in T .” The variety $\text{Spec } A$ is not “determined” by its vanishing locus over k , but it *is* determined by looking at solutions in all possible ring extensions. Here is a precise statement.

Lemma 2.1.2.1. *The functor $A \mapsto \text{Hom}_{\text{Aff}_k}(A, -)$ from the category of affine k -algebras to the category of set-valued functors on the category of affine k -algebras is fully-faithful and we can identify Aff_k as the full-subcategory consisting of (co-)representable functors.*

Proof. This is a special case of the Yoneda lemma. □

Example 2.1.2.2. Suppose given a morphism $\varphi : A \rightarrow B$ of k -algebras and suppose we fix presentations $A = k[x_1, \dots, x_m]/(f_1, \dots, f_r)$ and $B = k[y_1, \dots, y_n]/(g_1, \dots, g_s)$. We claim that a morphism as above is essentially the restriction of a polynomial map. Indeed, the composite morphism $k[x_1, \dots, x_m] \rightarrow B$ corresponds to specifying polynomials $\varphi(x_1), \dots, \varphi(x_m)$ in B satisfying the equations f_1, \dots, f_r . Moreover, because there is a surjection $k[y_1, \dots, y_n] \rightarrow B$, these elements can all be lifted to $k[y_1, \dots, y_n]$. A choice of such lifts then determines a homomorphism $k[x_1, \dots, x_m] \rightarrow k[y_1, \dots, y_n]$, which is precisely a morphism between affine spaces.

Example 2.1.2.3. It is even useful to consider “solutions” in non-reduced rings. E.g., suppose $T = k[\epsilon]/\epsilon^2$. Take $A = k[x, y]/(xy - 1)$. Suppose we would like to construct a homomorphism $k[x, y]/(xy - 1) \rightarrow k[\epsilon]/\epsilon^2$. First, we need to specify two elements x and y of $k[\epsilon]/\epsilon^2$; any element can be written as $a + b\epsilon$. So suppose we have two elements $x = a + b\epsilon$ and $y = a' + b'\epsilon$. Now, the equation $xy - 1$ imposes the relation $(a + b\epsilon)(a' + b'\epsilon) - 1 = 0$ in $k[\epsilon]/\epsilon^2$. In other words, $(aa' + (ab' + ba')\epsilon) - 1 = 0$. This means $aa' = 1$ and $ab' + ba' = 0$ or equivalently, $ab' = -ba'$. The first condition, corresponds simply to a solution of $xy = 1$ in k , i.e., a k -point on the graph.

The second condition can be interpreted as picking out the tangent space at (a, a') , i.e., we can think of a $k[\epsilon]/\epsilon^2$ -valued point as a k -point together with a tangent vector at that point.

Given two k -algebras A and B , we can form their tensor product $A \otimes_k B$. The k -algebras A and B are k -modules, and as a k -module, the tensor product is the usual tensor product. We give $A \otimes_k B$ a k -algebra structure by defining $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ then extending to $A \otimes_k B$ by linearity. Note that $k[x_1, \dots, x_m] \otimes_k k[y_1, \dots, y_n] \cong k[x_1, \dots, x_m, y_1, \dots, y_n]$. More generally, given presentations $A = k[x_1, \dots, x_m]/(f_1, \dots, f_r)$ and $B = k[y_1, \dots, y_n]/(g_1, \dots, g_s)$, the tensor product $A \otimes_k B$ can be identified with $k[x_1, \dots, x_m, y_1, \dots, y_n]/(f_1, \dots, f_r, g_1, \dots, g_s)$.

Remark 2.1.2.4. Note that $A \otimes_k B$ is a coproduct in the category of rings. More precisely, there are maps $A \rightarrow A \otimes_k B$ and $B \rightarrow A \otimes_k B$ such that if C is any k -algebra equipped with homomorphisms $A \rightarrow C$ and $B \rightarrow C$, then there exists a unique map $A \otimes_k B \rightarrow C$ making the relevant diagrams commute. Since Spec is a contravariant functor, it follows that $\text{Spec } A \otimes_k B$ is a product in the category of affine k -schemes: i.e., it is the product of $\text{Spec } A$ and $\text{Spec } B$ in the category of k -schemes. Note that the topology on $\text{Spec } A \otimes_k B$ is *not* the product topology in general. This can be seen already with $A = k[x]$ and $B = k[y]$! Nevertheless, we will still write $\text{Spec } A \times_{\text{Spec } k} \text{Spec } B$ for the product variety. If it is clear from context, we will drop the subscript $\text{Spec } k$ in the product. Thus, the functor Spec does not preserve products.

Fibers of a map

If $\varphi : A \rightarrow B$ is a ring homomorphism, then φ corresponds to a morphism $f : \text{Spec } B \rightarrow \text{Spec } A$. Given a point of $\text{Spec } A$, we may therefore consider the fiber of f over that point. There are several things we could mean by this idea. Generally, a T -point of A corresponds to a ring homomorphism $A \rightarrow T$. In that case we can form the tensor product $B \otimes_A T$; this comes equipped with a morphism $B \rightarrow B \otimes_A T$. A useful case to consider is when T is reduction modulo a maximal ideal $\mathfrak{m} \subset A$. In that case, $\mathfrak{m}B$ is an ideal in B , which no longer needs to be maximal. The scheme-theoretic fiber of f over the closed point corresponding to \mathfrak{m} coincides with the ring $B/\mathfrak{m}B$. Observe that A/\mathfrak{m} is a field κ by assumption, and thus $B/\mathfrak{m}B$ is automatically a κ -algebra.

2.2 Lecture 3: Naive \mathbb{A}^1 -invariants

We now attempt to transpose some of the ideas about homotopies between maps of topological spaces to the category of affine k -schemes for a fixed field k . First, we need an analog of the unit interval and to do this, we isolate some of the formal properties of I . The properties we use are as follows: (i) there are two distinguished points $0, 1 \in I$, (ii) there is a multiplication map $I \times I \rightarrow I$ that makes I into a topological monoid.

2.2.1 \mathbb{A}^1 -invariants

We claim that \mathbb{A}^1 is an analog of I in topology. In this case, there are two maps $0, 1 : \text{Spec } k \rightarrow \text{Spec } k[x]$ corresponding to evaluation at 0 and 1. To write down an analog of the monoid operation, we need a ring map $k[x] \rightarrow k[y_1] \otimes_k k[y_2]$. At the level of coordinate functions, the product map sends $(y_1, y_2) \mapsto y_1 y_2$. We can then take the induced pull-back map on functions sends $f(x)$ to

$f(y_1 y_2)$. Note that 1 is an identity for the product map. Indeed, if we evaluate $f(y_1 y_2)$ either at $y_1 = 1$ or $y_2 = 1$, the resulting ring map is the identity map. On the other hand, evaluation at 0 takes any function f to its constant term. Thus, we can think of the multiplication map as providing a homotopy, parameterized by \mathbb{A}^1 between the identity map and the map $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ corresponding to the constant map to a point, followed by the map $\text{Spec } k \rightarrow \mathbb{A}^1$ given by inclusion of 0.

Definition 2.2.1.1. Suppose \mathcal{C} is some category of abstract algebraic structures (e.g., groups, rings, etc.). A \mathcal{C} -valued invariant on Var_k is a functor $\mathcal{F} : \text{Aff}_k \rightarrow \mathcal{C}$. A \mathcal{C} -valued invariant \mathcal{F} on Aff_k is called \mathbb{A}^1 -invariant if, for any $X \in \text{Aff}_k$, the pullback along the projection $p_X : X \times \mathbb{A}_k^1 \rightarrow X$ induces a \mathcal{C} -isomorphism $\mathcal{F}(X) \rightarrow \mathcal{F}(X \times \mathbb{A}^1)$.

If $k = \mathbb{R}$ or \mathbb{C} , then one way to produce invariants as above is to use invariants from topology. For concreteness, fix $k = \mathbb{R}$ and suppose $X \in \text{Aff}_{\mathbb{R}}$. In that case, the set $X(\mathbb{R}) := \text{Hom}(\text{Spec } \mathbb{R}, X)$ can be equipped with the structure of a topological space in the usual sense. Indeed, if $X = \text{Spec } A$, and we fix a presentation $A = k[x_1, \dots, x_m]/(f_1, \dots, f_r)$ for A then we realize $X(\mathbb{R})$ as a closed subset of $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$ and we can view it as a topological space with the induced topology.

If we fix a different presentation $A = k[x_1, \dots, x_n]/(g_1, \dots, g_s)$ then we get an *a priori* different topological space $X'(\mathbb{R}) \rightarrow \mathbb{A}^n(\mathbb{R})$. However, since the two coordinate rings are abstractly isomorphic, we can fix an isomorphism $k[x_1, \dots, x_m]/(f_1, \dots, f_r) \cong k[x_1, \dots, x_n]/(g_1, \dots, g_s)$. By Example 2.1.2.2, this isomorphism corresponds to polynomial maps that restrict to real solutions of the respective systems of equations that are mutually inverse. Since polynomial maps are continuous, we conclude that $X(\mathbb{R})$ and $X'(\mathbb{R})$ are actually homeomorphic. In a similar vein, if $f : X \rightarrow Y$ is a morphism of affine algebraic varieties, then we conclude that the induced maps $X(\mathbb{R}) \rightarrow Y(\mathbb{R})$ are continuous. Therefore, we conclude that the assignment $X \mapsto X(\mathbb{R})$ yields a functor $\text{Var}_{\mathbb{R}}^{\text{aff}} \rightarrow \text{Top}$.

Remark 2.2.1.2. The category of affine algebraic varieties over the real numbers is very rich. There is a famous theorem of Nash-Tognoli: given a compact differentiable manifold M , there exists an integral $X \in \text{Var}_{\mathbb{R}}^{\text{aff}}$ and a diffeomorphism M and $X(\mathbb{R})$ [Tog73] (such an X is called an *algebraic model* of M). In fact, the situation is even more interesting: such representations are very far from unique. Since X is integral, it has a well-defined fraction field $k(X)$: this field is a finitely generated extension of k . Say that two integral affine schemes X and X' are birationally equivalent if $k(X) \cong k(X')$ as fields. One can even show that there are infinitely many birationally inequivalent models of M . In dimension 1 this is a fun exercise: if $n > 0$, the equation $x^{2n} + y^{2n} = 1$ has real points diffeomorphic to S^1 for every n , but the function fields of each of these varieties differs as n varies. More generally, any dimension 1 manifold is a disjoint union of circles. For every $n > 0$, the variety given by the equation $y^{2n} = -(x^2 - 1)(x^2 - 2) \cdots (x^2 - m)$ has graph consisting of m disjoint circles and the resulting varieties can be shown to be birationally inequivalent for different values of n .

Exercise 2.2.1.3. Show that $X \mapsto X(\mathbb{C})$ determines a functor $\text{Aff}_{\mathbb{C}} \rightarrow \text{Top}$.

With this in mind, one way to produce \mathbb{A}^1 -invariants is simply to take a homotopy invariant on Top and compose with the “realization” functors just described.

Example 2.2.1.4. Suppose k is a field, and assume we have an embedding $\iota : k \hookrightarrow \mathbb{R}$ (or similarly with \mathbb{R} replaced by \mathbb{C}). For example, we could take $k = \mathbb{Q}$. The choice of ι defines a functor $\text{Aff}_k \rightarrow \text{Top}$ that we will call a *realization* functor. If \mathcal{F} is any \mathcal{C} -valued invariant of Top , one obtains a corresponding \mathcal{C} -valued invariant of Var_k . If \mathcal{F} is a \mathcal{C} -valued homotopy invariant, then since $\mathbb{A}^1(\mathbb{R})$ is a contractible topological space, it follows that the composite functor $\text{Var}_k \rightarrow \mathcal{C}$ is \mathbb{A}^1 -invariant. Note that, in contrast to the situation explained above, inequivalent embeddings ι can yield varieties that are not just topologically inequivalent, but in fact *homotopy inequivalent!* The first example of this goes back to J.P.-Serre.

Remark 2.2.1.5. Take $k = \mathbb{C}$ and X an affine \mathbb{C} -variety. One natural question to ask is: what can you say about the homotopy type of $X(\mathbb{C})$? For example, when does $X(\mathbb{C})$ have the homotopy type of a finite CW complex. A classical result of Andreotti-Frankel [AF59], generalized independently by Karchyauskas [Kar77] and Hamm [Ham95] shows that this is always the case. Moreover, they show that if X is a dimension d complex affine variety, then $X(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\leq d$.

While the above examples are restricted to work over subfields of the real or complex numbers, it is also possible to produce \mathbb{A}^1 -invariants that are purely algebraic. We now describe an example that arises in elementary algebra.

Homotopy invariance of units

Set $\mathbf{G}_m = \text{Spec } k[t, t^{-1}]$. Suppose $X = \text{Spec } A \in \text{Aff}_k$ and suppose we are given a map $X \rightarrow \mathbf{G}_m$. Such an element corresponds to a homomorphism $\varphi : k[t, t^{-1}] \rightarrow A$. Such a homomorphism corresponds to an element $\varphi(t) \in A$ such that $\varphi(t)^{-1} \in A$ as well. In other words, $\varphi(t)$ is a unit. Conversely, given a unit $u \in A$, define a homomorphism $k[t, t^{-1}] \rightarrow A$ by sending $t \mapsto u$ and extending by linearity. We write A^\times for the set of units in A . This description of units shows that the assignment $A \mapsto A^\times$ is actually a functor. As a consequence there is always a group homomorphism $A^\times \mapsto A[x]^\times$. On the other hand, the evaluation at 0 homomorphism provides a section of $A \rightarrow A[x]$, i.e., a homomorphism $A[x] \rightarrow A$ such that the composite $A \rightarrow A[x] \rightarrow A$ is the identity. It follows that the map $A^\times \rightarrow A[x]^\times$ is injective.

We can analyze surjectivity of this map. Indeed, if $f \in A[x]$ is a unit, then we can write $f = a_0 + a_1x + \cdots + a_nx^n$ and what we just said shows that a_0 must be a unit. In that case, we write $a_0^{-1}f = 1 + \alpha_1x + \cdots + \alpha_nx^n$ where $\alpha_i = a_0^{-1}a_i$. This takes the form $1 + z$ where $z = \alpha_1x + \cdots + \alpha_nx^n$. In that case, an inverse is given by $\frac{1}{1+z} = \sum_{j \geq 0} (-1)^j z^j$. In order for this element to lie in $A[x]$, we require that $z^n = 0$ for all n sufficiently large, i.e., z^n is nilpotent. The following result characterizes the units in $A[x]$.

Proposition 2.2.1.6. *An element $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$ is a unit if and only if $a_0 \in A^\times$ and a_i is nilpotent for $i > 0$. In particular, the functor $X \mapsto \mathbf{G}_m(X)$ is \mathbb{A}^1 -invariant on $\text{Var}_k^{\text{aff}}$ (in particular, there are no non-constant morphisms $\mathbb{A}^1 \rightarrow \mathbf{G}_m$).*

Exercise 2.2.1.7. *Prove Proposition 2.2.1.6.*

1. Show that if A is a ring, and x is a nilpotent element of A then $1 + x$ is a unit in A .
2. Show that if $\alpha_0, \dots, \alpha_n$ are nilpotent elements of A , then $\sum_{i=0}^n \alpha_i x^i$ is a nilpotent element of $A[x]$.

3. If f is a unit in $A[x]$ and $g = \sum_{i=1}^m b_i x^i$ is an inverse of f , prove by induction on r that $a_n^{r+1} b_{m-r} = 0$ and conclude that a_n is nilpotent.

Remark 2.2.1.8. It is important to note that, in the above, homotopy invariance did not hold for the functor \mathbf{G}_m on *all* affine k -schemes, only for affine k -varieties. Indeed, this is a phenomenon of which we must be aware: even for “natural” functors, homotopy invariance need not hold for all affine k -schemes.

Remark 2.2.1.9. The proposition above highlights one difference between the algebraic and the continuous category. Indeed, $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$, while $\mathbf{G}_m(\mathbb{C}) = \mathbb{C}^\times$. While there are no non-trivial algebraic maps $\mathbb{A}^1 \rightarrow \mathbf{G}_m$, by evaluation on \mathbb{C} , observe that there are *continuous* maps $\mathbb{C} \rightarrow \mathbb{C}^\times$, e.g., the exponential map.

Algebraic singular homology

We now produce a “purely algebraic” version of singular homology for an arbitrary affine scheme. We begin by recalling a construction of “simplices” in algebraic geometry by naively transplanting the definitions from topology.

Example 2.2.1.10. If k is a field, then define $\Delta_k^n = \text{Spec } k[x_0, \dots, x_n] / (\sum_{i=0}^n x_i - 1)$. If $n = 0$, then Δ_k^0 is isomorphic to $\text{Spec } k$. If $n = 1$, then Δ_k^1 is the line $x_0 + x_1 = 1$ in \mathbb{A}_k^2 . More generally, Δ_k^n is isomorphic to \mathbb{A}_k^n (though the isomorphism with a polynomial ring depends on a choice). As in the topological setting, there are face morphisms $\Delta_k^n \rightarrow \Delta_k^{n-1}$ and degeneracy morphisms $\Delta_k^{n-1} \rightarrow \Delta_k^n$. These morphisms are defined by projection away from x_i and the inclusion of $x_i = 0$.

Definition 2.2.1.11. If X is an affine scheme over a base k , then the *algebraic singular simplicial set* attached to X , denoted $\text{Sing}^{\mathbb{A}^1} X(k)$, is the simplicial set whose n -simplices are the k -morphisms $\text{Hom}(\Delta_k^n, X)$ and where the face and degeneracy maps are induced by the structures just defined.

Remark 2.2.1.12. The affine n -simplex described above seems to have originally been considered by D. Rector in the 1970s [Rec71, Remark 2.5].

For another purely algebraic \mathbb{A}^1 -invariant, we can appeal to constructions involving $\text{Sing}^{\mathbb{A}^1} X$ for X a smooth affine k -variety.

Exercise 2.2.1.13. Show that there is an isomorphism $\text{Sing}^{\mathbb{A}^1} X \times_k Y \cong \text{Sing}^{\mathbb{A}^1} X \times \text{Sing}^{\mathbb{A}^1} Y$.

Indeed, mimicking the definition of ordinary singular homology, we observed that we define a chain complex as follows.

Definition 2.2.1.14. The *algebraic singular chain complex* of an affine scheme X is the chain complex $C_*^{\text{alg}}(X, \mathbb{Z})$ with $C_n^{\text{alg}}(X, \mathbb{Z}) := \mathbb{Z}(\text{Sing}^{\mathbb{A}^1}_n X)$ and with differential $d_i := \sum_{j=0}^n (-1)^j s_{n,i}^*$. The *algebraic singular homology* of X is the homology of the chain complex $H_i^{\text{alg}}(X, \mathbb{Z}) := H_i(C_*^{\text{alg}}(X, \mathbb{Z}))$.

Lemma 2.2.1.15. The functor $X \mapsto H_i^{\text{alg}}(X, \mathbb{Z})$ is \mathbb{A}^1 -invariant.

Example 2.2.1.16. These groups are often not that interesting. For example, if $X = \mathbf{G}_m$ is considered over a base ring k , then $\text{Sing}^{\mathbb{A}^1}_n \mathbf{G}_m = \mathbf{G}_m(k)$ for every integer n . In particular, one computes directly that $H_0^{\text{alg}}(\mathbf{G}_m, \mathbb{Z}) = \mathbb{Z}(\mathbf{G}_m(k))$, while the higher groups $H_i^{\text{alg}}(\mathbf{G}_m, \mathbb{Z})$ are all trivial. A similar statement holds for any (non-empty) proper open subset of \mathbb{A}^1 . This kind of example that suggests the construction of “interesting” \mathbb{A}^1 -invariants will require real work.

Remark 2.2.1.17. Just as in the topological situation, the functor sending X to its ring of functions is not \mathbb{A}^1 -invariant. This functor also has a nice description. Indeed if $X = \text{Spec } A$, then an element $a \in A$ determines a homomorphism $k[t] \rightarrow A$ by sending t to a . Conversely, given a homomorphism $k[t] \rightarrow A$ the homomorphism is uniquely specified by the image of t , i.e., an element of A . Therefore, $\text{Hom}_{\text{Var}_k^{\text{aff}}}(-, \mathbb{A}^1)$ represents the functor “functions on X ”.

2.2.2 Naive \mathbb{A}^1 -homotopies

We now transport the definitions from classical homotopy theory to the algebro-geometric setting.

Definition 2.2.2.1. If $f, g : X \rightarrow Y$ are two morphisms of affine k -schemes, then a *naive \mathbb{A}^1 -homotopy* between f and g is a morphism $H : X \times \mathbb{A}^1 \rightarrow Y$ such that $H(x, 0) = f$ and $H(x, 1) = g$; in this case we will say that f and g are connected by a naive \mathbb{A}^1 -homotopy.

Lemma 2.2.2.2. *If $F : X \times \mathbb{A}^1 \rightarrow Y$ is a naive \mathbb{A}^1 -homotopy between morphisms f and f' , and if $G : Y \times \mathbb{A}^1 \rightarrow Z$ is a naive \mathbb{A}^1 -homotopy between g and g' , then $g \circ f$ and $g' \circ f'$ are connected by a naive \mathbb{A}^1 -homotopy.*

Proof. If X and Y are affine k -schemes, and $F : X \times \mathbb{A}^1 \rightarrow Y$ is a naive \mathbb{A}^1 -homotopy between f and f' and $G : Y \times \mathbb{A}^1 \rightarrow Z$ is a naive \mathbb{A}^1 -homotopy between g and g' , then we can define a naive \mathbb{A}^1 -homotopy between $g \circ f$ and $g' \circ f'$ by taking $H(x, t) := G(F(x, t), t)$. \square

Notice that f is always connected to f by a naive \mathbb{A}^1 -homotopy (namely the composite of the projection map $X \times \mathbb{A}^1 \rightarrow X$ and the map $f : X \rightarrow Y$). Likewise, if f and g are connected by a naive \mathbb{A}^1 -homotopy, then g and f are connected by a naive \mathbb{A}^1 -homotopy. Indeed, consider the map $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ given by $\phi(x) = 1 - x$: this map is an isomorphism that sends 0 to 1 and 1 to 0. Therefore given $H : X \times \mathbb{A}^1 \rightarrow Y$, we can pre-compose with $\text{id} \times \phi : X \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1$ to obtain a new map $H' : X \times \mathbb{A}^1 \rightarrow Y$ with $H'(x, 0) = g$ and $H'(x, 1) = f$.

In topology, the fact that homotopy equivalence is transitive stems from the fact that setting copies of the unit interval “end-to-end”, i.e., taking $I \amalg I$ where we identify 1 in the first factor with 0 in the second factor, is homeomorphic to I itself. An explicit homeomorphism is gotten by identifying the first copy of I with $[0, \frac{1}{2}]$ and then the second copy of I with $[\frac{1}{2}, 1]$. Unfortunately, the rigidity inherent in algebraic varieties manifests itself in the fact that the relation that two maps are naively \mathbb{A}^1 -homotopic is not an equivalence relation. Even though the relation f is connected to g by a naive \mathbb{A}^1 -homotopy is reflexive and symmetric, the following example shows that it need not be transitive in general.

Example 2.2.2.3. Take X to be the affine scheme $\text{Spec } k[x, y]/(xy)$. Geometrically, this scheme consists of two copies of the affine line glued at the origin; as usual, refer to the line $y = 0$ as the x -axis and the line $x = 0$ as the y -axis. Consider the points $(1, 0)$, $(0, 0)$ and $(0, 1)$. While

each consecutive pair of points are naively \mathbb{A}^1 -homotopic, there is no morphism $\mathbb{A}^1 \rightarrow X$ that connects $(1, 0)$ and $(0, 1)$. We claim any non-constant morphism $\mathbb{A}^1 \rightarrow X$ factors through one of the components. Indeed, given a non-constant morphism $k[x, y]/(xy) \rightarrow k[t]$, the relation $xy = 0$ shows that at most one of x and y is sent to a non-constant element of $k[t]$.

The essential point of this example is that X is a reducible algebraic variety and one can envision more complicated examples. One can also envision situations in which naive \mathbb{A}^1 -homotopy is well-behaved: for example, if we consider a target variety Y , and any morphism from X to Y is naively \mathbb{A}^1 -homotopic to a morphism $\mathbb{A}^1 \rightarrow Y$, then we can sequentially replace “chains” of maps from \mathbb{A}^1 to a single morphism from \mathbb{A}^1 . For example, we can identify X with a closed subscheme of \mathbb{A}^2 and then identify \mathbb{A}^1 with the closed subvariety of \mathbb{A}^2 given by the equation $x + y = 1$. In that case, \mathbb{A}^2 itself can be thought of as a deformation of X to \mathbb{A}^1 . Then, given a morphism $f : X \rightarrow Y$, we can extend f to a morphism $\mathbb{A}^2 \rightarrow Y$, then we will obtain a suitable condition.

As a consequence of the example, we have to consider the equivalence relation generated by “ f is connected to g by a naive \mathbb{A}^1 -homotopy” (his seemingly innocuous distinction between the algebro-geometric and the topological categories is the source of many of the complications that will arise in our setting). We make the following definition.

Definition 2.2.2.4. Suppose $f, g : X \rightarrow Y$ are two morphisms of affine k -schemes. We will say that f and g are *naively \mathbb{A}^1 -homotopic* if they are equivalent for the equivalence relation generated by naive \mathbb{A}^1 -homotopy. Likewise, two affine k -schemes X and Y are *naively \mathbb{A}^1 -weakly equivalent* if there exist morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that the two composites are naively \mathbb{A}^1 -homotopic to the respective identity maps. We write $[X, Y]_N$ for the set of naive \mathbb{A}^1 -homotopy classes of maps from X to Y .

Exercise 2.2.2.5. If \mathcal{F} is a \mathcal{C} -valued \mathbb{A}^1 -homotopy invariant, and if f and g are naively \mathbb{A}^1 -homotopic maps, then $\mathcal{F}(f) = \mathcal{F}(g)$.

Example 2.2.2.6. If X is an affine k -scheme, then X and $\mathbb{A}^n \times X$ are naively \mathbb{A}^1 -weakly equivalent for any X . The composite $X \rightarrow \mathbb{A}^n \times X \rightarrow X$ is equal to the identity. To see that the other composite is naively \mathbb{A}^1 -homotopic to the identity, we use the map $\mathbb{A}^1 \times \mathbb{A}^n \rightarrow \mathbb{A}^n$ given by $(t, x) \mapsto tx$.

2.2.3 The naive \mathbb{A}^1 -homotopy category

We can try to formally define a “universal” \mathbb{A}^1 -homotopy invariant by constructing a new category whose objects are objects in Var_k^{aff} and whose morphisms are naive \mathbb{A}^1 -homotopy classes of morphisms between k -varieties. The following definition makes sense because composites of naively \mathbb{A}^1 -homotopic maps are naively \mathbb{A}^1 -homotopic.

Definition 2.2.3.1 (Naive \mathbb{A}^1 -homotopy category). The naive \mathbb{A}^1 -homotopy category over a field k is the category $\mathcal{N}(k)$ whose objects are those of Var_k^{aff} and whose morphisms are the sets of naive \mathbb{A}^1 -homotopy classes of maps between affine sk -varieties.

Lemma 2.2.3.2. If $X, Y \in \text{Var}_k^{aff}$, then the projection map $[X, Y]_N \rightarrow [X \times \mathbb{A}^1, Y]_N$ is a bijection.

Proof. The map in question is evidently split (via any inclusion $X \hookrightarrow X \times \mathbb{A}^1$) and therefore injective. Thus, it suffices to demonstrate surjectivity. Suppose $f : X \times \mathbb{A}^1 \rightarrow Y$ is a morphism. We want to show that f is \mathbb{A}^1 -homotopic to $f(x, 0)$. To this end, consider the product map $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Note that $\mu(t, 0) = 0$, while $\mu(t, 1) = 1$. Then, define a naive \mathbb{A}^1 -homotopy between f and $f(x, 0)$ by considering the map $X \times \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow Y$ given by $f(x, \mu(t, s))$. \square

Example 2.2.3.3. The functor sending X to $\mathbf{G}_m(X)$ is representable on $\text{Var}_k^{\text{aff}}$ by $[-, \mathbf{G}_m]_N$.

2.3 Lecture 4A: More homotopy invariants

In this lecture, we will make some sample computations of naive \mathbb{A}^1 -homotopy classes of maps and then we will begin building some more interesting \mathbb{A}^1 -homotopy invariants.

2.3.1 Naive \mathbb{A}^1 -homotopy calculations

We now give some examples to show that naive \mathbb{A}^1 -homotopy classes of maps can sometimes be determined in practice; we begin with a few exercises.

Exercise 2.3.1.1. Suppose $Z \subset \mathbb{A}^1$ is a closed subset defined by a polynomial f . Let $U \subset \mathbb{A}^1$ be the complement of Z , with coordinate ring $k[U] = k[x, \frac{1}{f}]$.

1. Show that $[\text{Spec } k, U]_N = U(k)$.
2. More generally, show that if X is any smooth affine scheme, then $U(X) = [X, U]_N$.

Example 2.3.1.2. Consider the variety $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$; this can be identified as the spectrum of the ring $k[x, x^{-1}, y]$. Even though there is a copy of the affine line passing through every point, there are no morphisms $\mathbb{A}^2 \rightarrow \mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$.

Exercise 2.3.1.3. Suppose G is an affine algebraic k -group (i.e., a group object in the category $\text{Var}_k^{\text{aff}}$). Show that for any $X \in \text{Var}_k^{\text{aff}}$, the set $[X, G]_N$ inherits a group structure making the map $G(X) \rightarrow [X, G]_N$ a homomorphism. Moreover, this group structure is functorial in both X and G .

Pick coordinates x_{ij} on the n^2 -dimensional affine space M_n of $n \times n$ -matrices. With this choice, if $X \in M_n$ is an $n \times n$ -matrix, then $\det X$ is a polynomial of degree n in the variables x_{ij} . In particular, we can define $GL_n = \text{Spec } k[x_{ij}, \det X^{-1}]$ and $SL_n = \text{Spec } k[x_{ij}]/(\det X = 1)$. The explicit formulas for matrix multiplication and matrix inversion show that GL_n and SL_n are affine algebraic groups (the identity is given by the $n \times n$ -identity matrix). A T -point of GL_n , for some test k -algebra T , is precisely an invertible $n \times n$ -matrix with coefficients in T . Likewise, a T -point of SL_n is an invertible $n \times n$ -matrix with coefficients in T and whose determinant is equal to 1.

Proposition 2.3.1.4. If k is any field, then $[\text{Spec } k, SL_n]_N = Id_n$.

Proof. Any “elementary matrix” gives rise to a matrix naively \mathbb{A}^1 -homotopic to the identity. Indeed, let e_{ij} be a matrix unit (i.e., an $n \times n$ -matrix such that $(e_{ij})_{kl} = 1$ if $i = k$ and $j = l$ and 0 otherwise). If $i \neq j$, then consider the matrix $E_{ij}(\alpha) := Id_n + \alpha e_{ij}$; this matrix is called an elementary matrix (or an elementary shearing matrix). Observe that $E_{ij}(t\alpha)$ provides a naive homotopy between

$E_{ij}(\alpha)$ and Id_n . The statement follows from the following observation: any element of $SL_n(k)$ can be written as a product of elementary matrices.

To begin, recall that any element of $GL_n(k)$ can be written as a product of an elementary matrix as in the previous paragraph, matrices of the form $Id_n + (\alpha - 1)e_{ii}$ for $1 \leq i \leq n, \alpha \in k^*$, and permutation matrices. We claim that every permutation matrix is a product of elementary matrices and matrices of the form $Id_n + (\alpha - 1)e_{ii}$. Indeed, any element of the symmetric group can be written as a product of transpositions. For $GL_2(k)$, we simply perform row operations to transform

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ into such a matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, by fixing different embeddings $GL_2(k) \rightarrow GL_n(k)$, we conclude that similar formulas hold for arbitrary elements. As a consequence, we see that every matrix $X \in GL_n(k)$ can be written as a product $\prod_i E_i$ where each E_i is either elementary or of the form $Id_n + (\alpha - 1)e_{ii}$ for $1 \leq i \leq n, \alpha \in k^*$. We now study the possible commutators of elements. Since elementary matrices act as row operations, and matrices of the form $Id_n + (\alpha - 1)e_{ii}$ act by multiplying a row by α , we conclude that the commutator of an elementary matrix and one of the form $Id_n + (\alpha - 1)e_{ii}$ is the identity unless i is one of the rows being acted upon by the elementary matrix. In the remaining case, one immediately verifies the following identities

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\alpha}{\beta} \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \alpha\beta \\ 0 & 1 \end{pmatrix}.$$

By taking transposes, we obtain similar formulas for lower triangular matrices. Using these observations, we conclude that we can write any element of $GL_n(k)$ as a product of a diagonal matrix D and a product of elementary matrices.

Now, suppose $X \in SL_n(k)$ and write $X = DE_1 \cdots E_n$. Observe that $\det D = 1$ (though we cannot necessarily assume that D is the identity matrix based on the way in which we formulated our algorithm above). However, if $D = \text{diag}(\alpha_1, \dots, \alpha_n)$, then we can write $D = \text{diag}(\alpha_1, \alpha_1^{-1}, 1, \dots, 1)(1, \alpha_1\alpha_2, \alpha_3, \dots, \alpha_n)$, i.e., any diagonal matrix with determinant 1 can be written as the product of a diagonal matrix in $SL_2(k)$ embedded in $SL_n(k)$ and a diagonal matrix in $SL_{n-1}(k)$ embedded in $SL_n(k)$. Thus, proceeding recursively, we see that any diagonal matrix of determinant 1 can be written as a product of diagonal matrices that are in the image of $SL_2(k)$. Now, it is straightforward to show that a diagonal matrix in $SL_2(k)$ can be written as a product of elementary matrices. For example, one can write:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1-\alpha}{\alpha} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}.$$

Taken together, we conclude that every element of $SL_n(k)$ can be written as a product of elementary matrices. Since every elementary matrix is naively \mathbb{A}^1 -homotopic to the identity, we conclude that $[\text{Spec } k, SL_n]_N = \{Id_n\}$ (i.e., as a group it is the trivial group). \square

Exercise 2.3.1.5. Show that the determinant homomorphism $\det : GL_n \rightarrow \mathbf{G}_m$ induces an isomorphism $[\text{Spec } k, GL_n] \rightarrow k^\times$.

Remark 2.3.1.6. One possible generalization of the units functor is the functor sending a ring R to the group $GL_n(R)$ of invertible $n \times n$ -matrices over R and we can investigate the \mathbb{A}^1 -homotopy invariance of this functor. As before since $GL_n(-)$ is a functor, there is an evident map $GL_n(R) \rightarrow GL_n(R[x])$ and evaluation at zero shows that this map is injective. Note that, if $n \geq 2$, this map is never surjective. Indeed, take any element f of $R[x]$ that is not in R and consider the elementary matrix $Id_n + fe_{ij}$ (with $i \neq j$): this is an element of $GL_n(R[x])$ that does not lie in the image of $GL_n(R)$.

Example 2.3.1.7. Suppose $X \in GL_n(R[x])$. The matrix $X(0)$ is in $GL_n(R)$. Setting $Y := X(0)^{-1}X$ produces a matrix such that $ev_0(X(0)^{-1}X) = Id_n$. Now, we can also consider the evaluation map $M_n(R[x]) \rightarrow M_n(R)$ and note that $ev_0(X(0)^{-1}X - Id_n) = 0$. Consider the matrix $Y(tx)$. For $t = 0$, this matrix is $Y(0) = Id_n$, while for $t = 1$ it is simply $Y(x)$. Therefore, the matrix Y is naively \mathbb{A}^1 -homotopic to the identity. It follows that the matrix X is naively \mathbb{A}^1 -homotopic to $X(0)$.

Further generalizations of these computations

Closely related naive \mathbb{A}^1 -homotopy classes to those studied above can be very interesting. Fix an integer n , and work over a field whose characteristic does not divide n . The center of SL_n is a affine algebraic group $\mu_n := \text{Spec } k[t, t^{-1}]/(t^n - 1)$; this affine algebraic group is also a subgroup of \mathbf{G}_m . Now, the group scheme μ_n acts by left multiplication on SL_n , i.e., there is a morphism $\mu_n \times SL_n \rightarrow SL_n$. We can form the quotient by this group action. More precisely, define $PGL_n := \text{Spec } k[SL_n]^{\mu_n}$, i.e., the spectrum of the ring of invariant functions. The elements of $PGL_n(k)$ are precisely the invertible $n \times n$ -matrices over k up to scaling.

Exercise 2.3.1.8. Show that $[\text{Spec } k, PGL_n]_N = k^*/(k^*)^n$, i.e., the quotient of the group k^* by the subgroup of n -th powers.

Chapter 3

Projective modules, gluing and locally free modules

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As mentioned in the introduction, by following parallels with topology, it is natural to study vector bundles on affine varieties. We develop in this chapter the basic properties of such objects, i.e., projective modules over a ring. Our ultimate goal is to study a collection of \mathbb{A}^1 -invariants that are constructed out of projective modules. Nevertheless, our goal here is show that theory of projective modules really does parallel the theory of vector bundles in topology.

3.1 Lecture 4B: Definitions and Basic Properties

3.1.1 Modules over a ring

We begin by developing the basic ideas in a slightly more general context than we studied before. Suppose R is an arbitrary commutative unital ring and write Mod_R for the category of *all* R -modules.

Definition 3.1.1.1. If R is a commutative ring, an R -module M is called

1. *finitely generated* if there is an epimorphism $R^{\oplus n} \rightarrow M$;
2. *finitely presented* if M is the cokernel of a map $R^{\oplus n} \rightarrow R^{\oplus m}$ (equivalently, M is finitely generated, and for some surjection $\varphi : R^{\oplus m} \rightarrow M$, the kernel $\ker \varphi$ is finitely generated as well).
3. *coherent* if M is finitely generated and any finitely generated (not necessarily proper) submodule is itself finitely presented.

It follows from the definitions that for any ring R that M coherent implies M finitely presented, and M finitely presented implies M finitely generated. For general rings R , the reverse implications need not hold. We write Mod_R^{fg} , Mod_R^{fp} and Mod_R^{coh} for the full subcategories of Mod_R consisting of finitely generated, finitely presented or coherent R -modules.

Example 3.1.1.2. Beware: over “big” rings strange things can happen. For example: a submodule of a finitely generated R -module need not be finitely generated. For instance, take $R = k[x_1, x_2, \dots]$ be a polynomial ring in infinitely many variables. You can check that the ideal $\langle x_1, x_2, \dots \rangle$ is an R -submodule of a free R -module of rank 1, yet fails to be finitely generated.

Lemma 3.1.1.3. *Let R be a commutative unital ring and suppose*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of R -modules. The following statements hold:

1. *Any extension of finitely generated R -modules is finitely generated, i.e., if M' and M'' are finitely generated, then so is M .*
2. *Any extension of finitely presented R -modules is finitely presented, i.e., if M' and M'' are finitely presented, so is M .*
3. *Any quotient of a finitely generated module is finite, i.e., if M is finitely generated, so is M'' .*
4. *Any quotient of a finitely presented module by a finitely generated submodule is finitely presented, i.e., if M is a finitely presented R -module, and M' is finitely generated, then M'' is finitely presented as well.*
5. *If M'' is finitely presented, and M is finitely generated, then M' is finitely generated as well.*

Proof. To be added. □

Remark 3.1.1.4. Our desire to work with arbitrary commutative rings is not generality for its own sake. If M is a compact manifold, then the rings $C(M)$ or $C^\infty(M)$ of complex-valued continuous functions on M or complex valued smooth functions on M *need not* be Noetherian rings.

The following fact is fundamental.

Theorem 3.1.1.5. *The category Mod_R , equipped with the usual structures of direct sum and tensor product is abelian and symmetric monoidal (see Appendices A.3.1 and A.2.3).*

Remark 3.1.1.6. In general, the categories Mod_R^{fg} and Mod_R^{fp} need not be abelian categories (the example above shows what can go wrong for finitely generated R -modules), but Mod_R^{coh} turns out to be an abelian category [Sta15, Tag 05CU]. When R is a Noetherian ring, all three notions are equivalent (reference?).

Remark 3.1.1.7. A ring R is coherent if it is coherent as a module over itself (i.e., all finitely generated ideals in R are finitely presented). Rings appearing in topology are rarely coherent. For example a result due to Neville [Nev90] characterizes those topological spaces for which the ring of continuous functions is coherent: such spaces are called *basically disconnected*. More precisely, this means that for any continuous function f , the closure of the open set $\{x \in X \mid f(x) \neq 0\}$ is again open.

One of the touchstone results of elementary algebra is the structure theorem for finitely generated modules over a PID: every finitely generated module can be written as a direct sum of a free module and a torsion module, and we can give a nice classification of the torsion modules in terms of the (non-zero) prime ideals of the ring. There is a straightforward generalization of torsion modules to arbitrary commutative rings.

Definition 3.1.1.8. A module M over a commutative unital ring R is called a *torsion R -module*, if there exists a regular element $r \in R$ (i.e., a non-zero divisor) such that $rM = 0$. An R -module M that is not a torsion R -module is called *torsion-free*.

As the “classification” of prime ideals for rings that are not PIDs is more complicated (the spectrum of the ring is a precise measure of the complexity), the structure of torsion modules for more general rings becomes more complicated. Moreover, even the “easy” part of the structure theorem is more complicated: if R is not a principal ideal domain, it is not necessarily the case that torsion free R -modules are themselves free.

3.1.2 Projective and flat modules

We now present some interesting torsion-free modules.

Definition 3.1.2.1. Suppose R is a commutative unital ring. An R -module M is called

1. *flat* if $- \otimes_R M$ is an exact functor on Mod_R , i.e., preserves exact sequences;
2. *projective* if $\text{Hom}_R(M, -)$ is an exact functor on Mod_R ,
3. *invertible* if $- \otimes_R M$ is an auto-equivalence of Mod_R .

Remark 3.1.2.2. Note that $- \otimes_R M$ is an exact functor if and only if $M \otimes_R -$ is exact since included in the statement that Mod_R is symmetric monoidal is a natural isomorphism between these two functors.

Definition 3.1.2.3. If $f : R \rightarrow S$ is a homomorphism of commutative rings, then we say that f is a *flat homomorphism* if S is a flat R -module.

Example 3.1.2.4. Any free R -module is projective (or flat). A free R -module of rank 1 is invertible. Any finitely generated projective (or flat) module over a principal ideal domain is necessarily free (this follows from the structure theorem). In particular, if k is a field, any finitely generated projective $k[t]$ -module is free.

Remark 3.1.2.5. In 1955, Serre posed the question of whether finitely generated projective $k[t_1, \dots, t_n]$ -modules (k a field) are free [Ser55]. This question stimulated much work in the theory of projective modules and was answered by Quillen and Suslin (independently) in 1976.

Example 3.1.2.6. If R is a ring, then we can consider $R \times 0$ as an $R \times R$ -module. This module is evidently a direct summand of a free module (namely $R \times R$), but is not itself free. Thus, there exist examples of projective modules that are not free.

Lemma 3.1.2.7. *An arbitrarily indexed direct sum of R -modules is flat (resp. projective) if and only if each summand is flat (resp. projective).*

Proof. Since arbitrary direct sums commute with tensor products in the category of R -modules, there is an isomorphism of functors $(\bigoplus_{i \in I} M_i) \otimes_R - \cong \bigoplus_{i \in I} (M_i \otimes_R -)$. Thus, the first functor is exact if and only if the second functor is exact. Likewise, there is an isomorphism of functors $\text{Hom}_R(\bigoplus_{i \in I} P_i, -) \cong \prod_i \text{Hom}_R(P_i, -)$ and the left-hand-side is exact if and only if the right hand side is exact. \square

Lemma 3.1.2.8. *An R -module L is invertible if and only if there exists an R -module L' such that $L \otimes_R L' \xrightarrow{\sim} R$.*

Proof. If $L \otimes_R -$ is an auto-equivalence, then the existence of L' is an immediate consequence of the fact that equivalences of categories are essentially surjective. In the other direction, if L' exists as in the statement, then $- \otimes_R L'$ is a quasi-inverse to $- \otimes_R L$ since $(- \otimes_R L) \otimes_R L' \cong - \otimes_R (L \otimes_R L') \cong - \otimes_R R$, which is the identity functor. \square

Injective modules

There is a notion of injective module that is dual to that of projective module. More precisely, one makes the following definition.

Definition 3.1.2.9. If R is a commutative unital ring, then an R -module M is injective if $\text{Hom}_R(-, M)$ is exact.

Concretely, an R -module M is injective if given any R -module map $j : N \rightarrow M$ and an injective R -module map $N \rightarrow N'$, there exists an R -module map $j' : N' \rightarrow M$ extending j , i.e., such that the composite $N \rightarrow N' \rightarrow M$ coincides with j .

Exercise 3.1.2.10. *Show that any product of modules is injective if and only if each factor is injective.*

Localizations are flat ring homomorphisms

The following elementary fact about localization will be used repeatedly in what follows.

Theorem 3.1.2.11. *Suppose R is a commutative unital ring, and $S \subset R$ is a multiplicative subset.*

1. *If M is an R -module, then $M[S^{-1}] = M \otimes_R R[S^{-1}]$.*
2. *The assignment $M \mapsto M[S^{-1}]$ is an exact functor $\text{Mod}_R \rightarrow \text{Mod}_{R[S^{-1}]}$.*
3. *In particular, $R \rightarrow R[S^{-1}]$ is a flat ring homomorphism.*

Proof. For Point (1). Consider the map $M \times R[S^{-1}] \rightarrow M[S^{-1}]$ given by $(m, \frac{r}{s}) \mapsto \frac{rm}{s}$. This map is R -bilinear by construction, and therefore there exists a map $M \otimes R[S^{-1}] \rightarrow M[S^{-1}]$ such that $m \otimes \frac{r}{s} \mapsto \frac{rm}{s}$. Define a map $M[S^{-1}] \rightarrow M \otimes R[S^{-1}]$ by the formula $\frac{m}{s} \mapsto m \otimes \frac{1}{s}$; we claim this is well-defined. Indeed, if $\frac{m'}{s'}$ presents the same element of $M[S^{-1}]$, then we can find t and $t' \in S$ such that $ms't = m's't'$. In that case, $m \otimes \frac{1}{s} = m \otimes \frac{s't}{ss't} = ms't \otimes \frac{1}{ss't} = m's't \otimes \frac{1}{ss't} = m' \otimes \frac{st}{ss't} = m' \otimes \frac{1}{s'}$. It is straightforward to check these two maps are inverses.

For Point (2), since tensoring is always right exact, it suffices to prove that if $M \rightarrow M'$ is an injective R -module map, $M[S^{-1}] \rightarrow M'[S^{-1}]$ remains injective. If we view M as a sub-module of M' , an element $\frac{x}{s} \in M'[S^{-1}]$ is zero if and only if there exists $t \in S$ such that $tx = 0$. However, the latter happens if and only if $\frac{x}{s} = 0$ in M itself.

The third statement is a consequence of the first two. □

3.2 Lecture 5: Projective modules and their properties

In this lecture we analyze further the basic properties of projective modules.

3.2.1 Properties of projective modules

Lemma 3.2.1.1. *Suppose R is a commutative unital ring. If P is an R -module, the following conditions on P are equivalent:*

1. *P is projective;*
2. *Any R -module epimorphism $M \rightarrow P$ is split.*
3. *P is a direct summand of a free R -module;*

Proof. (1) \implies (2). Suppose we are given a surjection $\varphi : A \rightarrow P$; we can complete this into an exact sequence $0 \rightarrow \ker(\varphi) \rightarrow A \rightarrow P \rightarrow 0$. Now, since P is projective, $\text{Hom}_R(P, -)$ is an exact functor, and applying it to the previous short exact sequence yields a short exact sequence of the form

$$0 \longrightarrow \text{Hom}_R(P, \ker(\varphi)) \longrightarrow \text{Hom}_R(P, A) \longrightarrow \text{Hom}_R(P, P) \longrightarrow 0.$$

In particular, we may lift the identity $1 \in \text{Hom}_R(P, P)$ to a morphism $P \rightarrow A$ that, by construction, splits the given epimorphism.

(2) \implies (3). By choosing generators of P , we may build an epimorphism from a free module $\varphi : F \rightarrow P$. By (2), such an epimorphism is split, and we obtain a morphism $P \rightarrow F$. Then, we conclude that $F \cong P \oplus \ker(\varphi)$.

(3) \implies (1). Suppose $P \oplus Q \cong F$, where F is a free module. Example 3.1.2.4 shows that F is projective, and then appeal to Lemma 3.1.2.7 allows us to conclude that any summand of a free R -module is also projective. \square

Remark 3.2.1.2. Recall that a projection operator on a k -vector space V is an endomorphism P such that $P^2 = P$. The projection onto a summand is an example of a projection operator. Given a finitely generated projective module M over a ring R , a direct sum decomposition $R^{\oplus n} \cong P \oplus Q$ the composite map $R^{\oplus n} \rightarrow P \hookrightarrow R^{\oplus n}$ is a projection operator on $R^{\oplus n}$, i.e., an idempotent matrix.

Example 3.2.1.3. Lemma 3.2.1.1 shows that any module M such that $M \oplus R^{\oplus n} \cong R^{\oplus N}$ is projective; such modules are called *stably free*. We now explain how to construct stably free R -modules. Suppose R is a ring and a_1, \dots, a_n is a sequence of elements in R . We will say that a sequence (a_1, \dots, a_n) is a *unimodular row* if there exist b_1, \dots, b_n such that $\sum_i a_i b_i = 1$. Given such a sequence, we can define an epimorphism $R^{\oplus n} \rightarrow R$ via multiplication by $\mathbf{a} := (a_1, \dots, a_n)^t$. Unimodularity ensures that this homomorphism is surjective. In that case, the kernel $P_{\mathbf{a}} := \ker \mathbf{a}$ is a projective R -module. In general, $P_{\mathbf{a}}$ need not be a free R -module, though it is sometimes difficult to prove this algebraically.

Exercise 3.2.1.4. Show that the projective module $P_{\mathbf{a}}$ associated with a unimodular row (a_1, \dots, a_n) is free if and only if (a_1, \dots, a_n) is the first row of an invertible $n \times n$ -matrix with determinant 1.

Example 3.2.1.5. Unfortunately, if $n = 2$ in the previous construction, one obtains a module P such that $P \oplus R \cong R^{\oplus 2}$. However, any unimodular sequence of length 2 is completable, so $P \cong R$. Indeed, if (a_1, a_2) is our unimodular row, then by definition we can find (b_1, b_2) such that $a_1 b_1 + a_2 b_2 = 1$. In that case, the matrix

$$\begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$$

has determinant $a_1 b_1 + a_2 b_2 = 1$.

Example 3.2.1.6. If M is any R -module, then we can consider the R -module dual $M^{\vee} := \text{Hom}_R(M, R)$; this has a natural R -module structure. Note that, if M is finitely generated free module, then so is $\text{Hom}_R(M, R)$. If P is a finitely generated projective module, then we claim $\text{Hom}_R(P, R)$ is projective. Indeed, if $P \oplus Q \cong R^{\oplus n}$, then $\text{Hom}_R(P \oplus Q, R) \cong \text{Hom}_R(R^{\oplus n}, R) \cong R^{\oplus n}$. Since for finite direct products of modules are also direct sums, it follows that $\text{Hom}_R(P, R)$ is a summand of $R^{\oplus n}$ as well.

Remark 3.2.1.7. For $n \geq 3$, it is more difficult to determine whether a given unimodular row of length n is completable. On the other hand, with the technology developed so far, it is also not clear whether there are any non-trivial examples.

Corollary 3.2.1.8. If R is a commutative unital ring, and M is an R -module, then following implications hold:

$$M \text{ is invertible} \implies M \text{ is projective} \implies M \text{ is flat.}$$

Proof. We leave the first implication as an exercise. For the second implication, since free R -modules are flat, direct summands of flat R -modules are flat, and since any projective R -module is a direct summand of a free R -module, it is necessarily flat as well. \square

Lemma 3.2.1.9. *Assume R is a commutative unital ring.*

1. *Any finitely generated projective R -module is finitely presented.*
2. *Any invertible R -module is finitely presented.*

Proof. Since projective modules are summands of free modules, it suffices to observe that any summand of a finitely generated free R -module is finitely presented. Indeed, suppose $R^{\oplus n} \cong P \oplus Q$ for two R -modules P and Q . In that case, we can view Q as the kernel of a surjection $R^{\oplus n} \rightarrow P$ and $R^{\oplus n}$ surjects onto Q as well.

For the second point, it suffices after the first point to show that invertible R -modules are finitely generated; this second statement is essentially a consequence of the definition of a tensor product in terms of finite sums of “pure” tensors. More precisely, suppose L is an invertible R -module and we are given an isomorphism $\varphi : L \otimes M \rightarrow R$. In that case, $\varphi^{-1}(1) = \sum_{i=1}^s x_i \otimes y_i$ for elements $x_i \in L$ and $y_i \in M$. Now, let $L' \subset L$ be the sub-module generated by x_1, \dots, x_s . In that case, the composite $L' \otimes M \rightarrow L \otimes M \rightarrow R$ is still surjective since $\varphi^{-1}(1) \in L' \otimes M$.

Now, we want to show that $L' \rightarrow L$ is actually an isomorphism. To this end, consider the exact sequence $0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$. Since M is invertible, we conclude that there is a short exact sequence of the form:

$$0 \longrightarrow L' \otimes_R M \longrightarrow L \otimes_R M \longrightarrow L'' \otimes_R M \longrightarrow 0$$

Since the first map in this exact sequence is surjective by what we asserted before, we conclude that $L'' \otimes M = 0$. However, by associativity (and commutativity) of tensor product, $(L'' \otimes_R M) \otimes_R L \cong L'' \otimes_R (M \otimes_R L) \cong L'' \otimes_R R \cong L''$. Thus, $L'' = 0$. \square

Interlude: compact objects

In this interlude, we give a categorical interpretation of the importance of finitely presented modules and in doing so we give a simple proof of the fact that invertible R -modules are finitely presented.

Definition 3.2.1.10. If \mathcal{C} is a category that admits filtered colimits, then an object $X \in \mathcal{C}$ is called *compact* if the functor $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \text{Set}$ preserves filtered colimits.

The category Mod_R admits filtered colimits (inherited from the category of sets), so it makes sense to speak of compact objects. In Lemma 3.2.1.9 we observed that direct summands of finitely generated R -modules are necessarily finitely presented; we use this observation together with the following exercise to better understand compact objects in Mod_R .

Exercise 3.2.1.11. *Show that any R -module M can be written as a filtered colimit of finitely presented modules.*

Lemma 3.2.1.12. *Any compact object in Mod_R is finitely presented.*

Proof. Suppose M is compact. We can write M as a filtered colimit of finitely presented R -modules $M = \text{colim}_i M_i$. Then, there are, by definition of compactness, a sequence of isomorphisms

$$\text{Hom}_R(M, M) \cong \text{Hom}_R(M, \text{colim}_i M_i) \cong \text{colim}_i \text{Hom}_R(M, M_i).$$

In particular, the identity map $M \rightarrow M$ factors through a map $M \rightarrow M_i$ for some i sufficiently large. It follows that the inclusion map $M_i \rightarrow M$ can be split, so M is a direct summand of the finitely presented module M_i . To conclude, we appeal to the fact that direct summands of finitely presented R -modules are finitely presented. \square

In fact, the converse to the above lemma holds.

Proposition 3.2.1.13. *The compact objects in Mod_R are precisely the finitely presented R -modules.*

Proof. We already saw that compact objects are finitely presented in Lemma 3.2.1.12, so it remains to establish that finitely presented R -modules are compact objects. To see this, we use a series of reductions. First, suppose $N = \text{colim}_i N_i$. Since $\text{Hom}_R(R, -)$ is the identity functor on Mod_R , we conclude that $\text{Hom}_R(R, \text{colim}_i N_i) = \text{colim}_i \text{Hom}_R(R, N_i)$. Next, we can observe that both the functors $\text{colim}_i \text{Hom}_R(-, N_i)$ and $\text{Hom}_R(-, N)$ commute with finite direct sums. Since filtered colimits are exact in the category of R -modules [Mat89, Theorem A2], both of the functors just mentioned are actually right exact. Now, if we pick a presentation $R^{\oplus m} \rightarrow R^{\oplus n} \rightarrow M \rightarrow 0$ for a finitely presented R -module, then combining the observations just made with the 5-lemma allows us to conclude that finitely presented R -modules are compact. \square

If \mathcal{C} is any monoidal category, then we can speak of *invertible objects* in \mathcal{C} :

Definition 3.2.1.14. If \mathcal{C} is any monoidal category, then an object $X \in \mathcal{C}$ is invertible if tensoring with X is an auto-equivalence of \mathcal{C} .

Remark 3.2.1.15. Invertible objects in Mod_R are precisely invertible R -modules.

Exercise 3.2.1.16. *An object X in a monoidal category \mathcal{C} is invertible if and only if there exists an object $X^* \in \mathcal{C}$ such that $X \otimes X^*$ is isomorphic to the unit object in \mathcal{C} .*

Lemma 3.2.1.17. *If \mathcal{C} is a monoidal category that admits filtered direct limits, then invertible objects are compact.*

Proof. Auto-equivalences of categories preserve filtered direct limits. \square

3.2.2 Tensor products and extension of scalars

We now study the behavior of these various kinds of modules under tensor product.

Lemma 3.2.2.1. *If R is a commutative unital ring, and M_1 and M_2 are R -modules then the following statement hold.*

1. *If M_1 and M_2 are flat, then $M_1 \otimes_R M_2$ is flat;*
2. *if M_1 and M_2 are projective, then $M_1 \otimes_R M_2$ is projective; and*
3. *if M_1 and M_2 are invertible, then $M_1 \otimes_R M_2$ is invertible.*

Proof. Exercise. \square

Lemma 3.2.2.2. *If $f : R \rightarrow S$ is any ring homomorphism, then “extension of scalars”, i.e., sending $M \rightarrow M \otimes_R S$ determines a functor $\text{Mod}_R \rightarrow \text{Mod}_S$. Extension of scalars sends*

1. flat R -modules to flat S -modules,
2. (finitely generated) projective R -modules to (finitely generated) projective R -modules, and
3. invertible R -modules to invertible R -modules.

Proof. For the first statement, observe that there is always an isomorphism of functors $(M \otimes_R S) \otimes_S - \cong M \otimes_R (S \otimes_S -)$. Thus, if M is a flat R -module, then $M \otimes_R S$ is a flat S -module.

For the second statement, if P is a (finitely generated) projective R -module, then $P \oplus Q \cong F$ for F a (finitely generated) free R -module. Then, $(P \oplus Q) \otimes_R S \cong P \otimes_R S \oplus Q \otimes_R S \cong F \otimes_R S$. Since $F \otimes_R S$ is a (finitely generated) free S -module, we conclude that $P \otimes_R S$ is (finitely generated) projective.

The third statement amounts to associativity of tensor product and is left as an exercise. \square

Flatness criteria

Flatness is a very technically useful criterion and we now introduce a criterion to check whether a module is flat.

Proposition 3.2.2.3 (Equational criterion for flatness). *If R is a commutative unital ring, then an R -module M is flat if and only if every relation in M is trivial.*

Remark 3.2.2.4. Over Noetherian rings, every finitely generated flat module is projective; this follows essentially from the equational criterion of flatness; see [Lam99, Theorem 4.38]. Over non-Noetherian rings, there may be finitely generated flat modules that are not projective. Indeed, if $R = C^\infty(\mathbb{R})$ the ring of real valued smooth functions on the real line, and \mathfrak{m} is the ideal of smooth functions vanishing at 0, then $R_{\mathfrak{m}}$ is the ring of germs of smooth functions at the origin. This module is flat because it is a localization. Set I to be the ideal of functions $f \in C^\infty(\mathbb{R})$ such that there exists $\epsilon > 0$ and $f(x) = 0$ for all $|x| < \epsilon$. One can check that, $R_{\mathfrak{m}} \cong R/I$, so $R_{\mathfrak{m}}$ is finitely generated as well. If R/I were projective, then the surjection $R \rightarrow R/I$ would split, and we could write $R \cong R/I \oplus I$. However, one can check that I is not even finitely generated.

3.2.3 Projective and locally free modules

Recall that if R is a commutative unital ring, then the Jacobson radical $J(R)$ is equal to the intersection of all maximal ideals of R (the intersection of the annihilators of simple R -modules).

Lemma 3.2.3.1 (Nakayama). *If M is a finitely generated R -module and $M/J(R) \cdot M = 0$, then $M = 0$.*

Example 3.2.3.2. We will essentially always apply Nakayama's lemma in the situation where R is a local ring with maximal ideal \mathfrak{m} . In that case, $J(R) = \mathfrak{m}$.

Projective modules over local rings

Using Nakayama's lemma, we can analyze finitely generated projective R -modules over local rings.

Proposition 3.2.3.3. *If R is a local ring, then every finitely generated projective R -module is free.*

Proof. Suppose \mathfrak{m} is the maximal ideal of R and $\kappa := R/\mathfrak{m}$ is the residue field. Assume P is a projective R -module. In that case, $P/\mathfrak{m} = P \otimes_R R/\mathfrak{m}$ is a finitely generated κ -module and thus a finite dimensional κ -vector space.

Fix a basis $\bar{e}_1, \dots, \bar{e}_n$ for P/\mathfrak{m} . I claim we can always choose $e_i \in P$ lifting \bar{e}_i . More generally, take any R -module M and a morphism $\varphi : M \rightarrow P$. Set $\bar{\varphi} : M \otimes_R R/\mathfrak{m} \rightarrow P/\mathfrak{m}$. Right exactness of tensoring shows that the $\text{coker}(\varphi) \otimes_R R/\mathfrak{m} \cong \text{coker}(\bar{\varphi})$. In particular, if $\bar{\varphi}$ is an epimorphism, then Nakayama's lemma shows that φ is an epimorphism as well.

On the other hand, since P is projective, $\ker(\varphi)$ is a direct summand of P and therefore also finitely generated. Since $\ker(\varphi)$ is trivial when reduced mod \mathfrak{m} again by Nakayama's lemma $\ker(\varphi)$ is trivial. Therefore, we conclude that φ is an isomorphism and thus P is free. \square

Tor and projective modules over a local ring

If R is a commutative unital ring, $I \subset R$ is an ideal and M is an R -module, then one can define $\text{Tor}_1^R(R/I, M)$ to be the kernel of the map $I \otimes M \rightarrow R \otimes M$. This definition is consistent with more general version of Tor that will be studied later.

Proposition 3.2.3.4. *Suppose R is a Noetherian local ring with maximal ideal \mathfrak{m} and residue field κ and P is a finitely generated R -module. The following conditions are equivalent:*

1. P is free;
2. P is projective; and
3. $\text{Tor}_1^R(R/\mathfrak{m}, P) = 0$.

Proof. That (1) \Rightarrow (2) is immediate. To see that (2) \Rightarrow (3) observe that projective modules are flat, and then apply $\otimes_R P$ to the exact sequence $0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow R/\mathfrak{m} \rightarrow 0$. That (3) \Rightarrow (1) is essentially a version of the argument involving Nakayama's lemma we gave above. By assumption $\mathfrak{m} \otimes_R P \rightarrow P$ is injective and therefore we have a short exact sequence of the form

$$0 \longrightarrow \mathfrak{m} \otimes_R P \longrightarrow P \longrightarrow R/\mathfrak{m} \otimes_R P \longrightarrow 0.$$

Nakayama's lemma tells us if P' is the sub-module of P generated by lifting a basis of $P/\mathfrak{m}P$, then $P = P'$ and thus by lifting a basis of $P/\mathfrak{m}P$ we conclude. \square

Local trivializations of projective modules

The next result is a key consequence of the fact that finitely generated projective modules are finitely presented, combined with the results above about freeness of finitely generated projective modules over local rings.

Proposition 3.2.3.5. *Assume R is a commutative unital ring, \mathfrak{p} is a prime ideal in R and P is a finitely generated projective R -module.*

1. The localization $P_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of some finite rank n .
2. There exists an element $s \in R \setminus \mathfrak{p}$ such that the localization of P away from s is free, i.e., $P[\frac{1}{s}]$ is a free $R[\frac{1}{s}]$ -module of rank n .
3. If \mathfrak{p}' is any prime ideal not containing s , $P_{\mathfrak{p}'}$ is a free $R_{\mathfrak{p}'}$ -module of rank n .

In particular, if L is an invertible R -module, then $L_{\mathfrak{p}}$ is free of rank 1.

Proof. Regarding point (1): since $R_{\mathfrak{p}}$ is a local ring, and $P_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$ -module it is necessarily free of some finite rank n .

For Point (2): we begin by observing that since P is finitely generated, it is finitely presented, and we can write P as the cokernel of a matrix M with coefficients in R . Now, to say that $P_{\mathfrak{p}}$ is free, is to say that we can find an invertible matrix with coefficients in $R_{\mathfrak{p}}$ such that the product of this invertible matrix (the matrix expressing the change from the standard basis of $P_{\mathfrak{p}}$ that is obtained from writing it as a quotient of a free module to the basis in which it is a direct summand) and the matrix M . By the definition of localization, each element of M can be written in the form f_{ij}/h_{ij} where $f_{ij} \in R$ and $h_{ij} \in R \setminus \mathfrak{p}$. Taking s to be the product of the h_{ij} , we see that $M \in R[\frac{1}{s}]$, but this is what we wanted to show.

Point (3) is a special case of point (2). For the final statement, since invertible R -modules are always finitely presented, and invertible R -modules over local rings are all free of rank 1, the final assertion is a consequence of the previous ones. \square

Remark 3.2.3.6. Kaplansky showed [Kap58] that projective modules over local rings are always free (without finite generation hypotheses). Along the way, Kaplansky established a remarkable structure theorem for projective modules: every projective module is a direct sum of countably generated projective modules. On the other hand, it is *not* the case projective modules that are not finitely generated are locally free. In fact, this latter statement fails even for countably generated projective modules. Indeed, there exists a countably generated ring R and a projective module M that is a direct sum of countably many locally free rank 1 modules such that M is not locally free [Sta15, Lemma 88.26.5 Tag 05WG]. For this reason (and due to many other pathologies that appear), we will typically avoid speaking about infinitely generated projective modules.

3.3 Lecture 6: Patching projective modules

In the previous lecture we showed that if P is a finitely generated projective module over a ring R , then P is locally free. We establish a slightly stronger version of that statement now.

3.3.1 Projective modules are locally free

Proposition 3.3.1.1. *If R is a commutative unital ring and P is a finitely generated projective R -module, then there is a integer r and finitely many elements $f_1, \dots, f_r \in R$ such that the family f_1, \dots, f_r generate the unit ideal in R and such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$ -module of finite rank for each i .*

Proof. This follows by combining the finite presentation of P and the fact that $\text{Spec } R$ is a quasi-compact topological space (see Exercise 2.1.1.32). Intuitively, fix a prime ideal \mathfrak{p} in R . By appeal to Proposition 3.2.3.5 we can find an element f_1 such that $P[\frac{1}{f_1}]$ is a free $R[\frac{1}{f_1}]$ -module. Now, pick a prime ideal in $R/(f_1)$ and consider the associated prime ideal in R and repeat the procedure. Altogether we obtain a sequence of elements f_1, f_2, \dots such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$ -module and the family f_i generate the unit ideal. However, since $\text{Spec } R$ is quasi-compact, it follows that a finite number of these modules already generate the unit ideal. \square

Remark 3.3.1.2. If X is a topological space, and $C(X)$ is the ring of real or complex valued continuous functions on X , then the notion of vector bundle over X we used earlier was precisely that arising from finitely generated projective $C(X)$ -modules.

Definition 3.3.1.3. An R -module M is called *locally free (of finite rank)* if there exists a family of elements $\{f_i\}_{i \in I}$ that generate the unit ideal such that the R_{f_i} -module M_{f_i} is free (of finite rank).

Remark 3.3.1.4. By Exercise 2.1.1.32 if the f_i generate the unit ideal, then by quasi-compactness of $\text{Spec } R$, we can pick a finite subset $I' \subset I$ that generates the unit ideal.

Corollary 3.3.1.5. *If R is a commutative unital ring, then finitely generated projective modules are locally free. In particular, invertible R -modules are always locally free of rank 1.*

We now study possible converses to Proposition 3.3.1.1: given a locally trivial module, when can we conclude that it is projective?

3.3.2 Zariski descent I: patching modules and homomorphisms

The following result establishes a compatibility for localizations of modules that will be useful in building a module out of localizations. We begin with the simplest case: suppose we have an affine scheme $X = \text{Spec } R$ covered by a pair of basic open sets $U_1 = \text{Spec } R_f$ and $U_2 = \text{Spec } R_g$, i.e., the elements f and g generate the unit ideal in R . From the topological point of view, $U_1 \amalg U_2$ forms an open cover of X ; this corresponds to a ring homomorphism $R \rightarrow R_f \times R_g$ (the product of rings corresponds to the coproduct of affine schemes). Extension of scalars gives a functor $\text{Mod}_R \rightarrow \text{Mod}_{R_f \times R_g}$. Every module over $R_f \times R_g$ can be written uniquely as $M_1 \times M_2$ for M_1 an R_f -module and M_2 an R_g -module. The image of the pullback functor consists of pairs of modules that are localizations of a single module.

Functoriality of localizations yields an additional compatibility condition on modules in the image of the pullback. In the case in which we are interested, the rings R_{fg} and R_{gf} are equal. If we begin with an R -module M , then we may consider M_f as an R_f -module and then $M_f \otimes_{R_f} R_{fg}$ as an R_{fg} -module. On the other hand, M_g is an R_g -module, and $M_g \otimes_{R_g} R_{gf}$. The associativity isomorphism for tensor product of R -modules yields a distinguished identification of the form:

$$(M \otimes_R R_f) \otimes_{R_f} R_{fg} \cong M \otimes_R (R_f \otimes_{R_f} R_{fg}) \cong M \otimes_R (R_{fg}) \cong M \otimes_R (R_g \otimes_{R_g} R_{gf}) \cong (M \otimes_R R_g) \otimes_{R_g} R_{gf}.$$

We use this distinguished identification to equate M_{fg} and M_{gf} in what follows.

Proposition 3.3.2.1. *Suppose R is a commutative unital ring and M is an R -module. Given an open cover of $\text{Spec } R$ by two basic open sets $D(f)$ and $D(g)$ whose intersection is $D(fg)$, we obtain modules M_f , M_g and M_{fg} over R_f , R_g and R_{fg} . Note that we can view M_f , M_g and M_{fg} as R -modules. The module M is the pullback of the diagram of R -modules:*

$$M_f \longrightarrow M_{fg} \longleftarrow M_g;$$

concretely, M is the submodule of $M_f \oplus M_g$ consisting of pairs (α, β) that localize to the same element of M_{fg} .

Proof. We first show that $M \rightarrow M_f \oplus M_g$ is injective. The statement that $D(f)$ and $D(g)$ cover $\text{Spec } R$ is equivalent to the statement that f and g generate the unit ideal, i.e., f and g are comaximal elements of R . Indeed, if $m \in M$ localizes to 0 in M_f and M_g , then it follows that $f^r m = 0 = g^r m$ for some r sufficiently large. However, the elements f^r and g^r are comaximal if f and g are so it follows that $m = 0$.

Since localization is functorial, we conclude that the image of the map $M \rightarrow M_f \oplus M_g$ consists of pairs of elements (α, β) that agree upon further localization to M_{fg} . To simplify the notation, set $S = \{f^n, n \geq 0\}$ and $T = \{g^n, n \geq 0\}$. Now, suppose $\frac{m}{s} \in M_f$ and $\frac{n}{t} \in M_g$ (i.e., $s = f^a$ and $t = g^b$) localize to the same element in M_{fg} . In that case, we know that there are elements $s' \in S$ and $t' \in T$ such that $s't'(tm - sn) = 0$. Therefore, $(tt')(s'm) = (ss')(t'n)$. Thus, after replacing $\frac{m}{s}$ by $\frac{s'm}{s's}$ and $\frac{n}{t}$ by $\frac{t'n}{t't}$, we may assume without loss of generality that $sn = tm$.

Now, since s and t are comaximal, we may find $x, y \in R$ and write $xs + yt = 1$. If we set $q = xm + yn$, then

$$sq = s(xm) + s(yn) = (xs)m + y(sn) = (xs)m + y(tm) = (xs + yt)m = m.$$

Similarly, $tq = n$ and therefore q localizes to $\frac{m}{s}$ and $\frac{n}{t}$, as required. \square

Remark 3.3.2.2. The assertion that the diagram in Proposition 3.3.2.1 is a pullback square is equivalent to the statement that the following sequence of R -modules is exact:

$$0 \longrightarrow M \longrightarrow M_f \oplus M_g \longrightarrow M_{fg};$$

the first map is induced by localization while the second map sends $(a, b) \mapsto a - b$ in the localization. In particular, M is precisely the kernel of the map $M_f \oplus M_g \rightarrow M_{fg}$.

In the special case where $M = R$, this is actually an exact sequence of flat R -algebras of the form

$$0 \longrightarrow R \longrightarrow R_f \oplus R_g \longrightarrow R_{fg}.$$

Thus, M is the zeroth cohomology group of the two-term complex $M \otimes (R_f \oplus R_g \rightarrow R_{fg})$ by simply taking the kernel.

Example 3.3.2.3 (Characteristic polynomials of endomorphisms). The results above show that *elements* of a module are “locally determined”. Here is an application of this fact. Suppose P is a finitely generated projective module over a commutative unital ring R of fixed rank n . Given α an endomorphism of P , we describe how to attach a characteristic polynomial to α . Suppose we can find a pair of elements $f, g \in R$ such that P_f and P_g are free R_f and R_g -modules. In that case, choosing a basis of P_f as an R_f -module, we can define the characteristic polynomial of α_f in the usual way as $\det(\alpha_f - \lambda Id_n) = P(\alpha_f, \lambda) \in R_f[\lambda]$ and similarly, we can define a characteristic polynomial of $P(\alpha_g, \lambda) \in R_g[\lambda]$ (and, as usual, the expression is independent of the choice of basis). Now, taking determinants of matrices commutes with extension of scalars. Since the modules $(P_f)_g$ and $(P_g)_f$ are isomorphic, the elements $P(\alpha_f, \lambda)$ and $P(\alpha_g, \lambda)$ necessarily coincide when viewed as elements of $R_{fg}[\lambda]$. Therefore, there we deduce that there is an element $P(\alpha, \lambda) \in R[\lambda]$ that restricts to $P(\alpha_f, \lambda)$ and $P(\alpha_g, \lambda)$. One can establish the existence of characteristic polynomials in general using an inductive argument and the fact that projective modules are locally free. Moreover, one can show by refining covers that the characteristic polynomial so defined is independent of the choice of cover. The characteristic polynomial defined in this fashion has all the

usual properties of the characteristic polynomial, e.g., the Cayley–Hamilton theorem holds, i.e., α satisfies $P(\alpha, \lambda)$.

Example 3.3.2.4. If α is an endomorphism of a rank n projective module over a ring R , then we can define $\text{tr}(\alpha)$, $\det(\alpha)$ and similar expressions. In particular, if P is a projective module of rank n , then there is a homomorphism $\text{Aut}_R(P) \rightarrow R^\times$ sending α to its determinant.

We now investigate the functoriality of the construction above.

Proposition 3.3.2.5. *Suppose R is a commutative unital ring and M and N are R -modules. Suppose f and g are comaximal elements of R and $\alpha_f : M_f \rightarrow N_f$ and $\alpha_g : M_g \rightarrow N_g$ are a pair of homomorphisms that localize to the same homomorphism $M_{fg} \rightarrow N_{fg}$.*

1. *There exists a unique R -module homomorphism $\alpha : M \rightarrow N$ that localizes to α_f and α_g .*
2. *The morphism α is an isomorphism (resp. monomorphism) if and only if α_f and α_g are isomorphisms (resp. monomorphisms)*

Proof. Since pullback is functorial, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M_f \oplus M_g & \longrightarrow & M_{fg} & \longrightarrow & 0 \\ & & & & \downarrow \alpha_f \oplus \alpha_g & & \downarrow \alpha_{fg} & & \\ 0 & \longrightarrow & N & \longrightarrow & N_f \oplus N_g & \longrightarrow & N_{fg} & \longrightarrow & 0 \end{array}$$

We define a morphism $\alpha : M \rightarrow N$ by diagram chase: given an element $m \in M$, the element $\alpha_f(m) \oplus \alpha_g(m)$ is contained in the kernel of the map $N_f \oplus N_g \rightarrow N_{fg}$ and therefore corresponds to a unique element $n \in N$ that we will call $\alpha(m)$. This map α is actually an R -module map (you should check this).

For Point (2), if the maps β_f , β_g and β_{fg} are injective (or isomorphisms), then an analogous diagram chase shows that β is injective (resp. an isomorphism) as well. \square

Remark 3.3.2.6. It is less clear (i.e., does not follow directly from diagram chasing) that if α_f and α_g are surjective, then α is surjective. Nevertheless, this statement is still true.

3.3.3 Zariski descent II: An equivalence of categories

We now prove our hoped-for converse: given modules over localizations together with a choice of isomorphism on the intersection, we can build modules over our ring itself. We formalize this idea by defining a category that parameterizes modules equipped with “patching data.”

Construction 3.3.3.1. Consider the category $\text{Mod}_R(f, g)$ where

objects are triples (M_1, M_2, α) , M_1 is an R_f -module, M_2 is an R_g -module, and α is an isomorphism of R_{fg} -modules $\alpha : (M_1)_g \xrightarrow{\sim} (M_2)_f$ and a morphism of triples consists of a pair of morphisms $\beta_1 : M_1 \rightarrow M'_1$, $\beta_2 : M_2 \rightarrow M'_2$ such that the following diagram commutes

$$\begin{array}{ccc} (M_1)_g & \xrightarrow{\alpha} & (M_2)_f \\ \downarrow \beta_1 & & \downarrow \beta_2 \\ (M'_1)_g & \xrightarrow{\alpha'} & (M'_2)_f \end{array}$$

We will refer to $\text{Mod}_R(f, g)$ as the category of patching data; this construction is an example of a fiber product of categories.

Given an R -module M , we can associate with M an object of $\text{Mod}_R(f, g)$ by sending M to (M_f, M_g, c) , where $c : (M_f)_g \rightarrow (M_g)_f$ is the distinguished isomorphism arising from localization (as described before Proposition 3.3.2.1). Functoriality of localization guarantees that, in this way, we obtain a functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R(f, g)$$

that we now study in greater detail.

Theorem 3.3.3.2. *Suppose R is a commutative unital ring and f, g are comaximal elements in R . The functor $\text{Mod}_R \rightarrow \text{Mod}_R(f, g)$ sending an R -module M to (M_f, M_g, c) is an equivalence of categories.*

Proof. We prove this result in a number of stages.

Step 1. We construct a candidate quasi-inverse as follows. Given a triple (M_1, M_2, α) , we define an R -module as the pullback of the diagram

$$M_1 \longrightarrow (M_1)_g \cong (M_2)_f \longleftarrow M_2.$$

Functoriality of pullbacks makes this a functor $\text{Mod}_R(f, g) \rightarrow \text{Mod}_R$.

Step 2. We study the composite $\text{Mod}_R \rightarrow \text{Mod}_R(f, g) \rightarrow \text{Mod}_R$. If we start with an R -module, consider the associated patching data as described before the proposition and then build the corresponding R -module by taking a fiber product, it follows from Proposition 3.3.2.1 that the resulting module is the simply the one we began with. Likewise, Proposition 3.3.2.5 shows that this composite acts as the identity on morphism sets of R -modules. In other words, the composite $\text{Mod}_R \rightarrow \text{Mod}_R(f, g) \rightarrow \text{Mod}_R$ is the identity functor on Mod_R .

Now, we proceed to analyze the other composite $\text{Mod}_R(f, g) \rightarrow \text{Mod}_R \rightarrow \text{Mod}_R(f, g)$.

Step 3. We analyze the other composite. Suppose (M_1, M_2, α) is an object in $\text{Mod}_R(f, g)$ and let M be the associated fiber product module.

Step 3a. The map $i : M \rightarrow M_1$ factors, by the universal property of localization, through a morphism $i' : M_f \rightarrow M_1$. Analogously, let $j : M \rightarrow M_2$ and $j' : M_g \rightarrow M_2$ be the corresponding morphisms for the other localization. We claim that i' is an isomorphism.

To see that i' is injective, suppose $m \in M$ and $i'(\frac{m}{f^n}) = 0 \in M_1$. In that case, $i(m) = 0$ and therefore by the definition of fiber products $j(m) \in M_2$ localizes to zero in $(M_2)_f$, i.e., $f^m j(x) = j(f^m x) = 0$ for some integer $m \geq 0$. Since $f^m i(x) = i(f^m x) = 0$, we conclude that $f^m x = 0$. Therefore, $\frac{x}{f^n} = 0 \in X_f$.

To see that i' is surjective, suppose $m_1 \in M_1$ and let $m_2/f^r \in (M_2)_f$ correspond to m_1 under the isomorphism α . In that case, $f^r m_1$ and m_2 localize to the same element of $(M_1)_g \cong (M_2)_f$. Therefore, there exists an element $m \in M$ such that $i(m) = f^r m_1$ and $j(m) = m_2$. In that case, $i'(\frac{m}{f^r}) = m_1$ as desired.

By symmetry, we conclude that $j' : M_g \rightarrow M_2$ is an isomorphism.

Step 3b. On the other hand, by localizing, we see that i' induces an isomorphism $(M_f)_g \rightarrow (M_1)_g$ and j' induces an isomorphism $(M_g)_f \rightarrow (M_2)_f$. The above computations show that the following diagram commutes:

$$\begin{array}{ccc} (M_f)_g & \xrightarrow{i'} & (M_1)_g \\ \downarrow c & & \downarrow \alpha \\ (M_g)_f & \xrightarrow{j'} & (M_2)_f. \end{array}$$

In other words, we have constructed an isomorphism between (M_f, M_g, c) and (M_1, M_2, α) in $\text{Mod}_R(f, g)$.

Step 3c. A careful analysis of the construction in Step 3b and the definition of morphisms in the category $\text{Mod}_R(f, g)$ shows that the isomorphism constructed is natural in the input data. In other words, we have actually constructed a natural isomorphism from the identity functor to the composite $\text{Mod}_R(f, g) \rightarrow \text{Mod}_R \rightarrow \text{Mod}_R(f, g)$. Thus, the functor we have constructed is actually a quasi-inverse. \square

Example 3.3.3.3. Just as in topology, even if M_f and M_g are free, the fact that we used an automorphism of M_{fg} in the above shows that M need not be free. For a really cheap example, consider the ring $R \oplus R$. In that case, we can take f and g to be idempotents, e.g., $f = (1, 0)$ and $g = (0, 1)$ and observe that $f + g = (1, 1)$, which is the unit of $R \oplus R$. In that case, take $M = Rf$, which is not free. The two localizations M_f and M_g are both free, but M is evidently not. Geometrically, $\text{Spec } R \oplus R$ is the disjoint union of two copies of $\text{Spec } R$ (thus the intersection is zero) so patching free modules can produce non-free modules.

3.3.4 The general case

So far, we have been treating patching for pairs of modules. However, by a straightforward induction procedure we can obtain patching results for open covers by basic open sets with more elements. Indeed, suppose R is a ring and f_1, \dots, f_r generate the unit ideal. In that case, we can find elements g_1, \dots, g_r such that $\sum_i f_i g_i = 1$. Therefore, if we set $f = f_1$ and $g = \sum_{i=2}^r f_i g_i$, then f, g evidently generate the unit ideal of R as well. It follows that any cover of $\text{Spec } R$ by basic open sets can always be replaced by a cover consisting of two basic open sets.

3.4 Lecture 7A: Building projective modules

3.4.1 Descent of properties of modules

Given a property P of modules that is stable by localization, we can ask the following: if $(M_1, M_2, \alpha) \in \text{Mod}_R(f, g)$ is such that M_1, M_2 have property P , does the object $M \in \text{Mod}_R$ obtained by the equivalence of Theorem 3.3.3.2, i.e., the fiber product module, have property P as well?

Definition 3.4.1.1. A property P for R -modules that is stable by localization will be called *local for the Zariski topology on $\text{Spec } R$* if an R -module M has property P if and only if for any pair of comaximal elements $(f, g) \in R$, M_f and M_g have property P .

Consider the homomorphism $R \rightarrow R_f \oplus R_g$. Since R_f and R_g are localizations, they are both flat R -modules, and we conclude that $R_f \oplus R_g$ is a flat R -module, i.e., the homomorphism $R \rightarrow R_f \oplus R_g$ is a flat R -module.

Lemma 3.4.1.2. *Set $R' = R_f \oplus R_g$, with f and g comaximal. A sequence $M_1 \rightarrow M_2 \rightarrow M_3$ is exact if and only if $M_1 \otimes_R R' \rightarrow M_2 \otimes_R R' \rightarrow M_3 \otimes_R R'$ is exact.*

Proof. Since $R_f \oplus R_g$ is a flat R -module, then the “only if” direction is immediate. Therefore we focus on the “if” direction. Since $\text{Spec } R' \rightarrow \text{Spec } R$ is an open cover, it follows that for any maximal ideal \mathfrak{m} of R , $R'/\mathfrak{m}R'$ is non-zero (indeed, any maximal ideal is either in R_f or R_g). Take an arbitrary sequence $M_1 \rightarrow M_2 \rightarrow M_3$ such that $M_1 \otimes_R R' \rightarrow M_2 \otimes_R R' \rightarrow M_3 \otimes_R R'$ is exact. Consider the R -module $H = \ker(M_2 \rightarrow M_3)/\text{im}(M_1 \rightarrow M_2)$, which measures the failure of exactness of M . By assumption

$$H \otimes_R R' \cong \ker(M_2 \otimes_R R' \rightarrow M_3 \otimes_R R')/\text{im}(M_1 \otimes_R R' \rightarrow M_2 \otimes_R R') = 0.$$

Now, take an element $x \in H$. There is an induced R -module map $R \rightarrow H$. If $I = \{r \in R \mid rx = 0\}$, then this map factors through an injection $R/I \subset H$. Now, $R/I \otimes_R R' \cong R'/IR' \subset H \otimes_R R' = 0$ again by flatness of $R \rightarrow R'$. If $I \neq R$, then there is a maximal ideal \mathfrak{m} containing I , which yields a contradiction. \square

We will need comparisons between homomorphisms of modules over R and over extensions. Given a ring homomorphism $R \rightarrow S$, functoriality of extension of scalars yields a natural map $\text{Hom}_R(M, N) \rightarrow \text{Hom}_S(M \otimes_R S, N \otimes_R S)$; this factors through a map

$$\text{Hom}_R(M, N) \otimes_R S \rightarrow \text{Hom}_S(M \otimes_R S, N \otimes_R S).$$

If S is a flat R -module, then $\otimes_R S$ is an exact functor.

Proposition 3.4.1.3. *If $\varphi : R \rightarrow R'$ is a flat ring homomorphism, and if M and N are R -modules with M finitely presented (resp. finitely generated), then the map*

$$\text{Hom}_R(M, N) \otimes_R S \rightarrow \text{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism (resp. monomorphism).

Proof. Exercise (for the time being). \square

The properties of being finitely generated, finitely presented, or finitely generated projective are all stable under localization.

Proposition 3.4.1.4. *The property that an R -module is finitely generated, finitely presented, or finitely generated projective is local for the Zariski topology on $\text{Spec } R$.*

Proof. Finite generation. Suppose (M_1, M_2, α) is a patching datum with M_1 and M_2 finitely generated. We claim that the pullback is finitely generated as well. Indeed, pick elements x_1, \dots, x_n of M such that $i(x_1), \dots, i(x_n)$ form R_{f_1} -generators for M_1 and y_1, \dots, y_m of M such that $j(y_1), \dots, j(y_m)$ form R_{f_2} -generators of M_2 . Let $M' \subset M$ be the submodule generated by

$x_1, \dots, x_n, y_1, \dots, y_m$. Since M' localizes to M_1 and M_2 , we conclude that $M' \rightarrow M$ is an isomorphism by the universal property of fiber product.

Finite presentation. Suppose (M_1, M_2, α) is a patching datum with M_1 and M_2 finitely presented. By the previous part, we know the pullback module M is already finitely generated. In particular, we can pick a free module $R^{\oplus r}$ and a surjection $R^{\oplus r} \rightarrow M$ with kernel K . Localizing we get an exact sequence $K_f \rightarrow R_f^{\oplus r} \rightarrow M_1$ and similarly for M_2 . We want to show that K_f is finitely generated. By assumption, we can choose a presentation $R_f^{\oplus n} \rightarrow R_f^{\oplus m} \rightarrow M_1$. By lifting elements, we can construct a homomorphism $R_f^{\oplus m} \rightarrow R_f^{\oplus r}$, and by chasing the diagram obtain a map $R_f^{\oplus n} \rightarrow K_f$ fitting into the following commutative diagram:

$$\begin{array}{ccccc} R_f^{\oplus n} & \longrightarrow & R_f^{\oplus m} & \longrightarrow & M_1 \\ \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & K_f & \longrightarrow & R_f^{\oplus r} \longrightarrow M_1. \end{array}$$

By diagram chasing you can show that cokernel of the left hand map coincides with the cokernel of the middle map. The cokernel of the middle map is finitely presented and therefore so is the cokernel of the left hand map. Therefore, we conclude that K_f is an extension of finitely presented modules and it suffices to check that any such extension is again finitely presented (in particular finitely generated). By symmetry, we conclude that K_g is finitely generated. Therefore, K_f and K_g are finitely generated and by appeal to the previous part, we conclude that K is finitely generated as well.

Projectivity. Let M be the module obtained by patching. Without loss of generality, we can assume that $M_1 = M_f$ and $M_2 = M_g$. Suppose we know that M_f and M_g are finitely generated projective. In that case M_{fg} is also finitely projective and we would like to conclude that M is finitely generated projective itself. We already saw that M is finitely generated if M_1 and M_2 are, so we may pick generators x_1, \dots, x_n for M , i.e., a surjection $R^{\oplus n} \rightarrow M$. Then, since M_f and M_g are projective, we know that M_f is a summand of $R_f^{\oplus n}$ and M_g is a summand of $R_g^{\oplus n}$, i.e., after localization at f and g the morphism $R^{\oplus n} \rightarrow M$ is split surjective. It suffices to establish that if $\varphi : M \rightarrow N$ is a homomorphism of finitely presented R -modules, then φ is split surjective if and only if the map $M \otimes_R (R_f \oplus R_g) \rightarrow N \otimes_R (R_f \oplus R_g)$ is split surjective; this is accomplished in Lemma 3.4.1.5. Intuitively, we can pick splittings of $R_f^{\oplus n} \rightarrow M_f$ and analogously for M_g . After localizing, these two splittings need not coincide as maps $R_{fg}^{\oplus n} \rightarrow M_{fg}$, but we are claiming that, up to changing bases, we can make them coincide so that they patch to give the required surjection. \square

Lemma 3.4.1.5. *Let $\varphi : M \rightarrow N$ be a homomorphism of R -modules with N finitely presented over R . The homomorphism φ is a split surjection if and only if the map $\varphi' : R_f \otimes_R M \oplus R_g \otimes_R M \rightarrow R_f \otimes_R N \oplus R_g \otimes_R N$ is a split surjection.*

Proof. Set $R' := R_f \oplus R_g$; the map in question is obtained by tensoring with R' . We observed in Lemma 3.4.1.2 that φ is surjective if and only if φ' is surjective. Thus, we just want to show that φ is split if and only if φ' is split. A splitting $N \rightarrow M$ of φ yields a lift of any endomorphism

of N to a homomorphism $N \rightarrow M$, i.e., the map $\text{Hom}_R(M, N) \rightarrow \text{Hom}_R(N, N)$ is surjective. Conversely, if $\text{Hom}_R(M, N) \rightarrow \text{Hom}_R(N, N)$ is surjective, then a lift of $1 \in \text{Hom}_R(N, N)$ is a splitting. Therefore, φ splits, if and only if $\text{Hom}_R(\varphi, N)$ is surjective. Now, again by Lemma 3.4.1.2, we see that $\text{Hom}_R(\varphi, N)$ is surjective if and only if $\text{Hom}_R(\varphi, N) \otimes R'$ is surjective. Then, $\text{Hom}_R(\varphi, N) \otimes R' \cong \text{Hom}_{R'}(\varphi \otimes_R R', N \otimes_R R')$ by Proposition 3.4.1.3. To conclude, we unwind this last surjectivity as being equivalent to $\varphi \otimes_R R'$ is splits, which is precisely what we wanted to show. \square

Corollary 3.4.1.6. *Every finite rank locally free R -module is a finitely generated projective R -module. Moreover, the categories of finitely generated projective R -modules and locally free R -modules are equivalent.*

Proof. Given a locally free R -module, combining Theorem 3.3.3.2, Proposition 3.4.1.4, and a straightforward induction argument, we conclude that every locally free R -module is finitely generated projective. The homomorphism sets in each case are simply R -module homomorphisms so the claimed equivalence is an immediate consequence of the fact that every finitely generated projective R -module is locally free. \square

Exercise 3.4.1.7 (Vaserstein's Serre-Swan theorem). *If X is a topological space, write $C(X, \mathbb{R})$ (resp. $C(X, \mathbb{C})$) for the set of real (resp. complex) valued continuous functions on X . Show that the functor assigning to a real (resp. complex) vector bundle $\pi : E \rightarrow X$, the $C(X, \mathbb{R})$ -module (resp. $C(X, \mathbb{C})$ -module) of sections yields an equivalence between the category of real (resp. complex) vector bundles on X (as defined in Lecture 1) and the category of finitely generated projective $C(X)$ -modules.*

3.4.2 Rank

If R is a ring and P is a projective R -module, then for any prime ideal $\mathfrak{p} \subset R$, we see that $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$. By Proposition 3.2.3.5, if P is furthermore finitely generated, then there is a Zariski open set contained in $\text{Spec } R \setminus \text{Spec } R/\mathfrak{p}$ on which the rank is constant. Using this observation, one deduces that sending a projective module to the integer n described above yields a continuous function from $rk : \text{Spec } R \rightarrow \mathbb{N}$ (the latter viewed as a discrete topological space). Note, in particular, that rk of a projective module is bounded, and locally constant. Thus, if R is a connected ring, then the rank of a projective module is simply an integer.

Exercise 3.4.2.1. *If R is a commutative unital ring, show that $\text{Spec } R$ is connected if and only if R has no non-trivial idempotents.*

Remark 3.4.2.2. Because of the conclusion of the previous exercise, we will call a commutative unital ring R *connected* if it has no non-trivial idempotents. In general, we can attempt to form a decomposition of R using commuting idempotents. However, without some finiteness hypothesis on R , it is possible that $\text{Spec } R$ has infinitely many connected components: e.g., take any (say connected) ring R and form the ring $\bigoplus_{n \in \mathbb{N}} R$. Nevertheless, if we focus attention on finitely generated projective modules, then the rank of any projective module takes only finitely many values. While we focus on connected rings for simplicity, the observation just mentioned will allow us to make statements about general disconnected rings as well.

Example 3.4.2.3. As we observed above, invertible modules always have constant rank 1.

When L is an invertible R -module, $\mathrm{Hom}_R(L, R)$ is again an invertible R -module. There is a canonical evaluation map $M \otimes_R M^\vee \rightarrow R$. Moreover, in this case, the evaluation map $L \otimes \mathrm{Hom}_R(L, R) \rightarrow R$ is an isomorphism: the identity map $\mathrm{Hom}_R(\mathrm{Hom}(L, R), \mathrm{Hom}(L, R))$ corresponds under the hom- \otimes -adjunction to the evaluation map $\mathrm{Hom}_R(\mathrm{Hom}(L, R) \otimes L, R)$. Alternatively, the evaluation map is evidently locally an isomorphism and therefore must be an isomorphism in general. Thus, L is an invertible module, there is a distinguished choice for a module L' such that $L \otimes L' \cong R$, namely $\mathrm{Hom}_R(L, R)$.

Exercise 3.4.2.4. If P and Q are projective R -modules of rank m and n , then $\mathrm{rk}(P \oplus Q) = m + n$ and $\mathrm{rk}(P \otimes Q) = mn$.

3.5 Serre's splitting theorem

In this section, we establish Serre's splitting theorem, which is perhaps the first result that suggests that the translation of ideas between vector bundles in topology and vector bundles on affine algebraic varieties is profitable. If V is a rank r vector bundle on a manifold M of dimension d , suppose $d \geq r$. If V splits as the sum of a trivial bundle and a bundle of rank $r - 1$, then a basis for the trivial summand gives a section of V that vanishes nowhere on M . Conversely, any nowhere vanishing section corresponds to a direct summand.

3.5.1 A topological splitting result

We begin by establishing a topological splitting result that motivated the result of Serre. The argument is essentially a "general position" result in topology. Recall that two subspaces V_1 and V_2 of a n -dimensional vector space are said to be in general position if $\dim V_1 \cap V_2 \leq \dim V_1 + \dim V_2 - n$. Analogously, one can say that two smooth submanifolds S_1 and S_2 of a Euclidean space of dimension n are in general position at a point x in their intersection if their tangent spaces $T_x S_1$ and $T_x S_2$ are in general position in the sense just described. If S_1 and S_2 are at general position at every point of their intersection, then $\dim S_1 \cap S_2 \leq \dim S_1 + \dim S_2 - n$ and we will say that S_1 and S_2 are in general position [Hir94, p. 78]. In particular, if we fix a submanifold S_1 , then a general submanifold of dimension $\dim S_2$ will intersect S_1 in the given dimension.

Theorem 3.5.1.1. *Let M be a manifold of dimension d . If $\pi : E \rightarrow M$ is a finite rank topological vector bundle on M of rank $r > d$, then E is isomorphic $E' \oplus F$, where E' is vector bundle of rank $\leq d$ and F is a free module.*

Proof. By induction, it suffices to show that E has a nowhere vanishing section under the dimension hypotheses. Since we only care about such a section up to fiberwise rescaling, we can consider the associated fiber bundle whose fiber over a point $x \in M$ is the projective space $\mathbb{P}(E_x)$, and our goal is to construct a section of this fiber bundle.

We can assume without loss of generality that E is the quotient of a trivial bundle of some rank. Indeed, such a presentation corresponds to giving a finite-dimensional vector space V of sections of E . In that case, we have a surjection

$$V \longrightarrow E \longrightarrow 0$$

and we let E' be the kernel of this map. Thus, it suffices to show that there is a subspace W of V such that $\dim W = r - 1$ and $W \cap E'_x = 0$ for all $x \in M$. Consider the subspace P'_x corresponding to the projective space of E'_x in $\mathbb{P}(V)$. The union $\cup_{x \in M} P'_x$ is a subspace Y of $\mathbb{P}(V)$. Note that $\dim Y = d + \dim P'_x = d + \dim V - 1 - r$. Now, general position (as discussed before the statement) tells us that there is a dimension 1 subspace not intersecting Y so long as $r > d$. \square

3.5.2 Nowhere vanishing sections

Suppose R is a commutative unital ring and P is a finitely generated projective R -module. In this section, we begin by linking non-vanishing sections of projective modules with direct summands, in a fashion identical to the topological situation sketched above. If \mathfrak{m} is a maximal ideal of R , then we know that R/\mathfrak{m} is a field, and P/\mathfrak{m} is a R/\mathfrak{m} -module. If \mathfrak{m} corresponds to a point x of $\text{Spec } R$, $\kappa_x := R/\mathfrak{m}$ and $P(x)$ for P/\mathfrak{m} , which we think of as the *fiber* of P at x . Given an element $s \in P$, we write $s(x)$ for the image of s in P/\mathfrak{m} . Given elements s_1, \dots, s_n in P , we will say these elements are *linearly (in)dependent at x* if the elements $s_i(x)$ are linearly (in)dependent in the κ_x -vector space P/\mathfrak{m} .

Lemma 3.5.2.1. *If s_1, \dots, s_n are elements of P , and $x \in \text{Spec } R$ is a closed point, the collection of closed points $x \in \text{Spec } R$ where $s_1(x), \dots, s_n(x)$ are linearly dependent is the set of closed points of a closed subscheme of $\text{Spec } R$.*

Proof. We first treat the case where P is a free R -module of finite rank. In that case, if we pick a basis e_1, \dots, e_n for P each s_i can be written as $s_i = \sum_j a_{ij}e_j$ for $a_{ij} \in R$. In that case, the matrix $A = \{a_{ij}\}$ can be evaluated at x to obtain a matrix $A(x)$. To say that the s_i are linearly dependent at x is to say that the matrix $A(x)$ does not have maximal rank; this is given by the vanishing of the minors of the matrix A .

Since P is projective, we know that $P \oplus Q \cong R$ and we can reduce to the case of a free module. \square

Lemma 3.5.2.2. *If P is a finitely generated projective R -module, and x_1, \dots, x_n are pairwise distinct closed points of $\text{Spec } R$, then given $v_i \in P(x_i)$, $1 \leq i \leq n$, there exists a section $s \in P$ such that $s(x_i) = v_i$ for every i .*

Proof. Since the points x_1, \dots, x_n are pairwise distinct, the corresponding maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are pairwise comaximal, and $\sum_i \mathfrak{m}_i = R$. Therefore, we can find elements $\xi_i \in \mathfrak{m}_i$ such that $1 = \sum_i \xi_i$. Now, choose representatives $s_i \in P$ of the v_i and set $s = \sum_i \xi_i s_i$. Then, $s(x_j) = \sum_i \xi_i(x_j) s_i(x_j) = \xi_j(x_j) s_j(x_j) = v_j$. \square

Lemma 3.5.2.3. *Suppose R is a commutative unital ring, P is a projective R -module and $s \in P$. The map $R \rightarrow P$ defined by $r \mapsto rs$ identifies R as a direct summand of P if and only if $s(x) \neq 0$ for every closed point $x \in \text{Spec } R$.*

Proof. If the map in question identifies R as a direct summand of P , then $s(x) \neq 0$ since $s(x)$ is a basis of a 1-dimensional subspace of $P(x)$ for every closed point x . Conversely, suppose $\varphi : R \rightarrow P$ is the homomorphism defined by s . Let R' be the image of φ in P , i.e., the R -submodule of P generated by s . The composite map $R(x) \rightarrow R'(x) \rightarrow P(x)$ is injective and $R(x) \rightarrow R'(x)$ is evidently surjective. Therefore $R(x) \rightarrow R'(x)$ is bijective and we conclude that $R_{\mathfrak{m}} \rightarrow R'_{\mathfrak{m}}$ is an isomorphism for every \mathfrak{m} and thus $R \rightarrow R'$ is an isomorphism... \square

3.5.3 Algebraic general position arguments

Next, Serre mimicked the general position argument in topology to establish the algebro-geometric analog of the topological splitting theorem.

Theorem 3.5.3.1 ([Ser58b, Théorème 1]). *If R is a Noetherian ring of Krull dimension d , then every projective module of rank $r > d$ can be written as $P \oplus F$ where P has rank $\leq d$ and F is free.*

We will deduce this result from the following more precise statement.

Theorem 3.5.3.2. *Suppose R is a Noetherian ring of Krull dimension d , and $Z \subset X := \text{Spec } R$ a closed subset, x_1, \dots, x_n are pairwise distinct closed points of $\text{Spec } R$ contained in Z and fix $v_i \in P(x_i)$. Assume (i) P is a finitely generated projective R -module and s_1, \dots, s_r are elements of P that are linearly independent at every closed point $x \in X$ and (ii) there is given an integer $k \geq 0$ such that $r + k \leq \text{rk}_x(P)$ for every closed point $x \in X$. Then, there is an element $s \in P$ and a closed subset $Z' \subset X$ such that*

1. $s(x_i) = v_i$ for $1 \leq i \leq n$;
2. s_1, \dots, s_r are linearly independent at every point $x \notin Z \cup Z'$;
3. Z' has height $\geq k$.

Chapter 4

Picard groups, normality and \mathbb{A}^1 -invariance

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In this section, we will study the Picard group in more detail. The basic goal is to show that the Picard group is a portion of a cohomology theory in algebraic geometry. Along the way, we will describe, a 2-term complex whose cohomology computes the group of units and the Picard group of a ring. This complex is the first incarnation of a Cousin or Gersten complex. Using this

complex, we will show that under suitable hypotheses, the Picard group is another example of an \mathbb{A}^1 -invariant. To close we will discuss actual “computations” of the Picard group, which suggest connections between the kinds of invariants that appear here and other subjects (e.g., arithmetic).

4.1 Lecture 7B: Picard groups and functoriality

4.1.1 Picard groups

If R is a commutative unital ring, then we can consider the set of isomorphism classes of invertible R -modules. By Lemma 3.2.2.1, the set of isomorphism classes of invertible R -modules has a product structure corresponding to tensor product of R -modules. Associativity and commutativity of tensor product (up to natural isomorphism) show that this product is commutative and associative and the free R -module R is a unit for this multiplication. The operation of taking the dual determines an inversion function, and therefore the set of isomorphism classes of invertible R -modules is naturally a group.

Definition 4.1.1.1. If R is a commutative unital ring, the *Picard group of R* , denoted $\text{Pic}(R)$, is the group of isomorphism classes of invertible R -modules with the multiplication and inversion operations described above.

Lemma 4.1.1.2. If $f : R \rightarrow S$ is any ring homomorphism, then extension of scalars defines a homomorphism $f^* : \text{Pic}(R) \rightarrow \text{Pic}(S)$.

Proof. This amounts to the fact that pullbacks commute with tensor products. \square

4.1.2 Invertible modules and line bundles

We now analyze the results of the previous section in the special case of invertible modules. Since invertible modules are always finitely generated projective, it follows from Corollary 3.4.1.6 that we can build invertible modules by gluing free rank 1 modules on localizations.

Construction 4.1.2.1 (Invertible modules are line bundles). Suppose $X = \text{Spec } R$ is an affine variety and L is an invertible R -module. Now, suppose we are given f_1, \dots, f_r that generate the unit ideal, such that L_{f_i} is a free R_{f_i} -module of rank 1. An isomorphism of free rank 1 modules over a ring is simply given by multiplication by a unit. A choice of isomorphism $R_{f_i} \xrightarrow{\sim} L_{f_i}$ for each i corresponds to a basis element $\sigma_i \in L_{f_i}$. The basis elements σ_i and σ_j both determine basis elements of $(L_{f_i})_{f_j} = (L_{f_j})_{f_i}$ and these two basis elements differ by an element of $R_{f_i f_j}^\times$. Write α_{ij} for the unit such that $\alpha_{ij}\sigma_j = \sigma_i$. Thus, $\alpha_{11} = 1$ by convention.

The units so obtained necessarily satisfy some compatibilities. Note first that $\alpha_{ij}\alpha_{ji} = 1$. On the other hand, given a triple i, j, k of indices, we obtain three basis elements σ_i, σ_j and σ_k . In that case, $\alpha_{ik}\sigma_k = \alpha_{ij}\alpha_{jk}\sigma_k$ so $\alpha_{ik} = \alpha_{ij}\alpha_{jk}$. Since multiplication of units is associative, all “higher” compatibilities may be deduced from this one. In this way, one attaches to any invertible module a sequence $(\alpha_{ij} \in R_{f_i f_j}^\times)$ satisfying $\alpha_{ij}\alpha_{ji} = 1$ and the cocycle condition $\alpha_{ij}\alpha_{jk}\alpha_{ki} = 1$. Conversely, such data uniquely determines an invertible R -module L .

Remark 4.1.2.2. If one chooses a different collection of g_1, \dots, g_s that generate the unit ideal and such that L_{g_i} is a free rank 1 R_{g_i} -module, then one obtains a different description of the module L . Thus, there is a lot of redundancy in the description given here.

Example 4.1.2.3 (Projective modules are vector bundles). A similar statement holds for projective R -modules of constant rank n . In that case, we fix f_1, \dots, f_r as above, we pick bases $\sigma_i^1 \cdots \sigma_i^n$ and we attach an $n \times n$ -matrix $X_{ij} \in GL_n(R_{f_i f_j})$ expressing change of basis from the basis σ_i to the basis σ_j . As above, these matrices satisfy $X_{ii} = Id$, $X_{ij} X_{ji} = Id_n$ and $X_{ij} X_{jk} X_{ki} = Id_n$ for all indices.

4.2 Lecture 8: Line bundles and divisors

At the end of the last lecture we gave a description of invertible R -modules in terms of line bundles. Now, we provide another equivalent description that links invertible R -modules with subvarieties of $\text{Spec } R$.

4.2.1 Cartier divisors

We begin by treating the case where $X = \text{Spec } R$ is an integral domain with fraction field K . If $X = \text{Spec } R$ is an integral domain, then $\text{Spec } R$ is connected. The restriction that R is an integral domain excludes rings of continuous functions on topological spaces, but we will explain later how to connect with that situation.

The map $R \hookrightarrow K$ is an inclusion. If P is a (finitely generated) projective module, then since P is flat we conclude that $P \otimes_R R \rightarrow P \otimes_R K$ is again an injective map of R -modules. Now, $P \otimes_R K$ is also a finitely generated K -module, i.e., a finite-dimensional K -module. In particular, if P has rank r , we can fix an isomorphism $P \otimes_R K \xrightarrow{\sim} K^r$. If we pick a surjection $R^{\oplus n} \rightarrow P$, then under the isomorphism $P \otimes_R K \xrightarrow{\sim} K^r$ it makes sense to speak of the image of these R -module generators in $K^{\oplus r}$ determines an $n \times r$ -matrix of elements of K .

We analyze the case where $P = L$ has rank 1 in more detail. In this case, we fix an identification $L \otimes_R K \xrightarrow{\sim} K$. Via this identification, a surjection $R^{\oplus n} \rightarrow L$ yields a sequence of n -elements in K , $\sigma_1, \dots, \sigma_n$. Each σ_i can be written as $\frac{r_i}{s_i}$ where $s_i \in R \setminus 0$. Therefore, L is the R -submodule of K generated by $\sigma_1, \dots, \sigma_n$. Now, if we multiply through by the product of the s_i (i.e., clear the denominators), then every element of sL is an element of R , i.e., sL is an ideal. For this reason, one makes the following definition.

Definition 4.2.1.1. If R is an integral domain, a *fractional ideal* I in R is an R -submodule $I \subset K$ such that there exists a (non-zero) element $r \in R$ with $rI \subset R$.

Remark 4.2.1.2. We have imposed no finiteness hypotheses on I in the above definition so that ideals are always examples of fractional ideals. If R is Noetherian, then since $rI \subset R$, we conclude that I is necessarily finitely generated.

So far, we have only use the property that L has rank 1, but not that L is actually an invertible R -module. Now, we know $\text{Hom}_R(L, R)$ is an invertible R -module and the evaluation map yields the isomorphism $L \otimes_R \text{Hom}_R(L, R) \rightarrow R$. If I' is the invertible ideal attached to L^\vee , then $I \otimes_R I' \cong R$.

Definition 4.2.1.3. If R is an integral domain, an *invertible fractional ideal* I in R is a fractional ideal I in R for which there exists an invertible ideal I' with $I \otimes_R I' \cong R$.

Remark 4.2.1.4. Note that invertible fractional ideals are automatically finitely presented ideals, since invertible modules are finitely presented by Lemma 3.2.1.9.

The tensor product of R -modules again equips the set of (isomorphism classes of) invertible fractional ideals with a group structure. We write $\text{Cart}(R)$ for the set of invertible fractional ideals on R with this group structure. Since invertible fractional ideals are evidently invertible R -modules it follows that there is a group homomorphism $\text{Cart}(R) \rightarrow \text{Pic}(R)$. Moreover, since we can attach an invertible fractional ideal to any invertible R -module, it follows that this homomorphism is surjective.

Remark 4.2.1.5. An alternative way to think about invertible fractional ideals, via the description provided above, is that they correspond to invertible R -modules L together with a choice of embedding $L \hookrightarrow K$. The map $\text{Cart}(R) \rightarrow \text{Pic}(R)$ then corresponds to the map that simply forgets the embedding.

Note that each choice of generators for an invertible R -module yields an a priori different invertible fractional ideal presentation, so we do not expect the homomorphism $\text{Cart}(R) \rightarrow \text{Pic}(R)$ to be surjective in general. However, we can describe the kernel of this homomorphism rather explicitly. Indeed, the identity element of $\text{Pic}(R)$ is the trivial R -module R . Thus, we want to describe invertible ideal structures on R itself. Given an invertible fractional ideal I and a chosen isomorphism $\varphi : I \rightarrow R$ as an R -module, $\varphi^{-1}(1)$ determines an element of I . The element $\varphi^{-1}(1)$ is a non-zero element f of K , i.e., a rational function on R , since $I \subset K$ is a non-zero R -submodule. Note that this fractional ideal is evidently invertible since $R \cdot f^{-1}$ is an inverse. In this way, we obtain a map

$$\text{div} : K^\times \longrightarrow \text{Cart}(R)$$

called the divisor map.

The image of the above map surjects onto the kernel of $\text{Cart}(R) \rightarrow \text{Pic}(R)$, but there is some redundancy: we must identify those invertible fractional ideals where bases differ by invertible change of coordinates. Now, two different bases $R \cdot f$ and $R \cdot f'$ are identified if and only if they differ by an element of R^\times . Therefore, the map $\text{div} : K^\times \rightarrow \text{Cart}(R)$ factors through an injection

$$0 \longrightarrow K^\times / R^\times \longrightarrow \text{Cart}(R) \longrightarrow \text{Pic}(R) \longrightarrow 0.$$

Alternatively, we have established the following result.

Proposition 4.2.1.6 (Units-Pic sequence). *If R is any integral domain, there is an exact sequence*

$$1 \longrightarrow R^\times \longrightarrow K^\times \longrightarrow \text{Cart}(R) \longrightarrow \text{Pic}(R) \longrightarrow 0.$$

Example 4.2.1.7. The theory of fractional ideals is probably most familiar from number theory. A number field K is a finite extension of \mathbb{Q} . The ring of integers \mathcal{O}_K in K is the integral closure of \mathbb{Z} in K . In this situation, the Picard group is more commonly known as the ideal class group and measures the failure of unique factorization. Take $K = \mathbb{Q}(\sqrt{-5})$. Note that unique factorization fails in $\mathbb{Q}(\sqrt{-5})$ since $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Let $J = (2, 1 - \sqrt{-5})$. One shows that $J^2 = (2)$, which is principal. In fact, $\text{Pic}(R)$ is cyclic of order 2 generated by J . More generally, it is a fantastic fundamental result in algebraic number theory that $\text{Pic}(\mathcal{O}_K)$ is always a finite abelian group.

Example 4.2.1.8. If R is a principal ideal domain, it follows from the structure theorem that $\text{Pic}(R) = 0$. In particular, $\text{Pic}(k[t]) = 0$. More generally, any localization of a PID is a PID, so we conclude that Picard groups of (non-empty) proper open subsets of \mathbb{A}_k^1 also have trivial Picard groups.

4.2.2 Cartier divisors vs. invertible fractional ideals

We now give another equivalent characterization of invertible R -modules. Suppose I is an invertible fractional ideal. Since I is an invertible R -module, we can think of I as a locally free rank 1 module equipped with an embedding $I \hookrightarrow K$. Now, suppose we find elements f_1, \dots, f_n that generate the unit ideal in R such that I_{f_i} is a free R_{f_i} -module of rank 1. In this case, the embedding $I \hookrightarrow K$ factors through an embedding $I_{f_i} \hookrightarrow K$. A choice of basis of I_{f_i} determines a section $\sigma_i \in I_{f_i}$, which we view as an element of K . As in the description of line bundles we obtain generating sections $\sigma_1, \dots, \sigma_n \in K$. Now, the equality $\alpha_{ij}\sigma_j = \sigma_i$ translates to the equality $\alpha_{ij} = \frac{\sigma_i}{\sigma_j}$. In other words, the rational function $\frac{\sigma_i}{\sigma_j}$ is an element of $R_{f_i f_j}^\times$; in this case we will say that $\frac{\sigma_i}{\sigma_j}$ is a unit on $U_i \cap U_j$ for each i, j .

Definition 4.2.2.1. If $X = \text{Spec } R$ is an integral affine variety, then a *Cartier divisor* D on X is a collection $\{U_i, f_i\}$, where U_i is an open cover of X by principal open sets and $\sigma_i \in K$ is a rational function such that σ_i/σ_j is a unit on $U_i \cap U_j$ for each i, j .

Conversely, given a Cartier divisor, by reversing the procedure described above, we can obtain an invertible fractional ideal. Indeed, given $\{U_i, \sigma_i\}$ as in the definition, we consider the R -submodule of K defined by $\sigma_1, \dots, \sigma_n$ and this submodule is, by construction, locally free of rank 1 and therefore invertible. In other words, we have established a bijection between Cartier divisors on X and invertible fractional ideals in R . This bijection can be enhanced to respect the group structures: one simply equips the set of Cartier divisors with the group structure obtained by transporting the group structure on invertible fractional ideals by means of the bijection.

First, in order to compare two Cartier divisors $D = \{U_i, \sigma_i\}$ and $E = \{V_j, \tau_j\}$, since the open covers $\{U_i\}$ associated with D and $\{V_j\}$ associated with E need not coincide, we will use the open cover given by $\{U_i \cap V_j\}$ which refines both. To say that two fractional ideal coincide if they coincide as invertible fractional ideals translates to the condition that if $\sigma_i \tau_j^{-1}$ is a unit on $U_i \cap V_j$. The sum $D + E$ is given by the functions $\{U_i \cap V_j, \sigma_i \tau_j\}$, and the inverse is given by $\{U_i, \sigma_i^{-1}\}$.

The map $\text{div} : K^\times \rightarrow \text{Cart}(R)$ can be described in terms of Cartier divisors as well: sending a rational function f to the Cartier divisor $\{X, f\}$ yields the required map. If D is a Cartier divisor, we write $\mathcal{O}_X(D)$ for the invertible R -module associated with D (this is obtained by the gluing construction). In this way, we obtain an alternative identification of the units-Pic sequence described in the previous section.

4.2.3 The units-Pic sequence for general commutative rings

In the discussion above, we restricted attention to integral domains, but this was only a technical convenience. Rings of continuous functions will not, in general, be integral domains. Moreover, typically they have many zero divisors (e.g., functions with bounded but disjoint supports). We now observe that with slightly more work, the theory developed above holds equally well for rings that are integral domains; we will keep rings of continuous functions in the back of our head.

Definition 4.2.3.1. If R is a commutative ring, the total quotient ring of R , denoted $\text{Frac}(R)$, is the localization of R at the multiplicative set of all non-zero divisors.

In this generality, $\text{Frac}(R)$ is no longer a field. Nevertheless, since we are inverting precisely the non-zero-divisors in R , the map $R \rightarrow \text{Frac}(R)$ is injective. Thus, if L is an invertible R -module, the map $L \rightarrow L \otimes_R \text{Frac}(R)$ remains injective. However, we can no longer assert anything about $L \otimes_R \text{Frac}(R)$; this may be a non-free $\text{Frac}(R)$ -module! We can still analyze the extension of scalars homomorphism $\text{Pic}(R) \rightarrow \text{Pic}(\text{Frac}(R))$. The kernel of this map consists precisely of those invertible R -modules such that $L \otimes_R \text{Frac}(R)$ is a free rank 1 $\text{Frac}(R)$ -module.

Suppose we are given an invertible R -module such that $L \otimes_R \text{Frac}(R)$ is a free rank 1 $\text{Frac}(R)$ -module. By choosing generators of L , one obtains an R -submodule I of $\text{Frac}(R)$ generated by finitely many elements $\sigma_1, \dots, \sigma_n$. Clearing the denominators, we conclude that $sI \subset R$.

Definition 4.2.3.2. If R is a commutative ring, then an invertible fractional ideal in R is an invertible R -submodule of $\text{Frac}(R)$, such that $L \otimes_R \text{Frac}(R)$ is a free rank 1 $\text{Frac}(R)$ -module. Write $\mathbf{I}(R)$ for the set of invertible fractional ideals.

As before the set of invertible fractional ideals is a group under tensor product of R -modules, and there is, by construction an exact sequence of the form

$$\mathbf{I}(R) \rightarrow \text{Pic}(R) \rightarrow \text{Pic}(\text{Frac}(R)).$$

The kernel of the map $\mathbf{I}(R) \rightarrow \text{Pic}(R)$ once again consists of invertible fractional ideal structures on the trivial R -module. A choice of basis of a free rank 1 R -submodule of $\text{Frac}(R)$ is uniquely determined by an element $u \in \text{Frac}(R)^\times$. Two such choices of basis differ by an element of R^\times and therefore, just as above one obtains an exact sequence of the form

$$1 \rightarrow R^\times \rightarrow (\text{Frac}(R))^\times \rightarrow \mathbf{I}(R) \rightarrow \text{Pic}(R) \rightarrow \text{Pic}(\text{Frac}(R)),$$

which no longer need be exact on the right.

Remark 4.2.3.3. If $\varphi : R \rightarrow S$ is any R -module map, then the kernel of $\text{Pic}(R) \rightarrow \text{Pic}(S)$ coincides precisely with the set of invertible R -modules L such that $L \otimes_R S \cong S$; we will call such objects invertible R -submodules of S and we write $\text{Pic}(\varphi)$ or $\text{Pic}(R, S)$ for the set of isomorphism classes of such objects. This set is a group with respect to tensor product of R -modules. Arguing as above, the kernel of the map $\text{Pic}(\varphi) \rightarrow \text{Pic}(R)$ corresponds to invertible R -submodule structures on the trivial module $R \otimes_R S$, which correspond to elements of S^\times modulo the image of R^\times (which need not inject in S^\times in general). In other words, one obtains an exact sequence of the form

$$R^\times \rightarrow S^\times \rightarrow \text{Pic}(\varphi) \rightarrow \text{Pic}(R) \rightarrow \text{Pic}(S)$$

If R is a subring of S , then we can even assert that the left hand map is injective. Functoriality of the resulting exact sequence is a consequence of functoriality of extension of scalars.

The identification of the above sequence in terms of Cartier divisors is slightly more complicated, but proceeds as before. Suppose I is an invertible R -submodule of $\text{Frac}(R)$ (such a thing is free of rank 1 as a $\text{Frac}(R)$ -module by “clearing the denominators”). We may choose a local

trivialization of I . In other words, we may find elements f_1, \dots, f_n such that I_{f_i} is a free R_{f_i} -module of rank 1 and such that $\{f_1, \dots, f_n\}$ generates the unit ideal. The map $R \mapsto R_{f_i}$ induces a homomorphism $\text{Frac}(R) \rightarrow \text{Frac}(R_{f_i})$. Now, there is a commutative square of the form

$$\begin{array}{ccc} R & \longrightarrow & R_{f_i} \\ \downarrow & & \downarrow \\ \text{Frac}(R) & \longrightarrow & \text{Frac}(R_{f_i}). \end{array}$$

From this it follows that there is a canonical isomorphism $I_{f_i} \otimes_R \text{Frac}(R_{f_i}) \cong (I \otimes_R \text{Frac}(R)) \otimes_{\text{Frac}(R)} \text{Frac}(R_{f_i})$. Since $I \otimes_R \text{Frac}(R)$ is a free $\text{Frac}(R)$ -module of rank 1, it follows that so is $I_{f_i} \otimes_R \text{Frac}(R_{f_i})$. In any case, our choice of trivialization determines an element $\sigma_i \in \text{Frac}(R_{f_i})$. As before, the formula $\alpha_{ij}\sigma_j = \sigma_i$ holds in $R_{f_i f_j}$. It follows that $\frac{\sigma_i}{\sigma_j}$ must be an element of $R_{ij}^\times \in \text{Frac}(R_{f_i f_j})^\times$. Therefore, if we define a Cartier divisor on $\text{Spec } R$ to be a collection $D = \{U_i, \sigma_i\}$ where U_i is an open cover by principal open sets and where $\sigma_i \in \text{Frac}(R_{f_i})$ such that σ_i/σ_j is a unit on $U_i \cap U_j$, then we see that there is a bijection between Cartier divisors and invertible R -submodules of $\text{Frac}(R)$ just as in the case where X is integral. The fundamental difference is that we may no longer be able to identify $\text{Frac}(R_{f_i})$ for different values of i .

Example 4.2.3.4. If X is a compact manifold, and we take $R = C(X, \mathbb{R})$ the ring of real-valued continuous functions on X . Given a partition of unity $\{f_i\}_{i=1, \dots, n}$, the f_i are typically zero divisors: if g is any compactly supported function with support disjoint from f_i , then $f_i g = 0$. For example, $f_i f_j$ might even be zero. If f_i is a zero divisor, then the $R \rightarrow R_{f_i}$ is not injective. It follows that the map $\text{Frac}(R) \rightarrow \text{Frac}(R_{f_i})$ is not injective in general either.

Picard groups of non-reduced rings

Now, suppose R is a connected commutative unital ring and N is the nilradical of R . The nilradical is always contained in the Jacobson radical $N \subset J(R)$. As a consequence, we may appeal to Nakayama's lemma to compare finitely generated R -modules and finitely generated R/N -modules.

Proposition 4.2.3.5. *If R is a commutative unital ring, and P, P' are projective R -modules, then if $P/N \cong P'/N$ then $P \cong P'$.*

Using this fact, we observe that if we want to study Picard groups of commutative rings, we can always assume that our rings are reduced by passing from R to R/N .

Corollary 4.2.3.6. *If R is a commutative unital ring, then the map $\text{Pic}(R) \rightarrow \text{Pic}(R/N)$ is an isomorphism.*

Example 4.2.3.7. Suppose X is a compact Hausdorff space and $R = C(X, \mathbb{R})$, the ring of real valued continuous functions. If f is a nilpotent element of $C(X)$, then $f^n = 0$ for some integer n . This means $f^n(x) = 0 \in \mathbb{R}$, which means $f(x) = 0$. In other words, in this case the ring of continuous functions on X is reduced. In fact, the maximal ideals in $C(X)$ have been characterized by Gelfand-Kolmogoroff (cf. [GJ60, Chapter 7]): they are parameterized by the points $x \in X$: \mathfrak{m}_x is the ideal of functions vanishing at a point. It follows that $\bigcap_{x \in X} \mathfrak{m}_x = 0$. Thus, in the case of continuous functions, the real differentiating feature is the presence of zero-divisors. The prime ideal structure of such rings is much more complicated (see, e.g., [Koh58],[GJ60, Chapter 14]).

4.3 Lecture 9: Curves and some geometry

We now attempt to compute Picard groups of some simple integral varieties. Since we view dimension as a reasonable measure of complexity of a variety, we start with low-dimensional examples. Suppose R is a commutative domain of Krull dimension 0. In that case, we know that (0) is the only prime ideal and furthermore that it is maximal. In other words, R is simply a field. If R is not a domain, then situation is more interesting, but perhaps less geometric, so we leave this for later. Arguably the first geometrically interesting case to consider is that were R has Krull dimension 1.

4.3.1 Dedekind domains: examples

Definition 4.3.1.1. A commutative unital ring R is called a *Dedekind domain* if it is a Noetherian integral domain, integrally closed in its field of fractions, and has Krull dimension 1.

Remark 4.3.1.2. To say that R has Krull dimension 1 (see Definition 2.1.1.25) is to say that every chain of prime ideals is of the form $\mathfrak{p}_0 \subset \mathfrak{p}_1$. Since R is an integral domain that has Krull dimension 1, then we know that (0) is a prime ideal, and therefore that any non-zero prime ideal is maximal.

Examples of Dedekind domains

Directly from the definitions, one sees that for any field k , $k[x]$ and \mathbb{Z} are Dedekind domains. We first establish a way to produce new Dedekind domains from old ones.

Proposition 4.3.1.3. *If R is a Dedekind domain with fraction field K and L is a finite separable extension of K , then the integral closure S of R in L is a Dedekind domain as well.*

Proof. First, we prove that S is a Noetherian domain. To this end, we will show that it is a sub- R -module of a finite rank free R -module and therefore Noetherian as well (that it is a domain is left as an exercise). For any extension L/K , we can consider the trace pairing $L \times L \rightarrow K$ given by $(x, y) \mapsto \text{Tr}_{L/K}(xy)$ (recall that we view L as a K -vector space and take the trace). If L/K is separable, then the trace pairing is non-degenerate.

We claim that for any element $x \in L$, if x is integral over R , then $\text{Tr}_{L/K}(x) \in R$. This follows from two facts: (i) the minimal polynomial of x has coefficients in R and (ii) if P is the minimal polynomial of x , d is the degree of P , and $[L : K] = ed$ for some integer e , then $\text{Tr}_{L/K}(x) = -ea_1$, where a_1 is the coefficient of x^{d-1} in the minimal polynomial. For (i), if we take any monic polynomial Q with coefficients in R satisfied by x (such a polynomial exists since x is integral over R), then the minimal polynomial P divides Q . In this case, one shows that the coefficients of P are integral over R (exercise!) and since R is integrally closed, must lie in R .

Now, pick $x_1, \dots, x_n \in L$ that are integral over R and that form a K -basis for L . The integral closure S of R is contained in the module $M := \{y \in L \mid \langle x_i, y \rangle \in R, i = 1, \dots, n\}$. There is an induced isomorphism $M \cong R^{\oplus n}$ and since $S \subset R^{\oplus n}$, since R is Noetherian, S is a finitely generated R -module. We conclude that S is also a Noetherian domain.

Since S is integrally closed in its field of fractions it remains to show that S has Krull dimension 1 if R has the same property. To this end, we analyze chains of prime ideals in S . Let \mathfrak{P} be a non-zero prime ideal of S and let $\mathfrak{p} = \mathfrak{P} \cap R$. We claim that \mathfrak{p} is non-zero. Indeed, if we pick a non-zero element x of \mathfrak{P} , then since x is integral over R , we see that x satisfies a monic polynomial with

coefficients in R and we can choose one f of minimal degree. This polynomial necessarily has non-zero constant term (if not, this would contradict minimality). Moreover, the equation shows that the constant term is in the ideal (x) .

Now, if $\mathfrak{P} \subset \mathfrak{Q}$ is a proper inclusion of prime ideals in S , then setting $\mathfrak{q} = \mathfrak{Q} \cap R$ we conclude that there is an inclusion $\mathfrak{p} \subset \mathfrak{q}$. One may check that this inclusion is proper as well. Thus, if R has Krull dimension 1, S must have Krull dimension 1 as well. \square

Remark 4.3.1.4. The following result is known as the Krull-Akizuki theorem [Mat89, Theorem 11.7]: if R is a Noetherian integral domain with field of fractions K and having Krull dimension 1, L is a finite algebraic extension of K and S is a ring with $R \subset S \subset L$, then S is a Noetherian ring of Krull dimension ≤ 1 . From this one deduces [Mat89, p. 85], that if R is any Noetherian integral domain of Krull dimension 1, and L is any finite algebraic extension of the fraction field of R , then the integral closure \bar{S} of R in L is a Dedekind domain. In particular, separability is not necessary in the statement.

Example 4.3.1.5. Since \mathbb{Z} is an integral domain, the integral closure \mathcal{O}_K of \mathbb{Z} in a finite extension K of \mathbb{Q} is a Dedekind domain. Likewise, $k[x]$ is a Dedekind domain for any field k . Given any finite separable extension E of $k(x)$ the integral closure of $k[x]$ in E is a Dedekind domain.

Definition 4.3.1.6. If R is a Noetherian integral domain of Krull dimension 1 that contains a field k , then $\text{Spec } R$ is called an *affine curve* over k .

Example 4.3.1.7. Suppose $f \in k[x]$ is a non-zero polynomial, and consider the equation $y^r - f(x)$. Assume r is invertible in k (i.e., r is coprime to the characteristic exponent of k). If f is a separable polynomial (i.e., f has no repeated roots upon passing to an algebraic closure of k), then you can check that $P := y^r - f(x)$ is irreducible over $k(x)$ and we can consider its splitting field E over $k(x)$. In that case, we can form the integral closure R of $k[x]$ in E .

Note that there is a ring homomorphism $k[x] \rightarrow R$ by definition. There is also a ring homomorphism $k[x, y]/(y^r - f(x)) \rightarrow R$ by construction. The fraction field of $k[x, y]/(y^r - f(x))$ coincides with E and you can check that $k[x, y]/(y^r - f(x))$ is integrally closed in its field of fractions.

The map $k[x] \rightarrow R$ factors as the inclusion $k[x] \rightarrow k[x, y] \rightarrow k[x, y]/(y^r - f)$. If we set $C = \text{Spec } k[x, y]/(y^r - f)$, then we have the composite map $p : C \rightarrow \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1$. The composite map is the inclusion followed by the “projection onto x ”. We now study the fibers of this map. If \mathfrak{m} is a maximal ideal of $k[x]$, then we let $\kappa := k[x]/\mathfrak{m}$. Observe that $R/\mathfrak{m}R$ is a κ -algebra, and we can describe this κ -algebra explicitly. Indeed, the element f has value $\bar{f} \in \kappa[x]/\mathfrak{m}$. In that case, the algebra $R/\mathfrak{m}R$ can be identified as $\kappa[y]/(y^r - \bar{f})$. Note that $\kappa[y]/(y^r - \bar{f})$ is an algebra of dimension precisely r . If $\bar{f} = 0$ (i.e., f vanishes at the closed point corresponding to \mathfrak{m}) then this algebra is $\kappa[y]/y^r$, i.e., it has only one closed point with “nilpotent fuzz.” On the other hand, if \bar{f} has an r -th root in κ , then $\kappa[y]/(y^r - \bar{f})$ is $\kappa \oplus \cdots \oplus \kappa$ as a κ -algebra, and the fiber has r distinct points. In general, the structure of the fiber depends on the roots of \bar{f} in κ .

The picture we are describing here is a slight refinement of the usual ideal from complex analysis that $p : C \rightarrow \mathbb{A}_k^1$ is a branched cover of \mathbb{A}_k^1 branched along the locus where f vanishes. Indeed, the description above shows the kind of additional information that is kept beyond just keeping track of the number of points in the fiber of p .

Example 4.3.1.8 (The cusp). Dedekind domains that contain a field give examples of affine curves over k . Not all curves over a field k are Dedekind domains. Indeed, take the ring $R = k[x, y]/(y^2 -$

x^3), which is usually called the cuspidal cubic curve (cusp for short). You can check R is a Noetherian domain that has Krull dimension 1. Since the cusp can be parameterized by $x(t) = t^2$ and $y(t) = t^3$ there is a ring homomorphism $k[x, y]/(y^2 - x^3) \rightarrow k[t]$ (identify the former with $k[t^2, t^3]$) and thus the fraction field of R is a subfield of $k(t)$. Since $y/x = t$, it follows that the field of fractions of R contains $k(t)$ as well and thus it is equal to $k(t)$.

The fact that t is not in R shows that R is not integrally closed in $k(t)$. Indeed, the equation $z^2 - y = 0$ is a monic polynomial that, since $y = t^2$, has the root t in $k(t)$ and consequently no root in R . The integral closure of R in $k(t)$ is obtained by adjoining t to $k[x, y]/(y^2 - x^3)$, i.e., it is $k[t]$. The map $R \rightarrow k[t]$ corresponds to a map $\mathbb{A}_k^1 \rightarrow \text{Spec } R$. This integral closure is a Dedekind domain.

4.3.2 Local Dedekind domains: equivalent characterizations

We studied projective/invertible modules from a “local–global” point of view: locally such modules are free, and we can obtain all such modules by patching together local information. We will study Dedekind domains in a similar fashion: we analyze local Dedekind domains first, characterize such things, and then attempt to patch the information together.

Local Dedekind domains

We now proceed to characterize local Dedekind domains. If (R, \mathfrak{m}) is a local Dedekind domain, then R has a unique non-zero ideal, which is necessarily the maximal ideal \mathfrak{m} . We begin by analyzing a special case, namely consider the localization of $k[x]$ at the maximal ideal \mathfrak{m}_0 ; this is certainly a local Dedekind domain with maximal ideal $\mathfrak{m}_0 k[x]_{\mathfrak{m}_0}$. We can take the polynomial x as a generator of the ideal \mathfrak{m}_0 . In that case, any element of $k[x]_{\mathfrak{m}_0}$ can be written uniquely as $x^r u$ and the number r is the order of vanishing of f at 0. This defines a function from $k[x]_{\mathfrak{m}_0} \rightarrow \mathbb{N}$ and if we restrict to non-zero elements, it is a surjective monoid homomorphism, i.e., $\text{ord}_0(fg) = \text{ord}_0(f) + \text{ord}_0(g)$. Furthermore, it satisfies $\text{ord}(f + g) \geq \min(\text{ord}(f), \text{ord}(g))$. Now, if $f \in k(x)^\times$, then either $f \in k[x]_{\mathfrak{m}_0}$ or $f^{-1} \in k[x]_{\mathfrak{m}_0}$ and therefore, we can extend ord to a (surjective) group homomorphism $k(x)^\times \rightarrow \mathbb{Z}$ preserving the additional inequality. It is convenient to define $\text{ord}_0(0) = \infty$ so that $\text{ord}(x + -x) = \infty \geq \min(\text{ord}(x), \text{ord}(-x))$ (this helps to remember the inequality). (Note: alternatively, we could have spoken about the order of pole of a function; in this case, the inequality would be reversed.) We now abstract these facts.

Definition 4.3.2.1. If K is a field, then a *discrete valuation (of rank 1) on K* is a surjective homomorphism $\nu : K^\times \rightarrow \mathbb{Z}$ such that $\nu(f + g) \geq \min(\nu(f), \nu(g))$. The valuation ring associated with a discrete valuation is $\mathcal{O}_\nu := \{x \in K \mid \nu(x) \geq 0\}$.

We will extend ν to a function on K by setting $\nu(0) = \infty$. In this case, an easy way to remember the inequality is that $\nu(0) \geq \min(\nu(f), \nu(-f))$ for any f .

Lemma 4.3.2.2. Suppose (R, ν) is a discrete valuation ring.

1. A non-zero element $r \in R$ is a unit if and only if $\nu(r) = 0$.
2. The function $\nu : R \setminus 0 \rightarrow \mathbb{Z}$ is a Euclidean norm, i.e., R is a Euclidean domain and thus a PID.

3. If π is any element with $\nu(\pi) = 1$ (a local uniformizing parameter), any element $r \in R \setminus 0$ (resp. K^\times) can be written uniquely as $\pi^n u$ for $n = \nu(r) \geq 0$ (resp. n an integer) and u a unit in R .
4. The only non-zero proper ideals of R are (π^n) with $n \geq 1$.
5. The elements $\{r \mid \nu(r) > 0\}$ form a maximal ideal in R and this together with the ideal (0) are the only prime ideals in R .

Proof. Since $1 \cdot 1 = 1$, it follows that $\nu(1) = \nu(1) + \nu(1)$, so $\nu(1) = 0$. For any non-zero element $u \in R$ since ν is a group homomorphism, we conclude that $\nu(u^{-1}) = -\nu(u)$. If u is a unit, then since $u, u^{-1} \in R$ we know that $\nu(u)$ and $\nu(u^{-1})$ are both ≥ 0 . Since $\nu(u^{-1})$ is both ≤ 0 and ≥ 0 it must be zero. Conversely, suppose $u \in R$ and $\nu(u) = 0$. In that case, u is non-zero and the inverse u^{-1} of u in K has valuation zero as well. Therefore, $u^{-1} \in R$.

We leave (2) as an exercise.

For the third statement, suppose $r \in R$ is a non-zero element. In that case $\nu(r) = n \geq 0$. Then, the element $r\pi^{-n}$ has valuation 0 and is therefore a unit, i.e., $u = r\pi^{-n}$. More generally, if $x \in K^\times$, then either $x \in R^\times$ or x^{-1} is in R^\times .

For the fourth statement if $I \subset R$ is a non-zero proper ideal, then there is an integer n such that $\nu(x) \geq n$ for every $x \in I$ (e.g., choose generators). In that case, $x = \pi^n u$ and we see that $(\pi^n) \subset I$. Now, if $y \in I$ since we can write $y = \pi^s v$ for a unit v and $s \geq n$, we conclude that $y \in (\pi^n)$. In other words, $I = (\pi^n)$. The last statement is a consequence of this one. \square

Theorem 4.3.2.3. *The following are equivalent:*

1. R is a local Dedekind domain;
2. R is a discrete valuation ring;
3. R is a local PID;
4. R is a local ring such that every non-zero ideal of R is invertible.

Proof. The fact that (2) \implies (1) is an immediate consequence of Lemma 4.3.2.2. In fact, that Lemma also shows that (2) \implies (3) and (2) \implies (4).

See [Mat89, Theorem 11.2] for the equivalences. \square

4.4 Lecture 10: Picard groups, Dedekind domains and Weil divisors

Having understood Dedekind domains locally (they are discrete valuation rings), we now attempt to understand how to patch this information together. We begin by studying how the property of being “integrally closed in the fraction field” behaves under localization.

4.4.1 Integral closure and localization

Lemma 4.4.1.1. *If R is a domain with fraction field K , and if R is integrally closed in K , then then any localization of R is integrally closed in K as well.*

Proof. Let R be as in the statement and suppose $S \subset R$ is a multiplicative subset. Suppose g is an element of the fraction field K that is integral over $R[S^{-1}]$. Let $P = x^d + a_{d-1}x^{d-1} + \cdots + a_0$ be a monic polynomial with coefficients in $R[S^{-1}]$ that is satisfied by g . Choose $s \in S$ that “clears the

denominators” of the a_i , i.e., such that $sa_i \in R$ for all i . Note that sg satisfies the monic polynomial $x^d + sa_{d-1}x^{d-1} + \cdots + s^d a_0$, which has coefficients $s^{d-j}a_j$ in R . Since R is integrally closed in its field of fractions, $sg \in R$ and thus $g \in R[S^{-1}]$. \square

Lemma 4.4.1.2. *If R is a domain with fraction field K , then the following statements are equivalent:*

1. *the domain R is integrally closed in K ;*
2. *for every prime ideal $\mathfrak{p} \subset R$, the local ring $R_{\mathfrak{p}}$ is integrally closed in K ; and*
3. *for every maximal ideal $\mathfrak{m} \subset R$, the ring $R_{\mathfrak{m}}$ is integrally closed in K .*

Proof. The implications (1) \implies (2) \implies (3) are all consequences of Lemma 4.4.1.1. For the implication (3) \implies (1), we proceed as follows. We claim that, for any domain R with fraction field K , $R = \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$ inside K ; the assertion follows immediately from this claim.

To prove the claim, first observe that since R is a domain, $R \rightarrow R_{\mathfrak{m}}$ is injective and thus $R \rightarrow \bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$ is injective. It thus remains to prove surjectivity. Suppose g is an element of $\bigcap_{\mathfrak{m}} R_{\mathfrak{m}}$ that is not in R . Consider the set $I = \{x \in R \mid xg \in R\}$. Note that I is an ideal, which is proper since $1 \notin I$. Therefore there exists a maximal ideal \mathfrak{m} containing I . If $g \in R_{\mathfrak{m}}$, then there exists $s \in R \setminus \mathfrak{m}$ such that $sg \in R$. Thus, $s \in I \subset \mathfrak{m}$, which is a contradiction. Thus, $I = R$. \square

Since the property of being integrally closed in the field of fractions is cumbersome, we introduce the following terminology.

Definition 4.4.1.3. A ring R will be called *normal* if for every prime ideal \mathfrak{p} , $R_{\mathfrak{p}}$ is a domain, integrally closed in its field of fractions.

Exercise 4.4.1.4. *Show that any UFD is normal.*

The beginning of the proof of Proposition 4.3.1.3 can be repeated to establish the following result.

Proposition 4.4.1.5. *If R is a normal Noetherian integral domain with fraction field K and L is a finite separable extension of K , then if R' is the integral closure of R in L , the map $R \rightarrow R'$ makes R' into a finitely generated R -module.*

4.4.2 Equivalent characterizations of Dedekind domains

Theorem 4.4.2.1. *The following conditions on a commutative integral domain R are equivalent:*

1. *the ring R is a Dedekind domain;*
2. *the ring R is Noetherian, and for each non-zero prime ideal $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is a discrete valuation ring.*

Proof. The implication (1) \implies (2) follows from combining the results established above. For the implication (2) \implies (1), note that R is a Noetherian domain by assumption, and integrally closed in its field of fractions by Lemma 4.4.1.2. Therefore, it suffices to show that R has Krull dimension 1. For any (non-zero) prime ideal \mathfrak{p} , the ring $R_{\mathfrak{p}}$ has precisely 2 ideals (0) and the ideal $\mathfrak{p}R_{\mathfrak{p}}$. The result follows. \square

Theorem 4.4.2.2. *The following conditions on a commutative integral domain R (that is not a field) are equivalent:*

1. R is a Dedekind domain;
2. every non-zero ideal of R is invertible;
3. every non-zero fractional ideal of R is invertible;

Proof. We will only prove the implications (1) \implies (2) and (1) \implies (3), since that is all we will use in what follows. We first treat the local case, i.e., suppose R is a local Dedekind domain. In that case, R is a discrete valuation ring and thus if I is a non-zero (fractional) ideal of R , then $I = (\pi^r)$ (possibly with $r < 0$) for some choice of uniformizing parameter π . Such ideals are evidently invertible.

Now, suppose R is a Dedekind domain and I is a non-zero ideal. Note that I is automatically finitely presented (since R is Noetherian). By the previous step, we know that $I_{\mathfrak{p}}$ is invertible for every prime \mathfrak{p} . Since it is finitely presented, we can even find a Zariski open set on which it is invertible. Since this is true for every non-zero prime \mathfrak{p} , we can certainly find an open cover of R on which I restricts to an invertible module. However, by Zariski patching, we conclude that I must have been invertible to begin with.

For the reverse implications we refer the reader to... □

4.4.3 Picard groups of Dedekind domains

Because of Theorem 4.4.2.2 we conclude that if R is a Dedekind domain, then $\text{Cart}(R)$ is simply the set of fractional ideals (not necessarily invertible!) in R . Now, given a fractional ideal I , as \mathfrak{p} varies through the prime ideals of R , we can consider the ideal $I_{\mathfrak{p}}$. This ideal can be written uniquely as (π^r) for some choice of uniformizing parameter.

Exercise 4.4.3.1. *If R is a Dedekind domain and I is a fractional ideal, show that $I_{\mathfrak{p}}$ is non-trivial for only finitely many primes $\mathfrak{p} \subset R$ (if we think in terms of rational functions, then a given rational function has only finitely many distinct poles).*

With this in mind, there is an induced homomorphism $\text{Cart}(R) \rightarrow \bigoplus_{\mathfrak{p} \text{ non-zero prime}} \mathbb{Z} \cdot \mathfrak{p}$ sending I to $(\mathfrak{p}, I_{\mathfrak{p}} \cong (\pi_{\mathfrak{p}}^r))$. This map is injective by construction. We claim that it is bijective and we show this by explicitly constructing an inverse homomorphism. Indeed, any homomorphism from a free group is determined by what it does on generators, so send \mathfrak{p} to the invertible ideal that corresponds to it; this makes sense because all non-zero ideals are invertible. Extending by linearity gives an inverse to the homomorphism in the other direction.

Putting everything together, we see that, for a Dedekind domain, the units-Pic sequence reads:

$$1 \longrightarrow R^{\times} \longrightarrow K^{\times} \longrightarrow \bigoplus_{\mathfrak{p} \text{ non-zero prime}} \mathbb{Z} \cdot \mathfrak{p} \longrightarrow \text{Pic}(R) \longrightarrow 0.$$

Alternatively, we have constructed a two-term complex

$$K^{\times} \longrightarrow \bigoplus_{\mathfrak{p} \text{ non-zero prime}} \mathbb{Z} \cdot \mathfrak{p}$$

whose cohomology coincides with the units of R and the Picard group of R ; this complex is an example of a *Cousin complex*, which is a notion that will be important later in \mathbb{A}^1 -homotopy theory.

Maps of curves

Exercise 4.4.3.2. *Suppose $\varphi : R \rightarrow S$ is a homomorphism of Dedekind domains. Show that $f : \text{Spec } S \rightarrow \text{Spec } R$ is either constant (i.e., has image a point) or factors through an open subset of $\text{Spec } R$.*

4.4.4 In what sense can we “actually” compute Picard groups?

While the above structural results are nice, they perhaps sidestep the question of what the Picard group actually looks like, even for Dedekind domains. Saying that a group is a quotient of a free abelian group of infinite rank by the image of a map that is difficult to understand is perhaps not so helpful. I state a few results that indicate how widely the Picard group can vary. The first result shows that Picard groups of curves over algebraically closed fields can be “very big”.

Theorem 4.4.4.1 (Grothendieck(?)). *If k is an algebraically closed field, and R is a Dedekind k -algebra then $\text{Pic}(R)$ is a divisible abelian group.*

Remark 4.4.4.2. The proof of this result uses techniques that are very different from those we study here. In fact, one shows that $\text{Pic}(R)$ is the set of points of an algebraic variety (the Picard variety) that has a natural abelian group structure; this structure is obtained essentially by studying integrals of differential forms. When $k = \mathbb{C}$, one observes that $\text{Pic}(R)$ is a compact topological space (in the usual topology), and is isomorphic as a complex manifold to a torus T . If k has positive characteristic, one produces a purely algebraic variant of this complex torus.

Theorem 4.4.4.3 (Mordell-Weil). *If k is a number field, and R is a Dedekind k -algebra, then $\text{Pic}(R)$ is a finitely generated abelian group.*

Remark 4.4.4.4. Even saying this raises questions: a finitely generated abelian group is a product of a free part and a torsion part. What does the torsion subgroup look like? What is the rank of the free part? Each of these questions is interesting, and the answers are largely conjectural. E.g., the Birch–Swinnerton-Dyer (BSD) conjecture asserts that the rank of the free part can be calculated purely analytically from an L -function one attaches to the abelian variety in a fashion that generalizes the analytic class number formula.

Finally, we state a result of Claborn, which shows that the Picard groups of Dedekind domains comprise all abelian groups.

Theorem 4.4.4.5 ([Cla66, Theorem 7]). *Given any abelian group A , there is a Dedekind domain D such that $\text{Pic}(D) \cong A$.*

Remark 4.4.4.6. We refer the reader to [Fos73, §14] for a detailed treatment of the above result and a discussion of the strategy of the proof.

4.5 Lecture 11: Weil divisors and Picard groups of higher dimensional varieties

4.5.1 Weil divisors

We now analyze the construction made above for rings of higher dimension. We will now assume that R is a Noetherian normal domain having Krull dimension d . If K is the fraction field of R , then we can study discrete valuations on K . Since discrete valuation rings are always of dimension 1, the kinds of local rings on R we will get will *not* be localizations at arbitrary prime ideals. The dimension of a localization at a prime ideal has an alternative name: the height of the prime ideal.

Definition 4.5.1.1. If \mathfrak{p} is a prime ideal in a ring R , the *height* of \mathfrak{p} , denoted $\text{ht}\mathfrak{p}$, is the maximum of the lengths of chains of prime ideals contained in P .

We use the following facts about normal domains.

Proposition 4.5.1.2. *If R is a Noetherian normal domain, and if $\mathfrak{p} \subset R$ is a height 1 prime ideal, then $R_{\mathfrak{p}}$ is a discrete valuation ring.*

Proof. We know that $R_{\mathfrak{p}}$ is Noetherian and normal since both the properties of being Noetherian and normal localize by Lemma 4.4.1.1 in the latter case. Since \mathfrak{p} has height 1, we conclude that $R_{\mathfrak{p}}$ is a local Noetherian normal domain having Krull dimension 1. Therefore, $R_{\mathfrak{p}}$ is a discrete valuation ring by the equivalent conditions of Theorem 4.3.2.3. \square

Theorem 4.5.1.3 (Krull). *Suppose R is a Noetherian normal domain.*

1. *The equality $R = \bigcap_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} R_{\mathfrak{p}}$ holds.*
2. *For any $f \in R \setminus 0$, there are only finitely many height 1 prime ideals containing f .*

Proof. See [Mat89, Theorem 12.4(i) p.88] \square

We now proceed to link the Picard group more closely with the geometry of closed subvarieties. To begin we revisit the notion of Cartier divisor. Suppose $X = \text{Spec } R$ is an integral affine scheme with fraction field K . In that case, a Cartier divisor $D = \{U_i, \sigma_i\}$ with $\sigma_i \in K$ is called *effective* if σ_i is a unit on U_i . Note that every Cartier divisor can be written as the difference of two effective divisors: if we write $\sigma_i = \frac{r_i}{s_i}$, with $r_i, s_i \in R$, then $\{U_i, r_i\}$ and $\{U_i, s_i\}$ are both effective Cartier divisors. Indeed, since $\frac{r_i s_j}{s_j r_j}$ is a unit on $U_i \cap U_j$, we conclude that both $\frac{r_i}{r_j}$ and $\frac{s_i}{s_j}$ are units on $U_i \cap U_j$. Thus, the group of Cartier divisors can be thought of in terms of formal differences in the monoid of all effective Cartier divisors.

Now, if $\{U_i, f_i\}$ is an effective Cartier divisor, then the vanishing of f_i determines a hypersurface in U_i . Let R_i be the ring of functions on U_i (some principal open subset of X). The compatibilities inherent in being a Cartier divisor mean that the local ideals $(f_i) \subset R_i$ patch together to determine an ideal $I(D) \subset R$; this ideal is *locally principal* by construction (i.e., there is a Zariski open cover of X by principal open sets on which this ideal is actually a principal ideal). This construction yields an equivalence between locally principal ideals in R and effective Cartier divisors on R . Thus, effective Cartier divisors on X correspond to certain closed subvarieties of X that are locally cut out by a single equation.

Given an ideal $I(D)$, if we consider the localization of R at a height 1 prime ideal, by Proposition 4.5.1.2 we obtain an ideal in a discrete valuation ring. Such an ideal is necessarily of the form $(\pi_{\mathfrak{p}})^{r_{\mathfrak{p}}}$ for some positive integer r and choice of local uniformizing parameter $\pi_{\mathfrak{p}}$. Therefore, we can attach to each ideal $I(D)$ a formal sum $\sum_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} r_{\mathfrak{p}} \cdot \mathfrak{p}$. As in the case of Dedekind domains, it follows from Theorem 4.5.1.3(2) that the integer $r_{\mathfrak{p}}$ is only non-zero for finitely many \mathfrak{p} . Since we can write any Cartier divisor as a formal difference, in this way we obtain a function

$$\text{Cart}(R) \longrightarrow \bigoplus_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p},$$

just as in the situation for Dedekind domains. By Theorem 4.5.1.3(1), this homomorphism is necessarily injective: indeed, if $I(D)$ is the ideal associated with an effective Cartier divisor, then if $I(D) \cap R_{\mathfrak{p}} = 0$ for all height 1 prime ideals, then $I(D) = 0$ already.

Definition 4.5.1.4. If R is a Noetherian normal domain, the group of *Weil divisors* (denoted $\text{Div}(R)$) is the free abelian group on height 1 prime ideals of R , i.e., $\text{Div}(R) := \bigoplus_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p}$.

If R is a Noetherian normal domain, then there is a canonical map $\text{div} : K^{\times} \rightarrow \text{Div}(R)$ that sends $f \rightarrow \sum_{\mathfrak{p}} \nu_{\mathfrak{p}}(f)$.

Definition 4.5.1.5. If R is a Noetherian normal domain, the class group $Cl(R)$ is defined as $\text{coker } \text{div}$.

For Dedekind domains, we showed the map $\text{Cart}(R) \rightarrow \text{Div}(R)$ is an isomorphism. The situation is slightly different in higher dimensions: while the map is always injective (by construction), there is an obstruction to surjectivity. The problem is this: the subvariety D constructed in the previous section is, by definition, always *locally cut out by a single equation*. On the other hand, given a height 1 prime ideal \mathfrak{p} , there is no reason to expect that the subvariety R/\mathfrak{p} is locally cut out by a single equation.

4.5.2 Triviality of the class group

Lemma 4.5.2.1. *If R is a Noetherian domain, then R is a unique factorization domain if and only if R is normal and $Cl(R) = 0$.*

Proof. We use the following fact from ring theory: if R is a Noetherian domain, then R is a UFD if and only if every height 1 prime ideal is principal [Sta15, Tag 034O Lemma 10.119.6] or [Mat89, Theorem 20.1] (one proof of this result uses Krull's principal ideal theorem: if R is a Noetherian ring, $x \in R$, and \mathfrak{p} is minimal among prime ideals in R containing x , then \mathfrak{p} has height ≤ 1 together with a bit of the theory of primary decomposition).

Suppose every prime ideal of height 1 is principal. In that case, if \mathfrak{p} is a prime ideal of height 1, we can choose a generator f . Then, f lies in the image of the divisor map. It follows that the map div is surjective, and thus that $Cl(R) = 0$.

We leave the other direction as an exercise. □

Corollary 4.5.2.2. *If k is a field, $Cl(k[x_1, \dots, x_n]) = 0$.*

Definition 4.5.2.3. A ring R is called *locally factorial* if for every prime ideal \mathfrak{p} , the localization $R_{\mathfrak{p}}$ is a unique factorization domain.

Proposition 4.5.2.4. *If R is a locally factorial Noetherian normal domain, then the map*

$$\text{Cart}(R) \longrightarrow \bigoplus_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p}$$

is an isomorphism. As a consequence, under these hypotheses, the induced map $\text{Pic}(R) \rightarrow \text{Cl}(R)$ is an isomorphism.

Proof. As before it suffices to construct an inverse map and to do this we proceed exactly as before: beginning with a height 1 prime ideal \mathfrak{p} , it suffices to show that \mathfrak{p} is actually an invertible ideal in R . Since R is Noetherian, \mathfrak{p} is automatically finitely presented, and therefore it suffices to check this after localization at every prime. The ideal \mathfrak{p} determines an ideal in $R_{\mathfrak{q}}$ as \mathfrak{q} ranges through the prime ideals of R . By assumption, $\text{Cl}(R_{\mathfrak{q}}) = 0$, so locally \mathfrak{p} is a principal fractional ideal. By the finite presentation assumption, we can find a Zariski open neighborhood containing $R_{\mathfrak{q}}$ on which \mathfrak{p} is principal, and we can cover $\text{Spec } R$ by such open sets. \square

Example 4.5.2.5. Note that if R is a Dedekind domain, it follows from Lemma 4.3.2.2 that R is normal and locally factorial. We will focus on systematically writing down other examples of such rings soon.

We have shown that, if $X = \text{Spec } R$ is a normal affine variety, then there is a two-term complex

$$K^{\times} \xrightarrow{\text{div}} \bigoplus_{\mathfrak{p}|\text{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p};$$

the cokernel of the map div is $\text{Pic}(R)$, while the kernel of div is R^{\times} .

The Picard groups of a UFD

It is possible to give a direct proof that the Picard group of a UFD is trivial (without passing through the identification afforded by Proposition 4.5.2.4).

Proposition 4.5.2.6. *If R is a UFD, then $\text{Pic}(R) = 0$.*

Proof. See [Sta15, Tag 0AFW Lemma 15.84.3]. \square

4.5.3 Scheme-theoretic images and dominant maps

Before moving forward we describe some ideal theoretic properties of the image of the morphism of schemes attached to $\varphi : R \rightarrow S$. We begin with some equivalent characterizations of the condition that a point lies in the image.

Lemma 4.5.3.1. *Suppose $\varphi : R \rightarrow S$ is a ring homomorphism. Given a prime ideal $\mathfrak{p} \subset R$, the following are equivalent:*

1. \mathfrak{p} is in the image of $\text{Spec } S \rightarrow \text{Spec } R$;

2. $S \otimes_R \kappa(\mathfrak{p}) \neq 0$;
3. $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \neq 0$;
4. $(S/\mathfrak{p}S)_{\mathfrak{p}} \neq 0$; and
5. $\mathfrak{p} = \varphi^{-1}(\mathfrak{p}S)$.

The above equivalent conditions justify making the following definition.

Definition 4.5.3.2. Given $\varphi : R \rightarrow S$, $\ker(\varphi)$ is an ideal in R . The affine scheme corresponding to $\text{Spec } R/\ker(\varphi) \subset \text{Spec } R$ will be called the *scheme-theoretic image* of $f : \text{Spec } S \rightarrow \text{Spec } R$.

Example 4.5.3.3. Consider the map $\mathbb{A}^1 \rightarrow \mathbb{A}^2$ given by “inclusion of the x -axis”; this corresponds to a ring map $k[x, y] \rightarrow k[x]$ given by projection onto x . The kernel of this ring map is the ideal (y) .

Definition 4.5.3.4. A morphism $\varphi : R \rightarrow S$ of affine schemes is called *dominant* if the scheme theoretic image of $f : \text{Spec } S \rightarrow \text{Spec } R$ is dense.

Example 4.5.3.5. Consider the map $SL_2 \rightarrow \mathbb{A}^2$ obtained by projection onto the first column. Ring theoretically, if we identify $k[SL_2] = k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{12}x_{21} - 1)$, then map in question is given by the inclusion of $k[x_{11}, x_{21}]$. This ring map is injective, so the morphism in question is dominant, but note that the morphism $SL_2 \rightarrow \mathbb{A}^2$ is *not* surjective. Indeed, the first column of an invertible 2×2 -matrix over any ring cannot be identically zero. Thus, the scheme-theoretic image of a ring map, and the image of the associated map of spectra need not coincide in general.

Lemma 4.5.3.6. A morphism $\varphi : R \rightarrow S$ of affine schemes is dominant if and only if $\ker(\varphi)$ is nilpotent. In particular, if R is reduced, then φ is dominant if and only if it is injective.

4.5.4 Dominant maps and pullbacks of divisors

We know that if $\varphi : R \rightarrow S$ is a ring homomorphism, then there are induced maps $R^\times \rightarrow S^\times$ and $\text{Pic}(R) \rightarrow \text{Pic}(S)$. It is natural to ask if, assuming R and S are normal affine varieties, we can actually define a map of complexes that induces the corresponding maps after taking kernels and cokernels.

Construction 4.5.4.1. Given a dominant morphism $\varphi : R \rightarrow S$, we can define a pullback of Cartier divisors as follows: Given an invertible R -module L , we obtain a Cartier divisor by considering $L \hookrightarrow L \otimes_R K \xrightarrow{\sim} K$. Now, if $R \rightarrow S$ is dominant with corresponding inclusion $K \hookrightarrow E$, then we see that $S \hookrightarrow E$ is injective as well. In that case, the specified isomorphism $L \otimes_R K \xrightarrow{\sim} K$ induces an isomorphism

$$(L \otimes_R K) \otimes_K E \cong L \otimes_R (K \otimes_K E) \cong L \otimes_R E \xrightarrow{\sim} E$$

and the composite $L \otimes_R S \rightarrow L \otimes_R E \xrightarrow{\sim} E$ yields a Cartier divisor. Alternatively, if we think in terms of the $\{U_i, \sigma_i\}$, then we can simply pullback the defining equations.

Proposition 4.5.4.2. Suppose R and S are integral domains, and $\varphi : R \rightarrow S$ is a dominant ring homomorphism with K the fraction field of R , and E the fraction field of S .

1. There is a commutative diagram of the form

$$\begin{array}{ccc} K^\times & \xrightarrow{\text{div}} & \text{Cart}(R) \\ \downarrow & & \downarrow \\ E^\times & \xrightarrow{\text{div}} & \text{Cart}(S), \end{array}$$

where the left hand map is inclusion $K^\times \rightarrow E^\times$ and the right hand map is the pullback on Cartier divisors from Construction 4.5.4.1.

2. If R and S are furthermore, locally factorial, Noetherian and normal, then we can replace $\text{Cart}(-)$ by $\text{Div}(-)$.
3. The induced maps of kernels coincides with the pullback map of unit groups.
4. The induced maps of cokernels coincides with the pullback map of Picard groups.

Proof. For the first statement, the only thing that has to be checked is commutativity of the diagram. Given an element $f \in K^\times$, it is sent to the Cartier divisor $Rf \subset K$. Unwinding the definitions, this is sent to $Sf \subset E$, as expected.

For the second statement, we appeal to the identification of Proposition 4.5.2.4 to transport the pullback on Cartier divisors to Weil divisors (this is compatible with the divisor map by construction).

The final two statements follow by unwinding the definitions. \square

4.6 Lecture 12: \mathbb{A}^1 -invariance, localization and Mayer-Vietoris

Having developed a fair amount of technology for studying Picard groups, we now reap the benefits and deduce various basic properties that show $\text{Pic}(-)$ acts like a cohomology theory: we show it is \mathbb{A}^1 -invariant, and a suitable form of Mayer-Vietoris holds.

4.6.1 Homotopy invariance

The inclusion map $R \rightarrow R[t]$ is a dominant ring homomorphism. Assuming R is a domain, then $R[t]$ is also a domain. Therefore, there is an induced pullback map $\text{Cart}(R) \rightarrow \text{Cart}(R[t])$. If we write K for the fraction field of R , then we can identify the fraction field of $R[t]$ with $K(t)$.

Exercise 4.6.1.1. If R is a locally factorial Noetherian normal domain, then $R[t]$ is as well.

In that case, there is a morphism of complexes of the following form:

$$(4.6.1) \quad \begin{array}{ccc} K^\times & \longrightarrow & \bigoplus_{\{\mathfrak{p} \subset R \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \\ \downarrow & & \downarrow \\ K(t)^\times & \longrightarrow & \bigoplus_{\{\mathfrak{p} \subset R[t] \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z}. \end{array}$$

The left vertical map is injective.

We now describe the height 1 prime ideals in $R[t]$ more geometrically. If \mathfrak{p} is a prime ideal in $R[t]$ that has height 1, then the pullback under the ring map $R \rightarrow R[t]$ is a prime ideal in R , which may not have height 1. For example, the homomorphism $\mathbb{Z} \rightarrow R$ induced by the unit, determines a homomorphism $\mathbb{Z}[t] \rightarrow R[t]$. Any irreducible polynomial in t with integral coefficients therefore defines an ideal in $R[t]$, which is a height 1 principal ideal. Note that the pullback of this ideal to R under $R \mapsto R[t]$ is R itself.

On the other hand, the evaluation map $R[t] \rightarrow R$ defines height 1 prime ideals of the form $\mathfrak{p}[t]$ in $R[t]$; the pullback of such an ideal to R under $R \rightarrow R[t]$ is precisely \mathfrak{p} . Only these latter prime ideals are in the image of the pullback map. We summarize this observation in the following result.

Lemma 4.6.1.2. *The pullback map*

$$\bigoplus_{\{\mathfrak{p} \subset R \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \longrightarrow \bigoplus_{\{\mathfrak{p} \subset R[t] \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z}$$

is injective and its image consists of those height 1 prime ideals of the form $\mathfrak{p}[t]$.

To study the homotopy invariance question, it suffices to show that all height 1 prime ideals in $R[t]$ differ from a sum of those of the form $\mathfrak{p}[t]$ by the divisor of a rational function. To this end, suppose $\mathfrak{q} \subset R[t]$ is a height 1 prime ideal. In that case, we can consider the image of \mathfrak{q} in $K[t] \supset R[t]$. Now, since K is a field, $K[t]$ is a principal ideal domain, so the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$ is necessarily principal. Choose a generator f of the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$. The element f yields an element of $K(t)$.

Now, we analyze the principal divisor attached to f . If we pick generators f_1, \dots, f_r of \mathfrak{p} , then we can write these in the form $f_i(t) = a_{0,i} + \dots + a_{n_i,i}t^{n_i}$ where each $a_i \in R$. The corresponding element of $K[t]$ is obtained by introducing denominators. Since the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$ is principal, that means after inverting coefficients, $f_i(t) = \alpha_i f(t)$ for $\alpha_i \in R$. From the form of these expressions, we can deduce that $\text{div}(f)$ differs from \mathfrak{q} by prime divisors in the image of the pullback map. Altogether, we have established the following fact:

Theorem 4.6.1.3 (Homotopy invariance). *Assume R is a locally factorial Noetherian normal domain.*

1. *the map $\text{Pic}(R) \rightarrow \text{Pic}(R[t])$ is an isomorphism;*
2. *the map of Equation 4.6.1 is a quasi-isomorphism of complexes, i.e., induces an isomorphism after taking cohomology.*

Proof. Point (1) is established by the discussion just prior to the statement. For Point (2), we simply describe the maps on cohomology. That the map of complexes induces an isomorphism on H^0 (i.e., after taking kernels) follows from Proposition 2.2.1.6 and an isomorphism on H^1 (i.e., after taking cokernels) follows from the conclusion of Point (1). Since the complexes in question have only 2 terms, there are no other possibly non-vanishing cohomology groups. \square

Example 4.6.1.4. As is the case with units, the Picard group is *not* \mathbb{A}^1 -invariant on all rings. Moreover, counterexamples to \mathbb{A}^1 -invariance exist even for reduced rings. For example, one may check that if $R = k[x, y]/(y^2 - x^3)$, then the map $R \mapsto R[t]$ is not an isomorphism on Picard groups.

Remark 4.6.1.5. Theorem 4.6.1.3 is not the best possible \mathbb{A}^1 -invariance result for Picard groups. Indeed, if R is a ring, then evaluation determines a homomorphism $\text{Pic}(R[t]) \rightarrow \text{Pic}(R)$ for any ring R . Therefore, \mathbb{A}^1 -invariance is equivalent to establishing the kernel of this map is trivial. The kernel corresponds to invertible $R[t]$ -modules L such $L/tL \cong R$, and thus one would like to show that if L is a module such that L/tL is trivial, then L is already trivial.

Fix an isomorphism $L/tL \cong R$. On the other hand, if R is a domain, with fraction field K , then we know that $L \otimes_{R[t]} K[t]$ is a trivial rank 1 module. Therefore, we can fix a trivialization $L \otimes_{R[t]} K[t] \cong K[t]$ as well. Because L is finitely presented, we can find $f \in K$ such that the above trivialization extends to an isomorphism $L \otimes_{R[t]} R_f[t] \cong R_f[t]$. In this way, we obtain an isomorphism $R_f \cong L/tL \otimes_R R_f \cong L_f/tL_f \cong R_f$, which corresponds to a unit in R_f . Modifying the trivialization of $L \otimes_R K[t]$ by this unit, we can extend the isomorphism $L/tL \cong R$ over $R_f[t]$. One then wants to show that under suitable hypotheses this isomorphism can be extended over all of $R[t]$. This can be accomplished, e.g., if R is a Noetherian normal domain. However, it holds even more generally for semi-normal rings, which essentially rule out precisely singularities of ‘‘cusp’’ type (cf. Example 4.6.1.4). See [Tra70] [Swa80] and [Coq06] for more details.

4.6.2 The localization sequence

Suppose R is an integral domain and $S \subset R$ is a multiplicative set. In that case, there is a restriction map $\text{Pic}(R) \rightarrow \text{Pic}(R[S^{-1}])$. The ring homomorphism $R \rightarrow R[S^{-1}]$ is dominant and the induced map of function fields is simply the identity map. If, furthermore, R is locally factorial, Noetherian and normal, then there is a morphism of complexes of the form:

$$\begin{array}{ccc} K^\times & \longrightarrow & \bigoplus_{\{\mathfrak{p} \subset R \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \\ \downarrow & & \downarrow \\ K^\times & \longrightarrow & \bigoplus_{\{\mathfrak{p} \subset R[S^{-1}] \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array}$$

The left hand map is an isomorphism. Since the prime ideals in $R[S^{-1}]$ are precisely those prime ideals $\mathfrak{p} \subset R$ such that $S \cap \mathfrak{p}$ is the empty set, by unwinding the definition of pullback (for Cartier divisors! and tracing through the identification with Weil divisors), we conclude that the right vertical map is surjective. The kernel of the right vertical map consists precisely of those prime ideals \mathfrak{p} of height 1 such that $\mathfrak{p} \cap S$ is non-empty.

Theorem 4.6.2.1 (Localization sequence). *Suppose R is a locally factorial Noetherian normal domain and $S \subset R$ is a multiplicative set.*

1. *There is a short exact sequence of complexes of the form:*

$$0 \longrightarrow \left(\begin{array}{c} 0 \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \mid \text{ht}(\mathfrak{p})=1, \mathfrak{p} \cap S \neq \emptyset\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R[S^{-1}] \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \longrightarrow 0.$$

2. There is an exact sequence of groups of the form:

$$1 \longrightarrow R^\times \longrightarrow R[S^{-1}]^\times \longrightarrow \bigoplus_{\{\mathfrak{p} \mid \text{ht}(\mathfrak{p})=1, \mathfrak{p} \cap S \neq \emptyset\}} \mathbb{Z} \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(R[S^{-1}]) \longrightarrow 0.$$

Proof. The first point is an immediate consequence of the analysis before the statement.

The second point follows immediately from the first by taking the long exact sequence in cohomology associated with a short exact sequence of complexes. \square

Remark 4.6.2.2. The geometric interpretation of the exact sequence is as follows: every invertible R -module on $R[S^{-1}]$ extends to an invertible R -module. The kernel of this surjection is described in terms of the free abelian group on the irreducible codimension 1 components of the complement.

4.6.3 Zariski patching and Mayer-Vietoris sequences

Suppose f and g are comaximal elements of a commutative unital ring R . As before, we will assume that R is a locally factorial, Noetherian normal domain. We observed that any R -module could be obtained by patching modules on R_f and R_g that agree upon restriction to R_{fg} . We now analyze the consequences of this observation for the Picard group. In the previous section we studied the morphism of complexes associated with the maps $R \rightarrow R_f$ and $R \rightarrow R_g$, we now study the map $R \rightarrow R_f \oplus R_g$ that arises in patching. Given an invertible R -module, we obtain by restriction modules on R_f and R_g that agree upon restriction to R_{fg} . At the level of invertible modules, we can define a map $\text{Pic}(R_f \oplus R_g) \rightarrow \text{Pic}(R_{fg})$ by $(L_1, L_2) \mapsto L_1 \otimes L_2^{-1}$. We now analyze a version of these maps at the level of complexes and we mimic the standard construction of the Mayer-Vietoris sequence in topology.

Of course, restriction determines an injective map of complexes

$$\left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R_f \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R_g \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right).$$

On the other hand, we can define a “difference” map

$$\left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R_f \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R_g \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{\mathfrak{p} \subset R_{fg} \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \end{array} \right)$$

as follows. Consider the map $K^\times \oplus K^\times \rightarrow K^\times$ given by $(f, g) \mapsto fg^{-1}$ and the map

$$\left(\begin{array}{c} \bigoplus_{\{\mathfrak{p} \subset R_f \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \\ \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} \bigoplus_{\{\mathfrak{p} \subset R_g \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \\ \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} \bigoplus_{\{\mathfrak{p} \subset R_{fg} \mid \text{ht}(\mathfrak{p})=1\}} \mathbb{Z} \\ \mathbb{Z} \end{array} \right)$$

gotten by projection onto those height 1 prime ideals appearing in both R_f and R_g and then, on components corresponding to height 1 prime ideals that appear in both R_f and R_g , send (x, y) to $x - y$. You can check that the map defined componentwise in this fashion is a homomorphism of complexes.

Theorem 4.6.3.1 (Mayer-Vietoris). *Suppose R is a locally factorial Noetherian normal domain, and f, g are comaximal elements in R .*

1. *Restriction and difference (as defined above) fit together to give a short exact sequence of complexes of the form:*

$$0 \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{p \subset R \mid \text{ht}(p)=1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{p \subset R_f \mid \text{ht}(p)=1\}} \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{p \subset R_g \mid \text{ht}(p)=1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^\times \\ \downarrow \\ \bigoplus_{\{p \subset R_{fg} \mid \text{ht}(p)=1\}} \mathbb{Z} \end{array} \right) \longrightarrow 0;$$

2. *There is an induced Mayer-Vietoris exact sequence of the form*

$$1 \longrightarrow R^\times \longrightarrow R_f^\times \oplus R_g^\times \longrightarrow R_{fg}^\times \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(R_f) \oplus \text{Pic}(R_g) \longrightarrow \text{Pic}(R_{fg}) \longrightarrow 0.$$

Proof. We leave this proof as an exercise. \square

Remark 4.6.3.2. Once again, these are not the weakest hypotheses under which we can guarantee the existence of Mayer-Vietoris exact sequences for Picard groups as in Theorem 4.6.3.1(2). Indeed, write $\text{Inv}(R)$ for the category whose objects are invertible R -modules and morphisms are R -module homomorphisms. Given comaximal elements f, g in a ring R , then Theorem 3.3.3.2 combined with Proposition 3.4.1.4 gives a fiber product diagram of categories of the form

$$\begin{array}{ccc} \text{Inv}(R) & \longrightarrow & \text{Inv}(R_f) \\ \downarrow & & \downarrow \\ \text{Inv}(R_g) & \longrightarrow & \text{Inv}(R_{fg}), \end{array}$$

i.e., there is an equivalence between the categories of invertible R -modules and that of pairs (L_1, L_2, α) , where L_1 is an invertible R_f -module, L_2 is an invertible R_g -module, and $\alpha : L_1 \otimes_{R_f} R_{fg} \rightarrow L_2 \otimes_{R_g} R_{fg}$ is an isomorphism. From this fact, one can directly construct the following portion of the Mayer-Vietoris sequence:

$$1 \longrightarrow R^\times \longrightarrow R_f^\times \times R_g^\times \longrightarrow R_{fg}^\times \longrightarrow \text{Pic}(R) \longrightarrow \text{Pic}(R_f) \oplus \text{Pic}(R_g) \longrightarrow \text{Pic}(R_{fg}).$$

Indeed, one constructs the sequence $1 \rightarrow R^\times \rightarrow R_f^\times \times R_g^\times \rightarrow R_{fg}^\times$ by analyzing units, observes that $\text{Pic}(R) \rightarrow \text{Pic}(R_f) \oplus \text{Pic}(R_g) \rightarrow \text{Pic}(R_{fg})$ is exact by descent. Finally, one splices these two sequences together via a map $R_{fg}^\times \rightarrow \text{Pic}(R)$ which is induced by identifying $L \in \text{Pic}(R)$ with a triple (L_1, L_2, α) and observing that one may multiply α by a unit (acting as an endomorphism of $L_1 \otimes_{R_f} R_{fg}$). This point of view of constructing the long exact sequence directly from the categorical point of view will be useful later.

Chapter 5

Grothendieck groups, regularity and \mathbb{A}^1 -invariance

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In this section, we introduce another invariant of rings coming from projective modules: the Grothendieck of isomorphism classes of finitely generated projectives. We connect this invariant with the Picard group studied in the previous section. We then introduce the notion of regularity of a ring and study some basic properties of this notion as a first step toward understanding “smooth-

ness” in algebraic geometry. We also begin a discussion of the homological theory of projective modules, along the lines initiated by Cartan–Eilenberg [CE99]. In particular, we will discuss projective dimension of rings, and study conditions that guarantee finite projective dimension; these notions are closely connected with regularity by classical results of Auslander-Buchsbaum-Serre.

5.1 Lecture 13: Grothendieck groups

5.1.1 Grothendieck groups

If R is a commutative unital ring, then we can consider the set of isomorphism classes of projective R -modules. This set has a monoid structure given by direct sum (the unit being the zero R -module), but also a product given by tensor product of R -modules. Unlike the case of invertible R -modules, elements need not have inverses for this group structure (e.g., if R is a field, the dimension of a direct sum of R -modules is the sum of the dimensions of the summands and the dimension is always ≥ 0). Nevertheless, this monoid is still commutative (since $M \oplus M' \cong M' \oplus M$, functorially in the inputs).

Grothendieck observed that there is a universal way to construct an abelian group from a commutative monoid, generalizing the way the integers are built from the natural numbers. More precisely, every integer can be viewed as a “formal difference” of natural numbers. More abstractly, a formal difference can be equated with an element of $\mathbb{N} \times \mathbb{N}$. We define an addition on the set of formal differences componentwise. However, many formal differences correspond to the same integer, thus we need to impose an equivalence relation on the set of pairs to get integers. Say that (a, b) and (a', b') are equivalent if there exists $k \in \mathbb{N}$ such that $a + b' + k = a' + b + k$. In this form, the procedure works more generally: given a monoid M , consider $M \times M$, define addition componentwise and define an equivalence relation on pairs by saying $(m, n) \sim (m', n')$ if there exists $k \in M$ such that $m + n' + k = m' + n + k$.

Exercise 5.1.1.1. *Suppose A is a commutative monoid.*

1. *The procedure just described defines an abelian group A^+ (the group completion of A); this procedure is functorial with respect to homomorphisms of abelian groups.*
2. *There is a monoid homomorphism $A \rightarrow A^+$ (send a to $(a, 0)$) and given any abelian group B and a monoid map $\varphi : A \rightarrow B$, there is a unique homomorphism $A^+ \rightarrow B$ such that φ factors as $A \rightarrow A^+ \rightarrow B$.*

Definition 5.1.1.2. If R is a commutative unital ring, then $K_0(R)$ is the Grothendieck group of the monoid of isomorphism classes of projective modules with respect to direct sum.

Remark 5.1.1.3. If X is a topological space, and $C(X)$ is the algebra of real-valued continuous functions, then $K_0(C(X))$ is the Grothendieck group of isomorphism classes of topological vector bundles on X . If X is compact and Hausdorff, this coincides with the notion of topological K-theory as studied by Atiyah [Ati89] using the Vaserstein-Serre-Swan theorem.

To really spell things out, consider the following result which explains when the isomorphism classes of projective modules agree in $K_0(R)$.

Lemma 5.1.1.4. *If R is a commutative unital ring and P, P' are finitely generated projective R -modules, the following statements are equivalent:*

1. $[P] = [P']$ in $K_0(R)$;
2. there is a finitely generated projective R -module Q such that $P \oplus Q \cong P' \oplus Q$, i.e., the modules P and P' are stably isomorphic;
3. there is an integer n such that $P \oplus R^{\oplus n} \cong P' \oplus R^{\oplus n}$.

Proof. Exercise. □

Lemma 5.1.1.5. *Tensor product of R -modules equips the group $K_0(R)$ with the structure of a commutative unital ring.*

Proof. Exercise. □

Example 5.1.1.6. We can compute $K_0(\mathbb{Z})$ from the definition: via the structure theorem for finitely generated modules, the monoid of isomorphism classes of projective R -modules is isomorphic to \mathbb{N} under addition (via the monoid map sending a projective module to its rank). Thus, $K_0(\mathbb{Z}) = \mathbb{Z}$. More generally, if R is a principal ideal domain, the same argument shows that $K_0(R) \cong \mathbb{Z}$.

Lemma 5.1.1.7. *If $f : R \rightarrow S$ is a ring homomorphism, then extension of scalars induces a ring homomorphism $f^* : K_0(R) \rightarrow K_0(S)$.*

Example 5.1.1.8. If R is any commutative unital ring, then the map $\mathbb{Z} \rightarrow R$ induces a homomorphism $K_0(\mathbb{Z}) \rightarrow K_0(R)$; this homomorphism sends $\mathbb{Z}^{\oplus n} \rightarrow R^{\oplus n}$ and is injective. Since any non-zero ring has a maximal ideal \mathfrak{m} , there is an induced map $K_0(R) \rightarrow K_0(R/\mathfrak{m})$. The composite map $\mathbb{Z} \rightarrow R/\mathfrak{m}$ induces an isomorphism $\mathbb{Z} = K_0(\mathbb{Z}) \rightarrow K_0(R/\mathfrak{m}) \cong \mathbb{Z}$ and therefore, we conclude that \mathbb{Z} is a summand of $K_0(R)$ for any non-zero commutative unital ring. Thus, $K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$ where $\tilde{K}_0(R)$ is called the *reduced K_0 of R* . Note that $\tilde{K}_0(R) = 0$ if and only if each projective R -module is stably free.

Exercise 5.1.1.9. *Show that if R is a commutative unital ring and $N \subset R$ is the nilradical, then $K_0(R) \rightarrow K_0(R/N)$ is an isomorphism.*

5.1.2 Determinants of projective modules

If M is any R -module, we can speak of exterior powers of M . Define the tensor algebra $T(M)$ to be the R -module $\bigoplus_{n \geq 0} M^{\otimes n}$ with multiplication given on pure tensor by the formula

$$(x_1 \otimes \cdots \otimes x_m) \otimes (y_1 \otimes \cdots \otimes y_n) = (x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n)$$

and extended linearly. Define the exterior algebra $\wedge M$ to be the quotient of the graded algebra $T(M)$ by the two-sided (graded) ideal generated by $x \otimes x \in T^2(M)$. The image of the pure tensor $x_1 \otimes \cdots \otimes x_n$ in $\wedge M$ is denoted $x_1 \wedge \cdots \wedge x_n$. The k -th graded piece of $\wedge M$ is denoted $\wedge^k M$ and called the k -th exterior power of M . It follows that $\wedge^0 M = R$ (since $M^{\otimes 0} = R$), $\wedge^1 M = M$, and $\wedge^k M$ as the quotient of the k -fold tensor product $M \otimes \cdots \otimes M$ by the submodule generated by terms $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$ (this encodes the “alternating” condition). Exterior powers define endo-functors of the category Mod_R . Moreover, one can establish the following fact using the compatibility of extension of scalars and tensor products.

Exercise 5.1.2.1. *If $\varphi : R \rightarrow S$ is a ring homomorphism, then $(\wedge^n M) \otimes_R S \xrightarrow{\sim} \wedge^n (M \otimes_R S)$.*

The exterior power functors have the following properties that we will find useful.

Lemma 5.1.2.2. *If R is a commutative unital ring, then the exterior power functor has the following properties:*

1. *The module $\wedge^n R^{\oplus r}$ is a free module of rank $\frac{r!}{n!(r-n)!}$.*
2. *If $M \oplus N$ is a direct sum decomposition, then there is a natural isomorphism*

$$\wedge^n(M \oplus N) \cong \bigoplus_{i=0}^n (\wedge^i M) \otimes (\wedge^{n-i} N).$$

Lemma 5.1.2.3. *Suppose P is a finitely generated projective R -module.*

1. *The module $\wedge^n P$ is a finitely generated projective R -module for any integer n .*
2. *If P has constant rank r , then $\wedge^r P$ is an invertible module that we will call $\det P$.*
3. *If P has constant rank r , then $\wedge^n P = 0$ for $n > r$.*

Proof. For Point (1), note that $\wedge^n P$ is finitely generated by assumption (as a quotient of a finitely generated module). Since exterior powers commute with tensor product, and projective modules are locally free, by localizing we can assume without loss of generality that P is free. Thus, by appeal to Lemma 5.1.2.2(1), we conclude that $\wedge^n P$ is also locally free.

Points (2) and (3) follow from Lemma 5.1.2.2(1) as well since $\wedge^n R^{\oplus r}$ has dimension 1 if $n = r$ and is trivial if $n > r$. \square

Lemma 5.1.2.4. *Assume R is a connected commutative unital ring.*

1. *The map sending a finitely generated projective R -module to its determinant extends to a group homomorphism $\det : K_0(R) \rightarrow \text{Pic}(R)$.*
2. *The homomorphism of Point (1) is functorial with respect to homomorphism of connected commutative unital rings.*

Proof. If P is a projective module of rank r , Lemma 5.1.2.3(3) tells us that $\wedge^n P = 0$ for $n > r$. If P and Q are projective R -modules of ranks m and n , then

$$\wedge^{m+n}(P \oplus Q) \cong \bigoplus_{i=0}^{m+n} \wedge^i P \otimes \wedge^{m+n-i} Q$$

by Lemma 5.1.2.2(2). Now, since i and $m+n-i$ are both ≥ 0 and $\leq m+n-i$, we conclude that either $\wedge^i P = 0$ or $\wedge^{m+n-i} Q = 0$ unless $i = m$. Thus, we conclude that $\wedge^{m+n}(P \oplus Q) \cong \wedge^m P \otimes \wedge^n Q$.

The functoriality statement is immediate from the fact that forming exterior powers commutes with extension of scalars. \square

Remark 5.1.2.5. Using local constancy of rank, one can define the determinant for projective modules with non-constant rank “componentwise” and drop the assumption that R is connected in the previous statement, but we leave it to the interested reader to work this out.

Theorem 5.1.2.6 (Cancellation for rank 1 modules). *Suppose R is a commutative unital ring.*

1. *If L and L' are stably isomorphic invertible R -modules, then $L \cong L'$.*

2. The map $L \mapsto [L]$ determines an injection $\text{Pic}(R) \rightarrow K_0(R)^\times$.

Proof. Suppose $L \oplus R^{\oplus n} \cong L' \oplus R^{\oplus n}$. In that case, we conclude that $\wedge^{n+1}(L \oplus R^{\oplus n}) \cong \wedge^{n+1}(L' \oplus R^{\oplus n})$. However, $\wedge^{n+1}(L \oplus R^{\oplus n}) \cong \wedge^1 L \otimes \wedge^n R^{\oplus n} \cong L$, and similarly for L' . Therefore, $L \cong L'$.

For Point (2), observe that the composite function $\text{Pic}(R) \rightarrow K_0(R) \xrightarrow{\det} \text{Pic}(R)$ is the identity after the conclusion of Point (1) (though note that the second map is a homomorphism with respect to the additive structure on K_0 while the first map uses the multiplicative structure, so the composite is not a group homomorphism). \square

Remark 5.1.2.7. A natural generalization of the Point (1) in Theorem 5.1.2.6 is the general “cancellation” problem: if P and Q are stably isomorphic projective R -modules (of the same rank) are P and Q isomorphic? A special case of the cancellation problem is: when are stably-free modules free? These problems motivated some of the early study of the groups $K_0(R)$.

The map $\det : K_0(R) \rightarrow \text{Pic}(R)$ is a surjective group homomorphism by the same argument as in Theorem 5.1.2.6(2). Therefore, if $K_0(R) \rightarrow K_0(R[t_1, \dots, t_n])$ is an isomorphism, we see $\text{Pic}(R) \rightarrow \text{Pic}(R[t_1, \dots, t_n])$ is an isomorphism too (it is always split injective and the statement about K_0 guarantees surjectivity). Thus, one cannot expect $K_0(R)$ to be \mathbb{A}^1 -invariant without a hypothesis on R at least as strong as (semi-)normality (cf. Theorem 4.6.1.3 and Remark 4.6.1.5).

5.2 Lecture 14: Regular local rings

If M is a smooth manifold of dimension d , one way to define the tangent space at a point $x \in M$ is as follows: consider the ideal $\mathfrak{m}_x \subset C^\infty(M)$ consisting of smooth functions vanishing at x . Since locally around x there is a neighborhood of x diffeomorphic to an open subset of Euclidean space \mathbb{R}^d , we can pick local coordinates x_1, \dots, x_d that generate the maximal ideal \mathfrak{m}_x . The choice of local coordinates then yields a basis of the real vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. The tangent space is then the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$, which is thus a real vector space of dimension d .

5.2.1 Regular local rings: definitions and examples

We first discuss the notion of regularity locally, essentially by directly reinterpreting the situation in topology. There is one basic problem: in algebraic geometry, if $X = \text{Spec } R$ is an affine k -variety, then given a k -point, there is no reason for one to be able to find an open neighborhood of x in X that can be identified with an open subset of affine space. For the sake of intuition, let us discuss the case $k = \mathbb{C}$, and let us think about the “embedded” point of view and identify $R = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_r)$. In that case, non-singularity can be tested using the Jacobian criterion: one writes down the matrix of partial derivatives and smooth points are precisely those where the rank of the matrix $(\frac{\partial f_i}{\partial x_j})$ is maximal, i.e., equal to r . One identifies the tangent space as a subspace of \mathbb{C}^n , and thus at points where the rank of the Jacobian drops, the dimension of the tangent space increases.

If R is a ring with a maximal ideal \mathfrak{m} , then we can consider the field $k = R/\mathfrak{m}$. There is a natural structure of k -vector space on $\mathfrak{m}/\mathfrak{m}^2$. We first establish a result that provides a purely algebraic version of the intuitive description given above.

Theorem 5.2.1.1. *If R is a Noetherian local ring of Krull dimension d , with maximal ideal \mathfrak{m} and residue field κ , then $\dim_{\kappa} \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$.*

Proof. This is a consequence of Krull's generalized principal ideal theorem: if R is a Noetherian ring, $(x_1, \dots, x_c) \in R$ and P is minimal among prime ideals of R containing x_1, \dots, x_c , then $\text{ht} P \leq c$ (in the special case where $c = 1$, this says that principal ideals always have height ≤ 1 , which is where the name comes from, and the general case can be reduced to this one).

Now, if R is a Noetherian local ring of Krull dimension d , then the ideal \mathfrak{m} has height d by definition and by the previous result cannot be generated by fewer than d elements. By Nakayama's lemma, a collection of elements generates \mathfrak{m} if and only if the images of these elements in $\mathfrak{m}/\mathfrak{m}^2$ generates this as a κ -vector space. \square

Remark 5.2.1.2. Krull's principal ideal theorem is known to hold in various non-Noetherian settings, e.g., for Krull domains (reference?). However, the generalized principal ideal theorem can fail for Krull domains (reference). While work has been done to understand situations in which it holds, this basic fact is one reason Noetherian assumptions are in place here.

Definition 5.2.1.3. A Noetherian local ring (R, \mathfrak{m}) with residue field κ is called *regular* if $\dim_{\kappa} \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Example 5.2.1.4. Every field is a regular local ring of dimension 0. Local rings like $k[\epsilon]/\epsilon^2$ are not regular: indeed (ϵ) is a non-zero maximal ideal here but $(\epsilon)/(\epsilon)^2$ is a 1-dimensional k -vector space. However, the ring $k[\epsilon]/\epsilon^2$ has dimension 0. More generally, suppose (R, \mathfrak{m}) is a regular local ring of Krull dimension 0 and residue field κ . In that case, $\dim_{\kappa} \mathfrak{m}/\mathfrak{m}^2 = 0$ as well, i.e., $\mathfrak{m} = \mathfrak{m}^2$. By induction, one concludes that $\mathfrak{m} = \mathfrak{m}^n$ for all $n \geq 0$. However, a Noetherian ring of Krull dimension 0 is automatically Artinian, and therefore $\mathfrak{m}^n = 0$ for n sufficiently large. Therefore, \mathfrak{m} must be the zero ideal, in which case R is a field.

Example 5.2.1.5. Any discrete valuation ring is a regular local ring of dimension 1.

Example 5.2.1.6. Any localization of a polynomial ring $k[x_1, \dots, x_n]$ at a maximal ideal is a regular local ring of dimension n .

5.2.2 Symmetric algebras and tangent spaces

Given a Noetherian local ring (R, \mathfrak{m}) with residue field κ , we considered the κ -vector space $\mathfrak{m}/\mathfrak{m}^2$. We want to enhance this κ -vector space to an actual variety. To this end, we begin by recalling the construction of symmetric powers of a module, and we do this in greater generality than we will need here.

Symmetric powers

Definition 5.2.2.1. If R is a ring and M is an R -module, then the symmetric algebra on M , denoted $\text{Sym} M$ is defined as the quotient

$$\text{Sym} M := T(M)/\langle x \otimes y - y \otimes x \mid x, y \in M \rangle.$$

As with the exterior algebra, $\text{Sym}M$ is a graded algebra, but it is commutative. By the universal property of the tensor algebra, if M is an R -module, and A is any commutative R -algebra, any R -module homomorphism $M \rightarrow A$ extends to an R -algebra homomorphism $T(M) \rightarrow A$. The commutativity of A ensures that this homomorphism factors through a homomorphism $\text{Sym}M \rightarrow A$. On the other hand, given an R -algebra map $\text{Sym}M \rightarrow A$, there is an induced R -module homomorphism $M \rightarrow A$. These two constructions are mutually inverse and yield a universal property characterizing the symmetric algebra, which we summarize in the following result.

Lemma 5.2.2.2. *If M is an R -module, and A is an R -algebra, then*

$$\text{Hom}_{\text{Mod}_R}(M, A) = \text{Hom}_{\text{Aff}_R}(\text{Sym}M, A).$$

The symmetric power has other properties analogous to the exterior power.

Lemma 5.2.2.3. *If R is a commutative unital ring, then the symmetric power functor has the following properties:*

1. *The module $\text{Sym}^n R^{\oplus d}$ is a free module of rank $\binom{d-1+n}{n}$.*
2. *If $M \oplus N$ is a direct sum decomposition, then there is a natural isomorphism*

$$\text{Sym}^n(M \oplus N) \cong \bigoplus_{i=0}^n (\text{Sym}^i M) \otimes (\text{Sym}^{n-i} N).$$

Corollary 5.2.2.4. *If M is a free R -module of rank n , then a choice of basis x_1, \dots, x_n of M determines an isomorphism $\text{Sym}M \xrightarrow{\sim} R[x_1, \dots, x_n]$.*

Tangent cones

If (R, \mathfrak{m}) is a Noetherian local ring with residue field κ , then R is *filtered* by the powers of \mathfrak{m} : more precisely, if $a \in \mathfrak{m}^r$ and $b \in \mathfrak{m}^s$, then $ab \in \mathfrak{m}^{r+s}$. The associated graded ring of this filtered ring is the κ -algebra $gr_{\mathfrak{m}}R := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$. The fact that the ring (R, \mathfrak{m}) is filtered implies that there is an induced ring homomorphism $R \rightarrow gr_{\mathfrak{m}}R$.

Example 5.2.2.5. If we take R to be the localization of a polynomial ring over a field k in d -variables x_1, \dots, x_d at the maximal ideal (x_1, \dots, x_n) , then it follows that $\mathfrak{m}^r / \mathfrak{m}^{r+1}$ can be identified with the vector space of homogeneous symmetric polynomials of degree r in n variables. In particular, $\mathfrak{m}^r / \mathfrak{m}^{r+1} \cong \text{Sym}^r \mathfrak{m} / \mathfrak{m}^2$, which has dimension $\binom{d-1+r}{r}$. In particular, the identification

$$\binom{d-1+r}{r} = \binom{d-1+r}{d-1}$$

shows that this function grows as a polynomial of degree $d-1$. More precisely, set

$$\binom{t}{n} := \frac{1}{n!} t(t-1) \cdots (t-n+1)$$

and observe that $\binom{d-1+r}{d-1}$ is the value at integers of $\binom{t+(d-1)}{d-1}$. Polynomials that take integer values at integers are called *numerical polynomials*.

We now analyze the situation discussed in the previous example in greater detail.

Definition 5.2.2.6. If (R, \mathfrak{m}) is a Noetherian local ring with residue field κ , then the *tangent cone* at \mathfrak{m} is the graded κ -algebra $\bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$.

Before discussing the relationship between this notion and regularity, we discuss some facts about graded rings. Each $\mathfrak{m}^n / \mathfrak{m}^{n+1}$ is a finite-dimensional κ -vector space, and we can consider its dimension $\dim_{\kappa} \mathfrak{m}^n / \mathfrak{m}^{n+1}$; this assignment defines a function $f(n) := \dim_{\kappa} \mathfrak{m}^n / \mathfrak{m}^{n+1}$. In Example 5.2.2.5, we observed that in one special case this function grew as a polynomial in n . Before studying the general case, we recall some simple facts about numerical polynomials.

Numerical polynomials

We begin with a brief review of integer polynomials. A polynomial $P(t)$ is called a numerical polynomial if it takes integer values at integers. The sum and product of any two numerical polynomials is integer valued, and since 0 and 1 are integer valued, it follows that numerical polynomials form a subring of $\mathbb{Q}[t]$. The following binomial polynomials give examples of numerical polynomials of arbitrary degree:

In fact, the above polynomials form a basis of the \mathbb{Z} -module of numerical polynomials. Indeed, every numerical polynomial can be written as a rational linear combination of such polynomials (since there is one of each degree). Therefore, if given an integer polynomial f and an equality

$$f = \lambda_0 + \lambda_1 t + \cdots + \lambda_n \binom{t}{n}$$

with $\lambda_i \in \mathbb{Z}$, we conclude first that since $f(0) = \lambda_0$, that $\lambda_0 \in \mathbb{Z}$. Then by induction we may conclude that λ_i are all integers. Therefore, we have established the following fact.

Lemma 5.2.2.7. *The \mathbb{Z} -submodule of $\mathbb{Q}[t]$ consisting of numerical polynomials has a basis consisting of the polynomials $\binom{t}{n}$.*

The following result generalizes the observation made in Example 5.2.2.5.

Proposition 5.2.2.8. *If R is a Noetherian local ring with maximal ideal \mathfrak{m} and residue field κ , then the assignment $n \mapsto \dim_{\kappa} \mathfrak{m}^n / \mathfrak{m}^{n+1}$ is a numerical polynomial φ . Moreover, the following are equivalent: $\dim R = d$, φ has degree $d - 1$ and \mathfrak{m} is generated by d elements.*

Proof. This is established by induction; see [Sta15, Tag 00KD Proposition 10.59.8] □

Tangent cones and regularity

The universal property of the symmetric algebra shows that the R -module map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow gr_{\mathfrak{m}}R$ induces a homomorphism $\text{Sym}\mathfrak{m}/\mathfrak{m}^2 \rightarrow gr_{\mathfrak{m}}R$. We now use this observation to give another characterization of regularity.

Proposition 5.2.2.9. *If R is a regular local ring with maximal ideal \mathfrak{m} , the map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow gr_{\mathfrak{m}}R := \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ induces an isomorphism $\text{Sym}^{\bullet} \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} gr_{\mathfrak{m}}R$.*

Proof. Suppose R is simply a Noetherian local ring of Krull dimension d and maximal ideal \mathfrak{m} . In that case, we get a map

$$\psi : \text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \longrightarrow gr_{\mathfrak{m}} R;$$

this map is a ring homomorphism by the construction of the product on both sides. The map ψ is surjective essentially by construction: indeed, since R is a Noetherian local ring of dimension d , the ideal \mathfrak{m} is generated by $\geq d$ generators x_1, \dots, x_r and we can then write down generators for $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ as homogeneous polynomials of degree n in the x_i . To see that ψ is injective it suffices to count dimensions. Indeed, we observed above that if R is regular of dimension d , then \mathfrak{m} is generated by exactly d elements (and cannot be generated by fewer elements). The assignment $n \mapsto \dim_{\kappa} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a numerical polynomial and one shows that its degree is precisely $d - 1$ by Proposition 5.2.2.8. The kernel of the map ψ is a graded ideal I and we can consider the dimensions of the graded pieces of $\text{Sym} \mathfrak{m}/\mathfrak{m}^2/I$. The dimensions of these graded pieces also form a numerical polynomial [Sta15, Tag 00K1 Proposition 10.57.7] whose degree $< d - 1$ (see [Sta15, Tag 00K3 Lemma 10.57.10]). \square

5.3 Lecture 15: Geometry of regular local rings

5.3.1 Structural properties of regular local rings

Suppose k is an algebraically closed field and consider the ring $k[x]$. If $\mathfrak{m} \subset k[x]$ is a maximal ideal, then we can write $\mathfrak{m} = (x - a)$ and $\mathfrak{m}^n = (x - a)^n$. There is an evident sequence of inclusions $\mathfrak{m}^n \subset \mathfrak{m}^{n-1}$. The elements of \mathfrak{m}^n are those functions that have a zero of order $\geq n$ at $x = a$. It is immediate from this observation that $\bigcap_n \mathfrak{m}^n = 0$. Now, if M is any $k[x]$ -module, one can consider the filtration on M by powers of \mathfrak{m} . Krull established the following far-reaching generalization of this observation.

Proposition 5.3.1.1. *If R is a Noetherian local ring, and $I \subset R$ is any proper ideal, then for any finitely generated R -module M , $\bigcap_n I^n M$.*

Proof. Set $N = \bigcap_n I^n M$; this is a finitely generated R -module. Note that $N = I^n M \cap N$ for any integer n , by definition. We claim that $I^n M \cap N \subset IN$ for n sufficiently large; this is a consequence of the Artin-Rees lemma (which states that if I is an ideal in a Noetherian ring R , M is a finitely generated R -module and $N \subset M$ is a submodule, then there exists an integer $k \geq 1$ such that for $n \geq k$ the equality $I^n M \cap N = I^{n-k}(I^k M \cap N)$ holds). Granting this, the result follows immediately from Nakayama's lemma. \square

The fact that regular local rings $(R, \mathfrak{m}, \kappa)$ have $gr_{\mathfrak{m}} R$ isomorphic to a polynomial ring (via Proposition 5.2.2.9) is very useful: we can use the ring map $R \rightarrow gr_{\mathfrak{m}} R$ to “lift” statements about polynomial rings to corresponding statements about R itself (this technique works well to study filtered rings whose associated graded rings are “easy to understand”, e.g., the universal enveloping algebra of a Lie algebra). Here is an example of this kind of argument.

Proposition 5.3.1.2. *If R is a regular local ring, then R is a normal domain.*

Proof. We first prove that R is a domain. As usual, let \mathfrak{m} be the maximal ideal of R . Take elements $f, g \in R$ such that $fg = 0$. By Proposition 5.3.1.1, since $\bigcap_n \mathfrak{m}^n = 0$, we can find a and b maximal such that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$. The product fg lies in \mathfrak{m}^{a+b} , but since it is zero, it lies in \mathfrak{m}^{a+b+1} as well. Thus, we can view $fg \in \mathfrak{m}^{a+b+1}/\mathfrak{m}^{a+b+2}$. Now, $\text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \rightarrow gr_{\mathfrak{m}} R$ is an isomorphism by Proposition 5.2.2.9 and $\text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2$ is isomorphic to a polynomial ring in d variables, and so is a domain. In particular, the condition $0 = fg$ for the images in $\text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2$ means that $f = 0$ or $g = 0$. If $f = 0$, then that means $f \in \mathfrak{m}^{a+1}$ as well and if $g = 0$, that means $g \in \mathfrak{m}^{b+1}$. In either case, we obtain a contradiction.

Now, we establish that R is integrally closed in its field of fractions. Let \mathfrak{m} be the maximal ideal of R , κ the residue field, and set K to be the fraction field of R . By Proposition 5.2.2.9 we know that $\text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2$ is isomorphic to a polynomial ring, and is therefore integrally closed in its field of fractions $\kappa(\mathfrak{m}/\mathfrak{m}^2)$. The idea is that one deduces inductively that R is integrally closed in its field of fractions. See [Bou98, V.1.4 Proposition 15] for this statement. \square

Example 5.3.1.3. Any regular local ring R of Krull dimension 1 is a discrete valuation ring. Indeed, Proposition 5.3.1.2 implies R is a local Noetherian normal domain of Krull dimension 1, so this follows immediately from Theorem 4.3.2.3.

Definition 5.3.1.4. A sequence of elements (x_1, \dots, x_n) in a ring R is called a *regular sequence* if the ideal $(x_1, \dots, x_n) \subset R$ is proper, and for each i , x_{i+1} is not a zero-divisor in $R/(x_1, \dots, x_n)$.

Corollary 5.3.1.5. *If R is a regular local ring with maximal ideal \mathfrak{m} , and x_1, \dots, x_d is a minimal set of generators of R , then x_1, \dots, x_d is a regular sequence.*

Proof. We proceed by induction on i with the base case being that R is regular. Assume inductively that $R/(x_1, \dots, x_i)$ is a regular local ring. The images of (x_{i+1}, \dots, x_d) form a minimal set of generators for $\mathfrak{m}/(x_1, \dots, x_i)/\mathfrak{m}$, which is the maximal ideal in the Noetherian local ring $R/(x_1, \dots, x_i)$. Indeed, if one of these elements was zero, then we would have a generating set with fewer than $d - i$ generators, which would contradict the conclusion of Theorem 5.2.1.1. Thus, Proposition 5.3.1.2 guarantees that $R/(x_1, \dots, x_i)$ is an integral domain, and therefore x_{i+1} is not a zero-divisor. \square

We have obtained a number of criteria for regularity of a local ring now, and the following result puts everything together.

Proposition 5.3.1.6. *Suppose R is a Noetherian local ring of Krull dimension d , with maximal ideal \mathfrak{m} and residue field κ . The following conditions are equivalent.*

1. R is a regular local ring of Krull dimension d ;
2. $\dim_{\kappa} \mathfrak{m}/\mathfrak{m}^2 = d$;
3. the ideal \mathfrak{m} admits a system of generators with precisely d elements;
4. the map $\text{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \rightarrow gr_{\mathfrak{m}} R$ is an isomorphism.
5. the ideal \mathfrak{m} admits a system of generators that is a regular sequence of length d .

Proof. That (1) \Leftrightarrow (2) was the definition. That (2) \Leftrightarrow (3) was Theorem 5.2.1.1. That (3) \Rightarrow (4) is Proposition 5.2.2.9. That (4) \Rightarrow (5) is Corollary 5.3.1.5. It is not hard to show that (5) \Rightarrow (2). \square

5.3.2 Regular rings

Definition 5.3.2.1. Suppose R is a Noetherian ring. Say that $X = \text{Spec } R$ is *regular* at a closed point $x \in \text{Spec } R$ corresponding to a maximal ideal \mathfrak{m} if $R_{\mathfrak{m}}$ is a regular local ring and *singular* otherwise. Say that R is *regular* if $\text{Spec } R$ is regular at all closed points.

Proposition 5.3.2.2. *If R is a Noetherian regular domain, then R is a normal domain.*

Proof. According to Definition 5.3.2.1, all the localizations of R at maximal ideals are regular local rings. Now, Proposition 5.3.1.2 allows to conclude that the localizations of R at maximal ideals are normal rings. Finally, appealing to Proposition 4.4.1.2, since R is a Noetherian domain, and every localization of R at a maximal ideal is normal, we can conclude that R is normal as well. \square

Example 5.3.2.3. A regular ring of dimension 0 is a product of fields. Indeed, any Noetherian ring of Krull dimension 0 is a product of Artin local rings. A regular domain of dimension 1 is precisely a Dedekind domain. Indeed, if R is a regular ring of Krull dimension 1, then R is a Noetherian normal domain of Krull dimension 1 by Proposition 5.3.2.2.

The following result gives the first geometric consequence of normality: singular points of normal varieties lie in codimension ≥ 2 .

Corollary 5.3.2.4. *If R is a Noetherian normal domain, then R is regular in codimension 1, i.e., for any height 1 prime ideal $\mathfrak{p} \subset R$, then $R_{\mathfrak{p}}$ is a regular local ring.*

Proof. Since discrete valuation rings are regular local rings, it suffices to observe that if R is a Noetherian normal domain, then $R_{\mathfrak{p}}$ is a discrete valuation ring. \square

Proposition 5.3.2.5. *If R is a regular ring of Krull dimension d , then $R[x_1, \dots, x_n]$ is a regular ring of Krull dimension $d + n$.*

Proof. By induction, it suffices to treat the case where $n = 1$. Suppose \mathfrak{M} is a maximal ideal of $R[x]$ and set $\mathfrak{m} = \mathfrak{M} \cap R$. In that case, $R[x]_{\mathfrak{M}}$ is a localization of $R_{\mathfrak{m}}[x]$ at the maximal ideal $\mathfrak{M}R_{\mathfrak{m}}[x]$, so we can assume without loss of generality that R is a regular local ring.

Assuming now that R is local with maximal ideal \mathfrak{m} , let $k = R/\mathfrak{m}$ and consider the homomorphism $R[x] \rightarrow k[x]$. The ideal generated by \mathfrak{M} in $k[x]$ is principal, generated by a monic irreducible polynomial \bar{f} . Therefore, we can find a monic polynomial $f \in R[x]$ lifting this element and such that $\mathfrak{M} = (\mathfrak{m}, f)$. Since R is regular, it is an integral domain, and therefore $R[x]$ is an integral domain as well. Since the ideal \mathfrak{M} is maximal, the element f cannot be zero and combining everything is not a zero-divisor. Therefore, $\text{ht } \mathfrak{M} = \text{ht } \mathfrak{m} + 1$. Since $R[x]$ has Krull dimension $d + 1$, it follows that we have constructed a minimal set of generators for \mathfrak{M} and thus $R[x]_{\mathfrak{M}}$ is a regular local ring. \square

Remark 5.3.2.6. Regularity is an interesting notion, but from what we said above it is not clear that it captures the intuitive notion of smoothness from differential geometry. For example, we had to work hard to prove that if $X = \text{Spec } R$ is a regular ring then $X \times \mathbb{A}^n$ is regular. One can construct examples to show that if X is regular, then $X \times X$ need not be regular. For example set $k = \mathbb{F}_p(t)$, and take $R = k[x]/(x^p - t)$. In this case, R is a field, namely the purely inseparable extension of $\mathbb{F}_p(t)$ obtained by adjoining a p -th root of t ; therefore R is regular. On the other hand $R \otimes_k R$ is a zero-dimensional local ring, which is not a field, and therefore not regular.

5.4 Lecture 16: Projective resolutions and Tor

5.4.1 Projective resolutions and K_0

If R is a ring, then we think of elements of $K_0(R)$ as formal differences $([P], [Q])$ of projective R -modules. We now give a more flexible *homological* approach to elements of $K_0(R)$. First, observe that given a short exact sequence of projective modules

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0,$$

since this exact sequence splits, we conclude that $P \cong P' \oplus P''$. We now extend this result slightly, but before doing so we make the following general definition.

Definition 5.4.1.1. If R is a commutative unital ring, and M is an R -module, then a (left) resolution of M is an exact sequence

$$\cdots \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

Such a resolution is called *finite* if there is an integer $r \geq 0$ such that $E_s = 0$ for all $s > r$, *free* if each E_i is free, and *projective* if each E_i is a projective R -module. We will frequently write $E_\bullet \rightarrow M$ is a resolution.

Lemma 5.4.1.2. If R is a commutative unital ring, and Q_\bullet is a finite resolution of a projective R -module P by finitely generated projective R -modules, i.e.,

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow P \longrightarrow 0,$$

then $[P] = \chi(Q_\bullet) = \sum_{i=0}^n (-1)^i [Q_i]$.

Proof. We proceed by induction on n . If $n = 0$, the result is obvious. Since the map $Q_0 \rightarrow P$ is surjective and P is projective, we conclude that $Q_0 \cong P \oplus \ker(Q_0 \rightarrow P)$. However, $\ker(Q_0 \rightarrow P)$ is projective, and the map $Q_1 \rightarrow \ker(Q_0 \rightarrow P)$ is surjective, so we obtain a resolution of $\ker(Q_0 \rightarrow P)$ of smaller length. Thus, $[Q_0] = [P] + [\ker(Q_0 \rightarrow P)]$ and the result follows from the induction hypothesis. \square

This easy observation shows that one way to obtain results about the structure of $K_0(R)$ is to show that all modules have projective resolutions by modules of a certain type.

Proposition 5.4.1.3. If $\varphi : R \rightarrow S$ is a ring homomorphism, then say an S -module M is extended from R if $M \cong M' \otimes_R S$ for some R -module M' . If every projective S -module admits a finite projective resolution by modules extended from R , then $K_0(R) \rightarrow K_0(S)$ is surjective.

5.4.2 Properties of Tor

Suppose now that R is a commutative unital ring and M is an R -module, we can consider the functor $M \otimes_R -$ on Mod_R . Since neither this functor, nor $- \otimes_R M$ preserves exactness, we are interested in measuring the failure of exactness. The functor Tor_R is cooked up to measure this failure.

Example 5.4.2.1. Suppose R is a ring and M is an R -module, in that case given an element $r \in R$, multiplication by r determines an R -module map $\cdot r : R \rightarrow R$. If r is not a zero-divisor, then this map is injective. In that case, there is the following exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/(r) \longrightarrow 0.$$

Now, if we tensor this exact sequence with M the multiplication by r map induces the multiplication by r map:

$$M \xrightarrow{\cdot r} M.$$

While the initial sequence was exact, this sequence fails to be exact: the cokernel of multiplication by r is $M/rM \cong M \otimes_R R/(r)$, but the kernel of the multiplication by r map consists of those elements $m \in M$ such that $rm = 0$, i.e., it is the r -torsion submodule of M , sometimes written $Tor_1^R(R/(r), M)$.

Example 5.4.2.2. More generally, given a finitely generated ideal $I \subset R$ and generators (r_1, \dots, r_n) of I , we can study the I -torsion in an R -module M by sequentially comparing torsion with respect to each r_i ; as in the previous example, we will need some condition on the r_i to ensure that we actually obtain a resolution. For example, if I is generated by 2 elements r_1 and r_2 , then we can consider the map $R^{\oplus 2} \rightarrow R$ given by multiplication by $(r_1, r_2)^t$. The cokernel of this map is I . We can analyze this a bit more systematically by tensoring the two complexes $R \xrightarrow{\cdot r_1} R$ and $R \xrightarrow{\cdot r_2} R$ to obtain a diagram of the form

$$\begin{array}{ccc} R & \xrightarrow{r_2} & R \\ \downarrow r_1 & & \downarrow r_1 \\ R & \xrightarrow{r_2} & R. \end{array}$$

By changing the signs slightly, this yields the following complex

$$R \xrightarrow{(r_2, -r_1)} R \xrightarrow{\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}} R,$$

(the composite is zero precisely because of the sign change). The cokernel of the last map is precisely $R/(r_1, r_2)$, and in good situations, this sequence actually yields a resolution of $R/(r_1, r_2)$. Indeed, if we know that r_1 and r_2 are not zero-divisors, then the first map is injective.

The kernel of the map $R \oplus R \rightarrow R$ consists of those pairs (a, b) such that $r_1a + r_2b = 0$. Thus, r_2b lies in the ideal (r_1) and the kernel of the first map can be described as those elements $r \in R$ such that $r(r_2) \subset (r_1)$. This collection of elements is an ideal, called the ideal quotient and often denoted $(r_1 : r_2)$. Since r_1 is assumed to not be a zero-divisor, it follows that a is uniquely determined by b . The image of $R \rightarrow R \oplus R$ consists of elements of the form $(r_2r, -r_1r)$. Now, such an element is contained in the kernel and the image in $(r_1 : r_2)$ is precisely (r_1) . Thus, if r_2 is not a zero-divisor in $R/(r_1)$, we conclude that the sequence is exact in the middle as well. Thus, if (r_1, r_2) is a *regular sequence* in the sense we studied earlier, then we obtain a free resolution of $R/(r_1, r_2)$.

Tensoring this sequence with M we obtain a complex that has non-trivial homology: the zeroth homology still computes $M/(r_1, r_2)M$, but there are higher homology terms. For example, the

map $M \oplus M \rightarrow M$ one obtains sends $(m_1, m_2) \mapsto r_1 m_1 + r_2 m_2$. Thus the first homology of the complex obtained by tensoring with M is the quotient of the submodule of M annihilated by (r_1, r_2) by certain relations.

Remark 5.4.2.3. The complex described in Example 5.4.2.2 is called the *Koszul complex*, and admits a generalization to regular sequences in an arbitrary commutative ring R . The specific example shows that if $(R, \mathfrak{m}, \kappa)$ is a 2-dimensional regular local ring, then the maximal ideal admits a finite free resolution. The second example points to an ambiguity: there are many possible sequences of generators for an ideal I in a ring R , and the cohomology groups obtained in the example might depend on these choices.

Suppose M is a fixed R -module. If $P_\bullet \rightarrow M$ is a projective (flat) resolution of M , then the tensor product $P_\bullet \otimes_R N$ has the structure of a complex of R -modules and thus we can consider the homology of this complex. If P'_\bullet is another projective (flat) resolution of M , then using sign changes in a fashion similar to Example 5.4.2.2, then one can build a complex $Tot(P_\bullet \otimes_R P'_\bullet)$ out of $P_\bullet \otimes_R P'_\bullet$ (see [Wei94, 2.7.1] for details). Since P_\bullet and P'_\bullet are resolutions, one shows that $Tot(P_\bullet \otimes_R P'_\bullet)$ is another resolution of M . Moreover, the maps $P_\bullet \rightarrow Tot(P_\bullet \otimes_R P'_\bullet)$ and $P'_\bullet \rightarrow Tot(P_\bullet \otimes_R P'_\bullet)$ induce morphisms of complexes after tensoring with some module N . One checks as in [Wei94, Theorem 2.7.6] that the maps on homology induced by the morphisms of complexes in the previous sentence are isomorphisms. Therefore, the following definition makes sense.

Definition 5.4.2.4. Suppose M is a fixed R -module. If N is an arbitrary R -module, define $\text{Tor}_i^R(M, N)$ as the i -th homology of the complex $P_\bullet \otimes_R M$ for any flat resolution $P_\bullet \rightarrow M$.

Lemma 5.4.2.5. *If M is a fixed R -module, then the following statements hold:*

1. the groups $\text{Tor}_i^R(M, N)$ can be computed using a projective resolution;
2. the groups $\text{Tor}_i^R(M, N) = 0$ if $i < 0$;
3. the group $\text{Tor}_0^R(M, N) = M \otimes N$;
4. there is an isomorphism $\text{Tor}_i^R(M, N) \cong \text{Tor}_i^R(N, M)$;
5. the groups $\text{Tor}_i^R(M, N)$ have a natural R -module structure, functorially in the input modules; moreover the map induced by multiplication by $r \in R$ on M is precisely multiplication by r , i.e., the functor $\text{Tor}_i^R(-, N)$ is an R -linear functor; and
6. given a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, there is a functorially associated short exact sequence of R -modules of the form

$$\cdots \longrightarrow \text{Tor}_i^R(M', N) \longrightarrow \text{Tor}_i^R(M, N) \longrightarrow \text{Tor}_i^R(M'', N) \longrightarrow \text{Tor}_{i-1}^R(M', N) \longrightarrow \cdots$$

Remark 5.4.2.6. When we actually use these results, we will assume R is Noetherian and study finitely generated modules. In this case, a resolution by finitely generated flat modules will automatically be a projective resolution, so we will later be sloppy about the distinction. However, without suitable finiteness hypotheses in place, we will need to be careful about the difference between flat and projective resolutions. For example, take $R = \mathbb{Z}$ and consider $M = \mathbb{Q}$. Note that \mathbb{Q} is a flat \mathbb{Z} -module and therefore $\text{Tor}_i^{\mathbb{Z}}(\mathbb{Q}, N) = 0$ for $i > 0$ and any \mathbb{Z} -module M . However, \mathbb{Q} is not itself a projective \mathbb{Z} -module (it is an injective \mathbb{Z} -module) and has projective dimension 1.

5.4.3 Change of rings

If $R \rightarrow S$ is a ring homomorphism, and M and N are R -modules, then we can extend scalars from R to S to view $M \otimes_R S$ and $N \otimes_R S$ as S -modules. Now if $P_\bullet \rightarrow M$ is a projective resolution of M , we can use it to compute $\mathrm{Tor}_i^R(M, N) = H_i(P_\bullet \otimes_R N)$. The tensor product $P_\bullet \otimes_R S$ is *not* in general a projective resolution of $M \otimes_R S$. However, if $R \rightarrow S$ is a flat ring homomorphism, then $P_\bullet \otimes_R S$ is a flat resolution of $M \otimes_R S$. In that case, we deduce the following result.

Lemma 5.4.3.1. *If $\varphi : R \rightarrow S$ is a flat ring homomorphism, and M is an R -module, and N is an S -module then there is a functorial isomorphism*

$$\mathrm{Tor}_i^R(M, N) \otimes_R S \longrightarrow \mathrm{Tor}_i^S(M \otimes_R S, N).$$

In particular, if S is a localization of R , it follows that $\mathrm{Tor}_i^S(M \otimes_R S, N \otimes_R S)$ is a localization of the R -module $\mathrm{Tor}_i^R(M, N)$.

When $R \rightarrow S$ is not flat, the Tor-groups are still suitably functorial with respect to change of rings.

Lemma 5.4.3.2. *If $\varphi : R \rightarrow S$ is a ring homomorphism, M and N are R -modules, then there is a natural R -module map*

$$\mathrm{Tor}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M \otimes_R S, N \otimes_R S).$$

Definition 5.4.3.3. If M is an R -module, then we say that M has projective (resp. flat) dimension $\leq d$ if M admits a projective (resp. flat) resolution of length $\leq d$. We write $\mathrm{pd}(M)$ (resp. $\mathrm{fd}(M)$) for the minimum of the lengths of finite projective (resp. flat) resolutions of d (or ∞ if no such resolution exists).

Lemma 5.4.3.4. *If R is a Noetherian ring, and M is an R -module, the following conditions are equivalent:*

1. $\mathrm{pd}(M) = d$;
2. $\mathrm{Tor}_i^R(M, N) = 0$ for $i > d$.

Proof. The second statement implies the first since Tor can be computed using projective resolutions. For the other direction, we leave this as an exercise (for the time being): use the facts (i) that finitely presented flat modules are finitely generated projective and (ii) an arbitrary R -module can be written as a filtered colimit of its finitely presented sub-modules. \square

Definition 5.4.3.5. If R is a ring, we say R has *finite global dimension* if $\sup\{\mathrm{pd}(M) \mid M \in \mathrm{Mod}_R\}$ is finite.

5.5 Lecture 17: Homological theory of regular rings

5.5.1 Regular local rings and finite free resolutions

Our goal will be to study projective resolutions over regular local rings. If R is a regular local ring of dimension 0, then R is a field, and therefore every R -module is automatically projective. Therefore,

regular local rings of dimension 0 have finite global dimension. If R is a regular local ring of Krull dimension 1, then R is a discrete valuation ring and therefore a principal ideal domain. In that case, every finitely generated module is the direct sum of a finitely generated free module and a finitely generated torsion module. Any finitely generated torsion module admits a free resolution of length 1 and therefore, we conclude that every finitely generated module admits projective resolutions of length ≤ 1 . By careful limit arguments, one can show that not necessarily finitely generated modules also admit projective resolutions of length ≤ 1 . We now analyze the global dimension of modules over regular local rings in general.

5.5.2 Minimal free resolutions

There are particularly nice free resolutions of finitely generated modules over Noetherian local rings. If M is finitely generated, then we can choose a minimal set of generators of M to obtain a surjection $F_0 \rightarrow M$. Continuing inductively, we can choose a minimal set of generators of $\ker(F_i \rightarrow F_{i-1})$ to build a free resolution of M . Such a resolution will be called a *minimal* free resolution. If $F_\bullet \rightarrow M$ is a minimal free resolution, then $F_i \rightarrow F_{i-1}$ is given by a matrix with coefficients in \mathfrak{m} . Now, the image of $F_i \rightarrow F_{i-1}$ surjects onto the kernel of $F_{i-1} \rightarrow F_{i-2}$. Now, the kernel of $F_{i-1} \rightarrow F_{i-2}$ consists of relations among generators of F_{i-1} . If such a relation is given by a unit in R , then it follows that the two basis vectors are redundant; in other words if a resolution is minimal, then there is no relation with coefficient that is a unit. Thus, the image of $F_i \rightarrow F_{i-1}$ is contained in $\mathfrak{m}F_{i-1}$. Using this observation, we deduce the following fact.

Theorem 5.5.2.1. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \kappa)$. The modules $\mathrm{Tor}_i^R(M, \kappa)$ are finite-dimensional κ -vector spaces and $\dim_\kappa(\mathrm{Tor}_i(M, \kappa))$ is the same rank as the rank of the i -th free module in a minimal free resolution of M . Moreover, the following statements are equivalent:*

- in a minimal free resolution F_\bullet of M , $F_{n+1} = 0$;
- the projective dimension of M is at most n ;
- $\mathrm{Tor}_{n+1}(M, \kappa) = 0$;
- $\mathrm{Tor}_i(M, \kappa) = 0$ for all $i \geq n + 1$.

It follows that a minimal free resolution is the shortest possible projective resolution of M . In particular, M has finite projective dimension if and only if a minimal free resolution is finite.

Proof. If M is a finitely generated module, we can compute $\mathrm{Tor}_i^R(M, \kappa)$ by taking a free resolution of M . Pick a *minimal* free resolution $F_\bullet \rightarrow M$. In that case, $F_\bullet \otimes_R \kappa$ is a complex of finite-rank κ -vector spaces and the finiteness of $\dim_\kappa(\mathrm{Tor}_i(M, \kappa))$ is immediate. Since the image of $F_i \rightarrow F_{i-1}$ is contained in $\mathfrak{m}F_{i-1}$, it follows that after tensoring with $\kappa = R/\mathfrak{m}$, the maps $F_i \otimes_R \kappa \rightarrow F_{i-1} \otimes_R \kappa$ are trivial. Therefore, $\dim_\kappa(\mathrm{Tor}_i(M, \kappa)) = \dim_\kappa F_i \otimes_R \kappa$.

Now, (1) \Rightarrow (2) since Tor can be computed by a projective resolution. The statement (2) \Rightarrow (3) is immediate from the definition of projective dimension. Note that (3) \Rightarrow (1) as well, since we can compute $\mathrm{Tor}_{n+1}(M, \kappa)$ using a minimal free resolution, and in that case, $0 = \dim_\kappa(\mathrm{Tor}_{n+1}(M, \kappa)) = \dim_\kappa F_{n+1} \otimes_R \kappa$, i.e., $F_{n+1} = 0$. Once one term in a minimal free resolution is zero, we conclude that all higher terms are zero as well.

Note also that (1) \Rightarrow (4) \Rightarrow (3) \rightarrow (1) by a similar argument and using the fact that there cannot exist a resolution of length shorter than a minimal free resolution. \square

5.5.3 Tor and regular sequences

If R is a ring and $x \in R$ is not a zero-divisor, then we begin by establishing a connection between Tor_i^R and $\text{Tor}_i^{R/x}$.

Proposition 5.5.3.1. *Let R be a ring and $x \in R$ an element.*

1. *Given an exact sequence Q_\bullet of modules*

$$\cdots \longrightarrow Q_{n+1} \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots$$

such that x is not a zerodivisor on all Q_n , the complex \bar{Q}_\bullet obtained by tensoring with R/xR , i.e.,

$$\cdots \longrightarrow Q_{n+1}/xQ_{n+1} \longrightarrow Q_n/xQ_n \longrightarrow Q_{n-1}/xQ_{n-1} \longrightarrow \cdots$$

is also exact.

2. *If x is not a zerodivisor in R and also not a zerodivisor on the module M , while $xN = 0$, then, for all i $\text{Tor}_i^R(M, N) \cong \text{Tor}_i^{R/xR}(M/xM, N)$*

Proof. For Point (1), observe that since x is not a zero-divisor, the multiplication by x map determines a short exact sequence $Q_n \rightarrow Q_n \rightarrow Q_n/xQ_n$ for every n . Thus, multiplication by x yields an exact sequence of chain complexes of the form:

$$0 \longrightarrow Q_\bullet \xrightarrow{x} Q_\bullet \longrightarrow \bar{Q}_\bullet \longrightarrow 0.$$

Consider the associated long exact sequence in homology for this chain complex. Since Q_\bullet is an exact sequence, it follows that $H_*(Q_\bullet) = 0$. Therefore, by the five lemma, we conclude that $H_*(\bar{Q}_\bullet) = 0$, i.e., \bar{Q}_\bullet is exact as claimed.

For Point (2), take a free resolution of $F_n \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$. By Point (1), this sequence remains exact after applying $\otimes_R R/xR$. Thus, we obtain a free resolution of M/xM over R/xR . Let F_\bullet be the complex obtained by forgetting M in our free resolution. In that case, the homology at the n -th spot of $F_\bullet \otimes_R N$ computes $\text{Tor}_n^R(M, N)$. Since x kills N , $(R/xR) \otimes_{R/xR} N \cong N$. Thus, $\text{Tor}_n(M, N)$ is the homology at the n -th spot of $(F_\bullet \otimes_R R/(x)) \otimes_{R/(x)} N$. Since $F_\bullet \otimes_R R/(x)$ is a free resolution of M/xM over $R/(x)$ it follows that this group coincides with $\text{Tor}_n^{R/(x)}(M, N)$. \square

Proposition 5.5.3.2. *If R is a regular local ring of Krull dimension $\leq d$, then any finitely generated R -module has projective dimension $\leq d$.*

Proof. Suppose R is a regular local ring; throughout we write \mathfrak{m} for the maximal ideal in R and κ for the residue field. We proceed by induction on the dimension.

If $\dim R = 0$, then R is a field by Example 5.2.1.4. In that case, every finitely generated R -module is already free and the result follows.

Now, suppose $\dim R \geq 1$ and fix a finitely generated R -module M . It suffices to prove that $\text{Tor}_n(M, \kappa) = 0$ for $n > d$ by the equivalent properties of Tor. Now, choose a projective module P and a surjection $P \rightarrow M$ and let M_1 be the kernel of this map so we have a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow P \longrightarrow M \longrightarrow 0,$$

where F is finitely generated and free. Since $M_1 \subset F$, if we choose a regular parameter $x \in M$, x is not a zerodivisor on M_1 as well.

Therefore by Proposition 5.5.3.1 we conclude that $\mathrm{Tor}_n^R(M_1, \kappa) = \mathrm{Tor}_n^{R/xR}(M_1/xM_1, \kappa)$.

The long exact sequences for Tor associated with the above short exact sequence show that $\mathrm{Tor}_{n+1}^R(M, \kappa) \cong \mathrm{Tor}_n(M_1, \kappa) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, \kappa)$ for $n \geq d$. Since R is a regular local ring with maximal ideal \mathfrak{m} and x is a regular parameter, we conclude that R/xR is again a regular local ring. \square

Proposition 5.5.3.3. *If R is a ring, the following conditions are equivalent:*

1. *the ring R has finite global dimension;*
2. *every cyclic module R/I has projective dimension $\leq d$;*
3. *every finitely generated R -module has projective dimension $\leq d$.*

Proof. That (1) \Rightarrow (2) is immediate from the definition.

To see that (2) \Rightarrow (3), first observe that every finitely generated R -module has a finite filtration by cyclic modules. Indeed, we proceed by induction on the number of generators of M . Let x_1, \dots, x_r be a minimal generating set of M . Set $M' = Rx_1 \subset M$. In that case, M/M' has $r - 1$ generators, and $M' \cong R/I_1$ with $I_1 = \{f \in R \mid fx_1 = 0\}$.

To see that (3) \Rightarrow (2), we use a limit argument and write M as a filtered limit of finitely generated sub-modules. See [Sta15, Tag 065T] for more details. \square

Theorem 5.5.3.4. *If R is a regular local ring of Krull dimension $\leq d$, then R has finite global dimension.*

Proof. By Proposition 5.5.3.3 it suffices to show that all finitely generated modules have projective dimension $\leq d$, but this follows from Proposition 5.5.3.2. \square

5.5.4 Globalizing

In this section, we globalize the results of the previous section and show that for an arbitrary regular ring R , every R -module admits a finite projective resolution.

Proposition 5.5.4.1. *If R is a Noetherian ring, then R has finite global dimension if and only if there exists an integer n such that for every maximal ideal $\mathfrak{m} \subset R$ the ring $R_{\mathfrak{m}}$ has global dimension $\leq n$.*

Lemma 5.5.4.2 (Schanuel's lemma). *Suppose R is a ring and M is an R -module. Given two short exact sequences $0 \rightarrow K \rightarrow P_1 \rightarrow M \rightarrow 0$ and $0 \rightarrow L \rightarrow P_2 \rightarrow M \rightarrow 0$ with P_1 and P_2 projective, $K \oplus P_2 \cong L \oplus P_1$.*

Proof. Consider the module defined by the short exact sequence $0 \rightarrow N \rightarrow P_1 \oplus P_2 \rightarrow M$, where the last map is the sum of the two maps $P_1 \rightarrow M$ and $P_2 \rightarrow M$. Since the kernel of the map $P_1 \rightarrow M$ is K and the kernel of the map $P_2 \rightarrow M$ is L , one checks that the composite $N \rightarrow P_1$ is surjective with kernel L and $N \rightarrow P_2$ is surjective with kernel K . However, since the P_i are projective, the statement follows. \square

Corollary 5.5.4.3. *Suppose R is a ring and M is an R -module of projective dimension d . Given $F_e \rightarrow F_{e-1} \rightarrow \dots \rightarrow F_0 \rightarrow M$ an exact sequence with F_i projective and $e \geq d - 1$, the kernel $F_e \rightarrow F_{e-1}$ is projective (or the kernel of $F_0 \rightarrow M$ is projective if $e = 0$).*

Proof. We proceed by induction on d . If $d = 0$, then M is projective so given a surjection $F_0 \rightarrow M$, we can choose a splitting and identify $F_0 \cong M \oplus \ker(F_0 \rightarrow M)$ with both summands projective. Thus, if $e = 0$, we are done. If $e > 0$, then replacing M by $\ker(F_0 \rightarrow M)$ we can decrease e so we conclude by induction.

Now assume $d > 0$. Let $0 \rightarrow P_d \rightarrow P_{d-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M$ be a minimal length finite resolution with P_i projective. By Schanuel's lemma 5.5.4.2 we see that $P_0 \oplus \ker(F_0 \rightarrow M) \cong F_0 \oplus \ker(P_0 \rightarrow M)$. Thus, the result is true if $d = 1$ and $e = 0$ since the right hand side is $F_0 \oplus P_1$, which is projective. Therefore, we may assume that $e > 0$. In that case, the module $F_0 \oplus \ker(P_0 \rightarrow M)$ has a finite projective resolution $0 \rightarrow P_d \oplus F_0 \rightarrow P_{d-1} \oplus F_0 \rightarrow \cdots \rightarrow P_1 \oplus F_0 \rightarrow \ker(P_0 \rightarrow M) \oplus F_0$ of length $d-1$. Thus, by induction on d , we conclude that $\ker(F_e \oplus P_0 \rightarrow F_{e-1} \oplus P_0)$ is projective. \square

Proof. First, we claim that it suffices to demonstrate the result in the case where M is finitely generated (add). Now, suppose M is finitely generated and $0 \rightarrow K_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow M$ is a resolution with each F_i finitely generated and free (since R is Noetherian we can always build such a resolution: pick generators of M and build a surjection $F_0 \rightarrow M$, take the kernel of this map, which is again finitely generated. In that case, since R is Noetherian, K_n is finitely generated. By Corollary 5.5.4.3 we conclude that $K_n \otimes_R R_{\mathfrak{m}}$ is projective for every \mathfrak{m} . However, since K_n is finitely generated and locally projective, it must be projective. In other words, we have constructed a finite projective resolution of M . \square

Corollary 5.5.4.4. *If R is a regular ring of Krull dimension d , then every finitely generated projective R -module has finite projective dimension. Moreover, R has finite global dimension.*

Proof. If R is a regular ring, then $R_{\mathfrak{m}}$ is a regular local ring of Krull dimension d by definition. Proposition 5.5.3.4 implies that regular local rings of Krull dimension d have projective dimension $\leq d$. Therefore Proposition 5.5.4.1 guarantees that R has finite projective dimension as well. \square

5.6 Lecture 18: \mathbb{A}^1 -invariance and Mayer-Vietoris for K_0 over regular rings

Our goal in this lecture is to finally establish homotopy invariance of K_0 over regular rings. If R is a regular ring, then $R[t]$ is a regular ring by Proposition 5.3.2.5, and by induction it suffices to establish that the map $K_0(R) \rightarrow K_0(R[t])$ induced by $R \rightarrow R[t]$ is an isomorphism. This map is split by the evaluation map $R[t] \rightarrow R$, so it is automatically injective and therefore we just need to establish surjectivity. By Proposition 5.4.1.3 it suffices to show that every projective $R[t]$ module has a resolution by modules that are extended from R . The argument we give is due to Swan [Swa68] as presented in [Lam06, II.5].

5.6.1 \mathbb{A}^1 -invariance and resolutions of projective modules over $R[t]$

Suppose N is a projective $R[t]$ -module. If N admits a finite free resolution, then M is automatically extended from R . However, only stably free $R[t]$ -modules admit finite free resolutions. Nevertheless, we can start building a free resolution and study the failure of extensibility from R . More precisely, by picking $R[t]$ -module generators, we obtain a surjection $R[t]^n \rightarrow M$, and the module

$R[t]$ is evidently extended from R . The kernel M of $R[t]^n \rightarrow N$ is thus a sub-module of an $R[t]$ -module that is extended from R . While M itself is not evidently extended from R , we will show that we can resolve it by (finitely generated) $R[t]$ -modules that *are* so extended. More precisely, following Swan, we establish the following result.

Lemma 5.6.1.1 (Swan). *If R is a Noetherian ring and M is an $R[t]$ -submodule of some finitely generated $R[t]$ -module N that is extended from R , then there exists a short exact sequence*

$$0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} M \longrightarrow 0$$

where X and Y are finitely generated R -modules that are extended from R .

Proof. Write $N = R[t] \otimes_R N_0$ for some $N_0 \in \text{Mod}_R^{fg}$. In that case, set $N_r := \sum_{i=0}^r R \cdot t^i \otimes_R N_0$ and set $M_r = M \cap N_r$, $r \geq 0$. Since $N_r \in \text{Mod}_R^{fg}$ and since R is Noetherian, we see that $M_r \in \text{Mod}_R^{fg}$ as well. Since $R[t]$ is also Noetherian and since $N \in \text{Mod}_{R[t]}^{fg}$, we know that $M \in \text{Mod}_{R[t]}^{fg}$.

Pick an integer n large enough so that M_{n+1} contains an $R[t]$ -module generating set of M and set $X = R[t] \otimes_R M_n$ and $Y = R[t] \otimes_R M_{n+1}$. Define a map $f : Y \rightarrow M$ by $f(t^i \otimes m) = t^i m$ for every $m \in M_{n+1}$ and extend by linearity. Note that M_{n+1} is contained in the image of f by construction and therefore f is automatically a surjective $R[t]$ -module map.

We construct an $R[t]$ -module homomorphism $g : X \rightarrow Y$ as follows. Observe that $tN_n \subset N_{n+1}$ and therefore $tM_n \subset M_{n+1}$ as well. Now, define g by means of the formula

$$g(t^i \otimes m) = t^{i+1} \otimes m - t^i \otimes tm, \quad m \in M_n,$$

and extend this by linearity to an $R[t]$ -module morphism.

It remains to check that the resulting sequence is exact. First, we claim that $fg = 0$. To see this, take $m \in M_n$ and compute:

$$fg(t^i \otimes m) = f(t^{i+1} \otimes m - t^i \otimes tm) = t^{i+1}m - t^{i+1}m = 0.$$

Next, we claim that g is injective. Take $x = t^i \otimes m + t^{i-1} \otimes m' + \dots$ with $m, m' \in M_n$ and $m \neq 0$. In that case,

$$\begin{aligned} g(x) &= g(t^i \otimes m + t^{i-1} \otimes m' + \dots) \\ &= (t^{i+1} \otimes m - t^i \otimes tm) + (t^i \otimes m' + t^{i-1} \otimes tm') \\ &= (t^{i+1} \otimes m) + t^i \otimes (m' - tm) + \dots, \end{aligned}$$

and $t^{i+1} \otimes m \neq 0$ so $g(x) \neq 0$.

To conclude, it remains to show that $\ker(f) = \text{im}(g)$. Suppose $y = \sum_{i=0}^r t^i \otimes m_i \in \ker(f)$, with $m_i \in M_{n+1}$. We will show that $y \in \text{im}(g)$ by induction on r . If $r = 0$, this is clear. Thus, assume $r \geq 0$. Write $m_i = \sum_{j=0}^{n+1} t^j \otimes a_{ij}$ where $a_{ij} \in N_0$. In that case,

$$\begin{aligned} 0 &= f(y) = \sum_{i=0}^r t^i m_i = \sum_{i=0}^r \sum_{j=0}^n t^{i+j} \otimes a_{ij} \\ &= t^{r+n+1} \otimes a_{r,n+1} + t^{r+n} \otimes (\dots) + \dots. \end{aligned}$$

Thus, $a_{r,n+1} = 0$, i.e., $m_r \in M_n$. Then,

$$y - g(t^{r-1} \otimes m_r) = y - t^r \otimes m_r + t^{r-1} \otimes tm_r = \sum_{i=0}^{r-1} t^i \otimes m'_i \dots$$

□

So far, we have not used regularity of R . Under this assumption, using Corollary 5.5.4.4 we may further resolve the X and Y as above by projective modules that are extended from R .

Proposition 5.6.1.2 (Swan). *If R is a regular ring, then every finitely generated $R[t]$ -module M admits a finite projective resolution $P_\bullet \rightarrow M$ with each P_i extended from R .*

Proof. Suppose M is a finitely generated $R[t]$ -module. As above, pick a surjection $R[t]^n \rightarrow M$ and let M' be its kernel. In that case, Lemma 5.6.1.1 guarantees the existence of an exact sequence of the form:

$$0 \longrightarrow X \xrightarrow{g} Y \longrightarrow M' \longrightarrow 0,$$

where X and Y are finitely generated $R[t]$ -modules that are extended from R .

If \bar{X} and \bar{Y} are finitely generated R -modules such that $X = \bar{X} \otimes_R R[t]$ and $Y = \bar{Y} \otimes_R R[t]$, then since R is regular, by appealing to Corollary 5.5.4.4, we can find a finite projective resolutions $\bar{X}_\bullet \rightarrow \bar{X}$ and $\bar{Y}_\bullet \rightarrow \bar{Y}$.

Next, note that $R[t]$ is free as an R -module (of countable rank). In particular, it follows that $R[t]$ is flat as an R -module and therefore that $R \rightarrow R[t]$ is a flat ring homomorphism. Therefore, it follows that $X_\bullet := \bar{X}_\bullet \otimes_R R[t] \rightarrow X$ and $Y_\bullet := \bar{Y}_\bullet \otimes_R R[t] \rightarrow Y$ are again projective resolutions of X and Y .

Since X_\bullet and Y_\bullet are projective, we can inductively lift the morphism g to a morphism of complexes (abusing terminology)

$$g : X_\bullet \longrightarrow Y_\bullet.$$

To obtain a resolution of M' , we form the *mapping cone* of g . More precisely, we define a new complex $C(g)_\bullet$ whose terms are $C(g)_i := X_{i-1} \oplus Y_i$ and where the differential $X_{i-1} \oplus Y_i \rightarrow X_{i-2} \oplus Y_{i-1}$ is given by the matrix

$$\begin{pmatrix} -d_X^{i-1} & 0 \\ g & d_Y^i \end{pmatrix}.$$

Now, one checks that $C(g)$ is actually a chain complex, and that $C(g)$ is exact except in degree 0 where the cohomology is M' . In other words, we have produced a resolution

$$0 \longrightarrow C(g) \longrightarrow R[t]^{\oplus n} \longrightarrow M \longrightarrow 0.$$

By assumption, the terms of $C(g)$ are projective $R[t]$ -modules extended from R , and the result follows. □

Putting everything together, we obtain the following result.

Theorem 5.6.1.3 (Grothendieck). *If R is a regular ring, then $K_0(R) \rightarrow K_0(R[t])$ is an isomorphism of rings.*

5.6.2 Mayer-Vietoris

We can also deduce a Mayer-Vietoris sequence just as for the Picard group. To begin, recall that if f and g are comaximal elements of a ring R , then there is a fiber product diagram of categories of the form:

$$\begin{array}{ccc} \text{Vec}(R_{fg}) & \longrightarrow & \text{Vec}(R_f) \\ \downarrow & & \downarrow \\ \text{Vec}(R_g) & \longrightarrow & \text{Vec}(R). \end{array}$$

The next result can be obtained directly from this patching result (and thus could have been established immediately after our definition of K_0).

Proposition 5.6.2.1 (Weak Mayer-Vietoris). *If R is a commutative unital ring and f and g are comaximal elements of R , then there is a short exact sequence of the form*

$$K_0(R) \longrightarrow K_0(R_f) \oplus K_0(R_g) \longrightarrow K_0(R_{fg}).$$

Proof. Suppose P is a projective R -module. Suppose we have projective R_f and R_g modules P_f and P_g whose classes in $K_0(R_{fg})$ agree. In that case, the modules $(P_f)_g$ and $(P_g)_f$ are stably isomorphic. Therefore, we can fix an isomorphism $(P_f)_g \oplus R_{fg}^n \xrightarrow{\sim} (P_g)_f \oplus R_{fg}^n$. Since the modules R_{fg} are free, they are obtained via restriction. Therefore, we can glue these modules together to get an R -module. Thus, the image of $K_0(R) \rightarrow K_0(R_f) \oplus K_0(R_g)$ surjects onto the kernel of the difference map. \square

Remark 5.6.2.2. In general, both the kernel of $K_0(R) \rightarrow K_0(R_f) \oplus K_0(R_g)$ and the cokernel of $K_0(R_f) \oplus K_0(R_g) \rightarrow K_0(R_{fg})$ are non-trivial. One of the original goals of K-theory was to measure the failure of surjectivity and to turn K-theory into a cohomology theory. For example, we can describe the kernel of $K_0(R) \rightarrow K_0(R_f) \oplus K_0(R_g)$ in terms of automorphisms of projective modules on R_{fg} , just by paying attention to patching. Originally, one built ad hoc groups that allowed one to extend the above (very) short exact sequence to the left. Quillen eventually gave a good definition of higher K-theory, but it took longer to obtain Mayer-Vietoris sequences in great generality.

5.7 K_1 , units and homotopy invariance

At the end of the previous section we observed the existence of a portion of the Mayer-Vietoris sequence for K_0 and we observed that the failure of injectivity of the first map was described, via patching ideas, in terms of automorphisms. We now make this more precise by introducing the functor K_1 .

5.7.1 K_1 of a ring: basic definitions

Suppose R is a commutative unital ring. Consider the inclusion maps $GL_n(R) \rightarrow GL_{n+1}(R)$ defined by the formula

$$X \longmapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array} \right),$$

and set

$$GL(R) := \operatorname{colim}_n GL_n(R),$$

where the colimit is formed in the category of groups. We will refer to $GL(R)$ as the stable or “infinite” general linear group.

If G is any group, recall that the commutator subgroup $[G, G]$ is subgroup generated by commutators $[g, h] = ghg^{-1}h^{-1}$. The quotient $G/[G, G] = G^{ab}$ is an abelian group. Moreover, if A is any abelian group, then given any homomorphism $\varphi : G \rightarrow A$, the composite map $[G, G] \rightarrow G \rightarrow A$ is trivial so φ factors through a map $G/[G, G] \rightarrow A$. In particular, the assignment $G \mapsto G/[G, G]$ is a left adjoint to the forgetful functor $\mathbf{Ab} \rightarrow \mathbf{Grp}$.

Definition 5.7.1.1. If R is a commutative unital ring, then

$$K_1(R) := GL(R)/[GL(R), GL(R)];$$

this is an abelian-group valued functor on the category of commutative unital rings.

If R is any ring, then $\det : GL_n(R) \rightarrow \mathbf{G}_m(R)$ is a homomorphism. Since \det commutes with the inclusion maps $GL_n(R) \rightarrow GL_{n+1}(R)$, it follows that there is an induced map $\det : GL(R) \rightarrow \mathbf{G}_m(R)$. Since $\mathbf{G}_m(R)$ is abelian, this map factors uniquely through a homomorphism

$$\det : K_1(R) \longrightarrow R^\times$$

The maps $GL_n(R) \rightarrow R^\times$ are split by the map sending u to the diagonal matrix $\operatorname{diag}(u, 1, \dots, 1)$. Since these maps are also compatible with stabilization, we conclude that there is an induced splitting $R^\times \rightarrow GL(R)$ and thus a splitting $R^\times \rightarrow K_1(R)$ of \det . In particular, \det is always surjective. If we write $SK_1(R) = \ker(\det : K_1(R) \rightarrow R^\times)$, then using the splitting we conclude that $K_1(R) \cong R^\times \oplus SK_1(R)$.

Proposition 5.7.1.2. If F is a field, then $\det : K_1(F) \rightarrow F^\times$ is an isomorphism.

Next, we develop the link between K_1 and projective modules. Begin by observing that if F is a finite rank free R -module, then the homomorphism $GL_n(R) \rightarrow K_1(R)$ shows that any automorphism of F gives rise to an element of $K_1(R)$. Suppose more generally that P is a finitely generated projective R -module. Since P is a summand of a finite rank free R -module, we can write $P \oplus Q = R^n$ for some projective module Q . Suppose $\alpha : P \rightarrow P$ is an automorphism of P . The choice of splitting allows us to extend α to the automorphism (α, id_Q) of R^n . We now claim that α has a well-defined class in $K_1(R)$ independent of the splitting $P \oplus Q = R^n$. Indeed, any other splitting differs from this one by an automorphism of R^n . Thus, if X is a matrix representing (α, id_Q) , then gXg^{-1} represents the new splitting. This defines an inner automorphism of $GL_n(R)$. Stabilizing, such automorphisms act trivially on the abelianization. Therefore, we conclude that for any f.g. projective R -module P there is a well-defined map $\operatorname{Aut}_R(P) \rightarrow K_1(R)$.

5.7.2 The Bass–Heller–Swan theorem

As before, we can consider the map $K_1(R) \rightarrow K_1(R[t])$ for any commutative unital ring R . This map is always split injective. We now observe that in the same situations as for K_0 , it is also

surjective. Because of the existence of the determinant homomorphism, by appeal to homotopy invariance for units, we see that a necessary condition for surjectivity is that R is a reduced ring. The following result, due to Bass–Heller–Swan [BHS64], has a proof very similar to that given for K_0 above.

Theorem 5.7.2.1. *If R is a regular ring, then $K_1(R) \rightarrow K_1(R[t_1, \dots, t_n])$ is an isomorphism.*

Chapter 6

Local triviality and smooth fiber bundles

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So far we have been studying constructions related to projective modules. We established that projective modules were locally free, and geometrically this corresponds with the fact that vector bundles are “algebraic fiber bundles” locally trivial in the Zariski topology. We now consider some

other kinds of “algebraic fiber bundles”, specifically covering spaces, and are immediately forced to allow finer notions of local triviality than local triviality in the Zariski topology. Eventually, these types of ideas lead to the notion of Grothendieck topologies and the study of sheaves in this setting.

To set the stage, suppose X is a manifold with fundamental group Γ and universal cover \tilde{X} . In this case, $\tilde{X} \rightarrow X$ is locally trivial in the classical topology. On the other hand, given a complex representation $\rho : \Gamma \rightarrow GL(V)$ one can define an “associated fiber space”: $\tilde{X} \times V/\Gamma$; the identification $\tilde{X}/\Gamma \rightarrow X$ induces a projection $\tilde{X} \times V/\Gamma \rightarrow X$. Since $\tilde{X} \rightarrow X$ is locally trivial, the map $\tilde{X} \rightarrow X$ is locally trivial and we can build a corresponding local trivialization of $\tilde{X} \times V/\Gamma \rightarrow X$.

We begin by analyzing a corresponding purely algebraic situation and we observe that in many situations above, the two constructions just described yield “algebraic fiber bundles” with different notions of local triviality. Indeed, for “covering spaces” local triviality in the Zariski topology fails, while associated vector bundles may *still* be shown to be locally trivial with respect to the Zariski topology. By the end of the section, we hope to come up with a reasonable notion of “algebraic fiber bundle” that will contain many useful examples.

6.1 Lecture 19: Local triviality and cyclic covering spaces

6.1.1 Covering spaces and failure of Zariski local triviality

Consider the morphism $\mathbf{G}_m \rightarrow \mathbf{G}_m$ given by $t \mapsto t^r$. If we think topologically about this morphism (i.e., take $k = \mathbb{C}$), then it is a covering space with cyclic fundamental group. At the level of rings, the induced map is an injective ring homomorphism $k[t, t^{-1}] \rightarrow k[t, t^{-1}]$; the source copy of $k[t, t^{-1}]$ is identified as the subring $k[t^r, t^{-r}]$. Let us analyze some scheme-theoretic fibers of this morphism. If \mathfrak{m} is a maximal ideal of $k[t, t^{-1}]$, say of the form $t - a$ for some element $a \in k^\times$, then the ideal in the target copy of $k[t, t^{-1}]$ generated by \mathfrak{m} is principal generated by $t^r - a$. Thus, the scheme-theoretic fiber over $t - a$ is simply $k[t, t^{-1}]/(t^r - a)$. If k is algebraically closed, then by factoring $t^r - a$ we see that $k[t, t^{-1}]/(t^r - a) \cong k \oplus \cdots \oplus k$ as a k -algebra. More generally, if we have any homomorphism $k[t, t^{-1}]$ to an algebraically closed field (we will such a homomorphism a *geometric point*), we see that the same thing happens: the geometric fiber is simply a collection of disjoint copies of $\text{Spec } L$. We can also understand the fiber at the generic point, i.e., corresponding to the zero ideal. In this case, the induced map on generic points is simply the map $k(t) \rightarrow k(t)$ given by $t \mapsto t^r$, which corresponds to viewing $k(t)$ as the subfield $k(t^r)$ of $k(t)$: this is a finite degree field extension.

Now, let us contrast the topological situation with the algebraic situation. Indeed, if $k = \mathbb{C}$, then the covering space $\mathbf{G}_m \rightarrow \mathbf{G}_m$ is locally trivial in the classical topology. We claim that this morphism is *not* locally trivial in the Zariski topology. Indeed, take $k = \mathbb{C}$. Since the fibers over closed points are isomorphic to an r -tuple of points, if there was a Zariski open set $U \subset \mathbf{G}_m$ on which the restriction was trivial, then we could write the preimage of U under the r -th power map as $U \coprod \cdots \coprod U$, which is disconnected. However, since all Zariski open subsets of \mathbf{G}_m are connected, we obtain a contradiction. Therefore, the map $\mathbf{G}_m \rightarrow \mathbf{G}_m$, even though it is topologically locally trivial, is not locally trivial in the Zariski topology.

6.1.2 Torsors under μ_r

We now generalize the construction of the cyclic covering space described above to obtain another algebro-geometric invariant of a ring. If we view the second copy of $k[t, t^{-1}]$ as a $k[t, t^{-1}]$ -module via the action $t \mapsto t^r$, then it breaks up as a direct sum of r -copies of a free $k[t, t^{-1}]$ module of rank 1. Indeed, we identified the source copy of $k[t, t^{-1}]$ as the subring of the target copy of $k[t, t^{-1}]$ of the form $k[t^r, t^{-r}]$. Then, the multiples of this subring by t^i for $0 \leq i \leq r - 1$ yield r -copies of $k[t, t^{-1}]$. In other words,

$$k[t, t^{-1}] \cong k[t^r, t^{-r}] \cdot 1 \oplus k[t^r, t^{-r}] \cdot t \oplus \cdots \oplus k[t^r, t^{-r}] \cdot t^{r-1}.$$

The free $k[t, t^{-1}]$ -module of rank r comes equipped with an algebra structure, which is completely encoded by the formula $t^r = 1$.

In this form, we can abstract the construction slightly. Suppose R is a commutative unital ring. We can equip the free rank r R -module $R^{\oplus r}$ with an algebra structure as follows. By linearity, it suffices to explain the product on a suitable basis. Assume we are given a basis of the form $1, e, \dots, e^{r-1}$. To specify the product, we need to specify the value of $e^i e^j$. If $i + j < r$, then we simply set $e^i e^j = e^{i+j}$. Fix $a \in R^\times$ such that $e^r = a$. In that case, the algebra structure is uniquely specified. In this case, we can view $R \cdot e^i$ as $(R \cdot e)^{\otimes i}$, and in this setting the construction can be generalized further.

Indeed, if L is an invertible R -module (in the case above L is a free rank 1 R -module) and we are given an isomorphism $L^{\otimes r} \xrightarrow{\sim} R$, then we can equip $\bigoplus_{i=0}^{r-1} L^{\otimes i}$ with the structure of an R -algebra by the formula

$$\begin{aligned} L^{\otimes i} \otimes L^{\otimes j} &\longrightarrow L^{\otimes i+j} && \text{if } i + j \leq r, \text{ and} \\ L^{\otimes i} \otimes L^{\otimes j} &\longrightarrow L^{\otimes(r+(i+j-r))} \xrightarrow{\varphi \otimes \text{id}} \otimes L^{i+j-r} && \text{if } i + j \geq r. \end{aligned}$$

This leads to the following definition.

Definition 6.1.2.1. A μ_r -torsor over a ring R consists of a pair (L, φ) , where L is an invertible R -module and $L^{\otimes r} \rightarrow R$ is an R -module isomorphism.

Remark 6.1.2.2. If (L, φ) is an μ_r -torsor over a ring R , then $\bigoplus_{i=0}^{r-1} L^{\otimes i}$ is a finitely generated projective R -module by assumption, and as a consequence, the map $R \rightarrow \text{Sym} L$ is a flat ring homomorphism. The induced map $\text{Spec} \bigoplus_{i=0}^{r-1} L^{\otimes i} \rightarrow \text{Spec} R$ is our analog of a cyclic covering. Even though line bundles are locally trivial in the Zariski topology, the examples above show that μ_r -torsors are, in general, *not* locally trivial in the Zariski topology. Here is another example of this phenomenon.

Example 6.1.2.3. If $R = k$ is a field, then all invertible R -modules are free. In that case, a μ_r -torsor is simply a 1-dimensional k -vector space L equipped with an isomorphism $L^{\otimes r} \xrightarrow{\sim} k$. In particular, L is a finite-dimensional commutative k -algebra. If we pick a basis x of L , then $L^{\otimes r}$ has basis x^r and k has the standard basis 1. The isomorphism of 1-dimensional k -vector spaces is given in terms of these bases by multiplication by an element of k^\times . Thus, it can be written as $x^r = a$ for $a \in k^\times$. If a primitive r -th root of a is not in k , then the vector space L/k can be identified as a cyclic field extension given by adjoining a root of $x^r - a$ to k . If r is not divisible by the characteristic of k ,

this is a separable field extension. If r is divisible by the characteristic of k it is not separable. In the former case, if we adjoin an r -th root of a to k , then we can factor the equation $(x^r - a)$. In other words, after extending scalars from k to L , the algebra structure on L becomes trivial: $L \otimes_k L$ splits as a direct sum of r copies of L .

Given a μ_r -torsor, by forgetting φ we obtain an invertible R -module. From this point of view, there is an evident group structure on the set $\text{Tors}_{\mu_r}(R)$ for the set of isomorphism classes of μ_r -torsors: given (L, φ) and (L', ψ) , the line bundle $L \otimes L'$ can be equipped with a μ_r -torsor structure using the isomorphism $\varphi \otimes \psi$.

Definition 6.1.2.4. We write $\text{Tors}_{\mu_r}(R)$ for the group of isomorphism classes of μ_r -torsors (with the group structure defined above).

The invertible R -module associated with a μ_r -torsor actually defines an r -torsion class in $\text{Pic}(R)$, i.e., with respect to the group structure just defined, there is an exact sequence of abelian groups

$$\text{Tors}_{\mu_r}(R) \longrightarrow \text{Pic}(R) \xrightarrow{\times r} \text{Pic}(R).$$

We would like to understand the kernel of this map, which consists of μ_r -torsor structures on a free rank 1 R -module.

If (L, φ) is a μ_r -torsor with L a free module of rank 1, then, generalizing the case of fields, by picking a basis element e of R , we obtain an algebra structure on $R \oplus R \cdot e \oplus \cdots \oplus R \cdot e^{r-1}$ with $e^r = a \in R^\times$. If we change the basis by $e \mapsto \lambda e'$ with $\lambda \in R^\times$, then the equation $e^r = a$ becomes $e'^r = \lambda^{-r} a$. In other words, the element a is unique up to multiplication by r -th powers of units, i.e., elements of $R^{\times r}$. Thus, there is a homomorphism

$$R^\times / R^{\times r} \longrightarrow \text{Tors}_{\mu_r}(R)$$

sending an element a to the cyclic R -algebra defined by $e^r = a$. The argument above, corresponding to choosing a basis, shows this homomorphism is injective. Putting everything together, we have deduce the following result.

Proposition 6.1.2.5. *For any commutative unital ring R , there is an exact sequence of the form*

$$R^\times \xrightarrow{\times r} R^\times \longrightarrow \text{Tors}_{\mu_r}(R) \longrightarrow \text{Pic}(R) \xrightarrow{\times r} \text{Pic}(R).$$

Proof. The kernel of $\text{Tors}_{\mu_r}(R) \rightarrow \text{Pic}(R)$ consists precisely of μ_r -torsor structures on the free R -module of rank 1. The result consists in unwinding the definitions. \square

Remark 6.1.2.6. The map

$$R^\times \xrightarrow{\times r} R^\times$$

considered above is not injective. The kernel consists of elements $x \in R$ such that $x^r = 1$, i.e., solutions of $x^r - 1$ in R . If we set $\mu_r := \text{Spec } \mathbb{Z}[t, t^{-1}]/(t^r - 1)$, then such a solution corresponds to a ring homomorphism $\mathbb{Z}[t, t^{-1}]/(t^r - 1) \rightarrow R$, i.e., an element of $\mu_r(R)$. The map $\mu_r(R) \rightarrow R$ is evidently injective, so we may extend the exact sequence in Proposition 6.1.2.5 to the left. There is an evident group structure on μ_r and the name “ μ_r -torsor” is meant to make you think “twisted versions of μ_r ”. We will explain this in more detail later.

Corollary 6.1.2.7. *If R is a normal domain, then the map $\text{Tors}_{\mu_r}(R) \rightarrow \text{Tors}_{\mu_r}(R[t])$ is a bijection.*

Proof. Since R is a normal domain, this follows from the 5-lemma combined with homotopy invariance of groups of units (i.e., Proposition 2.2.1.6) and the Picard group (i.e., Theorem 4.6.1.3). \square

Remark 6.1.2.8. In topology, if (X, x) is a sufficiently nice pointed topological space, cyclic coverings with group $\mathbb{Z}/r\mathbb{Z}$ correspond to homomorphisms $\pi_1(X, x) \rightarrow \mathbb{Z}/r\mathbb{Z}$. Since $\mathbb{Z}/r\mathbb{Z}$ is abelian, such a homomorphism factors through the abelianization map $\pi_1(X, x) \rightarrow H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}/r\mathbb{Z}$. On the other hand, $\text{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ is, by means of the universal coefficient theorem, identified with $H^1(X, \mathbb{Z}/r\mathbb{Z})$. Thus, we have built an algebro-geometric analog of the group $H^1(X, \mathbb{Z}/r\mathbb{Z})$.

Exercise 6.1.2.9. *If R is a commutative unital ring, and $f, g \in R$ are comaximal elements, then establish a Mayer-Vietoris sequence involving $\mu_r(-)$ and $\text{Tors}_{\mu_r}(-)$. (Hint: think about Mayer-Vietoris for units and the Picard group).*

6.1.3 Finite locally free morphisms

Suppose r is a number that is invertible in R and set $S = \bigoplus_{i=0}^r L^{\otimes i}$. If L is a free rank 1 R -module, then a choice of basis of L determines an isomorphism we can identify $S = R[t]/(t^r - a)$ for some unit $a \in R^\times$. In that case, $S \otimes_R S$ is $R[t]/(t^r - a) \otimes_R R[t]/(t^r - a)$. However, over the ring $R[t]/(t^r - a)$, there is a direct sum decomposition

$$R[t]/(t^r - a) \otimes_R R[t]/(t^r - a) \cong \bigoplus_{i=0}^{r-1} R[t]/(t^r - a),$$

essentially because $t^r - a$ factors over this ring.

Definition 6.1.3.1. A morphism $f : R \rightarrow S$ of rings is called a *finite ring map* if f makes S into a finitely generated R -module.

Definition 6.1.3.2. A morphism $f : R \rightarrow S$ of rings is called a *locally free ring map* if f makes S into a locally free R -module.

Here is a different way to phrase Zariski local triviality of a ring map $\varphi : R \rightarrow S$: we can find elements f_1, \dots, f_n generating the unit ideal in R such that after tensoring with the ring homomorphism $R \rightarrow \bigoplus_{i=1}^n R_{f_i}$, the tensor product $S \otimes_R \bigoplus_{i=1}^n R_{f_i}$ is isomorphic to a suitable product, then f is Zariski locally trivial. From this point of view, $R \rightarrow \bigoplus_{i=0}^r L^{\otimes i}$ is trivialized by a *finite, flat* ring extension. In other words, the μ_r -torsor (L, φ) becomes trivial after a finite locally free extension of scalars. Before moving forward, let us observe that finite morphisms have the following permanence properties.

Proposition 6.1.3.3 (Finite morphisms). *Suppose $f : R \rightarrow S$, $g : S \rightarrow T$ and $\varphi : R \rightarrow R'$ are ring homomorphisms.*

1. *If f is finite, then the map $f' : R' \rightarrow S \otimes_R R'$ induced by the universal property of tensor product is also finite.*
2. *If f is finite, then the fibers of $\text{Spec } S \rightarrow \text{Spec } R$ are finite as well.*
3. *If f and g are finite, then so is $g \circ f$.*
4. *If the composite $g \circ f : R \rightarrow T$ is finite, then so is g .*

5. If r_1, \dots, r_m are elements generating the unit ideal in R and if $f_i : R_{r_i} \rightarrow S \otimes_R R_{r_i}$ are finite ring homomorphisms, then f is finite as well.

Proof. For the first point, suppose S is a finitely generated R -module. In that case, there is a surjection $R^{\oplus n} \rightarrow S$. If $R' \rightarrow R$ is an arbitrary homomorphism, then there is an induced map $R'^{\oplus n} \cong R' \otimes_R (R^{\oplus n}) \rightarrow R' \otimes_R S$. Unwinding the definitions shows that this map is a surjective R' -module map. The second point follows immediately from the first. We leave the third and fourth points as exercises. The final point is a consequence of Zariski patching for R -modules. \square

Example 6.1.3.4. If r is an integer that is invertible in R then given a μ_r -torsor (L, φ) , set $S = \bigoplus_{i=0}^{r-1} L^{\otimes i}$. Since L is locally free, we can pick elements $f_1, \dots, f_r \in R$ such that L_{f_i} is a free rank 1 R_{f_i} -module. In that case, even though $R \rightarrow S$ may not decompose as a product after a finite flat ring map, it does decompose as a product after extending scalars along $R \rightarrow \bigoplus R_{f_i} \rightarrow \bigoplus S_{f_i}$ by the same argument described above.

6.2 Lecture 20: Unramified and étale maps

6.2.1 Unramified morphisms

Unfortunately, finite locally free morphisms do not quite correspond to the topological idea of covering spaces as the following example shows.

Example 6.2.1.1. Let $R = k[x]$, and consider the map $k[x] \rightarrow k[x]$ defined by $x \mapsto x^r$. We identify the second copy of $k[x]$ as a free $k[x]$ -module of rank 1 isomorphic to $k[x^r] \cdot 1 \oplus k[x^r] \cdot x \cdots k[x^r] \cdot x^{r-1}$. Once more, this is a finite flat morphism as the second copy of $k[x]$ is a finite rank free $k[x]$ -module of rank r . However, this morphism is fundamentally different from the examples above. Indeed, the fiber over 0 here is $k[x]/x^r$, which is a nilpotent k -algebra. Unlike the examples considered above where, if k is algebraically closed, the fibers over closed points always have r distinct points, the fibers in this example have r distinct points only if $x \neq 0$ and in that case the fiber is a fat point. Thus, the morphism described here is *branched* at $x = 0$.

Example 6.2.1.2. More generally, if R is a ring and $f \in R$ is an element, then we can consider the ring $R[t]/(t^r - f)$. Assume for simplicity that f is not a zero-divisor. In that case, if we pass to R_f , then since f is a unit in R_f , the ring map $R_f \rightarrow R_f[t]/(t^r - f)$ is a μ_r -torsor as studied above. We think of $R \rightarrow R[t]/(t^r - f)$ as a cover of $\text{Spec } R$ *branched* or *ramified* along f .

We now introduce a notion of *unramified* morphism, which removes the possibility of branching. While the fibers of a finite morphism are finite, the number of points in the fiber can change (though for finite flat morphisms, the number of points counted *with multiplicity* does not change). The notion of an unramified cover in the setting in which we consider it essentially goes back to Serre [Ser58a].

Suppose $f : A \rightarrow B$ is a ring map. For a prime ideal $\mathfrak{p} \subset B$, set $\mathfrak{q} = f^{-1}\mathfrak{p}$ and write $\mathfrak{m}_{\mathfrak{q}}$ for the maximal ideal of the local ring $A_{\mathfrak{q}}$. Let $\kappa_{\mathfrak{q}}$ be the residue field, i.e., the quotient $A_{\mathfrak{q}}/\mathfrak{m}_{\mathfrak{q}}$. There is an induced map $\kappa_{\mathfrak{q}} \rightarrow B_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{q}}B_{\mathfrak{p}}$. Now, in general the ideal $\mathfrak{m}_{\mathfrak{q}}B_{\mathfrak{p}}$ is contained in the maximal ideal $\mathfrak{m}_{\mathfrak{p}} \subset B_{\mathfrak{p}}$. In particular, there is an induced map $\kappa_{\mathfrak{q}} \rightarrow \kappa_{\mathfrak{p}}$.

Definition 6.2.1.3. Say that $f : A \rightarrow B$ is *unramified* at \mathfrak{p} if (notation as just given)

1. S is a finitely presented R -algebra;
2. $\mathfrak{m}_q B_{\mathfrak{p}}$ is the maximal ideal of $B_{\mathfrak{p}}$; and
3. the map of residue fields $\kappa_q \rightarrow \kappa_{\mathfrak{p}}$ is a separable extension.

If f is not unramified at \mathfrak{p} , we will say that it is *ramified*. Say that f is *unramified* (or that B is an *unramified A -algebra*) if it is unramified at \mathfrak{p} for every prime ideal $\mathfrak{p} \subset B$.

Example 6.2.1.4. Fix a base-field k . The map $z \mapsto z^r$ from $\mathbf{G}_m \rightarrow \mathbf{G}_m$ is unramified if r is not divisible by p . Indeed, if we take a maximal ideal in $k[t, t^{-1}]$, then we essentially saw this before. On the other hand, if we take the zero ideal, then the extension field is separable under our hypotheses. Similar checks can be performed for morphisms of Dedekind domains.

Example 6.2.1.5. Fix a base field k . The morphism $z \mapsto z^r$ from $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ is always ramified at the maximal ideal corresponding to 0. Indeed, in this case, we consider the ring map $k[z] \rightarrow k[z]$ given by $z \mapsto z^r$. In this case, the maximal ideal (z) in the target copy of $k[z]$ pulls back to the maximal ideal (z) (since the ring map is simply inclusion as $k[z^r]$). In this case, $k[z]_{\mathfrak{q}}$ is the local ring with residue field k , while the ideal generated by the maximal ideal of $k[z]_{\mathfrak{q}}$ in $k[z]_{\mathfrak{p}}$ is the ideal (z^r) . Thus, the quotient is not a separable field extension of k .

Example 6.2.1.6. Unfortunately, finite unramified morphisms *still* do not correspond to a nice notion of unramified covering. First, observe that if $I \subset R$ is any non-zero proper ideal, then the map $R \rightarrow R/I$ is unramified if I is a finitely presented ideal. We also have the following silly example of an unramified morphism whose associated map of affine schemes is surjective. If R is any commutative unital ring, and $I \subset R$ is a proper, non-zero ideal, then the map $R \rightarrow R \oplus R/I$ given by the identity on the first factor and the projection on the second factor is a finite, unramified morphism. Unlike our examples above, $R \oplus R/I$ is *not* a flat ring map in general.

Exercise 6.2.1.7. If (L, φ) is a μ_r -torsor over a ring R and r is invertible in R , then $S := \bigoplus_{i=0}^{r-1} L^{\otimes i}$ is an unramified R -algebra.

As before, we can establish various permanence properties of unramified morphisms.

Proposition 6.2.1.8. Suppose $f : A \rightarrow B$, $g : B \rightarrow C$, and $\varphi : A \rightarrow A'$ are ring homomorphisms.

1. If f and g are unramified, so is $g \circ f$.
2. If f is unramified, then the induced ring map $B \rightarrow B \otimes_A A'$ (formed using φ) is also unramified.
3. If $g \circ f$ is unramified, then f is unramified.

Proof. Composites of separable extensions of separable. The remaining two statements are left as exercises. \square

6.2.2 Étale ring maps

Definition 6.2.2.1. If $f : A \rightarrow B$ is a ring map, then we will say that f is *étale* if f is flat and unramified (in particular, this means that B should be a finitely presented A -algebra).

The following permanence properties of étale morphisms are immediate consequences of the corresponding properties for unramified and flat morphisms.

Proposition 6.2.2.2. Suppose $f : A \rightarrow B$, $g : B \rightarrow C$, and $\varphi : A \rightarrow A'$ are ring homomorphisms.

1. If f and g are étale, so is $g \circ f$.
2. If f is étale, then the induced ring map $B \rightarrow B \otimes_A A'$ (formed using φ) is also étale.

Example 6.2.2.3. If k is a field, and r is invertible in k , then the map $\mathbf{G}_m \rightarrow \mathbf{G}_m$ given by $z \mapsto z^r$ is étale. Indeed, this map is evidently finitely presented. Moreover, we checked above that it is both flat and unramified.

Exercise 6.2.2.4. If R is a commutative unital ring, and (L, φ) is a μ_r -torsor, then $S := \bigoplus_{i=0}^{r-1} L^{\otimes i}$ is an étale R -algebra if r is invertible in R .

Example 6.2.2.5. If R is a ring and $f \in R$ is a non-zero divisor, then the map $R \mapsto R_f$ is étale. Indeed, we can identify R with $R[x]/(xf - 1)$ to see that R_f is finitely presented. It is flat because localizations are flat. It is unramified because the relevant map on residue fields is always an isomorphism.

Example 6.2.2.6. If k is a field, then a finite product of separable extensions of k , say $\prod_i L_i$, is an étale k -algebra. In fact, the converse is true as we will show later.

6.2.3 Differentials and ramification

Recall that a polynomial P in one variable over a field k is called separable if it has no repeated roots. A convenient way to check whether P is separable is to check whether P and its derivative are coprime. There is a generalization of this “differential” criterion for separability that we now motivate. We begin by discussing differential forms in algebraic geometry.

If f is a polynomial in one variable, then its derivative at a point y is given by the limit $\lim_{x \rightarrow y} \frac{f(x) - f(y)}{x - y}$. Here is a ring-theoretic interpretation of this limit. We send a polynomial f to the difference $f(x) - f(y) \in k[x, y]$. The n -th order Taylor expansion of f is obtained by considering powers of the ideal $(x - y) \subset k[x, y]$. More precisely, the n -th order Taylor expansion is simply the image of $f(x) - f(y) \in k[x, y]/(x - y)^{n+1}$.

Now, $k[x, y] \cong k[x] \otimes_k k[y]$ by the map sending $x \otimes 1 \rightarrow x$ and $1 \otimes y$ to y . In terms of this identification, the map sending f to $f(x) - f(y)$ is the map sending f to $f \otimes 1 - 1 \otimes f$. Next, note that the ideal $(x - y)$ is the ideal generated by $x \otimes 1 - 1 \otimes x$. This ideal, call it I , is precisely the kernel of the multiplication map $k[x] \otimes_k k[x] \rightarrow k[x]$ (which sends $f \otimes g \mapsto fg$). Note that the element $f \otimes 1 - 1 \otimes f$ lies in I by construction. The derivative of f can then be identified as the class of $f \otimes 1 - 1 \otimes f$ in I/I^2 . Write $d : k[x] \rightarrow I/I^2$ defined by $f \mapsto f \otimes 1 - 1 \otimes f/I^2$.

Now, I is a $k[x] \otimes_k k[x]$ -module, and admits the structure of a $k[x]$ -module in two different ways corresponding to the two inclusions $k[x] \rightarrow k[x] \otimes_k k[x]$. We will consider it as a $k[x]$ -module with respect to the *left* inclusion. In the form just written, we can extend these definitions to an arbitrary ring homomorphism.

Definition 6.2.3.1. Suppose $R \rightarrow S$ is a ring homomorphism, let $m : S \otimes_R S \rightarrow S$ be the multiplication map and let I be its kernel. We view I as an S -module with respect to left multiplication. The module of relative differentials, denoted $\Omega_{S/R}$ is I/I^2 viewed as an S -module (with respect to the left factor in the tensor product).

Example 6.2.3.2. What we computed above shows that if $R = k$ and $S = k[x]$, then $\Omega_{S/R}$ is a free S -module of rank 1. The element $dx = x \otimes 1 - 1 \otimes x/I^2$ can be taken as a $k[x]$ -module generator.

6.3 Lecture 21: Differential forms, derivations and separability

Last time we defined Kähler differentials. In this lecture we study algebraic differential forms in more detail.

6.3.1 Differential forms and derivations

Let us begin by observing that the map $d : k[x] \rightarrow I/I^2$ is *not* a $k[x]$ -module homomorphism. Indeed, it is not $k[x]$ -linear.

$$\begin{aligned} d(fg) &= fg \otimes 1 - 1 \otimes fg / I^2 \\ &= (fg \otimes 1 - f \otimes g + f \otimes g - 1 \otimes fg) / I^2 \\ &= f(g \otimes 1 - 1 \otimes g) + (f \otimes 1 - 1 \otimes f)g \\ &= fd(g) + d(f)g. \end{aligned}$$

This computation is the product rule for d . One makes the following definition generalizing this situation.

Definition 6.3.1.1. If S is a ring, and M is an S -module, then an M -valued derivation is a map $D : S \rightarrow M$ such that $D(s_1 + s_2) = D(s_1) + D(s_2)$ and $D(s_1s_2) = s_1D(s_2) + s_2D(s_1)$. If S is an R -algebra for some ring R , then we will say that D is an R -derivation if the composite $R \rightarrow S \xrightarrow{D} M$ is zero.

The Leibniz property implies a number of standard facts, e.g., $D(1) = D(1 \cdot 1) = 1D(1) + D(1) \cdot 1$ and thus $D(1) = 0$. It follows that every derivation is a \mathbb{Z} -derivation. Note that $\text{Der}_R(S, M)$ is an S -module in a natural way: the sum of two R -derivations is automatically an R -derivation and, given $s \in S$ and an R -derivation D , sD is also an R -derivation. Precisely the same computation above (replacing k by R and $k[x]$ by S) yields the following more result.

Lemma 6.3.1.2. *If $R \rightarrow S$ is a ring homomorphism, the the map $d : S \rightarrow \Omega_{S/R}$ defined by $s \mapsto s \otimes 1 - 1 \otimes s$ is an R -derivation.*

There is an evident map $\text{Hom}_S(\Omega_{S/R}, M) \rightarrow \text{Der}_R(S, M)$. Indeed, given any S -module homomorphism $\Omega_{S/R} \rightarrow M$, the composite map $d : S \rightarrow \Omega_{S/R} \rightarrow M$ is easily checked to be an R -derivation.

Proposition 6.3.1.3. *For any ring homomorphism $R \rightarrow S$, and any S -module M the map*

$$\text{Hom}_S(\Omega_{S/R}, M) \rightarrow \text{Der}_R(S, M)$$

just described is an isomorphism of S -modules, functorially in M .

Proof. It suffices to construct an inverse. Suppose $D : S \rightarrow M$ is an R -derivation. Recall that one can define an S -algebra structure on $S \oplus M$ by means of the formulas $(s, m) + (s', m') = (s + s', m + m')$ and $(s, m) * (s', m') = (ss', sm' + m's)$; we write $S * M$ for $S \oplus M$ with this algebra structure; this is evidently a commutative algebra. Moreover, by identifying M as the collection of

elements of the form $(0, m)$, defines an ideal in $S * M$ that we will also call M . Note that $M^2 = 0$ by construction since we can compute the product of any two elements: $(0, m) * (0, m') = 0$.

Now, given an R -derivation $D : S \rightarrow M$, there is an induced map $\varphi_D : S \otimes_R S \rightarrow S * M$ sending $\varphi_D(s_1 \otimes s_2) = (s_1 s_2, s_1 D s_2)$. You can check that this is an R -algebra homomorphism. Next, you can check that the ideal $I \subset S \otimes_R S$ is mapped into M . More precisely, if $\sum_i x_i \otimes y_i$ is such that $\sum_i x_i y_i = 0$, then

$$\varphi_D\left(\sum_i x_i \otimes y_i\right) = \left(\sum_i x_i y_i, \sum_i x_i D y_i\right) = \left(0, \sum_i x_i D y_i\right).$$

Since $M^2 = 0$, it follows that the homomorphism $S \otimes_R S \rightarrow S * M$ defined above induces an R -module map $f_D : I/I^2 \rightarrow M$. Now, we simply check that the required factorization exists:

$$f_D(ds) = f_D(1 \otimes s - s \otimes 1/I^2) = \varphi_D(1 \otimes s) - \varphi_D(s \otimes 1) = (s, D(s)) - (s, D(1)) = (0, D(s)).$$

Now, one shows that the assignment $D \rightarrow f_D$ determines an S -module homomorphism $\text{Der}_R(S, M) \rightarrow \text{Hom}_S(\Omega_{S/R}, M)$. \square

Corollary 6.3.1.4. *If $R \rightarrow S$ is a surjective ring homomorphism, then $\Omega_{S/R} = 0$.*

Proof. Any R -derivation of S is necessarily the zero map. Therefore, $\text{Der}_R(S, M) = 0$ for any M and the result follows from Proposition 6.3.1.3 combined with the Yoneda lemma. \square

Example 6.3.1.5. Take $S = R[x_1, \dots, x_n]$ and consider $\Omega_{S/R}$. In that case, $\text{Der}_R(S, S) = \text{Hom}(\Omega_{S/R}, S)$. We see that $\text{Der}_R(S, S)$ is the free $R[x_1, \dots, x_n]$ -module generated by the derivations $\frac{\partial}{\partial x_i}$ and therefore $\text{Hom}(\Omega_{S/R}, S)$ is a free $R[x_1, \dots, x_n]$ -module of rank n on generators dx_1, \dots, dx_n dual to $\frac{\partial}{\partial x_i}$. Moreover, the differential $d : S \rightarrow \Omega_{S/R}$ sends an element f to $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

6.3.2 Fundamental exact sequences for differentials

Suppose we are given a commutative diagram of ring homomorphism of the form

$$(6.3.1) \quad \begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R' & \longrightarrow & S' \end{array}$$

In that case, there is an induced ring map $S \otimes_R S \rightarrow S' \otimes_{R'} S'$. Moreover, this homomorphism sits in a commutative diagram with the two multiplication maps $m_S : S \otimes_R S \rightarrow S$ and $m_{S'} : S' \otimes_{R'} S' \rightarrow S'$. Therefore, there are induced maps $\ker(m_S) \rightarrow \ker(m_{S'})$ and homomorphisms $\Omega_{S/R} \rightarrow \Omega_{S'/R'}$. Moreover, there is a commutative diagram of the form

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow d & & \downarrow d \\ \Omega_{S/R} & \longrightarrow & \Omega_{S'/R'} \end{array}$$

where the top horizontal arrow is the ring map $S \rightarrow S'$ and the bottom horizontal arrow is the homomorphism just constructed.

Lemma 6.3.2.1. *If given a commutative diagram of ring maps as in 6.3.1 with $S \rightarrow S'$ surjective, then $\Omega_{S/R} \rightarrow \Omega_{S'/R'}$ is also surjective.*

Proof. Take an element of $\Omega_{S'/R'}$, which is represented by an element of $\ker(m_{S'})/\ker(m_{S'})^2$. Since $S \rightarrow S'$ is surjective, we can simply pick a lift of this element in $S \otimes S$. \square

Proposition 6.3.2.2. *Given a sequence of ring homomorphisms $A \rightarrow B \rightarrow C$, there is an exact sequence of the form*

$$C \otimes \Omega_{B/A} \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

Proof. Surjectivity follows from Lemma 6.3.2.1 applied with $R = A$, $R' = B$ and $S = S' = C$. Let $m_{C/A}$ be the multiplication map $C \otimes_A C \rightarrow C$ and let $m_{C/B}$ be the multiplication map $C \otimes_B C \rightarrow C$. We leave exactness as an exercise. \square

Proposition 6.3.2.3. *If given a commutative diagram of ring maps as in 6.3.1 where $S \rightarrow S'$ is surjective with kernel I , and $R = R'$, there is an exact sequence of S -modules of the form*

$$I/I^2 \longrightarrow S' \otimes_S \Omega_{S/R} \longrightarrow \Omega_{S'/R} \longrightarrow 0;$$

here the left-most map is characterized by the property that $f \in I$ is sent to $1 \otimes df$.

Proof. Since $S \rightarrow S'$ is surjective with kernel I , the middle term is $S/I \otimes_S \Omega_{S/R}$. Given $f \in I$, we send f to $1 \otimes df$. If \bar{f} is the class of f in I/I^2 , we need to show that the induced map is well-defined. This follows from a direct computation using the Leibniz rule. The surjectivity of the rightmost map follows from Lemma 6.3.2.1. We leave the check of exactness as an exercise. \square

Remark 6.3.2.4. The module I/I^2 is called the *conormal module*, for reasons we now explain. If $\varphi : S \rightarrow S'$ is a surjective ring map with kernel I , then φ induces a map $\text{Spec } S' \rightarrow \text{Spec } S$ identifying $\text{Spec } S'$ with a closed subset of $\text{Spec } S$. In that case, there is a pullback map on differential forms $\Omega_{S/R} \rightarrow \Omega_{S'/R}$. If we take $R = \text{Spec } k$ for a field k , then the kernel of this map corresponds to differential forms on $\text{Spec } S$ that restrict to zero on $\text{Spec } S'$, which is usually how one thinks of the conormals in geometry.

Corollary 6.3.2.5. *If $R \rightarrow S$ is a finitely presented ring map, then $\Omega_{S/R}$ is a finitely presented S -module.*

Proof. To say that S is finitely presented is to say that $S = R[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Then, observe that if I is the kernel of $R[x_1, \dots, x_n] \rightarrow S$, then I/I^2 is generated by the images of f_1, \dots, f_r . On the other hand, $\Omega_{R[x_1, \dots, x_n]/R}$ is a finitely generated free $R[x_1, \dots, x_n]$ -module, and therefore so is $S \otimes \Omega_{R[x_1, \dots, x_n]/R}$. \square

Example 6.3.2.6 (Hypersurfaces). Suppose $f \in k[x_1, \dots, x_n]$ is an element and set $R := k[x_1, \dots, x_n]/(f)$. In this case, we see that $\Omega_{k[x_1, \dots, x_n]/k}$ is a free $k[x_1, \dots, x_n]$ -module of rank n and thus $\Omega_{R/k}$ is a quotient of a free R -module of rank n and there is an exact sequence of the form

$$(f)/(f)^2 \longrightarrow R \otimes_{k[x_1, \dots, x_n]} \Omega_{k[x_1, \dots, x_n]/k} \longrightarrow \Omega_{R/k} \longrightarrow 0.$$

Since f is sent to $1 \otimes df$, we conclude that $\Omega_{R/k}$ is the quotient of the free module on generators dx_1, \dots, dx_n modulo the subspace spanned by $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. In particular, we see that the

rank of $\Omega_{R/k}$ increases at points where all $\frac{\partial f}{\partial x_i}$ vanish. In particular, we conclude that if the partial derivatives $\frac{\partial f}{\partial x_i}$ are always non-vanishing at closed points of R , then $\Omega_{R/k}$ is a projective module of rank $n - 1$. On the other hand, in that case, $(f)/(f)^2$ is a free rank 1 R -module, generated by the image of f . The map $(f)/(f)^2 \rightarrow R \otimes_{k[x_1, \dots, x_n]} \Omega_{k[x_1, \dots, x_n]/k}$ is injective (since it is so at closed points). Thus, we conclude that $\Omega_{R/k} \oplus (f)/(f)^2$ is a free module of rank n , i.e., $\Omega_{R/k}$ is *stably free*.

6.3.3 Interlude: the algebraic de Rham complex

If $R \rightarrow S$ is a ring map, then we can define S -modules $\Omega_{S/R}^i := \wedge^i \Omega_{S/R}$ using the exterior power functor on modules. So far, we do not know that $\Omega_{S/R}$ is projective (and, in general, it need not be!), so these exterior powers can be badly behaved. We would like to extend the differential $d : S \rightarrow \Omega_{S/R}$ to a differential $\Omega_{S/R}^i \rightarrow \Omega_{S/R}^{i+1}$ for all $i \geq 1$.

First, we treat the case where S is a polynomial ring over R (in possibly infinitely many variables). Now, $\Omega_{S/R}^i$ is a free S -module (possibly of infinite rank), with a basis given by expressions of the form $dx_{a_1} \wedge \cdots \wedge dx_{a_i}$. Thus, any element of $\Omega_{S/R}^i$ can be written as $\sum_i f_i \omega_i$, and we define $d(\sum_i f_i \omega_i)$ by the formula $\sum_i df_i \wedge \omega_i$. In this way, one obtains an R -linear map $\Omega_{S/R}^i \rightarrow \Omega_{S/R}^{i+1}$. Moreover, the formula $d^{p+q}(\omega \wedge \eta) = d^p \omega \wedge \eta + (-1)^p \omega \wedge d^q \eta$ holds for $\omega \in \Omega_{S/R}^p$ and $\eta \in \Omega_{S/R}^q$.

To treat the general case, write $S = R'/I$ where R' is a polynomial ring over R . The surjection $R' \rightarrow R'/I$ yields an exact sequence of the form

$$I \rightarrow S \otimes \Omega_{R'/R} \rightarrow \Omega_{S/R} \rightarrow 0.$$

Now, note that $S \otimes \Omega_{R'/R}$ is a free S -module. Using this observation, we conclude that $\wedge^i \Omega_{S/R}$ is a quotient of the free module $\wedge^i(S \otimes \Omega_{R'/R})$ by the submodule generated by $d(I) \wedge \Omega_{R'/R}^{i-1}$.

Lemma 6.3.3.1. *The map d^i on $\Omega_{R'/R}^i$ maps $I\Omega_{S/R}^i$ and $d(I) \wedge \Omega_{S/R}^{i-1}$ to the sub-module generated by $d(I) \wedge \Omega_{S/R}^i$ and $I\Omega_{S/R}^{i+1}$.*

Proposition 6.3.3.2. *If $R \rightarrow S$ is any ring map, then the derivation $d : S \rightarrow \Omega_{S/R}$ extends uniquely to a map $d^i : \Omega_{S/R}^i \rightarrow \Omega_{S/R}^{i+1}$ such that*

1. $d^0 = d$;
2. the sequence

$$S = \Omega_{S/R}^0 \xrightarrow{d^0} \Omega_{S/R}^1 \xrightarrow{d^1} \Omega_{S/R}^2 \xrightarrow{d^2} \cdots$$

is a complex; and

3. if $\omega \in \Omega_{S/R}^p$ and $\eta \in \Omega_{S/R}^q$, then $d^{p+q}(\omega \wedge \eta) = d^p \omega \wedge \eta + (-1)^p \omega \wedge d^q \eta$.

Definition 6.3.3.3. If $R \rightarrow S$ is a ring homomorphism, then the *algebraic de Rham complex* is the complex $(\Omega_{S/R}, d)$ guaranteed to exist by the previous part. One defines algebraic de Rham cohomology groups by the formula

$$H_{dR}^i(S/R) := H^i(\Omega_{S/R}).$$

Proposition 6.3.3.4. *The multiplication maps $\Omega_{S/R}^i \otimes \Omega_{S/R}^j \rightarrow \Omega_{S/R}^{i+j}$ equip $H_{dR}^*(S/R)$ with the structure of a graded commutative ring.*

If $S \rightarrow S'$ is a morphism of R -algebras, then we saw that there is an induced map $\Omega_{S/R} \rightarrow \Omega_{S'/R}$ fitting into a suitable commutative diagram. The algebraic de Rham complex is functorial with respect to such homomorphisms.

Proposition 6.3.3.5. *If $f : S \rightarrow S'$ is a homomorphism of R -algebras, then there is an induced morphism of complexes of abelian groups $\Omega_{S/R} \rightarrow \Omega_{S'/R}$. This homomorphism induces a pullback map*

$$f^* : H_{dR}^*(S/R) \rightarrow H_{dR}^*(S'/R).$$

Theorem 6.3.3.6. *If k is a field, then $H_{dR}^*(k[x_1, \dots, x_n]/k) = 0$ for $* > 0$ and coincides with k when $i = 0$.*

Proof. In this case, $\Omega_{R[x_1, \dots, x_n]}$ is a free module of rank n generated by $dx_1 \wedge \dots \wedge dx_n$. We know that $R \subset \ker(d^0)$ and since no x_i is sent to 0 by d^0 , we conclude that $R = \ker(d^0)$. Now, $\Omega_{R[x_1, \dots, x_n]/R}^i$ is a free R -module of rank $\binom{n}{i}$ with basis given by i -fold wedge products $dx_{a_1} \wedge \dots \wedge dx_{a_i}$ with $a_1 < a_2 < \dots < a_i$. Now, if ω is an i -form with $d\omega = 0$, then the usual formula for a primitive yields one in this case. \square

Exercise 6.3.3.7. *Prove a Künneth theorem for de Rham cohomology of affine varieties over a field. To do this, show first that $\Omega_{A \otimes_k B}^1 \cong \Omega_{A/k}^1 \otimes_k B \oplus A \otimes_k \Omega_{B/k}^1$. Conclude that the de Rham complex of $A \otimes_k B$ can be identified as the total complex of the double complex $\Omega_{A/k} \otimes \Omega_{B/k}$.*

Exercise 6.3.3.8. *Show that if k is a field, and A is a finite-type k -algebra, then the pullback map $H_{dR}^*(A/k) \rightarrow H_{dR}^*(A[t_1, \dots, t_n]/k)$ is a bijection.*

Exercise 6.3.3.9. *If A is a finite type k -algebra and f and g are comaximal elements of A , show that there is a short-exact sequence of complexes of the form*

$$0 \longrightarrow \Omega_{A/k} \longrightarrow \Omega_{A_f/k} \oplus \Omega_{A_g/k} \longrightarrow \Omega_{A_{fg}/k} \longrightarrow 0.$$

6.3.4 Differential forms and separability

We now use elementary results from field theory to establish that finite extensions L/k are separable if and only if corresponding modules of differentials are zero.

Proposition 6.3.4.1. *If L/k is a finite extension of fields, then L/k is separable if and only if $\Omega_{L/k} = 0$.*

Proof. First, let us show that if $L = k(\alpha)$ is a simple extension and if f is the minimal polynomial of α , then $\Omega_{L/k}$ is zero if and only if f' and f are coprime in $k[x]$.

In that case, the exact sequence of Proposition 6.3.2.3 yields an exact sequence of the form

$$(f)/(f^2) \longrightarrow L \otimes_{k[x]} \Omega_{k[x]/k} \longrightarrow \Omega_{L/k} \longrightarrow 0.$$

Now, note that $\Omega_{k[x]/k}$ is a free $k[x]$ -module of rank 1, and therefore, $L \otimes_{k[x]} \Omega_{k[x]/k}$ is an L -vector space of dimension 1. On the other hand $(f)/(f^2)$ is a free L -module of rank 1 with a canonical

trivialization given by (f) . Now, as observed above, the map in question sends f to $1 \otimes df$. Thus, this map is surjective if and only if $1 \otimes df$ is non-zero in $L \otimes_{k[x]} \Omega_{k[x]/k}$. If f is separable, then we can write $df = f'(x)dx$ and therefore, $1 \otimes df = f'(x) \otimes dx$. Thus, surjectivity is equivalent to $f'(x)$ being non-zero in $k[x]/(f)$.

Now, if L/k is separable, then it is simple, and therefore we can find an element γ such that $L = k(\gamma)$. If f is the minimal polynomial of γ , then f is necessarily separable, and a standard result shows that if f is separable, then f and f' are coprime.

Conversely, we can factor any finite extension E/k as $k \subset L \subset E$ where L/k is separable and E/L is purely inseparable. In that case, there is an exact sequence of the form

$$E \otimes \Omega_{L/k} \longrightarrow \Omega_{E/k} \longrightarrow \Omega_{E/L} \longrightarrow 0$$

By the previous result, we know that $\Omega_{L/k} = 0$, so we conclude that $\Omega_{E/k} \cong \Omega_{E/L}$. Therefore, we just have to show that for a purely inseparable extension E/L $\Omega_{E/L}$ is non-zero. Now, since E/L is purely inseparable, each element $\alpha \in E$ has minimal polynomial of the form $X^{p^m} - a$ for $a \in L^\times$. Since L is finitely generated over K , writing $L = K(\alpha_1, \dots, \alpha_n)$, by induction we see that we can assume $L = K(\alpha)$. In that case, we can simply repeat the argument above to see that the map $(f)/(f^2) \rightarrow K(\alpha) \otimes_{K[x]} \Omega_{K[x]}$ is the zero map, which means that $\Omega_{K(\alpha)/K}$ is 1-dimensional generated by the minimal polynomial f of α . \square

6.4 Lecture 22: Differential forms and unramified morphisms

6.4.1 Localizing and base change for differential forms

Suppose $A \rightarrow B$ is a ring homomorphism. Since the properties of being unramified and étale are local properties, in order to link them to differentials, we want to analyze the behavior of differentials under localization. There are two possibilities here: we can localize A or B at a multiplicative subset.

Lemma 6.4.1.1. *Suppose $\varphi : A \rightarrow B$ is a ring map.*

1. *If $S \subset A$ is a multiplicative subset mapping to a set of invertible elements in B , then $\Omega_{B/A} = \Omega_{B/A[S^{-1}]}$.*
2. *If $S \subset B$ is a multiplicative subset, then $B[S^{-1}] \otimes_B \Omega_{B/A} = \Omega_{B[S^{-1}]/A}$.*

Proof. In both cases, it is convenient to think in terms of derivations. Indeed, by the Yoneda lemma, it suffices to show that for any M there is a bijection between A -derivations $D : B \rightarrow M$ and $A[S^{-1}]$ -derivations $B \rightarrow M$. The map $A \rightarrow A[S^{-1}]$ shows that any $A[S^{-1}]$ -derivation extends to an A -derivation, so it suffices to establish the converse. Suppose given an A -derivation $D : B \rightarrow M$. An element of $A[S^{-1}]$ is an expression of the form as^{-1} for $s \in S$. Now, by assumption $\varphi(s)$ is invertible in B and therefore, $\varphi(s)\varphi(s^{-1}) = 1 \in B$. Since $D(1) = 0$, we conclude that

$$0 = D(\varphi(s)\varphi(s^{-1})) = \varphi(s)D(\varphi(s^{-1})) + \varphi(s^{-1})D(\varphi(s)) = \varphi(s)D(\varphi(s^{-1})).$$

Then, since $\varphi(s)$ is invertible, we conclude that $D(\varphi(s^{-1})) = 0$, which is what we wanted to show.

In the second case, there is a homomorphism $\Omega_{B/A} \rightarrow \Omega_{B[S^{-1}]/A}$ that factors through a $B[S^{-1}]$ -module map $B[S^{-1}] \otimes_B \Omega_{B/A} \rightarrow \Omega_{B[S^{-1}]/A}$. Now, there is an A -derivation $B[S^{-1}] \rightarrow \Omega_{B[S^{-1}]/A}$

by definition, and we want to show that it factors through $B[S^{-1}] \otimes_B \Omega_{B/A}$. To this end, we simply use the quotient rule to define the extension: if $\frac{b}{s}$ is an element of $B[S^{-1}]$ then since $\frac{b}{s}s = b$ we conclude that $d(b) = d(\frac{b}{s}s) = sd(\frac{b}{s}) + \frac{b}{s}d(s)$ and therefore we can set

$$d\left(\frac{b}{s}\right) = \frac{sdb - bds}{s^2}.$$

By construction this is an A -derivation of $B[S^{-1}]$ to $B[S^{-1}] \otimes_B \Omega_{B/A}$. \square

Remark 6.4.1.2. Note that there is a sequence of ring maps $A \rightarrow B \rightarrow B[S^{-1}]$ and therefore, we obtain a sequence

$$B[S^{-1}] \otimes_B \Omega_{B/A} \longrightarrow \Omega_{B[S^{-1}]/A} \longrightarrow \Omega_{B[S^{-1}]/B} \longrightarrow 0.$$

The above result shows that $\Omega_{B[S^{-1}]/B} = 0$. If $B[S^{-1}]$ is a finitely presented B -algebra, we saw before the $B \rightarrow B[S^{-1}]$ is unramified and flat and therefore étale. Thus, in this case we see that these conditions imply that the module of relative differentials is zero.

Now, suppose $R \rightarrow R'$ and $R \rightarrow S$ are ring homomorphisms. In this case, there is an induced map $R' \rightarrow R' \otimes_R S =: S'$ fitting into a commutative diagram of the form

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

In that case, there is an induced homomorphism $\Omega_{S/R} \rightarrow \Omega_{S'/R'}$ and this homomorphism factors through an S' -module map

$$S' \otimes_S \Omega_{S/R} \longrightarrow \Omega_{S'/R'}.$$

Note that, by associativity of tensor products, there is a canonical identification $(R' \otimes_R S) \otimes_S \Omega_{S/R} = R' \otimes_R \Omega_{S/R}$.

Proposition 6.4.1.3. *If $R \rightarrow R'$ and $R \rightarrow S$ are ring homomorphisms, then setting $S' = R' \otimes_R S$, the induced S' -module map*

$$S' \otimes_S \Omega_{S/R} \longrightarrow \Omega_{S'/R'}$$

is an isomorphism.

Proof. First, we observe that $d' : S \rightarrow \Omega_{S/R}$ extends to a map $S' \rightarrow S' \otimes_S \Omega_{S/R}$. Indeed, the map $R' \times S \rightarrow R' \otimes_R \Omega_{S/R}$ sending (x, a) to $x \otimes da$ is R -bilinear and if $\sum_i x_i \otimes a_i \in R' \otimes_R S$, then we may define

$$d'\left(\sum_i x_i \otimes a_i\right) := \sum_i x_i \otimes da_i.$$

You may check that this is an R' -derivation.

We now show that it satisfies the correct universal property. Let $D' : S' \rightarrow M'$ be an R' -derivation. The composite map $S \rightarrow S' \rightarrow M'$ is an R -derivation and therefore yields an S -linear map $\varphi_D : \Omega_{S/R} \rightarrow M'$. This morphism factors through an S' -linear map

$$\varphi'_D : (R' \otimes_R S) \otimes_S \Omega_{S/R} \longrightarrow M',$$

and we leave it to you to check that this provides the correct factorization. \square

6.4.2 Relative differentials for unramified morphism

Proposition 6.4.2.1. *If $\varphi : R \rightarrow S$ is an unramified ring map, then $\Omega_{R/S} = 0$.*

Proof. By assumption $\varphi : R \rightarrow S$ is finitely presented. Under this assumption $\Omega_{S/R}$ is a finitely presented S -module. In addition, we know that for any prime $\mathfrak{q} \subset S$, setting $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, the ideal $\mathfrak{m}_{\mathfrak{p}}S_{\mathfrak{q}}$ is the maximal ideal in $S_{\mathfrak{q}}$ and the induced map of residue fields is an isomorphism.

Note that $S_{\mathfrak{q}}$ is an R -algebra as well and Lemma 6.4.1.1(2) shows that $S_{\mathfrak{q}} \otimes_S \Omega_{S/R} = \Omega_{S_{\mathfrak{q}}/R}$. Moreover, since $\Omega_{S/R}$ is finitely presented, it follows that $\Omega_{S_{\mathfrak{q}}/R}$ is finitely presented as well. On the other hand, any element of $R \setminus \mathfrak{p}$ is mapped into $S \setminus \mathfrak{q}$ and since elements of $S \setminus \mathfrak{q}$ are invertible in $S_{\mathfrak{q}}$, it follows that $R \rightarrow S \rightarrow S_{\mathfrak{q}}$ factors through $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$. Therefore, we may apply Lemma 6.4.1.1(1) to conclude that $\Omega_{S_{\mathfrak{q}}/R} = \Omega_{S_{\mathfrak{q}}/R_{\mathfrak{p}}}$. Therefore, to establish the result, it suffices to show that $\Omega_{S_{\mathfrak{q}}/R_{\mathfrak{p}}} = 0$ for all \mathfrak{q} as above. Thus, without loss of generality we can assume that R and S are both local and that φ is a local homomorphism.

Next, if we write \mathfrak{m} for the maximal ideal in R , we may consider the map $R \rightarrow R/\mathfrak{m} = \kappa$ and there is an induced ring map $R/\mathfrak{m} \rightarrow S \otimes_R R/\mathfrak{m} \cong S/\mathfrak{m}S$. By Proposition 6.4.1.3 we conclude that $\Omega_{S/\mathfrak{m}S/\kappa} \cong S/\mathfrak{m}S \otimes_S \Omega_{S/R}$.

Now, by assumption $S/\mathfrak{m}S$ is a separable extension field of κ . Therefore, by Proposition 6.3.4.1, we conclude that $\Omega_{S/\mathfrak{m}S/\kappa} = 0$. Therefore, $S/\mathfrak{m}S \otimes_S \Omega_{S/R} = 0$. In that case, since $\Omega_{S/R}$ is finitely presented, by Nakayama's lemma, we conclude that $\Omega_{S/R} = 0$, which is precisely what we wanted to show. \square

Theorem 6.4.2.2. *A ring homomorphism $R \rightarrow S$ is unramified if and only if $R \rightarrow S$ is finitely presented and $\Omega_{R/S} = 0$.*

Proof. It remains to prove the direction that $R \rightarrow S$ finitely presented and $\Omega_{R/S} = 0$ implies that $R \rightarrow S$ unramified. \square

Corollary 6.4.2.3. *A finitely presented morphism $R \rightarrow S$ is étale if and only if it is flat and $\Omega_{S/R} = 0$.*

Another key property of étale ring maps that we will use is the following.

Proposition 6.4.2.4. *If $R \rightarrow S$ and $R \rightarrow S'$ are étale ring maps, then any R -algebra map $S \rightarrow S'$ is étale.*

Proof. See [Sta15, Lemma 10.141.9]. \square

6.4.3 Étale descent for modules

Earlier, we showed that if R is a commutative unital ring, and f and g were comaximal elements of R , then modules on R could be patched together from modules on R_f and R_g together with an isomorphism along R_{fg} . Geometrically, $\text{Spec } R_f$ and $\text{Spec } R_g$ formed a cover of $\text{Spec } R$ for the Zariski topology and their intersection was $\text{Spec } R_{fg}$. In particular, if we set $S = R_f \oplus R_g$, then the map $\text{Spec } S \rightarrow \text{Spec } R$ is a surjective map of topological spaces. Note that the maps $R \rightarrow R_f$ and $R \rightarrow R_g$ are both examples of étale ring maps. Moreover, the map $R \rightarrow S$ is also an étale ring map.

Definition 6.4.3.1. A ring map $\varphi : R \rightarrow S$ is called an *affine étale cover* if φ is étale and the induced map $\text{Spec } S \rightarrow \text{Spec } R$ is surjective.

If $R \rightarrow S$ is an étale ring map, then we may consider $\text{Mod}_R \rightarrow \text{Mod}_S$ and our basic goal is to characterize its essential image.

Example 6.4.3.2. Recall that if $S = R_f \oplus R_g$, then an R -module M determines an S -module by pullback. To patch things together we needed information about what happens on $R_{fg} = R_f \otimes_R R_g$. In particular, there were *a priori* two pullbacks $(M \otimes_R R_f) \otimes_{R_f} R_{fg}$ and $(M \otimes_R R_g) \otimes_{R_g} R_{fg}$, but associativity of tensor product yielded an identification between these two pullbacks. In this case let us interpret this in terms of S . We can compute:

$$\begin{aligned} S \otimes_R S &= (R_f \otimes_R R_f) \oplus (R_f \otimes_R R_g) \oplus (R_g \otimes_R R_f) \oplus (R_g \otimes_R R_g) \\ &= R_f \oplus R_{fg} \oplus R_{gf} \oplus R_g \end{aligned}$$

Now, we interpret the two pullbacks mentioned above in terms of the two different S -algebra structures on $S \otimes_R S$: namely the maps $s \mapsto s \otimes 1$ and $s \mapsto 1 \otimes s$ (think about switching the order of f and g). Now, in this case the two pullbacks $(M \otimes_R S) \otimes_S (S \otimes_R S)$ corresponding to the two different algebra structures just mentioned have redundant information: namely $(M \otimes_R S) \otimes_S (S \otimes_R S) \cong M_f \oplus M_{fg} \oplus M_{gf} \oplus M_g$, which contains $M \otimes_R S$ as a summand.

Remark 6.4.3.3. Note that, unlike the map $R \rightarrow R_f$, if $R \rightarrow S$ is a general étale map, $S \otimes_R S$ need not be isomorphic to S . Indeed, if $S \rightarrow R$ is an affine étale cover given by a μ_r -torsor, then $S \otimes_R S$ determines a μ_r -torsor over S !

We observed in Remark 3.3.2.2 that there is an exact sequence of the form

$$0 \longrightarrow R \longrightarrow R_f \oplus R_g \longrightarrow R_{fg}$$

where the second map is given by $(a, b) \mapsto a - b$. This is an exact sequence of flat R -modules. After tensoring this sequence with a module M we obtained an exact sequence of the form

$$0 \longrightarrow M \longrightarrow M_f \oplus M_g \longrightarrow M_{fg}$$

and we observed that M could be recovered as the zeroth cohomology of the two-term complex $M_f \oplus M_g \rightarrow M_{fg}$.

Now, consider an affine étale cover $R \rightarrow S$. By extension of scalars we obtain a functor $\text{Mod}_R \rightarrow \text{Mod}_S$. Now, there are two maps $p_1, p_2 : S \rightarrow S \otimes_R S$ corresponding to maps $s \mapsto s \otimes 1$ and $1 \otimes s$. Thus, there are two extension of scalar maps $p_1^*, p_2^* : \text{Mod}_S \rightarrow \text{Mod}_{S \otimes_R S}$. If M is an R -module, then associativity of tensor product yields a distinguished isomorphism $\alpha : p_1^*(M \otimes_R S) \xrightarrow{\sim} p_2^*(M \otimes_R S)$, just as we explained above. In this way, given an R -module, we obtain a diagram of the form

$$M \otimes_R S \longrightarrow (M \otimes_R S) \otimes_S (S \otimes_R S) \cong (M \otimes_R S) \otimes_S (S \otimes_R S) \longleftarrow M \otimes_R S$$

where the isomorphism in the middle is the distinguished isomorphism described above.

Let us reinterpret this picture in terms of our comments above. Note that $R \rightarrow S$ is an injective ring map in this situation (recall that this is equivalent to $\text{Spec } S \rightarrow \text{Spec } R$ having dense image, but

($\text{Spec } S \rightarrow \text{Spec } R$ is surjective by assumption). On the other hand, there are two maps $S \rightarrow S \otimes_R S$. The difference of these two maps is $s \otimes 1 - 1 \otimes s$ which is the universal R -derivation on S , and therefore, we know that the composite map $R \rightarrow S \otimes_R S$ is zero. Therefore, we have the following sequence

$$0 \longrightarrow R \longrightarrow S \longrightarrow S \otimes_R S.$$

In analogy with the situation in the Zariski topology, it is natural to ask whether this sequence is exact.

6.5 Lecture 23: Faithfully flat descent I

6.5.1 The Amitsur complex

Unfortunately, unlike the Zariski topology, it is certainly not clear how to refine an arbitrary open cover by those consisting of 2 open sets. Thus, we will, from the beginning keep track of 3-fold and higher intersections. Higher intersections corresponds to higher tensor powers of S with itself over R . Above, we considered the pair of maps $S \rightarrow S \otimes_R S$. There are, similarly, three maps $S \otimes_R S \rightarrow S \otimes_R S \otimes_R S$ corresponding to the three possible ways of including a pair of factors. Let us write

$$\epsilon_i : S \otimes_R \cdots \otimes_R S \longrightarrow S \otimes_R \cdots \otimes_R S$$

for the map the includes a 1 at the i -th factor. We can a homomorphism $\delta_r : \sum_{i=1}^{r+2} (-1)^{i+1} \epsilon_i$. The following exercise generalizes the fact that the map $S \rightarrow S \otimes_R S$ is an R -derivation.

Exercise 6.5.1.1. Show that the composite $d_{r+1} \circ d_r = 0$.

Given the conclusion of the preceding exercise, we can make the following definition.

Definition 6.5.1.2. If R is a ring and $R \rightarrow S$ is a ring homomorphism, then the *Amitsur complex* is the complex

$$S \xrightarrow{\delta^0} S \otimes_R S \xrightarrow{\delta^1} S \otimes_R S \otimes_R S \xrightarrow{\delta^2} \cdots ;$$

the ring homomorphism $R \rightarrow S$ yields an augmentation of this complex.

We now establish a generalization of what we described above for a Zariski cover of a ring R by two principal open sets. If R is a ring, and $f, g \in R$ are comaximal elements, the key step in the preceding argument was the fact that: if $S = R_f \oplus R_g$, then an exact sequence of R -modules was exact if and only if it remained so after tensoring with R' . The fact that the sequence remained exact after tensoring was an immediate consequence of the fact that $R \rightarrow S$ was flat. To establish the converse required that $R \rightarrow S$ induced a surjective map $\text{Spec } S \rightarrow \text{Spec } R$. The following result is a generalization of this fact.

Lemma 6.5.1.3. Suppose $\varphi : R \rightarrow S$ is any flat ring homomorphism such that the induced map $\text{Spec } S \rightarrow \text{Spec } R$ is surjective. If $M_1 \rightarrow M_2 \rightarrow M_3$ is a sequence of R -modules, then $M_1 \rightarrow M_2 \rightarrow M_3$ is exact if and only if $M_1 \otimes_R S \rightarrow M_2 \otimes_R S \rightarrow M_3 \otimes_R S$ is exact as well.

Proof. We only need to prove the “if” direction since φ flat by assumption. Let $H = \ker(M_2 \rightarrow M_3)/\text{im}(M_1 \rightarrow M_2)$ be the cohomology at M_2 of $M_1 \rightarrow M_2 \rightarrow M_3$. By flatness, we conclude that $H \otimes_R S = 0$. Assume that H is nonzero. In that case, we can take $x \in H$ and let $I = \{f \in R \mid fx = 0\}$ be its annihilator. The map $R \rightarrow H$ defined by $r \mapsto rx$ factors through an injection $R/I \rightarrow H$. Therefore, we conclude that $S/IS \rightarrow H \otimes_R S$ is also injective, again by flatness.

By assumption $\text{Spec } S \rightarrow \text{Spec } R$ is surjective, and it follows that for any maximal ideal $\mathfrak{m} \subset R$, the ring $S/\mathfrak{m}S$ is non-zero. Now, if I from the previous paragraph is not equal to R , we can find a maximal ideal containing it, and therefore a non-zero submodule of $H \otimes_R S$, which is a contradiction. Thus, H must have been zero to begin. \square

Lemma 6.5.1.4. *If $\varphi : R \rightarrow S$ is a flat map such that $\text{Spec } S \rightarrow \text{Spec } R$ is surjective (e.g., if φ is an affine étale cover), then the sequence*

$$0 \longrightarrow R \longrightarrow S \xrightarrow{\delta^0} S \otimes_R S \xrightarrow{\delta^1} S \otimes_R S \otimes_R S \xrightarrow{\delta^2} \cdots$$

is exact. In other words, the cohomology of the Amitsur complex is R in degree 0 and trivial in other degrees.

Proof. In the Zariski case, we checked this statement locally, which amounts to asserting that $S \rightarrow R$ has a section. Thus, let us first assume that $S \rightarrow R$ has a section. We begin by checking exactness at the first stage. In that case, $S = R \oplus I$ for some ideal $I \subset S$. We know that R is contained in the kernel of $S \rightarrow S \otimes_R S$ (since δ_0 is an R -derivation) and we want to show that it surjects onto the kernel.

Now observe that $S \otimes_R S \cong (R \oplus I) \otimes_R (R \oplus I) \cong R \otimes_R R \oplus R \otimes_R I \oplus I \otimes_R R \oplus I \otimes_R I$. Then, our sequence reads

$$0 \longrightarrow R \longrightarrow R \oplus I \longrightarrow R \oplus R \otimes_R I \oplus I \otimes_R I \oplus I \otimes_R I \longrightarrow \cdots$$

where the image of the first map is the first factor in the direct sum. Now, a direct computation shows that $\delta^0(i) = i \otimes 1 - 1 \otimes i$, which means that no element of i is contained in the kernel. You can check exactness at other stages using this observation.

Next, we reduce to the previous case. If $R \rightarrow S$ is a flat ring map with $\text{Spec } S \rightarrow \text{Spec } R$ surjective, then we can check exactness after tensoring over R with S by Lemma 6.5.1.3. In that case, multiplication $S \otimes_R S \rightarrow S$ determines a section. \square

6.5.2 Descending elements and homomorphisms

More generally, suppose M_1, \dots, M_n are arbitrary R -module. Define maps $\epsilon_i : M_1 \otimes \cdots \otimes M_n \rightarrow M_1 \otimes \cdots \otimes M_{i-1} \otimes S \otimes M_i \otimes \cdots \otimes M_n$. Tensoring the Amitsur complex up by a module M again yields a complex

Corollary 6.5.2.1. *If M is an R -module, and $R \rightarrow S$ is a flat ring homomorphism such that $\text{Spec } S \rightarrow \text{Spec } R$ is surjective (e.g., an affine étale cover), then the sequence*

$$0 \longrightarrow M \longrightarrow M \otimes_R S \xrightarrow{1 \otimes \delta_0} (M \otimes_R S) \otimes_S (S \otimes_R S) \xrightarrow{1 \otimes \delta_1} \cdots$$

is exact. In other words, the complex $M \otimes_R S \xrightarrow{1 \otimes \delta_0} (M \otimes_R S) \otimes_S (S \otimes_R S) \xrightarrow{1 \otimes \delta_1} \cdots$ has cohomology M in degree 0 and trivial in other degrees.

Proof. Same argument as above. \square

Remark 6.5.2.2. This result shows that one can “descend elements of modules”.

Now, we turn our attention to homomorphisms of R -modules. Suppose M and M' are R -modules. In that case, we can consider $\text{Hom}_R(M, M')$. There is an induced homomorphism $\text{Hom}_R(M, M') \rightarrow \text{Hom}_S(M \otimes_R S, M' \otimes_R S)$. We would like to characterize those homomorphisms in the image of this map.

Proposition 6.5.2.3. *If M and M' are R -modules, and $R \rightarrow S$ is a flat ring map such that $\text{Spec } S \rightarrow \text{Spec } R$ is surjective, then*

$$0 \longrightarrow \text{Hom}_R(M, M') \longrightarrow \text{Hom}_S(M_S, M'_S) \longrightarrow \text{Hom}_{S \otimes_R S}(M_{S \otimes_R S}, M'_{S \otimes_R S})$$

is exact.

Proof. Consider the commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & M_S & \longrightarrow & M_{S \otimes_R S} \\ & & \downarrow f & & \downarrow f_S & & \downarrow \\ 0 & \longrightarrow & M' & \longrightarrow & M'_S & \longrightarrow & M'_{S \otimes_R S} \end{array}$$

Now, note that injectivity of the maps $M \rightarrow M_S$ and $M' \rightarrow M'_S$ implies that the induced map on Hom groups is injective. Therefore, it remains to demonstrate surjectivity. Suppose $g : M_S \rightarrow M'_S$ is a homomorphism and suppose the two induced maps $g \otimes 1$ and $1 \otimes g$ coincide. It suffices to check that the restriction of g to M is sent to M' , but this can be checked elementwise. If $x \in M$, then consider $g(x)$. By assumption the element x satisfies $x \otimes 1 - 1 \otimes x = 0$. However, it follows that $g(x) \otimes 1 - 1 \otimes g(x) = 0$ as well. \square

6.6 Lecture 24: Faithfully flat descent II: an equivalence

6.6.1 Descent of objects

In the Zariski setting, we identified the category of R -modules with a category of triples (M_1, M_2, α) . Already if we consider covers with three open sets, this data is not enough: we need compatibility on threefold intersections. Recall from our construction of vector bundles that this involved a “cocycle condition” over threefold intersections. We now formulate this slightly more abstractly.

Given an S -module N , we will write $N \otimes S$ and $S \otimes N$ for the two $S \otimes_R S$ -modules that we obtain by tensoring up to $S \otimes_R S$ with respect to the “left” and “right” algebra structures. If N is obtained by extending scalars from an R -module M , then, unwinding the definitions of the module structures, there is a distinguished isomorphism $\alpha : N \otimes S \cong S \otimes N$.

To analyze the compatibility conditions required of this module, recall that we first pass to “threefold” intersections, which in this case corresponds to looking at $S \otimes_R S \otimes_R S$. Now, any map $\varphi : N \otimes S \rightarrow S \otimes N$ yields after tensoring up to $S \otimes_R S \otimes_R S$ three maps, which we write as

$$\begin{aligned} p_{12}^* \varphi &: N \otimes S \otimes S \longrightarrow S \otimes N \otimes S \\ p_{23}^* \varphi &: S \otimes N \otimes S \longrightarrow S \otimes S \otimes N \\ p_{13}^* \varphi &: N \otimes S \otimes S \longrightarrow S \otimes S \otimes N. \end{aligned}$$

Note that if φ is an isomorphism, then each of these maps is an isomorphism. Note in particular, that the composite $p_{23}^*\varphi \circ p_{12}^*\varphi$ makes sense and induces another map $N \otimes S \otimes S \longrightarrow S \otimes S \otimes N$. One checks that, in the situation above, $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$.

Definition 6.6.1.1. Suppose $\varphi : R \rightarrow S$ is a ring map. The category of descent data relative to φ is the category whose objects are pairs (M, α) where M is an S -module, and $\alpha : M \otimes S \rightarrow S \otimes M$ is an isomorphism of $S \otimes_R S$ -modules satisfying the cocycle condition $p_{23}^*\alpha \circ p_{12}^*\alpha = p_{13}^*\alpha$ and where morphisms are morphisms of S -modules, commuting with the relevant maps. Write $\text{Mod}_R(\varphi)$ for the category of descent data relative to φ .

Now, note that extension of scalars defines a functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R(\varphi)$$

for any ring map φ . We now proceed to construct a candidate quasi-inverse for this functor.

Given an S -module M equipped with a descent datum α , we consider the map

$$\delta_\alpha^0 : M \longrightarrow M \otimes_S (S \otimes_R S)$$

defined by $m \mapsto m \otimes 1 - \alpha(1 \otimes m)$. Bearing in mind the case where $M = S$, recall that $\delta_0 : S \rightarrow S \otimes_R S$, is not an S -linear map, just an R -linear map. Nevertheless, $\ker(\delta_\alpha^0)$ has an R -module structure.

Theorem 6.6.1.2. *If $\varphi : R \rightarrow S$ is a flat ring map with $\text{Spec } S \rightarrow \text{Spec } R$ surjective, then the functor sending an R -module M to the S -module $(M \otimes_R S)$ equipped with the structures discussed above is an equivalence of categories with quasi-inverse given by $(M, \alpha) \mapsto \ker(\delta_\alpha^0)$.*

Proof. If M is an R -module, then the composite functor

$$\text{Mod}_R \longrightarrow \text{Mod}_R(\varphi) \longrightarrow \text{Mod}_R$$

is the identity functor combining our results on descent for modules and homomorphisms. Thus, it remains to show that the other composite is naturally isomorphic to the identity. We leave it as an exercise to show this fact. \square

6.6.2 Descending properties of modules and morphisms

Proposition 6.6.2.1. *If $\varphi : R \rightarrow S$ is a faithfully flat ring map, then if $(M, \alpha) \in \text{Mod}_R(\varphi)$ has the property that M is finitely generated, finitely presented or finitely generated projective, then the R -module obtained from (M, α) has the same property.*

Proof. The proof for finite presentation is essentially identical to that we gave before in the ‘‘Zariski’’ setting, so we leave this as an exercise. \square

Lemma 6.6.2.2. *Suppose $R \rightarrow S$ is a faithfully flat ring map and M is an R -module. The module M is a flat R -module if and only if $M \otimes_R S$ is a flat S -module.*

Proof. If M is flat, then we already saw that $M \otimes_R S$ is flat with no hypotheses on $R \rightarrow S$. Thus, suppose $M_S := M \otimes_R S$ is a flat S -module. Suppose $N_1 \rightarrow N_2 \rightarrow N_3$ is an exact sequence of R -modules. We want to show that $N_1 \otimes M \rightarrow N_2 \otimes M \rightarrow N_3 \otimes M$ is exact. Since $R \rightarrow S$ is flat, we know that $N_1 \otimes_R S \rightarrow N_2 \otimes_R S \rightarrow N_3 \otimes_R S$ is again exact. Then, flatness of M_S tells us that $N_1 \otimes_R S \otimes_S M_S \rightarrow N_2 \otimes_R S \otimes_S M_S \rightarrow N_3 \otimes_R S \otimes_S M_S$ is also exact. However by using commutativity and associativity of tensor product, we see that $N_1 \otimes_R S \otimes_S M_S = N_1 \otimes_R S \otimes_S (S \otimes_R M) = N_1 \otimes_R S \otimes_R M = N_1 \otimes_R M \otimes_R S$ and therefore our exact sequence may be rewritten as

$$N_1 \otimes_R M \otimes_R S \longrightarrow N_2 \otimes_R M \otimes_R S \longrightarrow N_3 \otimes_R M \otimes_R S.$$

Since $R \rightarrow S$ is faithfully flat (use the Lemma above), we conclude that this implies the exactness of $N_1 \otimes M \rightarrow N_2 \otimes M \rightarrow N_3 \otimes M$. \square

Corollary 6.6.2.3. *If $R \rightarrow S$ is a faithfully flat ring map and $R \rightarrow R'$ is a ring map such that $S \rightarrow S \otimes_R R' = S'$ is a finite locally free morphism (resp. free), then $R \rightarrow R'$ is also finite locally free (resp. flat).*

Proof. To see that R' is a finitely generated R -module it suffices by the above proposition to know that S' is a finitely generated S -module. Likewise, to check that R' is a finitely generated projective (resp. flat) R -module, it suffices to show that S' is a finitely generated projective (resp. flat) S -module. \square

Lemma 6.6.2.4. *Suppose $\varphi : R \rightarrow R'$ is a morphism, $R \rightarrow S$ is a faithfully flat morphism, and consider $\varphi' : S \rightarrow R' \otimes_R S =: S'$.*

1. *If φ' is unramified, so is φ .*
2. *If φ' is étale, so is φ .*

Proof. Suppose $\varphi : R \rightarrow R'$ is an unramified morphism. This is equivalent to asserting that the finitely presented module $\Omega_{S/R} = 0$. Suppose $R \rightarrow R'$ is a faithfully flat map and set $S' = S \otimes_R R'$. In that case, by base-change for differentials, we conclude that $\Omega_{S'/R'} = \Omega_{S/R} \otimes_S S'$. Now, if $\Omega_{S'/R'} = 0$, then it follows that $\Omega_{S/R}$ must have been trivial to begin with by the equivalence of categories above.

The second statement follows by combining the unramified statement with the flat statement. \square

Remark 6.6.2.5. Thus, whether or not a morphism is étale can be checked after faithfully flat base change.

Following Serre, we give a notion of fiber bundle that generalizes our notion of vector bundle and μ_r -torsor.

Definition 6.6.2.6 (Algebraic fiber bundle). Suppose $f : R \rightarrow S$ is a map of k -algebras. We will say that f is an algebraic fiber bundle with fiber a k -algebra A if there exists an affine étale cover $R \rightarrow R'$ together with an isomorphism $S' = R' \otimes_R S \cong A \otimes_k R'$.

6.7 Lecture 25: Differentials, regularity and smoothness

6.7.1 Regularity and differentials

Suppose k is a field, and R is a Noetherian local k -algebra of dimension d . In that case, the residue field κ of R is an extension of k . Let us suppose that $k \rightarrow \kappa$ is an isomorphism. In that case, there is an exact sequence of the form

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \kappa \otimes_R \Omega_{R/k} \longrightarrow \Omega_{\kappa/k} \longrightarrow 0.$$

Here κ/k is a finite extension of k . If the extension κ/k is separable, then we saw that $\Omega_{\kappa/k} = 0$ and we conclude that there is a surjective map of κ -vector spaces of the form:

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \kappa \otimes_R \Omega_{R/k}.$$

One way to guarantee the hypotheses above are satisfied is to assume that k is an algebraically closed field. The next result establishes a link between tangent spaces as we studied them before and differentials.

Proposition 6.7.1.1. *If k be a field, and $(R, \mathfrak{m}, \kappa)$ is a Noetherian local k -algebra such that $k \rightarrow \kappa$ is an isomorphism, then the map $\delta : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \kappa \otimes_R \Omega_{R/k}$ is an isomorphism.*

Proof. We know that this map is a surjective map of k -vector spaces. It remains to show that it is injective. To see this, it suffices to show that the induced map of dual vector spaces is surjective. However, the source of the map $\delta^\vee : \text{Hom}_k(\kappa \otimes_R \Omega_{R/k}, k) \rightarrow \text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ can be identified by means of adjunction with $\text{Hom}_R(\Omega_{R/k}, k)$. Since $\text{Hom}_R(\Omega_{R/k}, k) = \text{Der}(R, k)$, this fact can be viewed of an algebraic incarnation of the idea from differential geometry that the tangent space can be identified with “point derivations”.

Recall that the map of the statement sends $x \in \mathfrak{m} \rightarrow 1 \otimes dx$. Under the identification $\text{Hom}_R(\Omega_{R/k}, k) = \text{Der}(R, k)$, the map of the previous paragraph sends a derivation $D : R \rightarrow k$ to a linear map $\text{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. Given $D : R \rightarrow k$, by restriction to \mathfrak{m} , we obtain a k -linear map $\mathfrak{m} \rightarrow k$. Given generators x_1, \dots, x_n of \mathfrak{m} , then \mathfrak{m}^2 is generated by $x_i x_j$. In that case, $D(x_i x_j) = x_i D(x_j) + x_j D(x_i) + h.o.t.$ and thus $D(x_i x_j)$ is zero when reduced modulo \mathfrak{m} . In other words, a derivation $D : R \rightarrow k$ restricts to a k -linear map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow k$. Unwinding the definitions, we see that $\delta^\vee(D)$ is the k -linear map just described.

To show that δ^\vee is surjective, we must build a derivation given a k -linear map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow k$. Such a map extends uniquely to a k -linear map $\mathfrak{m} \rightarrow k$ by precomposition with the surjection $\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^2$. On the other hand, there is a short exact sequence $0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$. This exact sequence is split by the inclusion $k \rightarrow R$, and therefore, we obtain a direct sum decomposition $R \cong \mathfrak{m} \oplus k$. Then, any element of R can be written uniquely as (m, c) where $m \in \mathfrak{m}$ and c is its “constant term.” Now, given a linear map $f : \mathfrak{m}/\mathfrak{m}^2 \rightarrow k$, define a derivation D_f by the formula $D_f(c, m) = f(m \bmod \mathfrak{m}^2)$. One checks that this is a derivation and that it establishes the required surjectivity. \square

Proposition 6.7.1.2. *Suppose k is a field and R is a Noetherian local k -algebra of dimension d whose residue field is isomorphic to k . If $\Omega_{R/k}$ is projective (hence free) of rank d , then R is a regular local ring.*

Proof. We saw in Proposition 6.7.1.1 that the map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \kappa \otimes_R \Omega_{R/k}$ is an isomorphism of k -vector spaces under the stated hypotheses. However, if $\Omega_{R/k}$ is projective of rank d , then $\kappa \otimes_R \Omega_{R/k}$ is a k -dimensional vector space. Thus, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$ and R is regular. \square

Remark 6.7.1.3. In fact, there is a suitable converse to the above proposition. If R is a regular local k -algebra of dimension d such that the residue field is isomorphic to k , then $\Omega_{R/k}$ is projective of rank d . Now, we know the fiber of $\Omega_{R/k}$ over the residue field has dimension d in this case, thus we need to guarantee that it is projective. If R is regular, then it is an integral domain, as we established before. Therefore, it has a field of fractions K . It will suffice to show that $\Omega_{R/k} \otimes_R K$ has dimension k as well. Assume K/k is a finitely generated field extension. In that case, $\dim \Omega_{K/k} \geq \text{tr.deg} K/k$ and equality holds if and only if K is separably generated over k .

Remark 6.7.1.4. The same argument as above works if κ is separable over k since in that case $\Omega_{\kappa/k} = 0$. Thus, if k is a perfect field, and R is a Noetherian local k -algebra such that $\Omega_{R/k}$ is projective (hence free), then R is a regular local k -algebra.

Corollary 6.7.1.5. *If k is an algebraically closed field, and R is a Noetherian k -algebra of dimension d such that $\Omega_{R/k}$ is a projective R -module of rank d , then R is a regular k -algebra.*

Proof. Since differentials localize we see that for a maximal ideal $\mathfrak{m} \subset R$, we have $(\Omega_{R/k})_{\mathfrak{m}} = \Omega_{R_{\mathfrak{m}}/k}$. The result then follows immediately from Proposition 6.7.1.2. \square

6.7.2 Smooth morphisms

Definition 6.7.2.1. A ring map $\varphi : R \rightarrow S$ is called *smooth* if φ is finitely presented, flat and $\Omega_{S/R}$ is a projective module.

Remark 6.7.2.2. Note that étale morphisms are smooth since the zero module is projective.

There are permanence properties of smooth morphisms that are inherited from the properties of morphisms we have established so far and also from properties of differentials.

Lemma 6.7.2.3. *1. Composites of smooth ring maps are smooth.
2. Base changes of smooth ring maps are smooth.
3. Smoothness is étale local.*

Example 6.7.2.4. If R is a ring, then $R \rightarrow R[x_1, \dots, x_n]$ is a smooth ring map.

Example 6.7.2.5. If k is a field, then any finitely presented k -algebra is automatically a flat k -algebra. In that case, $\Omega_{R/k}$ is automatically finitely presented. Thus, to check it is projective, it suffices to check upon localization at a maximal ideal.

Theorem 6.7.2.6. *If $\varphi : R \rightarrow S$ is a smooth ring map, then φ is finitely presented, flat, and has geometric fibers that are regular schemes.*

Proof. Since smoothness is stable by base-change, suppose we pick a map $R \rightarrow k$, where k is an algebraically closed field. In that case, we $k \otimes_R S$ is a smooth k -algebra. Now S_k is a finitely presented k -algebra and therefore Noetherian. Moreover, by base-change for differentials we conclude that $\Omega_{S_k/k}$ is projective as well. Therefore, we conclude that S_k is a regular k -algebra, which is what we wanted to show. \square

Example 6.7.2.7. Note that our previous example of a regular ring whose self-product was not regular provides an example of a regular ring that is not smooth (indeed, the inseparability of the extension in question makes the module of relative differentials non-trivial).

Given the above notions, we can now make sense of the notion of a smooth fiber bundle.

Definition 6.7.2.8. A ring map $\varphi : R \rightarrow S$ will be called a *smooth algebraic fiber bundle* if φ is an algebraic fiber bundle and φ is smooth.

Exercise 6.7.2.9. Show that the schemes Q_{2n-1} and Q_{2n} are smooth over \mathbb{Z} .

6.7.3 Local structure of smooth schemes

Suppose R is a smooth k -algebra with k an algebraically closed field. We know that X is regular, in particular, all of the local rings of R are regular local rings. In that case, recall that one of the equivalent characterizations of a d -dimensional Noetherian local ring R being regular was that if \mathfrak{m} is the maximal ideal in R and κ the residue field, then $gr_{\mathfrak{m}}R$ is a polynomial ring in d -variables over κ .

Now, if k is algebraically closed, since R has finite type over k , we conclude that $\kappa = k$. In that case, since the map $\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is a surjective map of k -vector spaces. We can find a finite-dimensional subspace $V \subset \mathfrak{m}$ such that the composite map $V \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism. Indeed, this corresponds to choosing functions $f_1, \dots, f_d \in \mathfrak{m}$ whose differentials df_i span the vector space $\mathfrak{m}/\mathfrak{m}^2$.

By the universal property of the symmetric power, this k -vector space map $V \rightarrow R$ extends to a k -algebra map

$$\mathrm{Sym}V \longrightarrow R,$$

i.e., we can view R as an algebra over a polynomial ring. We will refer to this map as the *tangent map*. By construction, this map sends the maximal ideal in the polynomial ring corresponding to 0 to the maximal ideal \mathfrak{m} . Since $V \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is an isomorphism, we conclude this map is unramified at \mathfrak{m} .

Theorem 6.7.3.1. *The induced map from the localization of $\mathrm{Sym}V$ at 0 to the localization of R at \mathfrak{m} is an étale ring map.*

Proof. The map has finite presentation since all rings are Noetherian and it is unramified by the discussion before the statement. Therefore, it suffices to show the map is flat. In that case, we set $A = \mathrm{Sym}V$ and write $R = A[t_1, \dots, t_n]/(P_1, \dots, P_n)$ (the exact sequence for differentials shows that we can find n elements, at least locally). One checks that regularity of R implies that the elements P_1, \dots, P_n form a regular sequence. The flatness assertion then follows from the next proposition. \square

Proposition 6.7.3.2. *Suppose B is a flat A -algebra and $f \in B$. If the image of $f \in B/\mathfrak{m}B$ is not a zero-divisor for any maximal ideal $\mathfrak{m} \subset A$, then $B/(f)$ is a flat A -algebra.*

Remark 6.7.3.3. The statement of Theorem 6.7.3.1 can be interpreted as saying that “étale locally” any smooth scheme looks like an affine space.

Example 6.7.3.4. If k is an algebraically closed field and R is an étale k -algebra, then R is isomorphic to a finite product of copies of k . Indeed, by assumption R is a finite dimensional R -algebra, so it is both Artinian and Noetherian. In particular it is isomorphic to a finite direct product of local rings. Since R is a smooth k -algebra it is regular by the results above and therefore reduced. Therefore, R is isomorphic to a finite product of extensions of k and is therefore a product of fields.

Chapter 7

Schemes, sheaves and Grothendieck topologies

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We now have a large class of “reasonable” algebraic varieties (namely smooth affine varieties), together with a class of interesting \mathbb{A}^1 -homotopy invariants (Picard groups, Grothendieck groups, de Rham cohomology, etc.). With the goal of eventually building a reasonable homotopy category, we turn our attention briefly back to categorical properties of algebraic varieties. We begin by enlarging

the category of affine varieties slightly by allowing certain gluing constructions to be performed.

7.1 Lecture 24: Presheaves and sheaves

7.1.1 Sheaves on topological spaces

The notion of a sheaf on a topological space is useful for studying locally defined properties. Here is a motivating problem. Suppose X is a topological space, and $\{U_i\}_{i \in I}$ is an open cover of X . Given continuous functions $f_i : U_i \rightarrow \mathbb{C}^\times$, can we find a function $f : X \rightarrow \mathbb{C}^\times$ whose restriction to U_i coincides with f_i ? Some compatibility amongst the f_i is necessary: if U_i and U_j are open sets that intersect, then we can restrict f_i and f_j to $U_i \cap U_j = U_{ij}$ and they must coincide there. On the other hand, if $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, then we can define a function f on $U_i \cup U_j$ whose values at $x \in X$ are given by $f_i(x)$ if $x \in U_i$ and $f_j(x)$ if $x \in U_j$. This extended function is continuous and by induction, assuming compatibility we can build a function f . Note that the function f that we have built is necessarily unique. The notion of a sheaf abstracts this gluing procedure, which has appeared repeatedly in previous sections.

Presheaves on a topological space

Definition 7.1.1.1. If X is a topological space, then define a category $\text{Op}(X)$ as follows: objects are open sets of X for the given topology and given two open sets U and V , there is a unique morphism $U \rightarrow V$ if $U \subset V$.

Remark 7.1.1.2. Note that $\text{Op}(X)$ has an initial object, corresponding to the empty set, and a final object, given by X itself.

A presheaf on X is a rule that assigns some structure to each open set, together with suitable “restriction” maps connecting the structures associated to different open sets. As before, it will be convenient to think of “algebraic structures” as simply the objects of a category. In practice, the category will be taken to be Set , Ab , Grp , or something similar.

Definition 7.1.1.3. Suppose \mathbf{C} is a category and X is a topological space. A \mathbf{C} -valued presheaf on X is a functor

$$\mathcal{F} : \text{Op}(X)^\circ \longrightarrow \mathbf{C}.$$

A morphism of \mathbf{C} -valued presheaves on X is a natural transformation of functors. Write $\text{PShv}(X, \mathbf{C})$ for the category of \mathbf{C} -valued presheaves on X .

Remark 7.1.1.4. While having a definition this general affords us considerable flexibility, it does come with some drawbacks. For example, we need to be a bit careful with terminology: if U is an open subset of a topological space X , then $\mathcal{F}(U)$ is just an object of the category \mathbf{C} and need not have any “internal” structure: in particular, it does not make any sense to speak of elements of $\mathcal{F}(U)$. Often we will take $\mathbf{C} = \text{Set}$ or Ab . In either of these cases, elements of $\mathcal{F}(U)$ are themselves sets or abelian groups and it makes sense to talk about their elements (more generally, this makes sense in any “concrete category”, i.e., a category equipped with a faithful functor to the category of sets). In that case, elements of $\mathcal{F}(U)$ will be called *sections of \mathcal{F} over U* . While it

may not be immediately apparent, we will later want to work with categories that are not necessarily concrete, so the flexibility of the definition will become essential. Freyd showed that the homotopy category of pointed topological spaces \mathcal{H}_* cannot be equipped with a faithful functor to \mathbf{Set} and is therefore not concrete [Fre04], so even “down to earth” categories may fail to be concrete.

Remark 7.1.1.5. Furthermore, note that we have imposed no restriction on the functor \mathcal{F} . Hartshorne restricts attention to $\mathbf{C} = \mathbf{Ab}$ and requires that $\mathcal{F}(\emptyset) = 0$ (the final object of \mathbf{Ab}). Since a general category \mathbf{C} as above need not have a final object, Hartshorne’s definition does not even make sense in this generality.

Example 7.1.1.6. If \mathbf{C} is a category and $A \in \mathbf{C}$ is an object, the constant presheaf on a topological space X is the presheaf assigning to each $U \in \mathbf{Op}(X)$ the object A and to each morphism the identity morphism.

Sheaves of sets on a topological space

We now define sheaves of sets by imposing the condition that sections are “locally” determined. More precisely, suppose U is an open subset of a topological space X and $\{U_i\}_{i \in I}$ is an open cover of U . In this case, if \mathcal{F} is a presheaf on X , there are restriction maps $\mathcal{F}(U) \rightarrow \mathcal{F}(U_i)$ and we can take the product of these to obtain a function

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i).$$

If $s \in \mathcal{F}(U)$ is a section, this function sends s to $\{s_i\}_{i \in I}$, where s_i is, intuitively speaking, the restriction of s to U_i . Similarly, there are a pair of restriction maps of the form:

$$\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}).$$

Now, the locality condition can be phrased in two steps: (i) any section $s \in \mathcal{F}(U)$ is determined by its restriction to U_i , i.e., the first map above is injective, and (ii), given a family of sections $\{s_i\}_{i \in I}$ whose restrictions to two-fold intersections agree, there exists a (necessarily unique) section $s \in \mathcal{F}(U)$ whose restriction to U_i coincides with s_i . These two conditions can be phrased more categorically as follows.

Definition 7.1.1.7. If X is a topological space, \mathcal{F} is a \mathbf{Set} -valued presheaf on X , then say \mathcal{F} is a *Set-valued sheaf on X* if for any open set U and any open cover $\{U_i\}_{i \in I}$ of U , the sequence

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram.

Remark 7.1.1.8. As observed above, the empty set is the initial object of $\mathbf{Op}(X)$. The empty set also has a distinguished cover given by the empty cover. The indexing set for the empty cover of the empty set is the empty set as well. The empty product in a category is simply the final object. Therefore, implicit in our definition of a sheaf is the condition that $\mathcal{F}(\emptyset) = *$ (where $*$ is the singleton set).

Exercise 7.1.1.9. Show that if X is a topological space and \mathcal{F} is a Set-valued sheaf on X , and U and V are disjoint open subsets of X , then $\mathcal{F}(U \cup V) = \mathcal{F}(U) \times \mathcal{F}(V)$.

Example 7.1.1.10. The fundamental example of a presheaf (of sets) that is not a sheaf (of sets) is the constant presheaf assigning to $U \in X$ a non-singleton set S . Indeed, $\mathcal{F}(\emptyset) = S$, rather than $*$.

Example 7.1.1.11. Suppose S is a set and view S as a topological space with the discrete topology. If X is a topological space, define the constant sheaf S_X to be the sheaf $\text{Hom}_{\text{Top}}(U, S)$ (i.e., continuous maps from U to S). If U is connected, such functions are constant, but if U is disconnected, then $\text{Hom}(U, S)$ is only constant on connected components. Thus, S_X consists of “locally constant functions.” We will refer to S_X as the *constant sheaf* associated with S .

Sheaves valued in a general category

Now that we have a reasonable notion of sheaves of sets, there are several ways we can talk about \mathbf{C} -valued sheaves where \mathbf{C} is a more general category. A fundamental problem is that if \mathbf{C} is a general category, then the constructions being used to define “restriction” need not even make sense. For example, if $\prod_{i \in I} \mathcal{F}(U_i)$ may not exist, and even if it does, equalizers may not exist in the given category. Rather than necessitating the existence of all such products and equalizers in \mathbf{C} , we use the Yoneda embedding to allow us “reduce our problem” to only considering Set-valued sheaves. Indeed, we can identify \mathbf{C} as the full subcategory of Set-valued contravariant functors on \mathbf{C} of the form $\text{Hom}_{\mathbf{C}}(-, Y)$ for Y an object in \mathbf{C} .

Exercise 7.1.1.12. Show that, given an object $A \in \mathbf{C}$, the assignment $\mathcal{F}_A(U) := \text{Hom}_{\mathbf{C}}(A, \mathcal{F}(U))$ defines a presheaf of sets \mathcal{F}_A on X .

Definition 7.1.1.13. Suppose X is a topological space, \mathbf{C} is a category and \mathcal{F} is a \mathbf{C} -valued presheaf on X . We will say that \mathcal{F} is a \mathbf{C} -valued sheaf on X if for every object $A \in \mathbf{C}$, \mathcal{F}_A is a Set-valued sheaf on X . A morphism of sheaves is simply a morphism of the underlying presheaves, i.e., a natural transformation of functors. Write $\text{Shv}(X, \mathbf{C})$ for the category of \mathbf{C} -valued sheaves on X .

Exercise 7.1.1.14. Show that if all necessary products and equalizers exist in \mathbf{C} , the definition above is equivalent to requiring that the diagram from the definition of a sheaf is an equalizer diagram in \mathbf{C} .

Example 7.1.1.15. If X is any topological space, then X determines a Set-valued presheaf on X , i.e., $\text{Hom}_{\text{Op}(X)}(-, X)$. This presheaf is a sheaf. More generally, if Y is any topological space, then we can consider the presheaf that assigns to $U \subset X$ the set of continuous maps $U \rightarrow Y$. You can check that this presheaf is necessarily a sheaf as well. In particular, taking $Y = \mathbb{R}$ or \mathbb{C} equipped with its usual topology, one can speak of the sheaf of real or complex valued continuous functions on X . We write \mathbb{C}_X for this sheaf. If X happens to be a differentiable manifold, then we may also speak of the sheaf of smooth functions on X .

Example 7.1.1.16. If X is a topological space, and $\pi : \mathcal{E} \rightarrow X$ is a vector bundle on X , then assigning to $U \subset X$ the set of sections of $\mathcal{E}|_U$ defines a sheaf of modules over the sheaf of continuous functions on X .

Example 7.1.1.17. If X is a topological space, $x \in X$ and S is a set, then the skyscraper sheaf associated with x is defined as follows: $x_*S(U) = S$ if $x \in U$ and \emptyset if $x \notin U$.

7.1.2 Isomorphism, epimorphism and monomorphisms of sheaves

Since morphisms in $\text{PShv}(X, \mathbf{C})$ are simply natural transformations of functors, it follows immediately that monomorphisms, epimorphisms and isomorphisms are determined sectionwise. Let us first Detecting epimorphisms and isomorphisms of sheaves is more subtle as we now discuss.

Lemma 7.1.2.1. *If $\mathcal{F}_1, \mathcal{F}_2$ are \mathbf{C} -valued presheaves on X , then a morphism $\varphi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a monomorphism if and only if the induced maps $\mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U)$ are monomorphisms for every $U \in \text{Ob}(X)$.*

Proof. Unwind the definitions. □

Detecting epimorphicity of sheaf maps is more complicated because of the “local” nature of the definition of sheaves. If X is a topological space and $x \in X$ is a point, then a neighborhood of x in X is an open set $x \in U \subset X$. Note that every point $x \in X$ has a neighborhood, namely X itself. If U_1 and U_2 are neighborhoods of U , then $U_1 \cap U_2$ is also a neighborhood of U . It follows that the subcategory of $\text{Op}(X)$ consisting of neighborhoods of x is a partially ordered set, viewed as a category.

Definition 7.1.2.2. *If \mathcal{F} is a presheaf on a topological space and $x \in X$ is a point, then the stalk of \mathcal{F} at x , denoted \mathcal{F}_x is defined by the colimit*

$$\mathcal{F}_x := \text{colim}_{x \in U \subset X} \mathcal{F}(U),$$

assuming this colimit exists in \mathbf{C} .

Remark 7.1.2.3. Because the category indexing the colimit is filtered, we can give a very direct definition of the colimit for a presheaf of sets. Namely, \mathcal{F}_x consists of pairs (U, s) where $x \in U$ and $s \in \mathcal{F}(U)$ modulo the equivalence relation given by $(U, s) \sim (U', s')$ if the sections s and s' coincide after a suitable refinement, i.e., there exists an open set $U'' \subset U \cap U'$ such that s and s' coincide upon restriction to $\mathcal{F}(U'')$. The same thing holds for presheaves of (abelian) groups.

Remark 7.1.2.4. The construction of the stalk is functorial in the input presheaf.

Example 7.1.2.5. The object \mathcal{F}_x is a generalization of the the notion of a “germ of a function” at a point. More precisely, let $X = \mathbb{R}^n$ and consider the sheaf \mathcal{F} of real valued continuous functions on X . The stalk of \mathcal{F} at x consists precisely of germs of continuous functions at x .

Proposition 7.1.2.6. *If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then φ is an epimorphism (resp. isomorphism) if and only if the induced map on stalks is an epimorphism (resp. isomorphism).*

Proof. By the Yoneda lemma, we reduce attention to set valued presheaves.

In that case, if φ is an epimorphism, then φ is surjective on stalks by unwinding the definitions.

Conversely, suppose $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is an epimorphism for each $x \in X$. Let $\psi_i : \mathcal{G} \rightarrow \mathcal{H}$, $i = 1, 2$ be two further morphisms of sheaves and assume $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We want to show that $\psi_1 = \psi_2$. Since taking stalks is functorial, it follows that

$$(\psi_1)_x \circ \varphi_x = (\psi_1 \circ \varphi)_x = (\psi_2 \circ \varphi)_x = (\psi_2)_x \circ \varphi_x.$$

By assumption, the induced maps on stalks are epimorphisms, and therefore $(\psi_1)_x = (\psi_2)_x$ for every $x \in X$.

Now, suppose $s \in \mathcal{G}(U)$ and consider $(\psi_1)_U(s)$ and $(\psi_2)_U(s)$. At each point $x \in U$, we can find a neighborhood V of x such that $(\psi_1)_U(s)$ and $(\psi_2)_U(s)$ coincide upon restriction to V . Doing this for every point $x \in U$, we obtain a cover of U on which the two sections agree after restriction and therefore, they must agree. Thus $\psi_1 = \psi_2$. \square

Exercise 7.1.2.7. Describe the stalks of a skyscraper sheaf.

Example 7.1.2.8. A surjective map of sheaves need not be surjective on sections. Here is a rather small example. Take $X = P, Q, R$ with open sets $X, \emptyset, \{P, R\}, \{Q, R\}$ and $\{R\}$. Consider first the constant sheaf \mathbb{Z}_X on X . Define another sheaf on X by taking the sum of the skyscraper sheaves $P_*\mathbb{Z} \oplus Q_*\mathbb{Z}$. Restriction defines a map $\mathbb{Z}_X \rightarrow P_*\mathbb{Z} \oplus Q_*\mathbb{Z}$, but this morphism is not surjective on sections. Indeed, the map on sections sends \mathbb{Z} to the diagonal in $\mathbb{Z} \oplus \mathbb{Z}$, which is evidently not surjective. However, this map is an epimorphism of sheaves.

Example 7.1.2.9. Suppose \mathcal{F} is a sheaf of sets on X . If $U \in X$ is an open set, then restriction of sections induces a map $\mathcal{F}(U) \rightarrow \mathcal{F}_x$ for any $x \in U$. Therefore, there is an induced map

$$\mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x.$$

The uniqueness statement in the sheaf condition guarantees that this map is injective.

7.1.3 Building sheaves from a basis

Suppose X is a topological space and \mathcal{B} is a basis of open sets for the topology on X . (Recall this means that we provide a set of open sets of X such that the elements cover X and given any two open sets in the base, the intersection can be covered by elements of the base). Often, it is convenient to specify some construction on the basis and show that it extends to all of X . We will do this now for sheaves on X . We abuse notation and write \mathcal{B} for the full subcategory of $\text{Op}(X)$ spanned by elements of the basis.

Example 7.1.3.1. The fundamental example to keep in mind (at least for our immediate purposes) is the case where $X = \text{Spec } R$ for R a commutative unital ring. In this case, we have a good handle on a basis for the Zariski topology on $\text{Spec } R$ (arising from principal open sets). The subset \mathcal{B} defines a subcategory of $\text{Op}(X)$ consisting of those open sets that are contained in \mathcal{B} .

Definition 7.1.3.2. If X is a topological space and \mathcal{B} is a basis for the topology on X , then a \mathbf{C} -valued presheaf on \mathcal{B} is a contravariant functor from \mathcal{B} to \mathbf{C} .

Remark 7.1.3.3. Every presheaf on X determines a presheaf on \mathcal{B} , but there is no reason that a presheaf on \mathcal{B} should determine a presheaf on X . Nevertheless, we will see now that sheaves on X are determined by their restriction to \mathcal{B} .

Definition 7.1.3.4. If X is a topological space and \mathcal{B} is a basis for the topology of X , then a presheaf of sets \mathcal{F} on \mathcal{B} is a sheaf on \mathcal{B} if it satisfies the following additional property: for any $U \in \mathcal{B}$ and any covering $U = \cup_{i \in I} U_i$ with $U_i \in \mathcal{B}$ and any coverings $U_i \cap U_j = \cup_{k \in I_{ij}} U_{ijk}$ with

$U_{ijk} \in \mathcal{B}$ the sheaf condition holds, i.e., for any collection of sections $s_i \in \mathcal{F}(U_i), i \in I$ such that for all $i, j \in I$ and for all $k \in I_{ij}, s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = s|_{U_i}$ for all i .

Remark 7.1.3.5. If \mathcal{F} is a sheaf of sets on X , then \mathcal{F} determines a sheaf on the basis by restriction to \mathcal{B} . We now show that, conversely, there exists a unique extension of a sheaf on a basis to a sheaf on X . Given a sheaf on a basis, we begin by explaining how to describe sections of the extended sheaf over an arbitrary open set.

Lemma 7.1.3.6. *Suppose X is a topological space and \mathcal{B} is a basis for the topology of X . Let \mathcal{F} be a sheaf of sets on \mathcal{B} . Given $U \in \mathcal{B}$, the map (see Example 7.1.2.9)*

$$\mathcal{F}(U) \longrightarrow \prod_{x \in U} \mathcal{F}_x$$

identifies $\mathcal{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property that for any $x \in U$ there exists a $V \in \mathcal{B}$ with $x \in V$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$ the equality $s_y = (V, \sigma) \in \mathcal{F}_y$.

Proof. As observed in Example 7.1.2.9 the map $\mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_x$ is injective. To establish surjectivity, take any element $(s_x)_{x \in U}$ on the right hand side satisfying the condition of the statement. We can find an open cover $\{U_i\}_{i \in I}$ of U with $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from a section $s_i \in \mathcal{F}(U_i)$. For every $y \in U_i \cap U_j$, the sections s_i and s_j agree in \mathcal{F}_y . Therefore, we can find an open set $y \in V_{ij} \in \mathcal{B}$ such that s_i and s_j restricted to this open set agree. The sheaf condition then guarantees that the sections s_i can be patched to obtain a section of $\mathcal{F}(U)$. \square

Using this observation, one may extend sheaves defined on a base of open sets for the topology on X to sheaves on all of X .

Theorem 7.1.3.7. *Suppose X is a topological space and \mathcal{B} is a base for the topology of X .*

1. *If \mathcal{F} is a sheaf of sets on \mathcal{B} , then there exists a unique sheaf \mathcal{F}^{ex} on X such that $\mathcal{F}^{ex}(U) = \mathcal{F}(U)$ for all $U \in \mathcal{B}$ compatibly with restriction mappings.*
2. *The assignment $\mathcal{F} \rightarrow \mathcal{F}^{ex}$ provides a quasi-inverse to the restriction functor from sheaves on X to sheaves on \mathcal{B} , i.e., restriction determines an equivalence between the category of sheaves on X and the category of sheaves on \mathcal{B} .*

Proof. For an open subset U of X , define $\mathcal{F}^{ex}(U)$ to be the subset of $\prod_{x \in U} \mathcal{F}_x$ consisting of sections such that for any $x \in U$, there exists a $V \in \mathcal{B}$ containing x and a section $\sigma \in \mathcal{F}(V)$ such that for all $y \in V$, $s_y = (V, \sigma)$ in \mathcal{F}_y . Restriction equips this assignment with the structure of a presheaf of sets on X . By Lemma 7.1.3.6, we conclude that $\mathcal{F}^{ex}(U)$ coincides with $\mathcal{F}(U)$ for any $U \in \mathcal{B}$.

To see that \mathcal{F}^{ex} is a sheaf on X is a direct check. Suppose U is an open set and $\{U_i\}_{i \in I}$ is an open cover of U . It is immediate from the definitions that $\mathcal{F}^{ex}(U) \rightarrow \prod_{i \in I} \mathcal{F}^{ex}(U_i)$ is injective. Suppose we are given sections $s_i \in \mathcal{F}^{ex}(U_i)$. By definition, each s_i consists of $(s_i)_x, x \in U_i$. If these sections agree upon restriction to $\mathcal{F}^{ex}(U_i \cap U_j)$, we claim they patch together as required. We leave this as an exercise. \square

Example 7.1.3.8 (Sheaves on affine schemes). Suppose S is a commutative unital ring. Consider the functor on rings defined by $\text{Hom}(S, -)$. If $X = \text{Spec } R$ for R a commutative unital ring, then if $\text{Hom}(S, -)$ determines a sheaf on a basis for $\text{Spec } R$, then $\text{Hom}(S, -)$ extends uniquely to a sheaf on X . For example, if $S = \mathbb{Z}[t]$, then you can check that $\text{Hom}(k[t], -)$ determines a sheaf on a basis for X . The resulting sheaf on X will be denoted $\mathcal{O}_{\text{Spec } R}$. If $S = \mathbb{Z}[t, t^{-1}]$, the resulting sheaf will be denoted $\mathcal{O}_{\text{Spec } R}^\times$.

Example 7.1.3.9. Take $X = \mathbb{C}^n$. We know how to speak about holomorphic functions on X . An open disc in \mathbb{C} , centered at x is an open subset of the form $D_\epsilon(x)$ consisting of all points of distance at most ϵ from x . A polydisc in \mathbb{C}^n centered a point $x = (x_1, \dots, x_n)$ is a subset isomorphic to $D_{\epsilon_1}(x_1) \times \dots \times D_{\epsilon_n}(x_n)$. Polydiscs provide a basis for the topology on X . Moreover, it makes sense to speak of holomorphic functions on a polydisc. Using the procedure above, one can define a sheaf $\mathcal{O}_{\mathbb{C}^n}^{\text{hol}}$ of holomorphic functions on X . More generally, the same procedure works for any complex manifold X to produce a sheaf $\mathcal{O}_X^{\text{hol}}$ of holomorphic functions on X .

7.2 Lecture 25: Schemes

We just observed that to a commutative ring R , we may assign the sheaf of commutative rings $\mathcal{O}_{\text{Spec } R}$. We now define general schemes by gluing together affine schemes. To do this precisely, we need a category to house topological spaces equipped with sheaves of functions, we begin by defining the notion of ringed and locally ringed spaces which is invented precisely for this purpose. Just as with manifolds, the general notion of scheme can be viewed as a topological space equipped with a suitable sheaf of algebraic functions.

Here is a simple example that shows that in performing gluing constructions we cannot expect to stay within the category of affine schemes. In topology, one defines S^2 as glued from two copies of \mathbb{C} over the intersection, \mathbb{C}^\times . The maps defining the gluing are algebraic functions: if z is a coordinate on the first copy of \mathbb{C} and z^{-1} is a coordinate on the other, then the gluing map is defined on the intersection by $z \mapsto z^{-1}$. We can try to perform this construction in the category of rings. Namely, we want to obtain the fiber product of rings $k[z]$ and $k[z^{-1}]$ over $k[z, z^{-1}]$. However, the collection of functions (f_1, f_2) such that $f_1(z) = f_2(z^{-1})$ consists only of elements of k . Thus, the “gluing” in the category of affine schemes is $\text{Spec } k$, which is evidently not what we have in mind when we think of \mathbb{P}^1 . The “problem” is that we are only thinking about functions that are globally defined and in complex analysis one learns that a polynomial function on the Riemann sphere is constant. Thus, we must expand our view beyond the world of rings to obtain a reasonable notion of quotient.

7.2.1 Ringed and locally ringed spaces

Definition 7.2.1.1. A ringed space is a pair (X, \mathcal{O}_X) consisting of a topological space and a sheaf of commutative rings on X .

Example 7.2.1.2. The examples to keep in mind are: M a topological manifold and \mathcal{C}_M the sheaf of (say, real-valued) continuous functions on M , M a smooth manifold and \mathcal{C}_M^∞ the sheaf of (say, real-valued) smooth functions on M .

A map of manifolds induces a corresponding pullback of functions; this corresponds to a suitable map of sheaves, albeit on different topological spaces. We now introduce a notion to compare sheaves on different topological spaces. The following notion is a formalization of what happens to functions under pullback along a morphism of smooth or topological manifolds.

Definition 7.2.1.3. Suppose $f : X \rightarrow Y$ is a continuous map of topological spaces. If \mathcal{F} is a sheaf on X and \mathcal{G} is a sheaf on Y , then an f -map $\xi : \mathcal{G} \rightarrow \mathcal{F}$ is a collection of maps $\xi_V : \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V))$ indexed by open sets $V \subset Y$ that commutes with restriction in a suitable sense.

Definition 7.2.1.4. If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are ringed spaces, a morphism of ringed spaces is a continuous map $f : X \rightarrow Y$ and an f -map of sheaves of rings $\mathcal{O}_Y \rightarrow \mathcal{O}_X$.

In all the geometric situations we consider (e.g., topological and smooth manifolds, schemes), the sheaves of function rings on our topological spaces have stalks that are *local rings*. E.g., the stalk of \mathcal{C}_M at a point $x \in M$ is the ring of germs of continuous functions at x ; this ring is a local ring with maximal ideal those continuous functions vanishing at x . Moreover, a map of smooth manifolds sends points to points and therefore induces corresponding maps of stalks (by functoriality of stalks); the resulting maps of stalks are local homomorphisms of local rings. Generalizing this observation, one makes the following definition.

Definition 7.2.1.5. A ringed space (X, \mathcal{O}_X) is *locally ringed* if for each point $x \in X$, the stalks $\mathcal{O}_{X,x}$ are local rings. A morphism $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that, for any $x \in X$, the map $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local homomorphism of local rings.

Example 7.2.1.6. If R is a commutative unital ring, then $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ is a locally ringed space.

7.2.2 Schemes

Earlier, we defined the category of affine schemes over a base ring k to be the opposite of the category of commutative k -algebras. Above, we showed how to associate a locally ringed space with any commutative unital k -algebra. We now show that this assignment identifies the category of affine k -schemes with its image.

Proposition 7.2.2.1. *Sending a commutative unital k -algebra R to the locally ringed space $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ extends to a fully-faithful functor from the category of affine schemes to the category of locally ringed spaces.*

Proof. See [Sta15, Lemma 25.6.4]. □

Definition 7.2.2.2. We write Aff for the full subcategory of locally ringed spaces spanned by affine schemes. If k is a commutative unital ring, we write Aff_k for comma category of Aff consisting of affine schemes equipped with a morphism to $(\text{Spec } k, \mathcal{O}_{\text{Spec } k})$ (in particular, $\text{Aff} = \text{Aff}_{\mathbb{Z}}$).

Given our identification of affine schemes above, we may now give the general definition of a scheme: a scheme is a locally ringed space obtained by gluing together ringed spaces of the form $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$ for a commutative unital ring R . We formalize this in two steps.

Definition 7.2.2.3. A scheme is a locally ringed space (X, \mathcal{O}_X) that is locally isomorphic to an affine scheme, i.e., given any point $x \in X$ there is an open neighborhood U of $x \in X$ such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. We write Sch for the full subcategory of the category of locally ringed spaces consisting of schemes (i.e., a morphism of schemes is morphism of locally ringed spaces). If S is a base-scheme, we write Sch_S for the full subcategory of Sch consisting of schemes admitting a morphism to S .

Example 7.2.2.4. If (X, \mathcal{O}_X) is a scheme, and $U \subset X$ is an open subset of the topological space X , then U carries the structure of scheme by defining $\mathcal{O}_U = \mathcal{O}_X|_U$. We refer to this as the induced open subscheme structure on U . A morphism of schemes $f : X \rightarrow Y$ is called an *open immersion* if it induces an isomorphism of X with an open subscheme of Y .

Just as in differential geometry, we may construct schemes by gluing.

Example 7.2.2.5. Suppose (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are schemes and $U \subset X$ and $V \subset Y$ are open subsets. The subset U inherits the structure of a locally ringed space by setting $\mathcal{O}_U = \mathcal{O}_X|_U$. Suppose we are given an isomorphism of locally ringed spaces $\varphi : (U, \mathcal{O}_U) \xrightarrow{\sim} (V, \mathcal{O}_V)$. In that case, we define a new scheme with underlying topological space $W := X \amalg Y / (x \sim \varphi(x))$ (equipped with the quotient topology). Note that there are continuous maps $i_X : X \rightarrow W$ and $i_Y : Y \rightarrow W$ by the definition of W . Using these maps, we can define a structure sheaf \mathcal{O}_W by gluing: for $Z \subset W$, define $\mathcal{O}_W(Z)$ to consist of pairs (s_1, s_2) such that $s_1 \in \mathcal{O}_X(i_X^{-1}(Z))$ and $s_2 \in \mathcal{O}_Y(i_Y^{-1}(Z))$ such that $\varphi(s_1|_{i_X^{-1}(Z) \cap U}) = s_2|_{i_Y^{-1}(Z) \cap V}$. The pair (W, \mathcal{O}_W) is still a scheme because every point in W has a neighborhood isomorphic to an affine scheme.

Example 7.2.2.6. In an analogous fashion, we may glue morphisms of schemes.

Example 7.2.2.7. The category Sch has a terminal object, namely $(\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$. Indeed, this is clear if X is an affine scheme and we may glue morphisms to obtain the morphism for a general scheme from this one.

Exercise 7.2.2.8. Show that the category of schemes has finite products.

1. Show that the tensor product of rings equips the the category of affine schemes with a product.
2. Assuming X and Y are schemes, show that $X \times Y$ can be equipped with a natural scheme structure by gluing (first, assume Y is affine, and inductively use the gluing construction for a suitable open cover of X by affine schemes, then use gluing again to obtain a scheme structure on $X \times Y$).
3. Show that the scheme structure you obtained on $X \times Y$ in the previous part makes it a product.

Remark 7.2.2.9. At this point, we could spend a long time developing what one might call the “general topology” of schemes: we can define various properties of schemes and study their permanence properties under morphisms. However, our main goal is to get back to the

Example 7.2.2.10 (Punctured affine space). Suppose n is an integer ≥ 1 . We define a scheme $\mathbb{A}^n \setminus 0$ inductively as follows. For $n = 1$, we set $\mathbb{A}^1 \setminus 0$ to be $\text{Spec } \mathbb{Z}[t, t^{-1}]$, $\mathcal{O}_{\text{Spec } \mathbb{Z}[t, t^{-1}]}$. For $n = 2$, we glue the affine schemes $\mathbb{A}^1 \times \mathbb{A}^1 \setminus 0$ and $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$ along $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \setminus 0$ via the identity map. More generally we may define $\mathbb{A}^2 \setminus 0 \times \mathbb{A}^m$ for any $m \geq 0$ by gluing $\mathbb{A}^1 \times \mathbb{A}^1 \setminus 0 \times \mathbb{A}^m$ and $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \times \mathbb{A}^m$ along $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \setminus 0$. Inductively we define $\mathbb{A}^n \setminus 0$ by gluing $\mathbb{A}^{n-1} \setminus 0 \times \mathbb{A}^1$ and $\mathbb{A}^{n-1} \times \mathbb{A}^1 \setminus 0$ over $\mathbb{A}^{n-1} \setminus 0 \times \mathbb{A}^1 \setminus 0$.

7.2.3 Projective space and graded rings

Example 7.2.3.1 (Projective space). We can define projective space by gluing as well by mimicking the construction in topology: to obtain \mathbb{P}^1 simply glue two copies of the affine line over \mathbf{G}_m by means of the isomorphism $z \mapsto z^{-1}$. More precisely, consider the affine scheme associated with $\mathbb{Z}[t]$ and with $\mathbb{Z}[t^{-1}]$. Consider the isomorphism of affine schemes $\mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z}[t, t^{-1}]$ given by $t \mapsto t^{-1}$. More generally, we can define \mathbb{P}^n inductively by gluing $n + 1$ copies of \mathbb{A}^n as follows.

Example 7.2.3.2. A slight modification of the construction of the projective line produces an example that strays from our geometric intuition: glue two copies of the affine line over the identity map $\mathbb{A}^1 \setminus 0 \rightarrow \mathbb{A}^1 \setminus 0$. The result is a scheme that one usually draws as an affine line with “doubled” origin. From one point of view, this kind of example is pathological (and in gluing manifolds, one usually eliminates this kind of example!), but from another point of view it gives us flexibility in the constructions we can make.

Another useful way to think about the construction above is as follows. Recall that the classical way to construct projective space over a field k is via lines through the origin in an $n + 1$ -dimensional k -vector space V . If we fix a basis of the line, then we simply get a non-zero vector in V . Two non-zero vectors determine the same line if one can be obtained from the other up to scaling. Thus, alternatively, we can think in terms of elements of $V \setminus 0$ invariant under scaling: the scaling action determines an action on functions via pullback. If x_i is a coordinate function on V , $i = 0, \dots, n$, then $x_i(v)$ is the i -th coordinate of the vector v in terms of the standard basis e_0, \dots, e_n . Thus, $x_i(\lambda v) = \lambda x_i(v)$. This action induces a grading on $k[x_0, \dots, x_n]$.

If f is a homogeneous degree d polynomial, then f determines a graded ideal in the graded ring $k[x_0, \dots, x_n]$. By homogeneity, the vanishing locus of f determines a scaling invariant subset of V and therefore passes to a corresponding subset of projective space. More generally, if we have an ideal defined by homogeneous polynomials, then its vanishing locus is again a subset of V that is invariant under scaling and therefore passes to a subset of projective space.

7.3 Lecture 26: Schemes continued, and properties of sheaves

7.3.1 The Proj construction

Just as with Spec , we now associate a topological space with a graded ring. If S is a graded ring, we write S_+ for the subset of positively graded elements. For the most part, when we write “graded ring” we will mean positively graded. At a few points, we will need to consider \mathbb{Z} -graded rings, in which case we will make that clear by explicitly saying \mathbb{Z} -graded ring.

Definition 7.3.1.1. If S is a graded ring, define $\text{Proj } S$ to be the set of homogeneous prime ideals \mathfrak{p} of S such that $S_+ \not\subset \mathfrak{p}$. We view $\text{Proj } S \subset \text{Spec } S$ and equip it with the structure of a topological space via the induced topology.

Remark 7.3.1.2. The assignment $S \rightarrow \text{Proj } S$ is not as “well-behaved” as the assignment $S \rightarrow \text{Spec } S$ in a number of ways. First, Proj does not yield a functor from graded rings. Indeed, if we think classically, and consider a vector space map $V \rightarrow W$, then there is no induced map $\mathbb{P}(V) \rightarrow \mathbb{P}(W)$ in general. Indeed if $\varphi : V \rightarrow W$ is a surjective map, then any line L contained in

the kernel of φ is sent to $0 \subset W$, which does not correspond to a point in $\mathbb{P}(W)$. In ring-theoretic terms, given a graded ring map $\varphi : A \rightarrow B$, the inverse image $\varphi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may still contain A_+ . On the other hand, if $V \rightarrow W$ is an injective ring map, then there is an induced ring map $\mathbb{P}(V) \rightarrow \mathbb{P}(W)$. See Remark 7.3.1.9 below for more details.

In another direction, while $\text{Spec } R$ is always a quasi-compact topological space, there are graded rings S for which $\text{Proj } S$ is *not* a quasi-compact topological space.

If S is concentrated in degree 0, then $\text{Proj } S$ coincides with $\text{Spec } S$. If S is a graded ring, write S_0 for the subring of elements of degree 0. In this case, the inclusion map induces a continuous map $\text{Proj } S \rightarrow \text{Spec } S_0$. In some instances, this map is not very interesting (e.g., if $S = \mathbb{Z}[x_0, \dots, x_n]$ as above).

However, in projective space as described above, if we look at the complement of the vanishing locus of a homogeneous polynomial of positive degree, then we obtain a set that has many functions. To make this precise, let $S = \bigoplus_{d \geq 0} S_d$ be a positively graded ring. If $f \in S_d$ is a homogeneous degree d element, then set $S_{(f)}$ to be the subring of the localization S_f consisting of elements of the form $\frac{r}{f^n}$ with r homogeneous and where the degree of r is nd . Likewise, if M is a graded module, we define an $S_{(f)}$ -module $M_{(f)}$ as the submodule of M_f consisting of elements of the form $\frac{x}{f^n}$ with x homogeneous of degree nd .

Example 7.3.1.3. Consider the complement of the vanishing locus of x_i ; this corresponds to looking at $\mathbb{Z}[x_0, \dots, x_n]_{(x_i)}$ as just described. The elements $\frac{x_j}{x_i}, j \neq i$ are degree 0. Geometrically, the complement of $x_i = 0$ is an affine space with precisely the coordinates described via projection.

The following result generalizes this observation to the situation where we invert a homogeneous element f of positive degree.

Lemma 7.3.1.4. *If S is a \mathbb{Z} -graded ring containing a homogeneous invertible element of positive degree, then the set $G \subset \text{Spec } S$ of \mathbb{Z} -graded primes of S (with the induced topology) maps homeomorphically to $\text{Spec } S_0$.*

Proof. We show that the map is a bijection by constructing an inverse: given a prime \mathfrak{p}_0 of S_0 , we want to associate with it a \mathbb{Z} -graded prime of S . By assumption, we can find an invertible $f \in S_d, d > 0$. If \mathfrak{p}_0 is a prime of S_0 , then $\mathfrak{p}_0 S$ is a \mathbb{Z} -graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$. If $ab \in \mathfrak{p}_0 S$ with a, b homogeneous, then $\frac{a^d b^d}{f^{deg(a)deg(b)}} \in \mathfrak{p}_0$. Therefore, either $\frac{a^d}{f^{deg a}} \in \mathfrak{p}_0$ or $\frac{b^d}{f^{deg b}} \in \mathfrak{p}_0$, i.e., either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. Therefore, $\sqrt{\mathfrak{p}_0 S}$ is a \mathbb{Z} -graded prime ideal of S whose intersection with S_0 is \mathfrak{p}_0 . \square

Given this observation, we now define principal open sets in $\text{Proj } S$.

Definition 7.3.1.5. If $f \in S$ is a homogeneous element of degree > 0 , define $D_+(f) = \{\mathfrak{p} \in \text{Proj } S \mid f \notin \mathfrak{p}\}$. If $I \subset S$ is a homogeneous ideal, define $V_+(I) = \{\mathfrak{p} \in \text{Proj } S \mid I \subset \mathfrak{p}\}$. More generally, if E is any set of homogeneous elements, then we define $V_+(E) = \{\mathfrak{p} \in \text{Proj } S \mid E \subset \mathfrak{p}\}$.

Proposition 7.3.1.6. *Suppose $S = \bigoplus_{d \geq 0} S_d$ is a graded ring and $f \in S$ is a homogeneous element of positive degree.*

1. *The sets $D_+(f)$ are open subsets of $\text{Proj } S$.*
2. *The equality $D_+(ff') = D_+(f) \cap D_+(f')$ holds.*

3. If $g = g_0 + \cdots + g_m$ be an element of S with $g_i \in S_i$, then

$$D(g_0) \cap \text{Proj } S = \cup_{f \in S_d, d \geq 1}$$

4. The sets $D_+(f)$ form a basis for the topology on $\text{Proj } S$.

5. The localized ring S_f has a natural \mathbb{Z} -grading.

The ring maps $S \rightarrow S_f \leftarrow S_{(f)}$ induce homeomorphisms

$$D_+(f) \leftarrow \{\mathbb{Z} - \text{graded primes of } S_f\} \rightarrow \text{Spec}(S_{(f)}).$$

6. The sets $V_+(I)$ for I a homogeneous ideal are closed subsets of $\text{Proj } S$ and any closed subset of $\text{Proj } S$ is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.

We can define a structure sheaf on $\text{Proj } S$ using the sets $D_+(f)$ by assigning to $D_+(f)$ the ring $S_{(f)}$.

Proposition 7.3.1.7. Suppose S is a graded ring.

1. The assignment $D_+(f) \mapsto S_{(f)}$ is a sheaf of rings on the basis $D_+(f)$ of $\text{Proj } S$ and therefore extends uniquely to a sheaf of rings $\mathcal{O}_{\text{Proj } S}$ on $\text{Proj } S$.

2. The ring space $(\text{Proj } S, \mathcal{O}_{\text{Proj } S})$ is a scheme.

Definition 7.3.1.8. We define $\mathbb{P}_{\mathbb{Z}}^n = \text{Proj } \mathbb{Z}[x_0, \dots, x_n]$, where x_i has degree $+1$.

Remark 7.3.1.9. If $\varphi : A \rightarrow B$ is a homomorphism of graded rings, then we can define an open subscheme of $\text{Proj } B$ that maps to $\text{Proj } A$ as follows: take $U(\varphi)$ to be the union of $D(\varphi(f))$ as f ranges over the homogeneous elements of A_+ . There is a canonical map of schemes $U(\varphi) \rightarrow \text{Proj } A$.

7.3.2 Properties of morphisms

A manifold is typically a paracompact Hausdorff topological space that is locally Euclidean. So far, our schemes are simply spaces that are “locally affine” and we have introduced neither finiteness nor separation properties. Here are two natural finiteness properties.

Definition 7.3.2.1. A scheme X is *quasi-compact* if the underlying topological space of X is quasi-compact, i.e., every open cover has a finite subcover. A morphism $f : X \rightarrow S$ of schemes is quasi-compact if the pre-image of every quasi-compact open in S is quasi-compact in X .

Remark 7.3.2.2. We have already seen that $\text{Spec } R$ is always a quasi-compact topological space. We introduced the notion of finite-type and finitely presented algebras over a base commutative unital ring. We now introduce versions of these for morphisms of schemes.

Definition 7.3.2.3. A morphism of schemes $f : X \rightarrow S$ has finite type at $x \in X$, if there exists an affine open neighborhood $\text{Spec } A$ of x and an affine open neighborhood $\text{Spec } R$ of $f(x)$ such that f maps A a finite type R -algebra. A morphism of schemes has locally finite type if it has finite type at every point $x \in X$ and has finite type if f has locally finite type and f is quasi-compact.

Definition 7.3.2.4. A morphism of schemes $f : X \rightarrow S$ is finitely presented at $x \in X$, if there exists an affine open neighborhood $\text{Spec } A$ of x and an affine open neighborhood $\text{Spec } R$ of $f(x)$ such that f maps $\text{Spec } A$ to $\text{Spec } R$ and makes A into a finitely presented R -algebra. A morphism of schemes is locally of finite presentation, if it is finitely presented at $x \in X$ for every point $x \in X$.

Definition 7.3.2.5. A morphism of schemes $f : X \rightarrow S$ is smooth (resp. étale) at $x \in X$, if there exists an affine open neighborhood $\text{Spec } A$ of x and an affine open neighborhood $\text{Spec } R$ of $f(x)$ such that f maps $\text{Spec } A$ to $\text{Spec } R$ and makes A into a smooth (resp. étale) R -algebra. A morphism of schemes is smooth if it is smooth at every $x \in X$.

Finally, we can define a class of objects that will be of fundamental importance in \mathbb{A}^1 -homotopy theory.

Definition 7.3.2.6. If S is a fixed scheme, we write Sm_S for the category of schemes that are smooth and have finite type over S .

7.3.3 Coherent sheaves and vector bundles on schemes

We have been thinking about projective modules as algebro-geometric analogs of vector bundles on affine schemes. However, we have largely avoided the idea that a vector bundle is a geometric object itself. If R is a ring, and P is a finitely generated projective R -module, there is a straightforward way to build a geometric object out of P : one defines an R -algebra $\text{Sym} P^\vee$ and then there is an induced map

$$\text{Spec } \text{Sym} P^\vee \longrightarrow \text{Spec } R.$$

The choice of taking the R -module dual before applying the spectrum yields a covariant functor from the category of projective R -modules to the category of affine R -schemes.

Assume P is a rank n projective R -module. Since projective modules are locally free, if we choose elements f_1, \dots, f_r that generate the unit ideal and such that $P \otimes_R R[\frac{1}{f_j}]$ is free, then we identify $\text{Sym} P_f^\vee$ with $\mathbb{A}_{R[\frac{1}{f}]}^n$ where n is the rank of p . Thus, we obtain an open cover of $\text{Spec } R$ on which $\text{Spec } \text{Sym} P^\vee$ restricted to each of these open sets is isomorphic to an affine space. The important point here is that the gluing data on intersections of open sets is given by elements of $GL_n(P_{f_i f_j})$. Thus, the morphism $\text{Spec } \text{Sym} P^\vee \longrightarrow \text{Spec } R$ is locally trivial. With this in mind, we may make two definitions of *vector bundle* on a general scheme.

We begin by generalizing the definition of vector bundles as locally free modules over a ring.

Definition 7.3.3.1. If (X, \mathcal{O}_X) is a scheme, then by a *vector bundle* on X we will mean a sheaf \mathcal{F} of \mathcal{O}_X -modules that is *locally free*, i.e., every point $x \in X$ has an open affine neighborhood $U_x = \text{Spec } R$ such that $\mathcal{F}|_{U_x}$ is a free R -module.

We may also make a definition of *geometric vector bundle* as above.

7.4 Lecture 27: Sheaves as spaces

Last time we define schemes in general, and we talked about smooth schemes over a base. Unfortunately, the category of smooth schemes, much like the category of CW complexes, is “too small” to

perform various categorical constructions. For example, the quotient of a smooth scheme by a subspace *need not* be a smooth scheme (a simple example of this phenomenon is the map $\mathbb{A}^1 \rightarrow C$ where C is the curve $y^2 = x^3 + x$; the normalization is \mathbb{A}^1 and the map is a quotient map).

The Yoneda embedding gives an embedding from Sm_S to the category of set-valued contravariant functors on Sm_S . If X is a smooth S -scheme, then such a contravariant functor restricts to a functor on X itself. From this point of view we can think of set-valued contravariant functors on Sm_S as giving, simultaneously, presheaves on every smooth scheme. For this reason, we will write $\text{PShv}(\text{Sm}_S)$ for the category of set-valued contravariant functors on Sm_S and refer to such objects as presheaves on Sm_S . The category $\text{PShv}(\text{Sm}_S)$ has all limits and colimits: these can be computed section-wise.

7.4.1 Grothendieck topologies

Example 7.4.1.1. When we talked about sheaves on a topological space X , we began with contravariant functors on the category $\text{Op}(X)$. In order to describe the notion of sheaf, we simply had to introduce a notion of “cover” of an open subset.

Following Grothendieck, we can axiomatize the notion of cover as follows.

Definition 7.4.1.2. A *site* is a category \mathbf{C} equipped with a collection of distinguished morphisms $\text{Cov}(\mathbf{C})$ with fixed target $\{U_i \rightarrow U\}_{i \in I}$ called *coverings* of \mathbf{C} satisfying the following axioms:

1. (isomorphisms) If $V \rightarrow U$ is an isomorphism in \mathbf{C} , then $V \rightarrow U \in \text{Cov}(\mathbf{C})$;
2. (refinements) if $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathbf{C})$ and for each $i \in I$, we are given $\{V_{ij} \rightarrow U_i\}_{j \in J_i} \in \text{Cov}(\mathbf{C})$, then $\{V_{ij} \rightarrow U\} \in \text{Cov}(\mathbf{C})$;
3. (fiber products) If $\{U_i \rightarrow U\}_{i \in I} \in \text{Cov}(\mathbf{C})$ and $V \rightarrow U$ is a morphism in \mathbf{C} , then for all i the fiber product $U_i \times_U V$ exists in \mathbf{C} , and $\{U_i \times_U V \rightarrow V\}_{i \in I} \in \text{Cov}(\mathbf{C})$.

Definition 7.4.1.3. Let U be a scheme. A *Zariski covering* of U is a family of morphisms $\{j_i : U_i \rightarrow U\}$ such that each j_i is an open immersion and the j_i are jointly surjective. If X is a scheme, we write X_{Zar} for the category whose objects are open immersions $U \rightarrow X$ and where coverings are Zariski coverings.

Lemma 7.4.1.4. *The category X_{Zar} equipped with the Zariski coverings is a site (called the small Zariski site of X).*

Definition 7.4.1.5. Let U be a scheme. An *étale covering* of U is a family of morphisms of schemes $\{\varphi_i : U_i \rightarrow U\}$ such that each φ_i is étale and such that φ_i are jointly surjective. If X is a scheme, we write $X_{\text{ét}}$ for the category whose objects are étale morphisms $U \rightarrow X$ and where coverings are étale coverings.

Lemma 7.4.1.6. *The category $X_{\text{ét}}$ equipped with the étale coverings is a site (called the small étale site of X).*

Now, given the notion of site, we can speak of the notion of presheaves and sheaves.

Definition 7.4.1.7. If \mathbf{C} is a site, then a presheaf (of sets) on \mathbf{C} is a contravariant functor \mathcal{F} from \mathbf{C} to Set . A presheaf of sets \mathcal{F} on \mathbf{C} is a sheaf if for every object $U \in \mathbf{C}$ and every covering

$\{U_i \rightarrow U\}$ in \mathbf{C} , the diagram of sets

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer. We write $\text{PShv}(\mathbf{C})$ for the category of presheaves on \mathbf{C} and $\text{Shv}(\mathbf{C})$ for the full-subcategory of $\text{PShv}(\mathbf{C})$ consisting of sheaves.

Remark 7.4.1.8. As before, if \mathbf{D} is a category, we can speak of \mathbf{D} -valued presheaves: these are contravariant functors \mathcal{F} from \mathbf{C} to \mathbf{D} . If $A \in \mathbf{D}$ is an object, then we can consider the Set-valued functor $\mathcal{F}_A(U) := \text{Hom}_{\mathbf{D}}(A, \mathcal{F}(U))$; this functor is evidently a presheaf on \mathbf{C} . We will say that a \mathbf{D} -valued presheaf on \mathbf{C} is a sheaf if for every object $A \in \mathbf{D}$, the presheaf of sets \mathcal{F}_A is a sheaf of sets.

Remark 7.4.1.9. We will have to work harder to speak of stalks of sheaves defined on sites: the problem is that there is no a priori definition of a point. After we have discussed a bit more category theory, we will provide a definition of a “point” of a site. Unfortunately, it will not always be the case that isomorphisms of sheaves can be detected using “points.” Nevertheless, in the geometric situations we considered above, there will be perfect analogs of the notions of stalk of a sheaf etc.

7.4.2 Sheafification

If \mathbf{C} is a site, by construction there is a forgetful functor $\text{Shv}(\mathbf{C}) \rightarrow \text{PShv}(\mathbf{C})$. Sheafification is a universal procedure for assigning a sheaf with a given presheaf. We give two constructions of this functor.

A concrete construction

First, we give a concrete construction of the sheafification in the special case where $\mathbf{C} = \text{Op}(X)$ for X a topological space. Given the technology developed so far, it is unclear how one might extend this construction to more general sites.

Construction 7.4.2.1. Suppose \mathcal{F} is a presheaf of sets on a topological space X . Define \mathcal{F}^\sharp by the formula

$$\mathcal{F}^\sharp(U) = \{(s_u) \in \prod_{u \in U} \mathcal{F}_u \mid (*)\}$$

where $(*)$ is the condition: for every $u \in U$, there exists an open neighborhood $u \in V \subset U$ and a section $\sigma \in \mathcal{F}(V)$ such that for all $v \in V$, $s_v = (V, \sigma) \in \mathcal{F}_v$.

Remark 7.4.2.2. By construction, if $V \subset U \subset X$, then the projection maps $\prod_{u \in U} \mathcal{F}_u \rightarrow \prod_{v \in V} \mathcal{F}_v$ send $\mathcal{F}^\sharp(U) \rightarrow \mathcal{F}^\sharp(V)$, i.e., \mathcal{F}^\sharp is necessarily a presheaf.

Lemma 7.4.2.3. *If \mathcal{F} is a sheaf, then the map $\mathcal{F} \rightarrow \prod_{x \in X} \mathcal{F}_x$ arising in Example 7.1.2.9 has image consisting of sections satisfying the condition $(*)$ of Construction 7.4.2.1, i.e., the map $\mathcal{F} \rightarrow \prod_{x \in X} \mathcal{F}_x$ induces an isomorphism $\mathcal{F} \rightarrow \mathcal{F}^\sharp$.*

Theorem 7.4.2.4. *Suppose X is a topological space and \mathcal{F} is a presheaf of sets on X . The following statements hold.*

1. The presheaf \mathcal{F}^\sharp is a sheaf.
2. Moreover, for any $x \in X$, $\mathcal{F}_x = \mathcal{F}_x^\sharp$.
3. For any sheaf \mathcal{G} on X and any morphism of presheaves $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, there is a unique morphism of sheaves $\mathcal{F}^\sharp \rightarrow \mathcal{G}$ factoring φ .

Proof. Exercise. □

Remark 7.4.2.5. The theorem above shows that sheafification is a left adjoint to the forgetful functor from sheaves to presheaves. As such, it preserves colimits.

Sheafification for sites in general

Now, we give an alternative construction of sheafification that will be more useful for understanding categorical properties of presheaves/sheaves. If \mathcal{F} is any presheaf of sets on a site \mathbf{C} and suppose $\{U_i \rightarrow U\}$ is a covering family. We always have the sequence:

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1}).$$

In particular, there is always a map

$$\mathcal{F}(U) \longrightarrow \text{eq}\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i_0, i_1 \in I \times I} \mathcal{F}(U_{i_0} \times_U U_{i_1})\right),$$

which in general need not be either injective or surjective. For notational simplicity, we will write \mathcal{U} for the covering family $\{U_i\}_{i \in I}$ and $H^0(\mathcal{U}, \mathcal{F})$ for the equalizer, i.e., we will write the map above as $\mathcal{F}(U) \rightarrow H^0(\mathcal{U}, \mathcal{F})$. In these terms, a presheaf on \mathbf{C} is a sheaf if and only if for every covering \mathcal{U} of any object U , the map $\mathcal{F}(U) \rightarrow H^0(\mathcal{U}, \mathcal{F})$ is an isomorphism.

If we take another covering family $\mathcal{V} = \{V_j\}_{j \in J}$, then there one defines $\mathcal{F}(U) \rightarrow H^0(\mathcal{V}, \mathcal{F})$ as well. However, there is no map between $H^0(\mathcal{U}, \mathcal{F})$ and $H^0(\mathcal{V}, \mathcal{F})$ in general. In order to compare these two, we define a new covering family $\mathcal{W} = \{U_i \cap V_j\}_{i, j \in I \times J}$. Then restriction defines maps of the form:

$$H^0(\mathcal{U}, \mathcal{F}) \longrightarrow H^0(\mathcal{W}, \mathcal{F}) \longleftarrow H^0(\mathcal{V}, \mathcal{F}).$$

The basic idea of sheafification is then to replace $\mathcal{F}(U)$ by $H^0(\mathcal{U}, \mathcal{F})$ as we refine further and further.

Given an object U ; the set of covers of U forms a category $\text{Cov}(U)$ in a natural way: the objects of $\text{Cov}(U)$ are simply covers of U and there is a non-identity morphism for each refinement. Note that $\text{Cov}(U)$ is always non-empty (since the identity map $U \rightarrow U$ is a covering). Furthermore, given two objects \mathcal{U} and \mathcal{U}' of $\text{Cov}(U)$, we can find \mathcal{W} mapping to both \mathcal{U} and \mathcal{U}' . We now observe that $\text{Cov}(U)$ is *cofiltered* in the sense of Definition A.1.3.2.

Lemma 7.4.2.6. *If \mathbf{C} is a site, and $U \in \mathbf{C}$ is an object, then the category $\text{Cov}(U)$ is cofiltered.*

Proof. We have already established the first two properties of the definition. It remains to show that given two refinements of a given cover \mathcal{U} , i.e., two morphisms $\mathcal{V} \rightarrow \mathcal{U}$ we can find a common refinement $\mathcal{W} \rightarrow \mathcal{V}$ such that precomposing with this refinement, the two maps agree. To see this, let $\mathcal{V} = \{V_j \rightarrow U\}$ and $\mathcal{U} = \{U_i \rightarrow U\}$. Then, a morphism of covers corresponds to a function $\alpha : J \rightarrow I$ and maps $f_j : V_j \rightarrow U_{\alpha(j)}$. Thus, suppose we are given two such maps α, α' and f_j and f'_j . By taking suitable fiber products, we obtain the required refining map. □

Note that assigning to $\mathcal{U} \in \text{Cov}(U)$ the set $H^0(\mathcal{U}, \mathcal{F})$ is *contravariant*, i.e., a presheaf \mathcal{F} , yields a functor $\text{Cov}(U)^{op} \rightarrow \text{Set}$. The category $\text{Cov}(U)^{op}$ is *filtered* since its opposite category is cofiltered. Therefore,

$$\text{colim}_{\mathcal{U} \in \text{Cov}(U)^{op}} H^0(\mathcal{U}, \mathcal{F})$$

is a *filtered colimit*.

Next, if $V \subset U$, then any cover of U induces a cover of V by taking fiber products with V and therefore, we obtain a functor $\text{Cov}(U) \rightarrow \text{Cov}(V)$. Moreover, given $W \subset V \subset U$, we obtain three functors $\text{Cov}(U) \rightarrow \text{Cov}(V) \rightarrow \text{Cov}(W)$ that fit into a commutative triangle. From these observations, we deduce the following fact.

Lemma 7.4.2.7. *Suppose \mathcal{F} is a presheaf on site \mathbf{C} .*

1. *The assignment*

$$U \mapsto \text{colim}_{\mathcal{U} \in \text{Cov}(U)} H^0(\mathcal{U}, \mathcal{F})$$

defines a presheaf \mathcal{F}^+ on X .

2. *The map $\mathcal{F}(U) \rightarrow H^0(\mathcal{U}, \mathcal{F})$ induces a morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^+$.*
3. *The assignment $\mathcal{F} \rightarrow \mathcal{F}^+$ is functorial.*

Proof. The first fact follows from transitivity of pullbacks: if $W \rightarrow V \rightarrow U$ are morphisms, and if $\mathcal{U} = \{U_i\}_{i \in I} \rightarrow U$ is an open cover, then we simply observe that $W \times_V (V \times_U \mathcal{U}) = W \times_U \mathcal{U}$. \square

We leave the proof of the following result as an exercise in unwinding the definitions.

Theorem 7.4.2.8. *Suppose \mathcal{F} is a presheaf of sets on \mathbf{C} .*

1. *The presheaf \mathcal{F}^+ is separated.*
2. *If \mathcal{F} is separated, then \mathcal{F}^+ is a sheaf and the morphism of presheaves $\mathcal{F} \rightarrow \mathcal{F}^+$ is a monomorphism.*
3. *If \mathcal{F} is a sheaf, then $\mathcal{F} \rightarrow \mathcal{F}^+$ is an isomorphism.*

In particular, the map $\mathcal{F} \rightarrow \mathcal{F}^{++}$ is the initial map from \mathcal{F} to a sheaf, i.e., the assignment $\mathcal{F} \rightarrow \mathcal{F}^{++}$ is a left adjoint to the forgetful functor from sheaves to presheaves.

7.4.3 Čech cohomology

The discussion above suggests a general definition of Čech cohomology. Suppose \mathbf{C} is a site and \mathcal{F} is a sheaf of abelian groups on \mathbf{C} . If $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$ is a family of morphisms with a fixed target, then we define the Čech complex as follow: first, define

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_p) \in I^{p+1}} \mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_p}).$$

For an element $s \in C^p(\mathcal{U}, \mathcal{F})$, we write $s_{i_0 \dots i_p}$ for its component in the factor $\mathcal{F}(U_{i_0} \times_U \cdots \times_U U_{i_p})$. Then, we define a differential

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$$

by means of the formula

$$d(s)_{i_0 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_p} |_{U_{i_0} \times_U \dots \times_U U_{i_{p+1}}};$$

here the restriction is taken with respect to the map

$$U_{i_0} \times_U \dots \times_U U_{i_{p+1}} \longrightarrow U_{i_0} \times_U \dots \times_U \hat{U}_{i_j} \times_U \dots \times_U U_{i_{p+1}}$$

where the $(\hat{-})$ denotes omission of the enclosed factor.

Exercise 7.4.3.1. Show that $\delta^2 = 0$, i.e., $(C^p(\mathcal{U}, \mathcal{F}), \delta)$ is a complex of abelian groups.

Definition 7.4.3.2. If \mathbf{C} is a site, \mathcal{F} is a sheaf of abelian groups on \mathbf{C} and $\mathcal{U} = \{U_i \rightarrow U\}_{i \in I}$ is a covering with fixed target U , the Čech cohomology of \mathcal{F} with respect to \mathcal{U} is defined by the formula:

$$\check{H}^i(\mathcal{U}, \mathcal{F}) := H^i(C^*(\mathcal{U}, \mathcal{F})).$$

Remark 7.4.3.3. Given the definition above, in order to obtain a notion that is independent of the cover, one could form the filtered colimit over the category of covers of a fixed target.

7.5 Lecture 28: Sheaves and neighborhoods

In the previous lecture we studied sheaves on sites.

7.5.1 Limits and colimits in sheaf categories

We have already observed that limits and colimits exist in the category of presheaves (these are defined sectionwise and limits and colimits exist in the category of sets). Above, we showed that sheafification of set-valued presheaves can be obtained by iterating the $+$ -construction twice. The same construction generalizes to presheaves valued in “reasonable” categories, e.g., abelian groups or groups.

Remark 7.5.1.1. One knows that for rather general target categories \mathbf{C} , sheafification can be constructed by means of a slight modification of the above procedure. If one imposes suitable “finiteness” conditions (the category is locally finitely presentable) on the category \mathbf{C} , then sheafification can be constructed exactly as above. See [Ulm71, 13 Theorem p. 301] for more details.

On the other hand, sheafification is a left adjoint (to the forgetful functor). Therefore, the forgetful functor is a right adjoint. There are general facts about preservation of limits and colimits by such functors that we now record: right adjoints preserve limits but not colimits in general.

Theorem 7.5.1.2. If \mathbf{C} is a “reasonable” category (see above) that has small limits and colimits, then so do $\text{PShv}(X, \mathbf{C})$ and $\text{Shv}(X, \mathbf{C})$.

1. The forgetful functor $\text{Shv}(X, \mathbf{C}) \rightarrow \text{PShv}(X, \mathbf{C})$ commutes with the formation of small limits.

2. The forgetful functor $\mathrm{Shv}(X, \mathbf{C}) \rightarrow \mathrm{PShv}(X, \mathbf{C})$ does not, in general, commute with the formation of colimits (even finite colimits (think about cokernels)).
3. The colimit of a diagram in $\mathrm{Shv}(X, \mathbf{C})$ can be computed by computing the colimit in presheaves and then sheafifying.
4. The sheafification functor commutes with all colimits and with finite limits but does not commute with all limits.

Proof. All of these results follow from general categorical observations. The two fundamental facts we use are: sheafification is a filtered colimit and a left adjoint. Therefore, the forgetful functor from sheaves to presheaves is a right adjoint.

In general, right adjoints preserve limits and the first point is an immediate consequence. On the other hand, right adjoints do not, in general preserve colimits, so the second point is also an immediate consequence.

The third point follows from the fact that colimits commute.

The fourth point stems from the fact that filtered colimits commute with formation of finite limits in the category of sets. \square

7.5.2 Stalks and neighborhoods

Having established some general categorical properties of sheaves, we now return to a more concrete setting. Suppose \mathbf{C} is a site whose underlying category consists of schemes. In that case, we can ask if there are analogs of the notion of a stalk that will allow us to test whether a given map of sheaves is an epimorphism or isomorphism. Once more, the question of whether a given map of sheaves is a *monomorphism* can be checked on sections (this is because the equalizer is a limit and the forgetful functor commutes with limits. There are two observations that we need to make here. First, the notion of stalk in the setting of sheaves on a topological space was obtained by taking a filtered colimit over neighborhoods of a point $x \in X$.

The notion of neighborhood in the Zariski topology can be phrased as follows. Suppose X is a scheme and $x \in X$ is a point. A neighborhood of x consists of an open subscheme $U \subset X$ that contains x . In the étale topology, we replace open immersions by étale morphisms. Thus, rather than considering an open subscheme $U \subset X$, we consider an étale map $\varphi : V \rightarrow X$. However, we run into a basic problem: the topological space underlying V and X are no longer the same. There are two ways around this.

1. We could simply pick a point $v \in V$ that is mapped to x via φ . In that case, note that because the morphism φ is, in particular, unramified at v , the residue field $\kappa(v)$ is a finite separable extension of $\kappa(x)$. From this point of view, when we refine a neighborhood, we permit ourselves to make separable extensions of the residue field.
2. We could restrict our attention to those $\varphi : V \rightarrow X$ such that the induced map of residue fields is an isomorphism. Indeed, since every open immersion is an étale morphism, we there are certainly neighborhoods of this form.

From our point of view, each of these choices has benefits, and we codify them both as definitions.

Definition 7.5.2.1. Suppose X is a scheme and $x \in X$ is a point.

1. An *étale neighborhood* of x consists of an étale morphism $\varphi : U \rightarrow X$ and a point $u \in U$ such that $\varphi(u) = x$.

2. A Nisnevich neighborhood of x consists of an étale morphism $\varphi : U \rightarrow X$ and a point $u \in U$ such that $\varphi(u) = x$ and the induced map $\kappa(x) \subset \kappa(u)$ is an isomorphism.

This new definition presents us with a natural question: can any separable extension of the residue field be realized by an étale neighborhood?

Proposition 7.5.2.2. *If X is a scheme, $x \in X$ is a point with residue field κ and k/κ is a finite separable extension, then there exists an étale neighborhood U of x and a point $u \in U$ with residue field k realizing this extension.*

Proof. By replacing X by a Zariski neighborhood of x , we can assume that $X = \text{Spec } R$ for some ring R .

Since k/κ is separable, we can write $k = \kappa[x]/(f)$ for f the minimal polynomial of a primitive element α generating the extension. Here f is a monic irreducible polynomial with coefficients in κ . Now, the ring map $R \rightarrow k$ factors through the localization $R_{\mathfrak{p}}$ of R at some prime ideal $\mathfrak{p} \subset R$ and there is a surjective ring map $R \rightarrow R_{\mathfrak{p}} \rightarrow \kappa$. The polynomial f lifts to an element of $R_{\mathfrak{p}}[x]$ by construction. By clearing the denominators, it follows that after multiplying α by g we can assume that the coefficients of f lie in the image of $R \rightarrow \kappa$. Therefore, we can find a polynomial $\tilde{f} \in R[x]$ mapping to f by reduction. Thus, $R[x]/(f)$ realizes the required separable extension. However, the ring map $R \rightarrow R[x]/(f)$ need not be étale. We now show that by localizing further, we can assume it will be étale.

Since f is separable, we know that f is coprime to its derivative f' . In particular, $f'(\alpha) = u$ is a unit. We claim it suffices to simply invert \tilde{f}' , i.e., define a map $R[x, \frac{1}{\tilde{f}'}] \rightarrow k$ by sending x to α . Indeed, \tilde{f}' is mapped to f' under the homomorphism $R \rightarrow \kappa$. Therefore, $\frac{1}{\tilde{f}'}$ is mapped to u^{-1} and thus the map in question is étale. \square

Now, given a scheme X and a point $x \in X$, we collect all neighborhoods of X into a category: this is the category whose objects are neighborhoods $(\varphi : U \rightarrow X, u \in U)$ and where a morphism of neighborhoods is a morphism $f : U \rightarrow U'$ such that $f(u) = u'$. We write $\text{Neib}_x^{\text{ét}}(X)$ (resp. $\text{Neib}_x^{\text{Nis}}(X)$) for the category of étale (resp. Nisnevich) neighborhoods of $x \in X$. The category of étale (resp. Nisnevich) neighborhoods of a given point $x \in X$ are *cofiltered* categories (again, see Definition A.1.3.2).

Proposition 7.5.2.3. *Suppose X is a scheme and $x \in X$ is a point.*

1. *The category $\text{Neib}_x^{\text{ét}}(X)$ (resp. $\text{Neib}_x^{\text{Nis}}(X)$) is non-empty.*
2. *Given any two étale (resp. Nisnevich) neighborhoods (U, u) and (U', u') of x , there is a third neighborhood (U'', u'') and maps $(U'', u'') \rightarrow (U, u)$ and $(U'', u'') \rightarrow (U', u')$ (i.e., a refinement).*
3. *Given any pair of maps $f, g : (U, u) \rightarrow (U', u')$, we can find a neighborhood (U'', u'') and a morphism $h : (U'', u'') \rightarrow (U, u)$ such that the composites $f \circ h = g \circ h$.*

Proof. The first point is immediate since X is non-empty by assumption. For the remaining statements, we prove the results for étale neighborhoods, and simply observe that corresponding statements hold for Nisnevich neighborhoods by tracing through the arguments.

For the second point, consider the fiber product $U'' = U \times_X U'$. Since étale morphisms are stable by base-change, it follows that U'' is étale over both U and U' . Likewise, by the definition of a fiber product, we can find a point $u'' \in U''$ mapping to both u and u' .

For the third point, observe that f and g define a map $(f, g) : U \rightarrow U' \times_X U'$. On the other hand, consider the diagonal map $\Delta : U' \rightarrow U' \times_X U'$. The fiber product:

$$\begin{array}{ccc} U'' & \longrightarrow & U \\ \downarrow & & \downarrow (f,g) \\ U' & \xrightarrow{\Delta} & U' \times_X U' \end{array}$$

is the universal solution to our problem. It suffices to show that $U'' \rightarrow U'$ is étale. To this end, observe that $U' \times_X U' \rightarrow U'$ is étale, again using the fact that étale morphisms are stable by base-change. On the other hand, the composite map $U \rightarrow U' \times_X U' \rightarrow U'$, given by (f, g) followed by a projection onto one of the factors, coincides with either the map f or g , i.e., the composite map is étale. We claim that this implies that (f, g) is also étale; see [Sta15, Lemma 10.141.9] for details. From this, it follows that U'' is étale over U' (again by the base-change property) and $U'' \rightarrow U' \rightarrow X$ is étale because composites of étale morphisms are étale. \square

Example 7.5.2.4. If \mathcal{O}_X is the structure sheaf of X , then the stalk $\mathcal{O}_{X,x}$ is a local ring. Analogously, we may now form the filtered colimit:

$$\operatorname{colim}_{U \in \operatorname{Neib}_x^{\text{ét}}(X)^{\text{op}}} \mathcal{O}(U).$$

This ring has a name as well, it is called the *strict Henselization* of $\mathcal{O}_{X,x}$ and we write $\mathcal{O}_{X,x}^{\text{sh}}$ for the colimit. Analogously, we write $\mathcal{O}_{X,x}^{\text{h}}$ for the corresponding filtered colimit over Nisnevich neighborhoods. There are ring maps

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}^{\text{h}} \longrightarrow \mathcal{O}_{X,x}^{\text{sh}}$$

and we will later discuss the important universal properties of these rings.

Definition 7.5.2.5. If \mathcal{F} is an étale sheaf on X , then the stalk of \mathcal{F}_x of \mathcal{F} at a point $x \in X$ is defined by

$$\mathcal{F}_x := \operatorname{colim}_{U \in \operatorname{Neib}_x^{\text{ét}}(X)^{\text{op}}} \mathcal{F}(U)$$

Remark 7.5.2.6. Given this definition, we may, without loss of generality, assume that x is a geometric point of X .

While all of these definitions are, I think, intuitively reasonable, they still do not answer the question with which we began: can isomorphisms of étale sheaves be checked on stalks? I do not want to spend a lot of time going off on tangents, but the following result is important.

Theorem 7.5.2.7. *A morphism of sheaves $\mathcal{F} \rightarrow \mathcal{G}$ is an epimorphism (resp. isomorphism) if and only if the induced map of stalks is an epimorphism (resp. isomorphism).*

7.5.3 Stalks and points

We record here a categorical observation about taking stalks.

Proposition 7.5.3.1. *If X is a topological space, and if $x \in X$ is a point, then the operation $\mathcal{F} \rightarrow \mathcal{F}_x$ is left exact, i.e., preserves finite limits.*

Proof. Since stalks are obtained by forming filtered colimits, we appeal to the fact that filtered colimits commute with finite limits in the category of sheaves. \square

The following result is an immediate consequence of the above proof and the definitions given above.

Proposition 7.5.3.2. *If X is a scheme and $x \in X$ is a point, then the functor sending an étale sheaf \mathcal{F} to the stalk \mathcal{F}_x is left exact.*

Remark 7.5.3.3. In general, one may axiomatize the ideas above and talk about points of a site and corresponding stalks. Unfortunately, it is *not* always the case that epimorphisms and isomorphisms of sheaves are detected by stalks, though such situations will never confront us.

7.6 Lecture 29: Refined neighborhoods of points on smooth varieties

Our goal in this section is to refine some of our earlier results about the local structure of points on smooth varieties. In particular, we will establish an étale neighborhood result of Lindel [Lin82, Proposition 2], which will end up being a key tool in our study of \mathbb{A}^1 -homotopy invariance for vector bundles. We begin by recasting our earlier algebraic results in more geometric language (though the proof Lindel's theorem is essentially entirely algebraic).

7.6.1 Refined local structure of smooth varieties

Earlier we showed that if X is a finite-type smooth k -scheme of dimension d and $x \in X$ is a closed point with residue field isomorphic to k , then there is a polynomial ring $k[x_1, \dots, x_d]$ and an étale map $k[x_1, \dots, x_d]_{(0)} \rightarrow \mathcal{O}_{X,x}$. If we let \mathfrak{m} be the maximal ideal of $\mathcal{O}_{X,x}$, then the construction proceeds as follows. The k -vector space \mathfrak{m} had dimension d and we picked d -functions $x_1, \dots, x_d \in \mathfrak{m}$ whose differentials generated $\mathfrak{m}/\mathfrak{m}^2$. The functions x_i define an inclusion $k[x_1, \dots, x_d] \rightarrow \mathcal{O}_{X,x}$ and factor through a map $k[x_1, \dots, x_n]_{(0)} \rightarrow \mathcal{O}_{X,x}$. The ring map in question is finitely presented, so by clearing denominators we may actually find Zariski open neighborhoods U of 0 in \mathbb{A}^n and V of $x \in X$ such that the map in question lifts to a morphism of schemes $U \rightarrow V$. We showed the map $U \rightarrow V$ is étale at x . Moreover, by shrinking U and V if necessary, we may even assume that φ is an étale map. Therefore, our result can be translated as follows.

Proposition 7.6.1.1. *If X is a finite-type smooth k -scheme of dimension d and $x \in X$ is a closed point with residue field k , then there is a Nisnevich neighborhood of x isomorphic to an open subscheme of affine space.*

We can further refine such local structure results by asking whether we can control other geometric features (we have a fair amount of choice in the functions x_i). For example, suppose X is a finite type smooth k -scheme, $x \in X$ and $Z \subset X$ is a smooth subvariety of codimension c at x . In that case, one may pick coordinate functions as above so that Z is locally cut out by x_1, \dots, x_c .

If X is a scheme and $Z \subset X$ is a closed subscheme, it makes sense to talk about open neighborhoods of Z that contain X . Analogously, we can speak of Nisnevich neighborhoods of $Z \subset X$, generalizing the notion of Nisnevich neighborhood of a point.

Definition 7.6.1.2. If X is a k -scheme and $Z \subset X$ is closed subscheme, then a Nisnevich neighborhood of $Z \subset X$ consists of an étale morphism $U \rightarrow X$ together with a morphism $Z \rightarrow U$ lifting the original inclusion map, i.e., such that the composite $Z \rightarrow U \rightarrow X$ coincides with the original closed immersion.

Remark 7.6.1.3. As before, one way to study local geometry around a closed subscheme is to study the colimit over Nisnevich neighborhoods of the given subscheme. This is, in some ways, similar to working in tubular neighborhoods of a given subvariety.

The following definition appears in the commutative algebra literature.

Definition 7.6.1.4. Given a ring homomorphism $\varphi : R \rightarrow S$ and an element $f \in R$ mapping under φ to an element $\varphi(f)$, we will say that φ is an *analytic isomorphism along f* if φ induces an isomorphism $R/f \rightarrow S/\varphi(f)$ is an isomorphism.

7.6.2 Lindel's étale neighborhood theorem

We now establish a slightly different étale neighborhood theorem. Above, we showed that we could find a Nisnevich neighborhood of certain closed points on smooth schemes that are isomorphic to open subschemes of affine space. The following result appears in [Lin82, Proposition 2].

Proposition 7.6.2.1 (Lindel). *Let k be a field, and let A be a finitely generated regular k -algebra of dimension $d \geq 1$. Assume $\mathfrak{m} \subset A$ is a maximal ideal such that the residue field $\kappa = A/\mathfrak{m}$ is a simple separable extension of k . There exist elements $x_1, \dots, x_t \in A$ such that*

1. *A is generated as a k -algebra by x_1, \dots, x_t and $\mathfrak{m} = (f(x_1), x_2, \dots, x_t)$ with $f(x_1) \notin \mathfrak{m}^2$, where f is the minimal polynomial of the image of x_1 in κ (note we are not claim).*
2. *$A_{\mathfrak{m}}$ is a Nisnevich neighborhood of $B_{\mathfrak{n}}$ where $B = k[x_1, \dots, x_d]$ and $\mathfrak{n} = \mathfrak{m} \cap D$.*

Proof. The first point is much simpler if k is algebraically closed. In that case, $k = \kappa$ since A is a finitely generated k -algebra. In that case, any set of algebra generators whose images in A have differentials spanning $\mathfrak{m}/\mathfrak{m}^2$ does the job. If k is not algebraically closed, the argument is slightly more involved.

There exists an element $u \in A$ such that $\kappa = k(\bar{u})$ where $\bar{u} \in A/\mathfrak{m}$. Then, for a suitable integer t , we may find elements $u_2, \dots, u_t \in A$ such that C is generated by u, u_2, \dots, u_t . Since $\kappa = k(\bar{u})$, we may find polynomials $w_i \in \kappa[u]$ with $u_i - w_i \in \mathfrak{m}$, $2 \leq i \leq t$. Set $x_i = u_i - w_i$. In that case, A is still generated by u, x_2, \dots, x_t and the maximal ideal $\mathfrak{m} = (f(u), x_2, \dots, x_t)$ where f is the minimal polynomial of \bar{u} .

Now, if $f(u) \notin \mathfrak{m}^2$, we can choose $x_1 = u$ to obtain (i). Thus, suppose $f(u) \in \mathfrak{m}^2$. Since $d \geq 1$ we can assume without loss of generality that $t \geq 2$ and $x_2 \notin \mathfrak{m}^2$. Define $x_1 = u - x_2$. In that case,

$f(u) = f(x_1 + x_2) = f(x_1) + f'(x_1)x_2 \pmod{x_2^2}$ and hence $f(u) - f(x_1) = f'(x_1)x_2 \pmod{\mathfrak{m}^2}$. Because κ/k is separable, we know that $f'(x_1)$ is a unit in $A_{\mathfrak{m}}$ and therefore $f'(x_1) \notin \mathfrak{m}$. Therefore, $f(x_1) \notin \mathfrak{m}^2$. Thus, A is generated by x_1, \dots, x_t as a k -algebra and $\mathfrak{m} = (f(x_1), x_2, \dots, x_t)$ and we have established Point (1).

For Point (2), we proceed as follows. Since $f(x_1) \notin \mathfrak{m}^2$ we may assume that $f(x_1), x_2, \dots, x_d$ is a minimal system of generators of $\mathfrak{m}A_{\mathfrak{m}}$. In particular, this system of elements is algebraically independent over k . Note that the fraction field of $k[x_1, \dots, x_d]$ has transcendence degree d over k . Likewise A is regular and hence an integral domain. Since it also has dimension d it follows that the fraction field of A which is an extension of $k[x_1, \dots, x_d]$ has transcendence degree d over k . It follows that the induced extension of fraction fields is actually algebraic.

Let B' be the integral closure of B in A . Observe that the fraction field of B' necessarily coincides with that of A because the fraction field of A is an algebraic extension of that of B . Thus, we have

$$B \hookrightarrow B' \hookrightarrow A,$$

where the first map is a normalization, and the second map induces an isomorphism on fraction fields, i.e., it is a birational isomorphism. First we will show that B' is a finite B -algebra and then we show that after replacing $\text{Spec } A$ by a suitable neighborhood around \mathfrak{m} and $\text{Spec } B'$ by the pre-image of this neighborhood, that B' is isomorphic to A .

Unlike the situation for Dedekind domains, the normalization of a Noetherian integral domain in a finite extension of its fraction field can fail to be a Noetherian integral domain (examples were constructed by Krull and Akizuki and exist already in dimension 2). The problems arise from possible inseparability of the extension of fraction fields, so are not really issues if k has characteristic 0. In fact, we have already proven in Proposition 4.4.1.5 that normalization is finite if the extension of fraction fields is separable.

We may also deal with the situation for arbitrary extensions of fraction fields as follows. An integral domain B is called *Japanese* if its integral closure in any finite extension of its fraction field is a finitely generated B -module [Sta15, Definition 10.155.1]. If k is a field, then any finitely generated k -algebra is Japanese [Sta15, Proposition 10.156.16]. In particular, this applies to B and we conclude that B' is a finitely generated B -module.

For the next step, the idea is to appeal to Zariski's main theorem (even if the extension of fraction fields is separable). We have

$$\text{Spec } A \longrightarrow \text{Spec } B' \longrightarrow \text{Spec } B.$$

Zariski's main theorem [Sta15, Theorem 28.51.1] tells us that there is an open subscheme U of $\text{Spec } B'$ such that the map $\text{Spec } B \rightarrow \text{Spec } B'$ is an isomorphism from the preimage of U to U and, moreover, that the preimage of the locus of points on which the map is an isomorphism consists precisely of points where the map has finite fibers. Zariski's main theorem also shows that the locus of points where a map has finite fibers is actually open [Sta15, Lemma 28.51.2]. Now, we observed above that A and B' have isomorphic fraction fields, so the open subset U on which the map is an isomorphism is non-empty. If we set $\mathfrak{m}' = \mathfrak{m} \cap B'$, then it is straightforward to check that the $A/\mathfrak{m}'A \rightarrow B'/\mathfrak{m}'$ is finite. In particular, the open set U in question is non-empty and contains the closed point corresponding to the maximal ideal \mathfrak{m}' . Thus, replacing B' by any non-empty open

affine subscheme containing the closed point corresponding to \mathfrak{m}' and A by its pre-image, we can assume that $\text{Spec } B \rightarrow \text{Spec } B'$ is an isomorphism.

Now, it remains to show that the map $\text{Spec } B \rightarrow \text{Spec } A$ is flat and unramified at \mathfrak{m} and preserves residue fields in the claimed fashion. To this end, we may set $\mathfrak{n} = \mathfrak{m} \cap B$ and replace A by $A_{\mathfrak{m}}$ and B by $B_{\mathfrak{n}}$. We will thus assume that A is regular local with maximal ideal \mathfrak{m} and B is regular local with maximal ideal \mathfrak{n} .

To show that the map $\text{Spec } B \rightarrow \text{Spec } A$ is unramified at \mathfrak{m} we observe that, as $x_1 - u \in \mathfrak{m}$, $\mathfrak{m} = \mathfrak{n}A$ and the induced map $A/\mathfrak{m} \rightarrow B/\mathfrak{n}$ is an isomorphism. Since A and B are Noetherian and the map is finite, it is automatically finitely presented. Thus, if the map is flat, it will be a Nisnevich neighborhood.

It remains to show that $\text{Spec } A \rightarrow \text{Spec } B$ is flat. This follows from a general result, which invokes regularity. More precisely: if $(A, \mathfrak{m}, \kappa)$ and $(B, \mathfrak{n}, \kappa')$ are Noetherian local rings, $A \rightarrow B$ is a local homomorphism, $\dim B = \dim A + \dim B \otimes \kappa$, and A and $B \otimes \kappa = B/\mathfrak{m}B$ are regular, then B is flat over A and regular [?, 21.D Theorem 51 p. 155] (this uses the local criterion for flatness, which we have not established). \square

Chapter 8

Vector bundles and \mathbb{A}^1 -invariance

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In this section, we analyze the functor sending a smooth k -scheme X to the set of isomorphism classes of rank r vector bundles on X . We will observe a dichotomy. If X is not affine, then this functor can *fail* to be \mathbb{A}^1 -invariant (even though we haven’t proven it in this generality, this is unlike the case of Picard groups, K_0 or algebraic de Rham cohomology). However, we will also prove the Lindel–Popescu theorem: if R is a regular ring containing a field, then every finitely generated projective $R[x_1, \dots, x_n]$ module is extended from R . Along the way, we give Quillen’s solution to the Serre problem on freeness of projective modules over polynomial rings over fields (or, more generally, PIDs). The two key tools are “Horrocks’ theorem” and Quillen’s “local-to-global” principle. We begin by analyzing vector bundles over \mathbb{P}^1 : we can completely classify such vector bundles by a result due to Dedekind–Weber sometimes attributed to Birkhoff–Grothendieck–

Hilbert–Plemelj, etc..

8.1 Lecture 30: Vector bundles on \mathbb{P}^1

Suppose R is a base ring and consider $\mathbb{P}_{\text{Spec } R}^1$. The goal of this section is to analyze vector bundles over \mathbb{P}_R^1 in various cases. If R is a field, the answer is essentially classical. However, already if R is a principal ideal domain, many interesting phenomena can occur.

8.1.1 Vector bundles on the projective line over a field

Recall that \mathbb{P}_R^1 is obtained by gluing together $\text{Spec } R[t]$ and $\text{Spec } R[t^{-1}]$ along $\text{Spec } R[t, t^{-1}]$ by means of the automorphism $t \mapsto t^{-1}$. Using this observation, we may give a description of vector bundles on \mathbb{P}_R^1 in terms of cocycles. In that case, if we begin with a trivial bundle over $\text{Spec } R[t]$, i.e., a free $R[t]$ -module P_+ of rank r and a free $R[t^{-1}]$ -module P_- of rank r , by restriction we obtain two free $R[t, t^{-1}]$ -modules $P_+ \otimes_{R[t]} R[t, t^{-1}]$ and $P_- \otimes_{R[t^{-1}]} R[t, t^{-1}]$. If we fix an isomorphism

$$P_+ \otimes_{R[t]} R[t, t^{-1}] \xrightarrow{\sim} P_- \otimes_{R[t^{-1}]} R[t, t^{-1}]$$

that covers the automorphism $R[t, t^{-1}] \rightarrow R[t, t^{-1}]$ given by $t \mapsto t^{-1}$, then we may patch together these bundles to obtain a vector bundle on \mathbb{P}_R^1 . If we fix bases $\{e_i^+\}$ of P_+ and $\{e_i^-\}$ of P_- , then the isomorphism $P_+ \otimes_{R[t]} R[t, t^{-1}] \xrightarrow{\sim} P_- \otimes_{R[t^{-1}]} R[t, t^{-1}]$ is uniquely determined by an element of $GL_r(R[t, t^{-1}])$. More precisely, we write X for the matrix that changes bases from the “ $-$ -coordinates” to the “ $+$ -coordinates.” This matrix is well-defined up to change of bases in P_+ and P_- , which corresponds to left multiplication by an element of $GL_r(R[t])$ and right multiplication by an element of $GL_r(R[t^{-1}])$ (or vice-versa). Therefore, we have defined an injective function

$$GL_n(R[t]) \backslash GL_n(R[t, t^{-1}]) / GL_n(R[t^{-1}]) \longrightarrow \mathcal{V}_r(\mathbb{P}_R^1)$$

where the right hand side is the set of isomorphism classes of rank r vector bundles on \mathbb{P}_R^1 . We now show that, in certain situations, this function is bijective.

Proposition 8.1.1.1. *If k is a field, then there is a bijection*

$$\mathcal{V}_r(\mathbb{P}_k^1) \cong GL_n(k[t]) \backslash GL_n(k[t, t^{-1}]) / GL_n(k[t^{-1}]).$$

Proof. As above, cover \mathbb{P}_R^1 by \mathbb{A}_R^1 with coordinate t and \mathbb{A}_R^1 with coordinate t' . Suppose \mathcal{E} is a vector bundle. Suppose \mathcal{E} is a vector bundle on \mathbb{P}_R^1 . The restriction of \mathcal{E} to \mathbb{A}_R^1 is equivalent to giving a rank r projective $R[t]$ (resp. $R[t^{-1}]$ -module). By assumption, these modules are free. Choosing bases yields the required inverse function. \square

Remark 8.1.1.2. The only thing required to make the above proof work was knowing that projective modules over $R[t]$ are free. In studying the Picard group, we observed that assertions like this are non-trivial even for invertible $R[t]$ -modules. If R is (semi-normal), then $\text{Pic}(R) \cong \text{Pic}(R[t])$ and thus, $\text{Pic}(R[t])$ is trivial if and only if $\text{Pic}(R)$ is trivial. In particular, $\text{Pic}(R)$ is trivial only if R is a UFD.

It follows that any element of $GL_r(k[t, t^{-1}])$ determines a rank r vector bundle on \mathbb{P}_R^1 and, conversely, that every vector bundle on \mathbb{P}_k^1 can be represented by such a matrix. We will refer to a matrix representation of a bundle over \mathbb{P}_R^1 as a *clutching function* because of the similarity of this notion with the corresponding one in topology.

Remark 8.1.1.3. There is some ambiguity in the identification between clutching functions and line bundles corresponding to whether we specify the “change-of-basis” matrix or its inverse. For this reason, it is more convenient to specify the vector bundle rather than the clutching function.

We will now constrain the choice of the clutching function a bit. Every unit in $k[t, t^{-1}]$ can be written uniquely as at^r for some integer r and $a \in k^\times$. The determinant homomorphism $GL_n(k[t, t^{-1}]) \rightarrow \mathbf{G}_m(k[t, t^{-1}])$ is surjective, and thus any such matrix has determinant at^s for some integer s and some unit a . The determinant of an element of $GL_n(k[t])$ lies in $k[t]^\times = k^\times$ by homotopy invariance of units. By multiplication by an element of $GL_n(k)$ with the given determinant, we can always assume that the determinant is t^s for some integer s .

In particular, we can consider the matrices $\text{diag}(t^{a_1}, \dots, t^{a_r})$ for sequences a_1, \dots, a_r of integers. The vector bundles determined by these matrices split as a direct sum of line bundles, which we write as $\mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r)$. By permuting the entries, we may assume that $a_1 \geq a_2 \geq \dots \geq a_r$ and fixing the determinant corresponds to fixing the sum of these entries. By studying normal forms for matrices, we now show that the matrices just described give rise to vector bundles that span the set of isomorphism classes of vector bundles on \mathbb{P}_k^1 for k a field. This can be phrased in purely matrix-theoretic terms and we give a proof following Hazewinkel–Martin [HM82].

Proposition 8.1.1.4. *Let k be a field, and let $X(t, t^{-1})$ be an $r \times r$ -matrix over $k[t, t^{-1}]$ with determinant t^s for some $s \in \mathbb{Z}$. There exist $r \times r$ -matrices $Y(t) \in GL_n(k[t])$ and $Z(t^{-1}) \in GL_n(k[t])$ such that*

$$Y(t)X(t, t^{-1})Z(t^{-1}) = \text{diag}(t^{a_1}, \dots, t^{a_r})$$

with $a_1 \geq \dots \geq a_r$

Proof. We proceed by induction on the size of the matrix. Since any 1×1 -matrix is already of the form $\alpha \cdot t^r$ (as it is a unit), it suffices to multiply by α^{-1} to obtain the result. Assume now that the result is known for matrices of size $(n-1) \times (n-1)$. We perform some preparatory steps to simplify the form of the matrix.

First, suppose $X = X_{ij}$. Now, by clearing the denominators, we can assume that $X \in GL_n(k[t])$ and also that $\det X = t^s$ for some $s \geq 0$. Now, computing the determinant by expanding along the first row we see that:

$$\det X = \sum_{j=1}^n (-1)^j X_{1j} \det \min_{1j} X.$$

In particular, t^s divides each term of the right hand side, and t^{r_1} divides $\gcd(X_{11}, \dots, X_{1j})$. Find $\gcd(X_{1j})$. Therefore, by means of column operations (i.e., right multiplication by elements of $GL_n(k[t])$) we may assume that X takes the form

$$\left(\begin{array}{c|c} t^{a_1} & 0 \\ \hline X_{2*} & X' \end{array} \right).$$

Since the result is true for matrices of size $(n-1) \times (n-1)$, at the expense of modifying X_{2*} , by means of suitable row and column operations, we can further assume that $X' = \text{diag}(t^{a_2}, \dots, t^{a_r})$ with $a_2 \geq \dots \geq a_r$. Thus, it suffices to prove that we may remove X_{2*} by means of suitable operations where $X_{2*} \in k[t, t^{-1}]$. By assumption $a_1 \geq 0$, so by suitable row operations, we may even assume that $X_{2*} \in k[t]$.

Now, observe that among matrices of the above form equivalent to our original matrix, we may assume that a_1 is maximal. We may achieve this by simply permuting the rows before we begin, repeating the procedure above, and searching through the resulting list of matrices... \square

Combining Proposition 8.1.1.4 with Proposition 8.1.1.1 we deduce the following result.

Corollary 8.1.1.5. *If k is a field, every vector bundle on \mathbb{P}_k^1 is isomorphic to a unique vector bundle of the form $\mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_r)$ with $a_1 \geq \dots \geq a_r$.*

Remark 8.1.1.6. The above result has been established many times by many different authors. The result was perhaps first proven by Dedekind–Weber [DW82] in the algebraic setting. Birkhoff proved [Bir13, p. 129] a corresponding result for matrices of complex analytic functions. Apparently the result was also rediscovered by D. Hilbert [?, ??] and J. Plemelj [Ple08, ??] around the same time as Birkhoff. Nowadays, the result is often attributed to Grothendieck [Gro57], who actually proved a slightly more general result about principal bundles.

8.1.2 Line bundles on \mathbb{P}^1

We now analyze line bundles on \mathbb{P}^1 over a ring that arise by the construction above.

Example 8.1.2.1. The structure sheaf \mathcal{O} is patched together from free modules of rank 1 over $\text{Spec } R[t]$ and $\text{Spec } R[t^{-1}]$. If we take basis 1 on each side, then the transition map is just given by the 1×1 identity matrix. The sections here can be described rather easily as well: if f is a function of t , then $f(t^{-1})$ is a function of t^{-1} and we want both $f(t)$ and $f(t^{-1})$ to be polynomial in t .

Example 8.1.2.2. There are two tautological line bundle on \mathbb{P}^1 . Over a field, they admit the following geometric description. If we take a trivial bundle of rank 2 over \mathbb{P}^1 , we can consider the sub-bundle whose fiber over a point $p \in \mathbb{P}^1$ is the line $L \subset V$ corresponding to that point or the quotient vector space V/L . In more ring-theoretic terms, if we pick coordinates in x_0, x_1 on V , then the line $L \subset V$ is determined by its slope x_0/x_1 if $x_1 \neq 0$ and x_1/x_0 if $x_0 \neq 0$. Unwinding the definitions, this corresponds to the clutching function given by multiplication by t . Thus, the tautological “sub-bundle” is $\mathcal{O}(-1)$. Dually, the tautological quotient bundle is $\mathcal{O}(-1)$.

Example 8.1.2.3. We can describe the cotangent bundle of \mathbb{P}^1 as well. The module of differential forms over $k[t]$ has a basis dt and over $k[t^{-1}]$ has basis $d(t^{-1})$. Now, $d(t^{-1}) = -\frac{1}{t^2}dt$ so the cotangent bundle is simply the line bundle $\mathcal{O}(-2)$. Dually, the vector field $\frac{d}{dt}$ is a basis for tangent vectors on $k[t]$. By the chain rule, $\frac{d}{dt} = \frac{d}{dt^{-1}} \frac{dt^{-1}}{dt} = -\frac{1}{t^2} \frac{d}{dt^{-1}}$. Therefore, the tangent bundle corresponds to $\mathcal{O}(2)$.

We can generalize these constructions to describe a basis of global sections of $\mathcal{O}(n)$ over \mathbb{P}^1 .

Proposition 8.1.2.4. *If t is a coordinate on \mathbb{A}^1 over a base ring R , then*

1. $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is a trivial R -module if $n \leq 0$; and

2. $H^0(\mathbb{P}^1, \mathcal{O}(n))$ is a free module with basis of sections $1, t, \dots, t^n$ if $n \geq 0$.

Proof. For $\mathcal{O}(n)$, the transition function going from the “+–coordinates” to the “––coordinates”, which is the inverse of the transition function we wrote down above, corresponds to multiplication by $t \mapsto t^n$. If $\{e^+\}$ is a basis of sections over $R[t]$, and $\{e^-\}$ is a basis of sections over $R[t^{-1}]$, then any section can be written as $f(t) \cdot e^+$. Such a section is transformed to $f(t^{-1})t^{-n} \cdot e^-$. Since any f can be written as a sum of monomials, the condition that a section be a global is that both $f(t)$ and $f(t^{-1})t^{-n}$ are regular functions of t . If $f = t^a$, then we want both t^a and t^{-a+n} to be regular. In particular, we see that for $0 \leq a \leq n$, both t^a and t^{-a+n} are regular. \square

8.1.3 Vector bundles on the projective line over a PID

If R is no longer a field, the situation regarding vector bundles over \mathbb{P}_k^1 can be rather different.

Definition 8.1.3.1. An object of an additive category \mathbf{C} is called *indecomposable* if it is not isomorphic to a direct sum of two non-zero objects.

The following example is perhaps the simplest example of the failure of \mathbb{A}^1 -invariance for the functor $\mathcal{V}_r(X)$ for smooth varieties in general.

Example 8.1.3.2. Suppose R is $k[x]$. In that case, we may build a rank 2 bundle on $\mathbb{P}_R^1 = \mathbb{P}^1 \times \mathbb{A}^1$ by considering the clutching function

$$\begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}.$$

When $x = 0$, this matrix corresponds to bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$. On the other hand, if $x = 1$, then since

$$\begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix} \begin{pmatrix} t & 0 \\ 1 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we conclude that the fiber over 1 is isomorphic to $\mathcal{O}(0) \oplus \mathcal{O}(0)$. Since every bundle on $\mathbb{P}^1 \times \mathbb{A}^1$ that is pulled back from \mathbb{P}^1 is a direct sum of line bundles, we conclude that this bundle could not be pulled back from \mathbb{P}^1 . Therefore, the function

$$\mathcal{V}_2(\mathbb{P}^1) \longrightarrow \mathcal{V}_2(\mathbb{P}^1 \times \mathbb{A}^1)$$

fails to be surjective in general. We will see later that it can also fail (rather badly) to be injective.

In fact, the phenomenon above is rather general. Suppose R is a discrete valuation ring with fraction field K , residue field κ and uniformizing parameter π . By slightly modifying the example above (replacing x by π) we can obtain many indecomposable bundles on \mathbb{P}_R^1 .

In that case, we can build a rank 2 bundle on \mathbb{P}_R^1 by considering the clutching function

$$\begin{pmatrix} t & 0 \\ \pi & t^{-1} \end{pmatrix}.$$

If we look at its image of this function in $GL_2(\kappa[t, t^{-1}])$ (by reduction mod (π)) we get the diagonal matrix $\text{diag}(t, t^{-1})$. On the other hand, if we look at the image in $GL_2(K[t, t^{-1}])$, then by Corollary 8.1.1.5 we see that the matrix must be equivalent to a diagonal matrix by suitable left

and right multiplications. In this case, since π is a unit in K , as above the bundle corresponding to our clutching function is isomorphic to $\mathcal{O}(0) + \mathcal{O}(0)$ over K . In particular, we have obtained an indecomposable bundle over \mathbb{P}_R^1 . The following result generalizes this construction.

Proposition 8.1.3.3. *Assume R is a discrete valuation ring. For integer every $r \geq 2$, there exist infinitely many pairwise non-isomorphic indecomposable rank r vector bundles on \mathbb{P}_R^1 .*

Proof. Consider the vector bundle defined by the transition function

$$\begin{pmatrix} t^a & 0 \\ \pi & t^b \end{pmatrix}$$

where $a > 0$ and $b < 0$. Reducing mod π this bundle is equivalent to $\mathcal{O}(-a) \oplus \mathcal{O}(-b)$. After inverting π , this bundle is equivalent to $\mathcal{O} \oplus \mathcal{O}(-a-b)$. \square

Remark 8.1.3.4. In general, vector bundles over \mathbb{P}^1 over a general ring can be rather complicated.

8.2 Lecture 31: Horrocks' theorem

In this section, we investigate further vector bundles on \mathbb{P}^1 over a general ring. We begin by studying Cech cohomology of line bundles on \mathbb{P}^1 .

8.2.1 Cech cohomology of bundles on the projective line

In this section, we compute the Cech cohomology of the bundles $\mathcal{O}(n)$ on \mathbb{P}^1 .

Proposition 8.2.1.1. *Suppose R is a fixed commutative unital ring. Let V be the 2-dimensional vector space of R . The following formula hold:*

1. $\check{H}^i(\mathbb{P}_R^1, \mathcal{O}(n)) = 0$ if $i \neq 0, 1$;
2. $\check{H}^0(\mathbb{P}_R^1, \mathcal{O}(n)) = R[x_0, x_1]^{(n)}$ if $n \geq 0$ and vanishes otherwise.
3. $\check{H}^1(\mathbb{P}_R^1, \mathcal{O}(n)) = R[x_0, x_1]^{-n-2}$ if $n \leq -2$ and vanishes otherwise.

Proof. In this case, we may compute Cech cohomology with respect to the open cover of \mathbb{P}^1 by two open sets isomorphic to \mathbb{A}^1 with intersection \mathbb{G}_m . If we choose coordinates $R[x]$ and $R[x^{-1}]$ then, the differential is given by... \square

Proposition 8.2.1.2. *If R is a Noetherian ring, and if \mathcal{F} is a locally free sheaf on \mathbb{P}_R^1 , then $\check{H}^i(\mathbb{P}_R^1, \mathcal{F})$ is a finitely generated R -module.*

Proof. We proceed by descending induction on i . For $i > 1$ $\check{H}^i(\mathbb{P}_R^1, \mathcal{F}) = 0$ by definition of the Cech complex. Now, we know the result is true for finite direct sums of bundles of the form $\mathcal{O}(i)$ by the previous proposition. Therefore, we deduce the result for any quotient of $\bigoplus_{i=1}^r \mathcal{O}(a_i)$ as follows. Indeed, the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow 0 \bigoplus_{i=1}^r \mathcal{O}(a_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

yields a long exact sequence in cohomology of the form

$$H^i(\mathbb{P}_R^1, \bigoplus_{i=1}^r \mathcal{O}(a_i)) \longrightarrow H^i(\mathbb{P}_R^1, \mathcal{F}) \longrightarrow H^{i+1}(\mathbb{P}_R^1, \mathcal{K}).$$

The induction hypothesis guarantees that $H^{i+1}(\mathbb{P}_R^1, \mathcal{K})$ and therefore we conclude that $H^i(\mathbb{P}_R^1, \mathcal{F})$ is finitely generated as well.

Therefore, to conclude it suffices to know that \mathcal{F} can be written as a quotient of a finite direct sum of modules of the form $\mathcal{O}(i)$. Indeed, restrict \mathcal{F} to $\text{Spec } R[t]$ and $\text{Spec } R[t^{-1}]$; we obtain finitely generated free modules M_+ and M_- over each of these open sets. Pick surjections $R^{\oplus n} \rightarrow M_+$ and $R^{\oplus m} \rightarrow M_-$. By including more generators if necessary, we may assume that $n = m$. \square

8.2.2 Horrocks' theorem

As we have seen above, the description of vector bundles on \mathbb{P}^1 over a local ring can be complicated even when the local ring is regular of dimension 1. Suppose R a Noetherian local ring and consider $R[x]$. View $\text{Spec } R[x]$ as an open subscheme of \mathbb{P}_R^1 as the complement of the section at ∞ . If we begin with a finite rank vector bundle on $\text{Spec } R[x]$, equivalently a projective $R[x]$ -module, when does this module arise as the restriction of a vector bundle on \mathbb{P}_R^1 ? Of course, any free $R[x]$ -module extends (in many ways) to \mathbb{P}_R^1 and any free $R[x]$ -module is necessarily extended from R . In [Hor64], Horrocks analyzed the converse to this statement. What can we say about a finitely generated projective $R[x]$ -module that extends to \mathbb{P}_R^1 ?

Theorem 8.2.2.1 ([Hor64, Theorem 1]). *Suppose R is a Noetherian local ring. If \mathcal{E} is a vector bundle on $\mathbb{A}_{\text{Spec } R}^1$ and \mathcal{E} extends to a bundle on $\mathbb{P}_{\text{Spec } R}^1$, then \mathcal{E} is a trivial bundle.*

Proof. Suppose \mathfrak{m} is the maximal ideal of R and κ is the residue field. Let us suppose that \mathcal{E} extends to a vector bundle \mathcal{G} on \mathbb{P}_R^1 . The restriction $\mathcal{G}|_{\text{Spec } \kappa}$ is a vector bundle on \mathbb{P}_κ^1 . Therefore, by Corollary 8.1.1.5, the bundle $\mathcal{G}|_{\text{Spec } \kappa}$ is a direct sum of line bundles over κ . On the other hand, tensoring by a line bundle on $\mathbb{P}_{\text{Spec } R}^1$ will not affect the form of the restriction to $\text{Spec } R[x]$. Therefore, we may assume that $\mathcal{G}|_{\text{Spec } \kappa} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$ where $a_r \geq 0$.

The proof proceeds by induction on the rank of \mathcal{E} . Evidently a rank 0 projective module is extended from $\text{Spec } R$, so assume \mathcal{E} has rank > 0 . In that case, observe that there is an exact sequence of the form

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{G}|_{\text{Spec } \kappa} \longrightarrow \mathcal{G}|_{\kappa}/\mathcal{O} \longrightarrow 0.$$

The map $\mathcal{O} \rightarrow \mathcal{G}|_{\text{Spec } \kappa}$ is precisely a nowhere vanishing section. If we can extend this section to a nowhere vanishing section of \mathcal{G} , then we obtain a short exact sequence of modules of the form

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{O} \longrightarrow 0,$$

where \mathcal{G}/\mathcal{O} is locally free. The restriction of this exact sequence to $\mathbb{A}_{\text{Spec } R}^1$ then yields a short exact sequence of projective modules, which necessarily splits by the definition of projectivity.

Thus, we will try to lift a non-vanishing section of $\mathcal{G}|_{\text{Spec } \kappa}$ to \mathcal{G} . We do this in two steps. First, we can filter R by powers of \mathfrak{m} . In doing this, we obtain exact sequences of the form

$$0 \longrightarrow \mathfrak{m}^i/\mathfrak{m}^{i+1} \longrightarrow R/\mathfrak{m}^{i+1} \longrightarrow R/\mathfrak{m}^i \longrightarrow 0.$$

The maps $R \rightarrow R/\mathfrak{m}^{i+1}$ induce maps $\mathrm{Spec} R/\mathfrak{m}^{i+1} \rightarrow \mathrm{Spec} R$ and we obtain corresponding maps

$$\mathbb{P}_{\mathrm{Spec} R/\mathfrak{m}^{i+1}}^1 \longrightarrow \mathbb{P}_{\mathrm{Spec} R}^1.$$

Since \mathcal{G} is locally free, tensoring with the exact sequence above yields an exact sequence

$$0 \longrightarrow \mathcal{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1} \longrightarrow \mathcal{G} \otimes_R R/\mathfrak{m}^{i+1} \longrightarrow \mathcal{G} \otimes_R R/\mathfrak{m}^i \longrightarrow 0.$$

Taking cohomology of this short exact sequence yields a long exact sequence; we examine the portion of the sequence

$$H^0(\mathcal{G} \otimes_R R/\mathfrak{m}^{i+1}) \longrightarrow H^0(\longrightarrow \mathcal{G} \otimes_R R/\mathfrak{m}^i) \longrightarrow H^1(\mathcal{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

However, $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite-dimensional κ -vector space and thus by simply choosing a basis we obtain isomorphisms $H^1(\mathcal{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1}) \cong H^1(\mathcal{G}|_{\kappa}) \otimes \mathfrak{m}^i/\mathfrak{m}^{i+1}$. In particular, if $H^1(\mathcal{G}|_{\kappa})$ vanishes, then we may lift at each stage. By our assumptions $\mathcal{G}|_{\kappa}$ is a direct sum of bundles of the form $\mathcal{O}(n)$ with $n \geq 0$. In particular, the first Čech cohomology of each such bundle vanishes, and we deduce the required surjectivity.

Now, there are maps $H^0(\mathcal{G}) \rightarrow H^0(\mathcal{G} \otimes_R R/\mathfrak{m}^{i+1})$ and taking the inverse limit, we obtain a map

$$H^0(\mathcal{G}) \longrightarrow \lim_n H^0(\mathcal{G} \otimes_R R/\mathfrak{m}^{i+1})$$

On the other hand, $H^0(\mathcal{G})$ is an R -module and thus has a topology induced by powers of \mathfrak{m} . If we complete this R -module, we obtain a module $\widehat{H^0(\mathcal{G})}$. The map above factors as

$$H^0(\mathcal{G}) \longrightarrow \widehat{H^0(\mathcal{G})} \longrightarrow \lim_n H^0(\mathcal{G} \otimes_R R/\mathfrak{m}^{i+1}).$$

It is a special case of Grothendieck’s theorem on formal functions that the right hand map is an isomorphism (though in this case, we may simply check everything by hand).

Now, $H^0(\mathcal{G})$ is a finite generated R -module. By basic properties of completion, we conclude that

$$H^0(\mathcal{G}) \longrightarrow \widehat{H^0(\mathcal{G})} \longrightarrow H^0(\mathcal{G} \otimes_R \kappa)$$

is surjective. However, since $H^0(\mathcal{G} \otimes_R \kappa)$ has a nowhere vanishing section, we conclude that \mathcal{G} also has a nowhere vanishing section, but this is precisely what we wanted to show. \square

Remark 8.2.2.2. While the proof of Horrocks’ theorem is rather short and intuitive in this setting, it requires some algebro-geometric machinery. In applications, Quillen used a closely related algebraic version of the result. It is possible to give a purely algebraic proof of this algebraic version of the result: see [Lam06, Chapter IV] for more details. We have chosen to give Horrocks’ original proof since we found it geometrically appealing.

8.3 Lecture 32: The Quillen–Suslin theorem

It was observed by Murthy that a global version of Horrocks’ theorem would imply a solution to the Serre problem about triviality of projective modules over polynomial rings over a field.

8.3.1 Extending vector bundles from \mathbb{A}_R^1 to \mathbb{P}_R^1

We may use Horrocks’ theorem to effectively give a criterion to study when modules are extended. Suppose we begin with a vector bundle on \mathbb{A}_R^1 (for what we are about to say, it will not be necessary to assume that R is a Noetherian local ring). If we would like to extend this vector bundle to \mathbb{P}_R^1 , then we do this by attempting to glue. In order to glue, it suffices to extend \mathbb{A}_R^1 to a Zariski open cover of \mathbb{P}_R^1 . The simplest possible situation would be if we could find an open cover by two sets. The easiest open cover is, of course, the usual open cover by $\text{Spec } R[t]$ and $\text{Spec } R[t^{-1}]$. However, it would suffice to take *any* Zariski open subset of $t^{-1} = 0$ inside $\text{Spec } R[t]$. Now, we give an easy and useful criterion for gluing.

Lemma 8.3.1.1. *Suppose X is a scheme and we have a Zariski open cover of X by two open subschemes U and V . If \mathcal{E} is a rank n vector bundle on U , such that $\mathcal{E}|_{U \cap V}$ is trivial, then \mathcal{E} extends to a vector bundle on X .*

Proof. Take the trivial bundle $\mathcal{O}_V^{\oplus n}$ and observe that by assumption $\mathcal{O}_V^{\oplus n}|_{U \cap V} \cong \mathcal{E}|_{U \cap V}$. Gluing these two vector bundles, we obtain the required extension. \square

Rather than attempt to make a choice of a Zariski open subset of $\text{Spec } R[t^{-1}]$ that contains the section at “ ∞ ”, we will look at all possible refinements of Zariski neighborhoods of ∞ . To make this clearer, set $s = t^{-1}$. We want to consider Zariski open subsets of $\text{Spec } R[s]$ that contain $s = 0$. We look at elements of $R[s]$ of the form $1 + sR[s]$: such elements have constant term 1 and therefore evidently avoid $s = 0$. We consider the localization:

$$V_\infty := \text{Spec } R[s][(1 + sR[s])^{-1}]$$

Note that the map $R[s] \rightarrow R[s](1 + sR[s])^{-1}$ is a localization and thus flat. Essentially V_∞ is the intersection of all open sets that contain ∞ . We can therefore cover \mathbb{P}_R^1 by the two open sets $\text{Spec } R[t]$ and V_∞ . We now give a description of this intersection.

Proposition 8.3.1.2. *The intersection $\text{Spec } R[t] \cap V_\infty = \text{Spec } R\langle t \rangle$, where $R\langle t \rangle$ is the localization of $R[t]$ at the multiplicative set of all monic polynomials.*

Combining these two results, we deduce the following criterion for extensibility.

Corollary 8.3.1.3. *If P is a projective $R[t]$ -module, then if $P \otimes_{R[t]} R\langle t \rangle$ is free, then P extends to \mathbb{P}_R^1 .*

Proof. If P is a projective $R[t]$ -module and $P \otimes_{R[t]} R\langle t \rangle$ is free, then there exists a monic irreducible polynomial f such that P_f is a free $R[t]_f$ -module. We can view $R[t]_f$ as the intersection with \mathbb{A}_R^1 of an open subset of \mathbb{P}_R^1 containing the section at ∞ . \square

In order to make this result useful, we need to better understand $R\langle t \rangle$ -modules. To this end, observe that if $R = k$ is a field, then $R\langle t \rangle = k(t)$, thus $k(t)$ has smaller dimension than $R[t]$. We now observe that this phenomenon is general.

Lemma 8.3.1.4. *Suppose R is a Noetherian ring of Krull dimension d .*

1. *The ring $R\langle t \rangle$ has Krull dimension d .*

2. If R is PID (resp. a field), then so is $R\langle t \rangle$.

Proof. We understand prime ideals in $R[t]$ rather well and we know $R[t]$ has Krull dimension $d+1$. To show that $R\langle t \rangle$ has Krull dimension d , we have to show that every prime ideal $\mathfrak{P} \subset R[t]$ of height $d+1$ localizes to the unit ideal in $R\langle t \rangle$. Equivalently, we have to show that \mathfrak{P} contains a monic polynomial. Following [Lam06, IV Proposition 1.2], we give an elementary proof of this fact.

Set $\mathfrak{p} = \mathfrak{P} \cap R$. One knows that if \mathfrak{P} has height $d+1$, then \mathfrak{P} is not pulled back from R , and thus $\mathfrak{p}[t]$ is a proper subset of \mathfrak{P} while \mathfrak{p} has height d . Therefore, \mathfrak{p} is a maximal ideal in R . Now, suppose $f \in \mathfrak{P}$ is some polynomial with coefficients in R that lies outside of $\mathfrak{p}[t]$. Say $f = a_n t^n + \cdots + a_0$. We want to modify f by an element of \mathfrak{B} to be monic.

Without loss of generality, we may assume that a_n does not lie in \mathfrak{p} . Now, since \mathfrak{p} is maximal, we can find $c = a_n b - 1 \in \mathfrak{p}$. In that case, $b \cdot f - ct^n$ is a monic polynomial contained in \mathfrak{B} , but this is precisely what we wanted to show.

For the second point, it suffices to observe that if R is a UFD, then $R[t]$ is also a UFD and then $R\langle t \rangle$ is also a UFD. Since $\dim R[t] = \dim R$, if R has dimension 1, it suffices to observe that UFDs are normal. \square

Combining this result with Horrocks' result, we may establish a preliminary result about projective modules over rings that are not principal ideal domains.

Corollary 8.3.1.5. *If R is a discrete valuation ring, then every finitely generated projective $R[t]$ -module is free.*

Proof. By assumption R is a local PID. Thus, by the lemma above $R\langle t \rangle$ is also a PID. In particular, every f.g. projective module over $R\langle t \rangle$ is free. Now, suppose P is a f.g. projective $R[t]$ -module. By what we just said $P \otimes_{R[t]} R\langle t \rangle$ is a free $R\langle t \rangle$ -module. Therefore, by the proposition above, we may extend P to a vector bundle over \mathbb{P}_R^1 . In that case, it follows from Horrocks' theorem, that P is extended from an R -module P_0 . But since R is a local PID, it follows that P_0 is free itself. Therefore, P is also free. \square

Remark 8.3.1.6. This discussion makes it clear that if one has a “global” version of Horrocks' theorem, then one would inductively be able to understand vector bundles on polynomial rings over a PID. That this is true, was more-or-less observed by Murthy shortly after Horrocks' theorem was published. Quillen's solution to the Serre problem proceeds precisely in this fashion by allowing one to prove a global version of Horrocks' theorem.

8.3.2 Quillen's patching theorem

Following Quillen, we now search for a global version of Horrocks' theorem. Recall that descent theory tells us that if R is a commutative unital ring and f and g are a pair of comaximal elements of R , then one way to build a projective R -module is by specifying projective R_f and R_g -modules together with suitable gluing data. Suppose we would like to tell if a given R -module is trivial. Know that the associated R_f and R_g -modules obtained by localization are trivial is certainly not sufficient to guarantee triviality. However, we could ask if, perhaps, one can modify the isomorphism over R_{fg} to guarantee that the glued module is trivial. Quillen's local-to-global principle precisely addresses this problem.

Theorem 8.3.2.1. *If M is a finitely presented $R[T]$ -module, and $M_{\mathfrak{m}}$ is an extended $R_{\mathfrak{m}}[t]$ -module for each maximal ideal $\mathfrak{m} \subset R$, then M is extended.*

Proof. Our argument follows the presentation of [Lam06, Theorem V.1.6]. Let $Q(M)$ be the set of $f \in R$ such that M_f is an extended $A_f[t]$ -module. We claim that $Q(M)$ is an ideal in A . We must show that if $f_0, f_1 \in Q(M)$, then $f = f_0 + f_1$ is also in $Q(M)$. After replacing R by R_f , we may assume that f_0 and f_1 are comaximal in R . If we set $N = M/tM$, then we will try to show that $M \cong N[t]$.

We can assume that M_{f_i} is extended from $N_{f_i}[t]$ and thus we may fix automorphisms $u_i : M_{f_i} \rightarrow N_{f_i}[t], i = 0, 1$. After composing with a suitable automorphism of $N_{f_i}[t]$ if necessary, we may assume that u_i reduces modulo t to the identity map of N_{f_i} . Pictorially, we have the following situation:

$$\begin{array}{ccccc}
 M_{f_0} & \xrightarrow{\quad\quad\quad} & M_{f_0f_1} & \xleftarrow{\quad\quad\quad} & M_{f_1} \\
 \downarrow & & \swarrow & \xrightarrow{(u_1)_{f_0}} & \downarrow \\
 N_{f_0}[t] & \xrightarrow{\quad\quad\quad} & N_{f_0f_1}[t] & \xrightarrow{\theta} & N_{f_1}[t] \\
 & & \swarrow & \xrightarrow{(u_0)_{f_1}} & \swarrow
 \end{array}$$

If $(u_0)_{f_1} = (u_1)_{f_0}$, then by Zariski descent, these two isomorphism patch together to give a module isomorphism $M \cong N[t]$ and we are done.

Quillen’s idea is to modify the choices of u_0 and u_1 by suitable automorphisms to guarantee that we may patch. Note that the element

$$\theta = (u_1)_{f_0} \circ ((u_0)_{f_1})^{-1} \in \text{End}_{R_{f_0f_1}[t]}(N)_{f_0f_1}[t] \cong \text{End}_R(N)_{f_0f_1}[t].$$

Set $E = \text{End}_R(N)$. By assumption θ reduces to the identity modulo t , i.e., $\theta \in (1 + tE_{f_0f_1}[t])^\times$. Therefore, it suffices to show that θ may be rewritten as $(v_1)_{f_0}^{-1} \circ (v_0)_{f_1}$ for suitable $v_i \in E_{f_i}[t]$. Granting this for the moment (it will be established in Lemma 8.3.2.2) then $(v_0u_0)_{f_1} = (v_1u_1)_{f_0}$ and so after replacing u_i by v_iu_i , we may patch together our local extensions as observed above.

To establish the result, it suffices then to show that $Q(M)$ is the unit ideal. Set $M' = R[t] \otimes_R M/tM$; this is a finitely presented $R[t]$ -module that is extended from M . For any maximal ideal $\mathfrak{m} \subset R$, there exists an isomorphism $\varphi : M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$. Since φ is a map of finitely presented modules, by clearing the denominators we conclude that there is an element $g \in R \setminus \mathfrak{m}$ such that φ is the localization of an isomorphism of $R_g[t]$ -modules $M_g \rightarrow M'_g$. In that case, $g \in Q(M) \setminus \mathfrak{m}$ and therefore $Q(M)$ is an ideal that is not contained in \mathfrak{m} which means that $Q(M) = R$. \square

Lemma 8.3.2.2. *Let R be a commutative unital ring, and suppose E is an R -algebra (not necessarily commutative!). If $f \in R$ and $\theta \in (1 + TE_f[T]^\times)$, then there exists an integer $k \geq 0$ such that for any $g_1, g_2 \in R$ with $g_1 - g_2 \in f^kR$, there exists $\psi \in (1 + TR[T]^\times)$ such that $\psi_f(T) = \theta(g_1T)\theta(g_2T)^{-1}$.*

Proof. To be added. For the moment, see [Lam06, Corollary V.1.2-3] \square

8.3.3 Globalizing Horrocks’ theorem and the Quillen–Suslin theorem

Combining the results so far, we may give the “global” version of Horrocks’ theorem.

Corollary 8.3.3.1. *If M is a finitely generated projective $R[t]$ -module that is the restriction of a vector bundle on $\mathbb{P}_{\text{Spec } R}^1$, then M is extended.*

Proof. To check whether M is extended, it suffices to check whether M is extended after localizing at every maximal ideal $\mathfrak{m} \subset R$. However, if M is a vector bundle on $R_{\mathfrak{m}}[t]$ that extends to $\mathbb{P}_{R_{\mathfrak{m}}}^1$, then M is extended by Horrocks’ theorem. Therefore, M is extended. \square

Finally, we may establish the Quillen–Suslin theorem.

Theorem 8.3.3.2 (Quillen–Suslin). *If R is a principal ideal domain, then $\mathcal{V}_r(R) \rightarrow \mathcal{V}_r(R[x_1, \dots, x_n])$ is an isomorphism.*

Proof. We proceed by induction on n . The result is true if $n = 0$ by the structure theorem. Take $A = k[t_1, \dots, t_{n-1}]$ and set $t = t_n$. Then, $B = A \otimes_{R[t]} R\langle t \rangle$ is a polynomial ring in $n - 1$ -variables over $R\langle t \rangle$. However, since R is a principal ideal domain, so is $R\langle t \rangle$. Therefore, $M \otimes_R [t]R\langle t \rangle$ is free over B by the induction hypothesis. Thus, $M \otimes_{A[t]} A\langle t \rangle$ is free over $A\langle t \rangle$ and therefore M is free by the results above. \square

8.4 Lecture 33: Lindel’s theorem on the Bass–Quillen conjecture

In this section, we turn our attention to \mathbb{A}^1 -invariance of the functor $\mathcal{V}_r(X)$. We already know that if $X = \mathbb{P}^1$, then \mathbb{A}^1 -invariance fails. However, buoyed by the Quillen–Suslin theorem, we consider the problem of \mathbb{A}^1 -invariance for $X = \text{Spec } R$ with R a regular ring. That this should be true was conjectured by Bass and became known as the Bass–Quillen conjecture. We begin by giving a mild strengthening Quillen’s patching theorem 8.3.2.1. Using this version of patching, to establish the Bass-Quillen conjecture in general, it suffices to establish it for a regular local ring. Beyond Quillen’s patching theorem, Lindel’s key idea was to reduce the result to the case of polynomial rings by using his étale neighborhood theorem and a refined étale descent result for vector bundles (though in the form we will state the result it will not be a special case of étale descent).

8.4.1 Quillen’s patching revisited and Roitman’s “converse”

Once again, our treatment follows [Lam06, Theorem 1.6]. We generalize Quillen’s theorem to treat two special cases of the Bass–Quillen conjecture.

Theorem 8.4.1.1. *If R is a commutative unital ring, and M is a finitely presented $R[t_1, \dots, t_n]$ -module, then following statements hold.*

(A_n) *The set $Q(M)$ consisting of elements $g \in R$ such that M_g is extended from an R_g -module is an ideal in R (sometimes called the Quillen ideal).*

(B_n) *If $M_{\mathfrak{m}}$ is extended from an $R_{\mathfrak{m}}$ -module for every maximal ideal $\mathfrak{m} \subset R$, then M is extended.*

Proof. To be added. \square

Corollary 8.4.1.2. *If R is a Dedekind domain, then every $R[t_1, \dots, t_n]$ -module is extended from R .*

Proof. By Quillen’s patching theorem, it suffices to prove this when R is a local Dedekind domain, i.e., when R is a local PID, but this follows immediately from the Quillen–Suslin theorem. \square

The above result admits a rather strong generalization, due to Roitman (without assuming the Quillen–Suslin theorem).

Theorem 8.4.1.3 (Roitman). *Suppose R is a commutative unital ring and $S \subset R$ is a multiplicative set. Fix an integer $n \geq 1$. If every finitely generated projective $R[t_1, \dots, t_n]$ -module is extended from R , then every finitely generated $R[S^{-1}][t_1, \dots, t_n]$ -module is extended from $R[S^{-1}]$.*

Proof. This is [Roi79, Proposition 2] (see also [Lam06, Theorem V.1.11]). By induction on n , it suffices to treat the case where $n = 1$. Therefore, assume every finitely generated projective $R[t]$ -module is extended from R and suppose P is a finitely generated projective $R[S^{-1}][t]$ -module. By Quillen's patching theorem, we may replace $R[S^{-1}]$ by $(R[S^{-1}])_{\mathfrak{m}}$ for \mathfrak{m} a maximal ideal of $R[S^{-1}]$. Equivalently, we may find a prime $\mathfrak{p} \subset R$ such that $(R[S^{-1}])_{\mathfrak{m}} = R_{\mathfrak{p}}$ and therefore, we may assume without loss of generality that $R[S^{-1}] = R_{\mathfrak{p}}$.

Thus, suppose P is a finitely generated projective $R_{\mathfrak{p}}[t]$ -module. We want to show that P is free. Since P is a direct summand of a finitely generated free module $P = R_{\mathfrak{p}}[x]^{\oplus n}$, it is determined by a projection operator on $R_{\mathfrak{p}}[t]^{\oplus n}$. We want to show that this projection operator is conjugate in $\text{Aut}_{R_{\mathfrak{p}}[t]}(R_{\mathfrak{p}}[x]^{\oplus n})$ to the matrix $\text{diag}(1, \dots, 1, 0, \dots, 0)$ where the number of 1s that appear is given by the rank of P .

Write $e(t)$ for the projection operator associated with P . Since $R_{\mathfrak{p}}$ is local, the module P/tP is free, and therefore $e(0)$ is conjugate in $\text{Aut}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\oplus n})$ to $\text{diag}(1, \dots, 1, 0, \dots, 0)$. Thus, by conjugation by an element of $\text{Aut}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\oplus n})$ we can assume without loss of generality that $e(0)$ is equal to the standard projection operator.

By clearing the denominators, we may find an element $r \in R \setminus \mathfrak{p}$ such that $e(rt)$ lies in the image of

$$M_n(R[t]) \rightarrow M_n(R_{\mathfrak{p}}[t])$$

(the constant terms $e(0)$ are 0 or 1, which already lie in R). Thus, we may fix $e_0(t) \in M_n(R[t])$ that localizes to $e(rt)$ such that $e_0(0)$ is the standard operator above. Since $e(rt)$ is a projection operator, $e(rt)^2 - e(rt) = 0$. Therefore, since $e_0(t)$ localizes to $e(rt)$, we conclude that $e_0(t)^2 - e_0(t)$ localizes to zero and therefore is killed by some element $s \in R \setminus \mathfrak{p}$.

Since $e_0(0)^2 = e_0(0)$, we conclude that $e_0(t)^2 - e_0(t)$ has the form $t\epsilon(t)$ for some matrix $\epsilon(t)$ in $R[t]$. Now, t is not a zero-divisor in $R[t]$. Therefore, $ste(t) = 0$ implies $s\epsilon(t) = 0$. Therefore, $s\epsilon(st) = 0$ as well. Thus,

$$e_0(st)^2 - e_0(st) = st\epsilon(st) = 0 \in M_n(R[t]),$$

and therefore, $e_0(st)$ determines a finitely generated projective $R[t]$ -module as well. Because every $R[t]$ -module is extended from R , it follows that this module is extended from R as well. Therefore, we may find $\sigma(t) \in \text{Aut}_{R[t]}(R[t]^{\oplus n})$ such that

$$\sigma(t)^{-1}e_0(st)\sigma(t) = e_0(0).$$

Localizing to $M_n(\mathfrak{A}_{\mathfrak{p}}[t])$, this becomes

$$\sigma(t)^{-1}e(rst)\sigma(t) = e(0),$$

and dividing by rs yields the formula we want:

$$\sigma\left(\frac{t}{rs}\right)^{-1}e(t)\sigma\left(\frac{t}{rs}\right) = e(0).$$

□

Combining Roitman's theorem and the Quillen–Suslin theorem, we may deduce another special case of the Bass–Quillen conjecture: the conjecture holds for R the localization of a polynomial ring over a field or PID.

Corollary 8.4.1.4. *If R is a principal ideal domain (or a field), and A is a localization of a polynomial ring over R , then any finitely generated projective $A[t_1, \dots, t_n]$ -module is extended.*

Proof. By the Quillen–Suslin theorem, if B is a polynomial ring over R , then every finitely generated projective B -module is free. Thus, every finitely generated $B[t_1, \dots, t_n]$ -module is extended from B , since every such module is free, again by the Quillen–Suslin theorem. Now, write A as a localization of suitable B and apply Roitman's theorem. □

8.4.2 Lindel's patching theorem

Essentially, Lindel's approach to the Bass–Quillen conjecture was to try to reduce it to the two results established above. The key step in this reduction was a patching result that rests on Lindel's Nisnevich neighborhood theorem: this is the place where, unlike the proof of the Quillen–Suslin theorem, one is forced to assume that one is considering regular rings containing a field. Indeed, Lindel's theorem shows that any regular local ring containing a field (such that the residue field is separable over the base) is a Nisnevich neighborhood the localization of a polynomial ring at a maximal ideal. The idea is then to use induction on the dimension combined with validity of the conjecture over localizations of polynomial rings to conclude. There is one further technical issue that arises: we cannot, without some restrictions, guarantee that residue field extensions at maximal ideals are always separable: one way to guarantee this is to assume one is working with regular varieties over a perfect field. It is possible to remove this assumption, but we treat this afterwards so as not to complicate the essential geometric idea of the proof.

Suppose k is a perfect field, R is a localization of a finite-type regular k -algebra of dimension d at a maximal ideal \mathfrak{m} . If κ is the residue field of R at \mathfrak{m} , we may find a polynomial ring $\kappa[x_1, \dots, x_d] \subset R$ such that, setting $\mathfrak{n} = \kappa[x_1, \dots, x_d] \cap \mathfrak{m}$ and $S = \kappa[x_1, \dots, x_d]_{\mathfrak{n}}$, the map $S \rightarrow R$ is an étale neighborhood. In fact, without too much work we may refine this neighborhood to a covering.

Lemma 8.4.2.1. *Let R be an étale neighborhood of a local ring S . There exists an element $f \in \mathfrak{n}$ such that*

$$\begin{array}{ccc} S & \longrightarrow & S_f \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_f \end{array}$$

is an affine étale cover.

Proof. This is a consequence of local structure of étale morphisms. Essentially we may factor $\text{Spec } R \rightarrow \text{Spec } S$ as the composite $\text{Spec } R \hookrightarrow \text{Spec } S[t] \rightarrow \text{Spec } S$ where the first map is a closed immersion defined by a polynomial $h(t) \in S[t]$ such that $h(0)$ lies in the maximal ideal of S and $h'(0)$ is a unit. In that case, we may take $f = h(0)$ and it suffices to check the remaining properties are satisfied. \square

If we take any non-zero element f of nS and we invert it, then the resulting rings S_f and R_f have dimension *smaller* than d . Note also that R_f is actually regular as the localization of a regular k -algebra is again regular (unfortunately, we have not proven this statement in this generality). Thus, we have the following picture:

$$\begin{array}{ccc} S & \longrightarrow & S_f \\ \downarrow & & \downarrow \\ R & \longrightarrow & R_f \end{array}$$

is an affine étale cover of S by S_f and R . Similarly, for any integer $n \geq 0$, we obtain an affine étale cover $S[t_1, \dots, t_n]$ of the form

$$\begin{array}{ccc} S[t_1, \dots, t_n] & \longrightarrow & S_f[t_1, \dots, t_n] \\ \downarrow & & \downarrow \\ R[t_1, \dots, t_n] & \longrightarrow & R_f[t_1, \dots, t_n]. \end{array}$$

Therefore, by étale descent, we may build projective $S[t_1, \dots, t_n]$ -modules by patching together projective $R[t_1, \dots, t_n]$ -modules and projective $S_f[t_1, \dots, t_n]$ -modules that agree upon extension of scalars to $R_f[t_1, \dots, t_n]$.

A finitely generated projective $R[t_1, \dots, t_n]$ -module P determines an $R_f[t_1, \dots, t_n]$ -module P' . If we work inductively with respect to the dimension of R , we may assume that P' is extended from an R_f -module P'_0 . Note that $P'_0 \cong P'/(t_1, \dots, t_n)P'$. We claim that P' is actually free. To see this, observe that $P'_0 \cong P'/(t_1, \dots, t_n)P' \cong (P/(t_1, \dots, t_n)P)_f$. Since R is local, $(P/(t_1, \dots, t_n)P)$ is already a free R -module. On the other hand, finitely generated projective $S_f[t_1, \dots, t_n]$ -modules are always extended from S_f by the corollary to Roitman's theorem established above. In fact, such modules are free. Therefore, étale descent tells us that we may glue P and a free $S_f[t_1, \dots, t_n]$ -module to obtain an $S[t_1, \dots, t_n]$ -module \tilde{P} . However, S is the localization of a polynomial ring and therefore, again by appeal to Roitman's theorem, we conclude that \tilde{P} is again extended from an S -module \tilde{P}_0 . Since $P \cong \tilde{P} \otimes_{S[t_1, \dots, t_n]} R[t_1, \dots, t_n]$, we conclude by associativity of tensor product that $P/(t_1, \dots, t_n)P \cong \tilde{P}_0 \otimes_S R$, i.e., that P is extended as well. Thus, putting everything together with Quillen's patching theorem, we have established the following fact.

Theorem 8.4.2.2 (Lindel). *If k is a perfect field, and R is a finite-type regular k -algebra, then every finitely generated $R[t_1, \dots, t_n]$ -module is extended from R .*

8.4.3 The Bass–Quillen conjecture: the geometric case and beyond

Finally, we eliminate the hypothesis on perfection of the base field.

Theorem 8.4.3.1. *Suppose R is a regular k -algebra, essentially of finite type over k . For any integer $r \geq 0$ and any integer $n \geq 0$, the map*

$$\mathcal{V}_r(R) \longrightarrow \mathcal{V}_r(R[t_1, \dots, t_n])$$

is a bijection.

Proof. It suffices to reduce to the case where k is perfect; this reduction was sketched by Mohan Kumar. Let k_0 be the prime field of k . We may write R as a quotient of a polynomial algebra over k : $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. Since P is a finitely generated projective R -module, it is the image of an idempotent endomorphism of a free R -module of finite rank. Let k' be the subfield of k generated by the coefficients of f_1, \dots, f_r and of the entries of α . Set $R' = k'[x_1, \dots, x_n]/(f_1, \dots, f_r)$. By construction, there is a projective module P' such that P is obtained by extending scalars from P' . Note also that $R' \rightarrow R' \otimes_{k'} k = R$ is a faithfully flat ring map. Since R is regular, it follows that R' is regular as well (use faithfully flat descent to show that it has finite global dimension). Since k'/k_0 is a finite extension, it follows that R' has essentially finite type over k_0 as well. Thus, replacing R by R' , we may assume that the base field is perfect, in which case the result follows from the version of Lindel's theorem established above. \square

8.4.4 Popescu's extension of Lindel's theorem

Popescu explained how to use approximation theorems to establish the Bass-Quillen conjecture in certain mixed-characteristic situations. In particular, if R is a Dedekind domain with perfect residue fields, he generalized Lindel's étale neighborhood theorem in a fashion that it could be applied to certain regular R -algebras A . We now state and prove the Lindel–Popescu's étale neighborhood theorem.

Theorem 8.4.4.1 ([Pop89, Proposition 2.1]). *Let R be a discrete valuation ring, p a local parameter in R , and (A, \mathfrak{m}) a regular local R -algebra, essentially of finite type. Set $\kappa = A/\mathfrak{m}$, and $k = \text{Frac}(R/(\mathfrak{m} \cap R))$. If*

1. $k \subset \kappa$ is separable;
2. $p \notin \mathfrak{m}^2$; and
3. $\dim A \geq 2$,

then A is an étale neighborhood of a localization of a polynomial R -algebra.

8.5 Lecture 34: Grassmannians and naive \mathbb{A}^1 -homotopies

Our goal in this section is to show that Lindel's theorem may be translated into a statement about naive \mathbb{A}^1 -homotopy classes of maps to a suitable Grassmannian variety. To begin, we recall the construction of Grassmannian varieties in algebraic geometry. One key point here is that we describe maps from an arbitrary affine scheme to a Grassmann variety.

8.5.1 Finite-dimensional Grassmannians

Classically, the Grassmannian is an object of linear algebra. Fix a field k , and let V be an n -dimensional vector space over a field k . As a set $Gr_{n,N}$ parameterizes n -dimensional quotients (or

sub-spaces) of an N -dimensional k -vector space. We begin by giving a construction of $Gr_{n,N}$ as a scheme (over $\text{Spec } \mathbb{Z}$, since this adds no additional complication). The idea of the construction can be thought of as a generalization of homogeneous coordinates, analogous to the construction of projective space. We will show that $Gr_{n,N}$ can be obtained by gluing together copies of affine space. The construction follows one of the standard constructions in differential geometry and simply observes that all the defining maps are given by polynomials.

We would like to show that $Gr_{n,N}$ is naturally the set of k -rational points of a smooth projective k -scheme. To this end, we follow the usual description of coordinate charts. Fix a basis e_1, \dots, e_N of V . If $W \subset V$ is an n -dimensional subspace, then by picking a basis w_1, \dots, w_n of W and writing v_i in terms of the basis e_1, \dots, e_n , we may associate with W an $n \times N$ -matrix of rank precisely n . Now, the space of $n \times N$ -matrices of rank precisely n is an algebraic variety: it is an open subscheme of \mathbb{A}^{nN} whose closed complement is defined by the vanishing of the $n \times n$ -minors.

We define $V_{n,N}$ to be the open subscheme of \mathbb{A}^{nN} complementary to the closed subscheme whose ideal is given by the vanishing of $n \times n$ -minors of V . Observe that many different $n \times N$ -matrices of rank n give rise to the same subspace: indeed, the redundancy is precisely the choice of basis of W . At the level of k -points, the change of basis of W corresponds to left multiplying by an element of $GL_n(k)$. However, by means of such multiplications, we can always reduce an $n \times N$ -matrix to one where a fixed $n \times n$ -minor is the identity. Thus, we look at the closed subscheme of \mathbb{A}^{nN} with coordinates X_{ij} where a fixed $n \times n$ -minor is the identity matrix. The resulting subscheme is isomorphic to $\mathbb{A}^{n(N-n)}$ (with coordinates given by the non-constant entries). Set-theoretically, these subsets form a cover of $Gr_{n,N}$. We may explicitly write down the transition maps on overlaps using matrix inverses and by Cramer's rule, these maps are algebraic. Even better, they are polynomial and all coefficients are $0, \pm 1$. Gluing these copies of $\mathbb{A}^{n(N-n)}$ together gives $Gr_{n,N}$ the structure of a scheme.

Note that $Gr_{n,N}$ carries several “tautological” vector bundles γ_n of rank n . Geometrically: if V is an N -dimensional vector space, we may consider the quotient bundle of $Gr_{n,N} \times V$ whose fiber over $x \in Gr_{n,N}$ is the quotient R_x of V corresponding to x . This definition does not suffice to build a vector bundle over the scheme $Gr_{n,N}$, but it is easy to soup it up to define such a bundle. We build a geometric vector bundle by gluing copies of the trivial bundle of rank n over each open set in the previous section.

Now, if X is any scheme, a map $X \rightarrow Gr_{n,N}$ defines a rank n vector bundle on X together with an epimorphism from a trivial bundle of rank N . If $X = \text{Spec } R$ is affine, this corresponds to a rank n projective R -module, together with a set of N R -module generators. In fact, we claim this map is a bijection: given a rank n vector bundle on X together with a surjection from a rank N trivial bundle, we can reconstruct the map $X \rightarrow Gr_{n,N}$.

Suppose we are given a surjection $R^{\oplus N} \rightarrow P$. Pick a Zariski cover of $\text{Spec } R$ over which P and Q trivialize. In that case, if we fix a subset of $1, \dots, N$ of size n , then there is an induced inclusion map $R^{\oplus n} \rightarrow R^{\oplus N}$, and we can ask that the composite map $R^{\oplus n} \rightarrow P$ is an isomorphism. This determines a collection of regular functions on R_f and thus a map $R_f \rightarrow \mathbb{A}^{n(N-n)}$. Varying through the subsets of size n , we obtain maps that may be glued to obtain a map $R_f \rightarrow Gr_{n,N}$. Varying through the open cover, we may patch to determine our map $X \rightarrow Gr_{n,N}$.

The construction we have just outlined goes by many different names and is very robust. In differential geometry, the construction above is sometimes called the “Gauss map attached to a vector bundle” and it is one step in a standard argument relating isomorphism classes of vector

bundles to homotopy classes of maps to Grassmannians. In our context, we have just described the “functor of points” of the Grassmannian $Gr_{n,N}$.

8.5.2 Infinite Grassmannians

If V is an N -dimensional vector space, and V' is an $N + 1$ -dimensional vector space, then any injective map $V \rightarrow V'$ defines a map $Gr_{n,N} \rightarrow Gr_{n,N+1}$. These maps may be defined scheme-theoretically. Indeed, we simply want to specify a rank n -vector bundle on $Gr_{n,N}$ together with a surjection from a trivial bundle of rank $N + 1$. However, we may simply take the universal bundle γ_n equipped with its standard surjection and add an additional trivial summand that maps trivially to γ_n . Now, we would like to define an analog of the infinite Grassmannian that appears in topology.

We set

$$Gr_n := \operatorname{colim}_N Gr_{n,N},$$

but we need to work a bit to make sense of the object on the right hand side. For our purposes here, we may view $Gr_{n,N}$ as a presheaf on the category of schemes. In that case, we may take the colimit in the category of presheaves.

8.5.3 Naive homotopy classification

More precisely, suppose we define Gr_n to be the ∞ -Grassmannian. There is a rank n vector bundle on Gr_n . Given any smooth affine scheme X , by the definition of the colimit, a morphism $X \rightarrow Gr_n$ corresponds to a morphism $X \rightarrow Gr_{n,N}$ for N sufficiently large. Since $X = \operatorname{Spec} R$ is affine, such a morphism corresponds to a rank n projective module P over R together with a surjection $R^{\oplus N} \rightarrow P$, i.e., N generators of P . Thus, there is an evident surjective map

$$\operatorname{Hom}(X, Gr_n) \longrightarrow \mathcal{V}_n(X).$$

Here, the left hand side corresponds to natural transformations of functors. Now, the right hand side is \mathbb{A}^1 -invariant. The left hand side is evidently not \mathbb{A}^1 -invariant: if we take two different sets of N generators of a given projective module of rank n over R yield different maps to the Grassmannian. Therefore, we would like to form the quotient of the left hand side by the relation generated by naive \mathbb{A}^1 -homotopy.

Theorem 8.5.3.1. *If k is a field and X is a smooth affine k -scheme, then the map*

$$\operatorname{Hom}(X, Gr_n) / \sim_{\mathbb{A}^1} \longrightarrow \mathcal{V}_n(X)$$

that sends a map $X \rightarrow Gr_n$ to its naive \mathbb{A}^1 -homotopy class is a bijection.

Proof. It suffices to demonstrate injectivity. Therefore, consider two maps $\varphi : X \rightarrow Gr_n$ and $\varphi' : X \rightarrow Gr_n$ that yield the same vector bundle. The map φ corresponds to a pair (P, e_1, \dots, e_r) where P is a rank n projective module and e_1, \dots, e_r are r -generators of P , while the map ψ corresponds to (P, f_1, \dots, f_s) . We want to show that the two resulting maps are naively \mathbb{A}^1 -homotopic. By adding copies of 0, we may view φ and ψ as N -generated projective modules where $N = r + s$. Thus, we want to construct a homotopy between the generators $(e_1, \dots, e_r, 0, \dots, 0)$ and $(0, \dots, 0, f_1, \dots, f_s)$.

To this end, consider the $R[t]$ -modules $P[t]$ obtained by extending scalars to $R[t]$. The set of elements e_1, \dots, e_r defines a set of generators for the $R[t]$ -module Pt and so does f_1, \dots, f_s . However,

$$(e_1, \dots, e_r, tf_1, \dots, tf_s) \text{ and } ((1-t)e_1, \dots, (1-t)e_r, f_1, \dots, f_s)$$

also define generators of $P[t]$. These two maps define a naive \mathbb{A}^1 -homotopy connecting the two different sets of generators, which is precisely what we wanted to prove. \square

This result may be improved in several different ways. Given a commutative unital ring R , it is not clear there is a uniform bound on the number of generators of a projective R -module of a fixed rank. If R is not finitely generated, such a bound need not exist for a given module. However, if R is finitely generated, then rank + dimension of $\text{Spec } R$ is a bound by a result of Forster-Swan. Passing to a larger Grassmannian was essential in the argument about to build a homotopy. As a consequence, the naive \mathbb{A}^1 -homotopy classes of maps to a fixed finite dimensional Grassmannian, do not obviously coincide with maps to the infinite Grassmannian.

Naive homotopy classes of maps to spheres: unimodular rows and complete intersection ideals...

Chapter 9

Symmetric bilinear forms and endomorphisms of \mathbb{P}^1

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In this section, we examine a surprising connection between symmetric bilinear forms and naive \mathbb{A}^1 -homotopy classes of maps to the projective line. The goal of this section is to expose a result of C. Cazanave [Caz12], building on ideas of J. Barge, J. Lannes [BL08], M. Ojanguren [Oja90] and F. Morel [Mor12]. In effect, this theory is the simplest non-trivial case of one that is still to be worked out: if X is a pointed smooth variety, then we may consider pointed morphisms $\mathbb{P}^1 \rightarrow X$; such maps are called *pointed rational curves on X* . Homotopic ideas suggest that the set of such morphisms might admit an *a priori* graded monoid structure and it would be interesting to understand the associated \mathbb{A}^1 -homotopy classes of maps.

9.1 Lecture 35: Pointed rational functions

Fix a base commutative unital ring R . In this section, we restrict our attention to \mathbb{P}_R^1 . Topologically $\mathbb{P}_\mathbb{C}^1$ is like a sphere, and the homotopy classification of self-maps of the sphere in terms of the Brouwer degree was an important result in homotopy theory. We now consider an algebro-geometric

analog of this question: can we give a homotopy classification of morphisms $\mathbb{P}_R^1 \rightarrow \mathbb{P}_R^1$ that sends the section at infinity to the section at infinity. We begin by describing a scheme that parameterizes such maps.

9.1.1 The scheme of pointed rational functions

Recall from the previous section that any endomorphism of \mathbb{P}^1 corresponds to specifying a line bundle \mathcal{L} on \mathbb{P}_R^1 together with two generating sections. We begin by restricting attention to $R = k$ is a field and write \mathbb{P}^1 for \mathbb{P}_k^1 . In that case, we saw $\text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$.

Suppose we are given a morphism $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ that sends ∞ to ∞ . If the map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ is not constant, then we may describe it as follows. Consider $\mathbb{A}^1 \subset \mathbb{P}^1$ in the target. Then $\varphi^{-1}(\mathbb{A}^1) = U \subset \mathbb{A}^1$. Choose a coordinate x on U so that we may write $U = k[x]_g$ for a suitable monic polynomial (whose vanishing defines $\mathbb{A}^1 \setminus U$ as a reduced scheme). The morphism φ is then uniquely determined by an element of $k[x]_g$, i.e., an expression of the form $\frac{f}{g}$. Since φ is pointed, we must have that $\deg f > \deg g$ and without loss of generality, we may assume that f and g have no common factor. Thus, adding zero coefficients if necessary, we may write

$$f = t^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad g = b_{n-1}x^{n-1} + \cdots + b_0.$$

for suitable elements $(a_{n-1}, \dots, a_0, b_{n-1}, \dots, b_0)$. The coefficients of this polynomial define a point in \mathbb{A}^{2n} . Thus, we have defined a function from the set of pointed morphisms $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ to an infinite disjoint union of copies of \mathbb{A}^{2n} .

The condition that f and g have no common factor may also be phrased purely algebraically: given two polynomials f and g in 1 variable, their resultant $\text{res}(f, g)$ admits an algebraic expression. If $\frac{f}{g} = \frac{A}{B}$ for two polynomials A and B of smaller degree, then $fB - gA = 0$. In other words, if f has degree m and g has degree n , then we consider the space of polynomials of degree $< m$ and of degree $< n$. Then, sending $(A, B) \rightarrow fB - gA$ defines a linear operator from $k[x]_{<m} \oplus k[x]_{<n} \rightarrow k[x]_{m+n-1}$. The condition that f and g have a common factor is thus that the kernel of this operator is non-zero. In terms of the basis given by powers of t , and the situation we considered above, we may write a formula for this operator: it is given by a so-called Sylvester matrix:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & b_{n-1} & 0 & \cdots & 0 & 0 \\ a_{n-1} & 1 & \cdots & 0 & 0 & b_{n-2} & b_{n-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & a_1 & 0 & 0 & \cdots & b_0 & b_1 \\ 0 & 0 & \cdots & 0 & a_0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix}$$

This is an $(2n + 1) \times (2n + 1)$ -matrix. The vanishing of its determinant determines a closed subscheme of \mathbb{A}^{2n} consisting of polynomials as above where f and g have a common factor; we write $\text{res}(f, g)$ for this determinant. Therefore, the complement of $\text{res}(f, g) = 0$ defines an open subscheme of \mathbb{A}^{2n} consisting of pairs (f, g) such that $\deg f = n$, $\deg g < n$, f is monic, and f and g have no common factor.

Definition 9.1.1.1. The scheme Rat_n of pointed rational functions of degree n is the open subscheme of \mathbb{A}^{2n} with coordinates $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}$ defined as the open complement of the closed subscheme defined by the vanishing of $\text{res}(f, g) = 0$.

Remark 9.1.1.2. By construction, the condition that $\text{res}(f, g) \neq 0$ implies that $Bf - Ag = 1$ where A and B are polynomials of degree $\leq n - 1$ and $\leq n - 2$ respectively. Moreover, these polynomials are necessarily unique.

Example 9.1.1.3. Note that Rat_0 consists precisely of the constant map, while Rat_1 contains the identity map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$; this is given by $(0, 1) \in \mathbb{A}^2(k)$. More generally, the map $\lambda x = \frac{x}{\lambda}$ for any $\lambda \in k^\times$ defines a degree 1 map; this corresponds to the element $(0, \lambda^{-1}) \in \mathbb{A}^2(k)$.

Naive \mathbb{A}^1 -homotopy classes

By definition, a naive \mathbb{A}^1 -homotopy of pointed rational functions is a family of pointed rational functions parameterized by \mathbb{A}^1 , i.e., a morphism $H : \mathbb{P}^1 \times \mathbb{A}^1 \rightarrow \mathbb{P}^1$ satisfying a suitable base-point condition. We saw above that for each integer n , there is a scheme of pointed rational functions of degree n , thus a better way to define such a homotopy is as follows. A naive homotopy between pointed rational functions of degree n is a morphism $\mathbb{A}^1 \rightarrow \text{Rat}_n$, i.e., an element of $\text{Rat}_n(k[t])$.

Example 9.1.1.4. Let us observe that there are non-trivial such homotopies. Suppose we have two polynomials (f, g) as above. Then $(ta_0, ta_1, \dots, ta_{n-1}, 1, 0, \dots, 0)$ yields a homotopy between $(f, 1)$ and the polynomial $(x^n, 1)$ (here checking that the resultant condition is satisfied is easy). Similarly, we may always simplify denominators if the numerator is x^n . However, if we write a general (f, g) , writing homotopies that stay within Rat_n can be rather difficult.

Definition 9.1.1.5. The set of pointed naive \mathbb{A}^1 -homotopy endomorphisms of \mathbb{P}^1 identifies with $\coprod_{n \geq 0} \text{Rat}_n / \sim_{\mathbb{A}^1}$.

9.1.2 A graded monoid structure on pointed endomorphisms

We are now going to equip the set of pointed naive \mathbb{A}^1 -homotopy endomorphisms of \mathbb{P}^1 with a monoid structure. We will not consider the one that arises from composition of endomorphisms. Rather, we want to consider one that is slightly more geometrically subtle. If we believe \mathbb{P}^1 is like a sphere, then there should be a monoid structure corresponding to the cogroup structure; in particular, the degree should be additive. The identity map for this monoid structure is the constant map (of degree 0) and to define the operation algebraically, we need to specify a morphism of schemes of the form

$$\text{Rat}_m \times \text{Rat}_n \longrightarrow \text{Rat}_{m+n}$$

for every pair of integers $m, n \geq 0$.

Suppose given two pairs (f_1, g_1) and (f_2, g_2) of degree m and n respectively. In this case, we may find unique polynomials (b_1, a_1) and (b_2, a_2) together with Bezout relations of the form $f_1 b_1 - a_1 g_1 = 1$ and $(f_2 b_2 - a_2 g_2) = 1$. (If one of m or n is zero, this is not really accurate: in that case, $f_i = (0, 1)$ and the Bezout-relation is given by $a_1 = 1, b_1 = 0$, i.e., one must make a convention about polynomials of negative degree.) We put each pair $f_i b_i - a_i g_i$ into the 2×2 -matrix

$$\begin{pmatrix} f_i & a_i \\ g_i & b_i \end{pmatrix};$$

and view this matrix as an element of $SL_2(k[t])$. Note that, under these identifications, the degree 0 map corresponds to the identity matrix in $SL_2(k[t])$.

Lemma 9.1.2.1. *If (f_1, g_1) and (f_2, g_2) are pointed rational functions of degree m and n respectively, then setting*

$$(f_1, g_1) \oplus (f_2, g_2) := \begin{pmatrix} f_1 & a_1 \\ g_1 & b_1 \end{pmatrix} \begin{pmatrix} f_2 & a_2 \\ g_2 & b_2 \end{pmatrix}$$

defines another pointed rational function (f_3, g_3) .

Proof. The product matrix is

$$\begin{pmatrix} f_1 f_2 + a_2 g_2 & f_1 a_2 + a_1 b_2 \\ g_1 f_2 + b_1 g_2 & g_1 g_2 + b_1 b_2 \end{pmatrix}.$$

And we simply set $(f_3, g_3) = (f_1 f_2 + a_2 g_2, g_1 f_2 + b_1 g_2)$. Now, simply observe that f_3 is a monic polynomial of degree $m + n$ since f_1 and f_2 are monic polynomials of degrees m and n , and $a_2 g_2$ has degree strictly smaller than $m + n$. \square

Proposition 9.1.2.2. *The binary operation on $\coprod_{n \geq 0} \text{Rat}_n$ just defined equips it with a graded monoid structure.*

Proof. It remains to check associativity of the operation, but this follows immediately from associativity of matrix multiplication. \square

Remark 9.1.2.3. Since matrix multiplication is not commutative, the operation \oplus is *not* evidently commutative. The monoid structure of the previous proposition induces a graded monoid structure on the set of naive \mathbb{A}^1 -homotopy classes as well.

Example 9.1.2.4. Consider the pointed rational function $(x, 1)$. In this case, the unique Bezout relation $bf - ag = 1$ takes the form $b = 0$ and $a = 1$, so the corresponding 2×2 -matrix is given by

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

Squaring this matrix gives $(x, 1) \oplus (x, 1) = (x^2 + 1, x)$. Similarly, we may produce “interesting” homotopies of rational functions by taking products of “uninteresting” homotopies as discussed above.

9.1.3 (Bezout) Symmetric bilinear forms

Suppose we are given a pointed rational function (f, g) . We may associate with (f, g) a symmetric bilinear form defined as follows. Suppose (f, g) has degree n . In that case, consider the expression

$$\text{B}_{f,g}(x, y) = \frac{f(x)g(y) - g(x)f(y)}{x - y} =: \sum_{1 \leq p, q \leq n} b_{p,q} x^{p-1} y^{q-1},$$

and observe that the matrix $\text{Bez}_{f,g} := [b_{p,q}]$ is a symmetric $n \times n$ -matrix.

Proposition 9.1.3.1 (Bezout). *If (f, g) is a pointed rational function of degree n , then*

$$\det \text{Bez}_{f,g} = (-1)^{\frac{n(n-1)}{2}} \text{res}(f, g).$$

In particular, if (f, g) is a pointed rational function, the form $\text{Bez}_{f,g}$ is a non-degenerate symmetric bilinear form.

Before moving forward, we recall some general facts about symmetric bilinear forms over a fields. If V is a k -vector space, then a symmetric bilinear form is a bilinear form $\varphi : V \times V \rightarrow k$ such that $\varphi(x, y) = \varphi(y, x)$. By duality, φ defines a map $\varphi^{ad} : V \rightarrow V^\vee$. We will say that φ is non-degenerate if φ^{ad} is an isomorphism. If we choose a basis of V , then we may identify φ^{ad} with an $n \times n$ -matrix over k , and the condition that φ^{ad} is an isomorphism is equivalent to the condition that the determinant of this matrix is non-zero. These conditions are all basis independent, but we can identify the space of symmetric bilinear forms with the space of $n \times n$ -symmetric matrices with non-zero determinant: this is an open subscheme of an affine space of dimension $\frac{n(n+1)}{2}$. Write S_n for the space of symmetric bilinear forms on an n -dimensional k -vector space.

Lemma 9.1.3.2. *The construction of Bézout defines a morphism of schemes*

$$\text{Rat}_n \longrightarrow S_n.$$

Proof. This follows from Bézout's formula 9.1.3.1. □

Since the block sum of symmetric matrices is again symmetric, it follows that block sum of matrices equips $\coprod_{n \geq 0} S_n$ with a monoid structure. Since $\coprod_{n \geq 0} \text{Rat}_n$ admits the structure of a graded monoid, it is natural to ask if the function just constructed actually is compatible with the monoid structure.

Example 9.1.3.3. Let us consider the Bézout form of the pointed rational function $(x, 1)$: in this case $f(x)g(y) - g(x)f(y) = x - y$ and thus the Bézout form is simply the 1×1 -identity matrix. On the other hand, we observed that if we consider the square of this map with respect to the monoid structure, then we obtained the pointed rational function $(x^2 + 1, x)$. The Bézout form of the latter is

$$B_{x^2+1,x} = \frac{(x^2 + 1)y - (y^2 + 1)x}{x - y} = -1 + xy,$$

and thus $\text{Bez}_{f,g} = \text{diag}(-1, 1)$. Note, in particular, that this matrix is *not* the 2×2 -identity matrix (check signs). Thus, the morphisms above is *not* a monoid map. Nevertheless, we may still ask whether it is a monoid map *up to homotopy*.

9.2 Lecture 36: Symmetric bilinear forms and Grothendieck-Witt groups

In the previous section, we observed that there was a “classical” connection between pointed endomorphisms of the projective line and symmetric bilinear forms over a field. Since we could construct a monoid structure on pointed rational functions, it is natural to ask whether there is a relationship between corresponding monoid structures on symmetric bilinear forms. Following the early work of Witt, we now discuss monoid structures in the latter case. For later use, we will develop this theory in greater generality than is necessary for applications to the ideas of the previous section.

9.2.1 Bilinear forms and inner product spaces

Suppose R is a commutative unital ring and M is an R -module. A bilinear form on M is a map $\beta : M \times M \rightarrow R$ such that β is R -linear in each variable separately. We will refer to a pair (M, β) as a *bilinear R -module*. Given bilinear modules (M, β) and (M', β') , a morphism of bilinear modules is an R -module map $\varphi : M \rightarrow M'$ such that $\beta'(\varphi(m_1), \varphi(m_2)) = \beta(m_1, m_2)$. We write Bil_R for the category whose objects are bilinear R -modules and whose morphisms are morphisms of such objects. An isomorphism of bilinear R -modules is called an isometry.

Remark 9.2.1.1. There is an evident forgetful functor $\text{Bil}_R \rightarrow \text{Mod}_R$. Note, however, that there is no additive structure on Bil_R compatible with additive structure on Mod_R simply because the sum of the linear maps underlying two morphisms of bilinear modules will not be a morphism of bilinear modules in general.

Given an element $m \in M$, $\beta(m, -)$ and $\beta(-, m)$ are R -linear maps $M \rightarrow R$. Moreover, sending $m \in M$ to $\beta(m, -)$ or $\beta(-, m)$ determines an R -linear map $M \rightarrow M^\vee$. We will say that a bilinear R -module (M, β) is *non-degenerate* if the two maps $M \rightarrow M^\vee$ just described are isomorphisms. A pair (M, β) consisting of an R -module M and a bilinear form β will be called an *inner product module*.

Definition 9.2.1.2. An inner product module (M, β) with M a finitely generated projective R -module will be called an *inner product space*.

The bilinear form β is called *symmetric* if for every $m, m' \in M$, $\beta(m, m') = \beta(m', m)$, i.e., the composite of β with the swap map $M \times M \rightarrow M \times M$ coincides with β . Similarly, a bilinear form β is called *skew-symmetric* if $\beta(m, m') = -\beta(m', m)$ for every $m, m' \in M$ and *symplectic* or *alternating* if $\beta(m, m) = 0$.

Remark 9.2.1.3. Note that every alternating bilinear form is skew-symmetric. Indeed, if (M, β) is an alternating bilinear form, then

$$0 = \beta(x + y, x + y) = \beta(x, x) + \beta(x, y) + \beta(y, x) + \beta(y, y) = \beta(x, y) + \beta(y, x).$$

Therefore, $\beta(x, y) = -\beta(y, x)$. On the other hand, if 2 is invertible in R , then any skew-symmetric form is alternating. Indeed, for any element $m \in M$, $\beta(m, m) = -\beta(m, m)$ and thus $2\beta(m, m) = 0$. Thus, $\beta(m, m) = 0$ for every $m \in M$. Conversely, if β is an alternating form, then $0 = \beta(x + y, x + y) = \beta(x, y) + \beta(y, x)$ and thus $\beta(x, y) = -\beta(y, x)$.

Definition 9.2.1.4. If (M, β) is a symmetric or skew-symmetric inner product space, then a pair of elements $x, y \in M$ are said to be *orthogonal* with respect to β , if $\beta(x, y) = 0$.

Lemma 9.2.1.5. If (M, β) is a symmetric or skew-symmetric inner product space, then an element $x \in M$ is zero if and only if x is orthogonal to every element of M is zero.

If $R \rightarrow S$ is a ring homomorphism, then there are extension of scalars functors $\text{Bil}_R \rightarrow \text{Bil}_S$.

Direct sums, tensor products and Grothendieck groups

If (M, β) and (M', β') are bilinear R -modules, then the direct sum $M \oplus M'$ can be equipped with a bilinear form that we will denote $\beta \oplus \beta'$. The pair $(M \oplus M', \beta \oplus \beta')$ will be called the orthogonal sum of (M, β) and (M', β') . This operation defines a coproduct in the category Bil_R . Direct sum equips the set of isomorphism classes on Bil_R with the structure of a monoid: the unit is given by the module 0 equipped with the trivial bilinear form.

Definition 9.2.1.6. If R is a ring, we write $W^{\text{mon}}(R)$ for the monoid of isomorphism classes of symmetric bilinear forms over R (with respect to orthogonal sum).

Likewise, there is a notion of tensor product of bilinear R -modules; we summarize the construction in the following result.

Lemma 9.2.1.7. *If (M, β) and (M', β') are bilinear R -modules, then there is a unique bilinear form $\beta \otimes \beta'$ on $M \otimes M'$ such that*

$$\beta \otimes \beta'(m_1 \otimes m'_1, m_2 \otimes m'_2) = \beta(m_1, m_2)\beta(m'_1, m'_2).$$

There is an evident unit for this bilinear structure which is the trivial R -module of rank 1 equipped with the multiplication map $R \times R \rightarrow R$. This tensor product of bilinear R -modules equips the category Bil_R with a symmetric bilinear structure. As a consequence of these structures, we may form the Grothendieck group of the category of symmetric bilinear forms.

Definition 9.2.1.8. If R is a commutative unital ring, then the Grothendieck-Witt ring of R , denoted $GW(R)$, is the Grothendieck group of the monoid of symmetric inner product spaces over R under the sum operation induced by direct sum and with product induced by tensor product of bilinear R -modules.

Remark 9.2.1.9. If $f : R \rightarrow S$ is a ring homomorphism, then there is an induced map $GW(R) \rightarrow GW(S)$.

Example 9.2.1.10 (Hyperbolic forms). If P is a finitely generated projective R -module, then there is always an evaluation isomorphism $P \times P^\vee \rightarrow R$, which induces a distinguished identification $P \rightarrow P^{\vee\vee}$. On the other hand, given a finitely generated projective R -module, specifying a non-degenerate symmetric bilinear form β on P is equivalent to specifying the isomorphism $P \xrightarrow{\sim} P^\vee$ (defined by $m \mapsto \beta(m, -)$). Combining these two observations, observe that there is always an isomorphism $P \oplus P^\vee \rightarrow P^\vee \oplus P^{\vee\vee} \cong P^\vee \oplus P$ and therefore $P \oplus P^\vee$ carries a distinguished symmetric bilinear form. We write $H(P)$ for $P \oplus P^\vee$ equipped with this form, which is called a *hyperbolic form*.

Example 9.2.1.11 (Diagonal forms). If $u \in R^\times$, we write $\langle u \rangle$ for the free R -module of rank 1 equipped with the (symmetric) bilinear form whose matrix representation is u . By what we just observed, $\langle u \rangle \cong \langle u' \rangle$ if and only if $u = a^2 u'$ for a some unit. More generally, we may consider the symmetric bilinear form $\langle u_1 \rangle \oplus \cdots \oplus \langle u_n \rangle$.

The assignment $P \mapsto H(P)$ determines a functor from the category of finitely generated projective R -modules to the category of symmetric inner product spaces over R .

Corollary 9.2.1.12. For any ring R , the group homomorphism $K_0(R) \xrightarrow{H} GW_0(R)$ has image an ideal in $GW_0(R)$.

Note that the group homomorphism above is *not* a ring homomorphism since it is not unital.

Definition 9.2.1.13. The Witt ring of a ring R , denoted $W(R)$ is the cokernel of the map $K_0(R) \xrightarrow{H} GW(R)$, i.e., there is a short exact sequence

$$K_0(R) \longrightarrow GW(R) \longrightarrow W(R) \longrightarrow 0.$$

9.2.2 Local rings: diagonalization and stabilization

In this section, we restrict attention to the case where R is a local ring and we investigate the structure of the Witt monoid and the Grothendieck-Witt group. If R is local, then every projective R -module is free. Thus, if (V, φ) is an inner product space over R , then we may choose a basis e_1, \dots, e_n over R and express φ in this basis; we write B for the matrix representation of φ in this basis. If we change bases, then the matrix representation B becomes ABA^t , where A is the change of basis matrix. Since $\det(ABA^t) = \det(A)^2 \det(B)$, it follows that the determinant of the matrix B is well-determined up to squares. Thus, we obtain a function

$$\text{disc} : \text{Bil}_R \longrightarrow R^\times / R^{\times 2}$$

called the discriminant of the bilinear form. Note that this function is evidently compatible with the direct sum operation since the matrix representation of the direct sum is the block-sum of matrices, and the determinant of a block matrix is simply the product of the determinants of the blocks.

Remark 9.2.2.1. One may generalize the discriminant construction as follows. Suppose R is an arbitrary commutative unital ring and (P, φ) is an inner product space, such that P has rank n . We may construct symmetric and exterior powers of bilinear R -modules using the tensor product described above. If (P, φ) is symmetric, then so is $(\wedge^n P, \wedge^n \varphi) =: (\det P, \det \varphi)$. The rank 1 form is called the *determinant of (P, φ)* .

Since $\wedge^n P$ is locally free of rank 1, we may find a Zariski open cover $\text{Spec } R_{f_i}$ of R over which P trivializes. If e_i is a basis of sections over R_{f_i} , then $(\det P, \det \varphi)$ is locally given by $\langle u_i \rangle$ for some unit u_i . Note that u_i is well-defined up to squares. In particular, $(\wedge^n P)^{\otimes 2}$ is locally isomorphic to the free module of rank 1. Using gluing, we conclude that $(\det P, \det \varphi)$ has the property that $\det P^{\otimes 2} \xrightarrow{\sim} R$. Fixing such an identification, we see that $\det P$ yields a μ_2 -torsor that we will call the *discriminant module* of (P, φ) .

Over the real numbers, Sylvester's inertia theorem implies that every symmetric bilinear form may be diagonalized with diagonal entries ± 1 . The following result generalizes this to arbitrary local rings and is often known as Sylvester's inertia theorem.

Theorem 9.2.2.2. Suppose (V, φ) is a symmetric bilinear form over a local ring R .

1. There is an orthogonal sum decomposition of the form:

$$V \cong \langle u_1 \rangle \oplus \cdots \oplus \langle u_n \rangle \oplus N$$

where the u_i s are units in R and $\varphi(x, x)$ is a non-unit for every $x \in N$.

2. If 2 is a unit in R , then $N = 0$, i.e., there is an orthogonal basis for V .

Example 9.2.2.3. In the special case where $P = R$ is free of rank 1 with basis e , then if e^\vee is a dual basis, we see that $H(R)$ is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The structure of this form depends on whether or not 2 is a unit in R . Indeed, if 2 is a unit in R , then we may use the basis $\frac{e+e^\vee}{2}$ and $\frac{e-e^\vee}{2}$ to see that $H(R)$ is isometric to $\langle 1 \rangle \oplus \langle -1 \rangle$. However, if 2 is not a unit in R , then one can not make such an identification. Indeed, every $h \in H(R)$ satisfies $h \cdot h \equiv 0 \pmod{2R}$.

Corollary 9.2.2.4 (Witt's cancellation theorem). *Suppose X , Y and Z are inner product spaces over a local ring R in which 2 is a unit, then $X \oplus Z \cong Y \oplus Z$ implies $X \cong Y$.*

Example 9.2.2.5. Witt's cancellation theorem is false when 2 is not a unit in R . Indeed, $\langle -1 \rangle \oplus \langle -1 \rangle \oplus \langle 1 \rangle \cong \langle -1 \rangle \oplus H(R)$. Indeed, if e_1, e_2 and e_3 are orthogonal vectors with $e_1 \cdot e_1 = e_2 \cdot e_2 = -1$ and $e_3 \cdot e_3 = 1$, then $e_1 + e_2 + e_3$, $e_1 + e_3$ and $e_2 + e_3$ form a new basis with inner product matrix $\langle -1 \rangle \oplus H(R)$. Since 2 is not a unit in R , $H(R)$ is not isometric to a direct sum of units as we saw above.

The above examples prompt the following definition.

Definition 9.2.2.6. We write $\text{WM}(R)$ for the (graded by the rank) monoid of *stable isometry classes* of symmetric bilinear forms over a ring R and $\text{WM}_n(R)$ for the degree n component of $\text{WM}(R)$.

Remark 9.2.2.7. The Grothendieck-Witt group can be identified as the Grothendieck group of the monoid of stable isometry classes of symmetric bilinear forms over R as well.

Proposition 9.2.2.8. *The discriminant factors through a monoid map*

$$\text{disc} : \text{WM}(R) \longrightarrow R^\times / R^{\times 2}.$$

9.2.3 Naive homotopy classes of symmetric bilinear forms I

Theorem 9.2.3.1. *Suppose k is a field.*

1. *For any integer $n \geq 0$, the quotient map $S_n(k) \rightarrow \text{WM}(k)$ factors through $[\text{Spec } k, S_n]_N$.*
2. *Consider the map*

$$\coprod_{n \geq 0} [\text{Spec } k, S_n]_N \longrightarrow \coprod_{n \geq 0} \text{WM}_n(k) \times_{k^\times / k^{\times 2}} k^\times$$

defined by the product of the quotient map and the determinant (the fiber product is taken via the discriminant). If the right hand side is equipped with the monoid structure given by orthogonal sum in $\text{WM}(k)$ and product in k^\times , then the aforementioned map is an isomorphism of graded monoids.

The above result may be deduced from the following results. To establish the above result, we must build naive \mathbb{A}^1 -homotopies among symmetric bilinear forms. We begin by observing that every symmetric bilinear form is homotopic to a diagonal form.

Lemma 9.2.3.2. *Let $B \in S_n(k)$ be an $n \times n$ -symmetric matrix. There exist units $u_1, \dots, u_n \in k^\times$ such that B is naively homotopic (in S_n) to the diagonal matrix $\text{diag}(u_1, \dots, u_n)$.*

Proof. If k has characteristic unequal to 2, then Theorem 9.2.2.2 implies that we may find an element $A \in SL_n(k)$ such that A^tBA is diagonal. In that case, writing A as a product of elementary matrices, we obtain the required \mathbb{A}^1 -homotopy. If k has characteristic 2, one has to work a bit harder. \square

Remark 9.2.3.3. Note that this diagonal form is *not* unique. Since any element of $SL_2(k)$ is \mathbb{A}^1 -homotopic to the identity, if we multiply a diagonal matrix by $\text{diag}(1, \dots, 1, u, u^{-1}, 1, \dots, 1)$ then we obtain another symmetric diagonal matrix naively homotopic to the original one.

Proposition 9.2.3.4. *Suppose n is a positive integer and suppose $B(T)$ is a non-degenerate $n \times n$ -symmetric matrix with coefficients in $k[t]$. There exists a matrix $P(T) \in SL_n(k[t])$ such that $P(t)^tB(t)P(t)$ is block diagonal where the blocks are either 1×1 -blocks corresponding to elements of k^\times or 2×2 -blocks of the form*

$$\begin{pmatrix} 0 & 1 \\ 1 & \alpha(t) \end{pmatrix}$$

with $\alpha(t) \in k[t]$.

Using this result we may state Cazanave’s result.

Theorem 9.2.3.5. *If k is a field, there is an isomorphism of graded monoids*

$$[\mathbb{P}^1, \mathbb{P}^1]_N \cong \coprod_{n \geq 0} \text{WM}_n(k) \times_{k^\times / k^{\times 2}} k^\times$$

9.3 Lecture 37: Symmetric bilinear forms and naive homotopy classes

The goal of this section is to sketch (half of) the proof of Cazanave’s result. We consider the map

$$\coprod_{n \geq 0} \text{Rat}_n(k) \longrightarrow \coprod_{n \geq 0} S_n(k).$$

The right hand side is precisely the monoid of stable isomorphism classes of symmetric bilinear forms with a specified orthogonal basis. While this map is not a monoid map, we will show that it becomes a monoid map after passing to naive \mathbb{A}^1 -homotopy classes. Since we know that every symmetric bilinear form over k may be diagonalized, in loose outline, we proceed as follows: we begin by observing that the Grothendieck-Witt ring may be presented by generators and relations, which tells us what relations we expect to appear on the right hand side upon passing to naive homotopy classes; since the Witt monoid is generated by diagonal forms, we then identify rational functions that realize each of these forms (this establishes surjectivity of the relevant map). Finally, to establish injectivity using the fact that the relations in the Witt monoid arise in “degree 2”, we reduce the proof of injectivity to analyzing the case $n = 2$ above, which may be treated by hand. We restrict attention to the case where k has characteristic $\neq 2$ for ease of exposition.

9.3.1 Presenting the Witt/Grothendieck-Witt rings

If R is a local ring having characteristic unequal to 2, then we know that every isomorphism class of symmetric bilinear forms is isomorphic to a diagonal form $\langle a_1, \dots, a_n \rangle$ for units $a_i \in R^\times / R^{\times 2}$ and some integer n . Sending a unit $u \in R^\times$ to the form $\langle u \rangle$ thus defines a function $R^\times / R^{\times 2} \rightarrow GW(R)$. There is an induced group homomorphism

$$\mathbb{Z}[R^\times / R^{\times 2}] \longrightarrow GW(R);$$

the diagonalizability of forms guarantees that this homomorphism is surjective. This observation allows us to give a presentation of the Grothendieck-Witt ring $GW(R)$.

Theorem 9.3.1.1. *If k is a field having characteristic unequal to 2, then $GW(k)$ is the quotient of the free commutative ring on $\mathbb{Z}[k^\times / k^{\times 2}]$ by the ideal generated by the following relations:*

- $\langle 1 \rangle = 1$;
- $\langle uv \rangle = \langle u \rangle \langle v \rangle$, for square classes u, v .
- $\langle u \rangle + \langle v \rangle = \langle u + v \rangle (1 + \langle uv \rangle)$, for $u, v, u + v$ units.

Proof. That the first two relations are satisfied is straightforward. The third is a slightly more subtle relation, but what is important all the relations involve only a pair of units. See [Lam05, Theorem 4.1] for a proof. The result can also be formulated in characteristic 2, but it requires a bit more effort. \square

9.3.2 Surjectivity and compatibility with the monoid structure

Since we have seen that the Witt monoid is generated by diagonal forms, to establish surjectivity, it suffices to build a pointed rational function that corresponds to a given unit. Indeed, if $u \in k^\times$, then consider the rational function (x, u) . Note that associated Bézout form is precisely $\langle u \rangle$. Observe that the unique Bézout relation in this case is $0x + u^{-1}u = 1$, so this rational function yields the matrix

$$\begin{pmatrix} x & u^{-1} \\ u & 0 \end{pmatrix}.$$

Note that Rat_1 is the closed subscheme of \mathbb{A}^2 given by polynomials $f = x + a_0, g = b_0$ with non-zero resultant. We now observe that bijectivity holds in degree 1 after passing to naive \mathbb{A}^1 -homotopy classes.

Lemma 9.3.2.1. *The map*

$$[\text{Spec } k, \text{Rat}_1]_N \rightarrow [\text{Spec } k, S_1]_N$$

is a bijection.

Proof. The homotopy $f = x + tv, g = u$ with $u \neq 0$ shows that an arbitrary degree 1 pointed rational function is homotopic to the function (x, u) . The rational functions (x, u) and (x, u') are not homotopic unless $u = u'$. As observed above, the Bézout form of (x, u) is the form u . Here the non-zero symmetric 1×1 -matrices are isomorphic to \mathbf{G}_m as a scheme, and thus there are no non-trivial \mathbb{A}^1 -homotopies. \square

Since every symmetric bilinear matrix may be diagonalized, to establish surjectivity it suffices to show that every pointed rational function is homotopic to a “diagonal” such form. We have already seen this is true in degree 1. For a sequence of units u_1, \dots, u_n , based on the discussion above, a natural candidate for “diagonal” rational functions are those of the form:

$$(x, u_1) \oplus \cdots \oplus (x, u_n).$$

We now show that every form is homotopic to such a form.

Lemma 9.3.2.2. *Every degree n pointed rational function is naively homotopic to one of the form $(x, u_1) \oplus \cdots \oplus (x, u_n)$ for suitable units u_1, \dots, u_n .*

Proof. We proceed by induction on n . The result has already been established for $n = 1$. Thus, assume $n \geq 2$.

First, we treat the simple case where $(f, g) = (x^n, u)$ for some unit u . The naive homotopy $(x^n, tx^{n-1} + u)$ show that (x^n, u) is naively homotopic to $(x^n, x^{n-1} + u)$. Now, simply observe that the formula

$$\begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^{n-1} + u & A \\ -xu & B \end{pmatrix} = \begin{pmatrix} x^n & A' \\ x^{n-1} + u & B' \end{pmatrix}$$

for suitable A, A', B, B' shows that $(x^n, x^{n-1}, u) = (x, 1) \oplus (x^{n-1} + u, -xu)$, where $(x^{n-1} + u, -xu)$ has degree $n - 1$ and we conclude by the induction hypothesis.

Therefore, to complete the proof, it suffices to show that any pointed degree n rational function is homotopic to (x^n, u) . We establish this in two steps. First, note that if f is a polynomial of degree n , then we showed that $f = x^n + ta_{n-1}x^{n-1} + \cdots + ta_0$ yields a homotopy from (f, u) to (x^n, u) . Second, we claim that every pointed degree n rational function can be written as a sum of expressions (f_i, u) for suitable polynomials f_i . To see this, we observe first that

$$(f, u) + (f', g') = (f'f - \frac{g'}{u}, uf').$$

Using this fact, one concludes by induction on the degree.

$$(f, g) = (f_1, u_1) \oplus \cdots \oplus (f_n, u_n)$$

for suitable monic polynomials f_i and units u_i . This decomposition yields a “twisted” continued fraction expansion of our pointed rational function. Combining these two observations, we complete the proof. \square

Now, it suffices to check that the “diagonal” pointed rational functions described above realize, up to naive homotopy, all the diagonal symmetric bilinear matrices.

Lemma 9.3.2.3. *Suppose (f, g) is a pointed rational function of degree n and $u \in k^\times$ is a unit. The Bézout form of $(x, u) \oplus (f, g)$ is naively homotopic to the form $\langle u \rangle \oplus \text{Bez}_{f,g}$.*

Proof. Observe that

$$(x, u) \oplus (f, g) = (xf + u^{-1}g, uf).$$

Now, to compute the Bézout form of the right hand side we write

$$\frac{(xf(x) + u^{-1}g(x))uf(y) - (yf(y) - u^{-1}g(y))uf(x)}{x - y} = uf(x)f(y) + \frac{f(x)g(y) - f(y)g(x)}{x - y}.$$

This form is evidently diagonal in the basis $1, \dots, x^{n-1}, f(x)$. Thus, the two symmetric matrices are conjugate by an element of $SL_{n+1}(k)$. Since every element of $SL_{n+1}(k)$ is naively homotopic to the identity, this yields the required homotopy. \square

Proposition 9.3.2.4. *The map*

$$[\mathrm{Spec} k, \coprod_{n \geq 0} \mathrm{Rat}_n]_n \longrightarrow [\mathrm{Spec} k, \coprod_{n \geq 0} S_n]_N$$

is a surjective map of monoids.

Proof. Observe that the monoids in both cases are generated in degree 1 and combine the above results. \square

9.3.3 Injectivity

Finally, it remains to establish injectivity. To do this, we analyze the final relation in the Grothendieck–Witt ring at the monoidal level. We will say that two diagonal forms $\mathrm{diag}(u_1, \dots, u_n)$ and $\mathrm{diag}(v_1, \dots, v_n)$ are directly *chain* equivalent if there exist pairs (u_i, u_{i+1}) and $v_i, v_{i+1} \dots$ To be added.

Chapter 10

Punctured affine space and naive homotopy classes

In this section, we establish a result of Vorst, building on results of Suslin, that gives a K_1 -analog of Lindel's theorem. We then apply this result to study naive homotopy classes of maps with target $\mathbb{A}^n \setminus 0$ for some integer $n \geq 0$.

Chapter 11

Milnor K-theory

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In this section, we analyze one generalization of the “units” as studied earlier. In addition to producing yet another \mathbb{A}^1 -invariant, we aim to introduce a key ingredient in our discussion of homotopy sheaves.

11.1 Milnor K-theory: definitions, residue map and transfer maps

11.1.1 Definitions, basic properties and examples

Definition 11.1.1.1. If R is a ring, then the *Milnor K-theory ring of R* , denoted $K_*^M(R)$ is the quotient of the tensor algebra $T_{\mathbb{Z}}(R^\times)$ by the two-sided ideal generated by $[x] \otimes [1 - x] = 0$, $x \in R^\times \setminus 1$. The n -th Milnor K-theory group is the degree n graded piece of $K_*^M(R)$.

Example 11.1.1.2. For any ring R , $K_0^M(R) = \mathbb{Z}$ and $K_1^M(R) = R^\times$.

Example 11.1.1.3. For any field F , $K_n^M(F) = F^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} F^* / \{ \langle a_1 \otimes \cdots \otimes a_n \mid a_i + a_{i+1} = 1, 1 \leq i \leq n-1 \rangle \}$.

We will write $[a_1, \dots, a_n]$ for the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$; this element is called a symbol of length n . We write the group structure additively and, in that case, $K_n^M(F)$ is the quotient of the free abelian group on symbols $[a_1, \dots, a_n]$ subject to the relations

1. $[a_1, \dots, a_i, \dots, a_n] + [a_1, \dots, a'_i, \dots, a_n] = [a_1, \dots, a_i a'_i, \dots, a_n]$, and

2. $[a_1, \dots, a_n] = 0$ if $a_i + a_{i+1} = 1$ for some $1 \leq i \leq n - 1$.

The product in the graded ring $K_n^M(F)$ is given by concatenation, i.e., $[a_1, \dots, a_m] \cdot [b_1, \dots, b_n] = [a_1, \dots, a_m, b_1, \dots, b_n]$.

Example 11.1.1.4. Suppose F is a totally ordered field, i.e., if F is equipped with a total order \leq such that if $a \leq b$, then for any c , $a + c \leq b + c$ and if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$. The group $K_n^M(F)$ is non-trivial. These conditions imply that $1 > 0$. Now define a function $K_n^M(F) \rightarrow \mathbb{Z}/n$ by sending $[a_1, \dots, a_n] \rightarrow 0$ if $a_i > 0$ for some i and $[a_1, \dots, a_n] \rightarrow 1$ if $a_i < 0$ for all i and one checks that this is a well-defined homomorphism. In particular, $K_n^M(\mathbb{R})$ is non-trivial for every $n \geq 0$.

Theorem 11.1.1.5. *If F is a finite field with q elements, then $K_n^M(F) = 0$ for $n \geq 2$.*

Proof. It suffices to show that $K_2^M(F) = 0$ since each generator can be written $[a_1, a_2][a_3, \dots, a_n]$. Now, recall that any finite subgroup of the multiplicative group of a field is cyclic. It follows that if F is a finite field, then F^\times is cyclic. Therefore, $F^\times \otimes F^\times$ is cyclic. Since any quotient of a cyclic group is cyclic, we conclude that $K_2^M(F)$ is also cyclic. If ξ is a cyclic generator of F^\times , then we can write any element of $K_2^M(F) = [\xi^i, \xi^j] = ij[\xi, \xi]$. Therefore, it suffices to establish that $[\xi, \xi] = 0$. We treat two separate cases: $p = 2$ and p odd.

First, we claim that $[\xi, -\xi] = 0$ in K_2 of any field. Indeed, in that case, observe that

$$-x = \frac{-x(x-1)}{x-1} = \frac{1-x}{1-x^{-1}},$$

and therefore

$$[x, -x] = [x, \frac{1-x}{1-x^{-1}}] = [x, 1-x] - [x, 1-x^{-1}] = [x, 1-x] + [x^{-1}, 1-x^{-1}],$$

and the latter is trivial by the defining relations.

Now, if $p = 2$, then $\xi = -\xi$ so $[\xi, \xi] = [\xi, -\xi] = 0$ by the above computation. If p is odd, then... \square

Theorem 11.1.1.6. *If F is an algebraically closed field, then $K_n^M(F)$ is uniquely divisible for $n \geq 2$.*

Proof. If F is an algebraically closed field, then F^\times is a divisible group. Indeed, if m is an integer, and $x^m = a$ for $a \neq 0$, then a has an m -th root. Since $K_n^M(F)$ is a quotient of $F^\times \otimes_{\mathbb{Z}} \dots \otimes_{\mathbb{Z}} F^\times$, and tensor products and quotients of divisible groups are divisible, it follows that $K_n^M(F)$ is divisible. \square

reciprocity, Fundamental theorem

11.1.2 Residue maps

If (F, ν) is a discretely valued field, the valuation is a map $\text{ord}_\nu : F^\times \rightarrow \mathbb{Z}$; if we identify $K_1^M(F) = K^\times$ and $K_0^M(F) = \mathbb{Z}$, then we can view ord_ν as a map $K_1^M(F) \rightarrow K_0^M(F)$. The next result is a significant generalization of this map.

Lemma 11.1.2.1. *Suppose (F, ν) is a discretely valued field with valuation ring \mathcal{O}_ν and residue field κ , π is a local uniformizing parameter, and $i \geq 1$ is an integer. There is a unique homomorphism*

$$\partial_\nu : K_i^M(F) \longrightarrow K_{i-1}^M(\kappa_\nu)$$

characterized by the property that given $\{x_1, \dots, x_n, \pi\}$ with $x_i \in \mathcal{O}_\nu^\times$, $\partial_\nu(\{x_1, \dots, x_n, \pi\}) = \{\bar{x}_1, \dots, \bar{x}_n\}$, where \bar{x}_i is the image of x_i in κ_ν .

Definition 11.1.2.2. *Suppose (F, ν) is a discretely valued field, and $i \geq 1$ is an integer, then the tame symbol or residue map*

$$\partial_\nu : K_i^M(F) \longrightarrow K_{i-1}^M(F)$$

is the unique map guaranteed to exist by Lemma 11.1.2.1.

Lemma 11.1.2.3. *The kernel of ∂_ν is generated by $\{x_1, \dots, x_n\}$ such that $x_i \in \mathcal{O}_\nu^\times$*

11.1.3 Transfer maps

11.2 Gersten resolution

Theorem 11.2.0.1. *If X is an excellent scheme, then there is a complex of the form*

$$\cdots \longrightarrow \bigoplus_{x \in X_{(i)}} K_n^M(\kappa_x) \longrightarrow \bigoplus_{x \in X_{(i)}} K_{n-1}^M(\kappa_x) \longrightarrow \cdots \bigoplus_{x \in X_{(0)}} K_{n-i}^M(\kappa_x)$$

11.3 Unramified Milnor K-theory sheaves

11.3.1 Definition and basic properties

Definition 11.3.1.1. *If U is a normal integral scheme with fraction field F , then we define the unramified Milnor K-theory of U , denoted $K_i^M(F)$, to be the intersection of the kernels of the residue maps $K_i^M(U) := \bigcap_{\nu \in U^{(1)}} \partial_\nu$.*

Proposition 11.3.1.2. *If $f : U \rightarrow U'$ is a morphism of normal integral schemes, then there is an induced morphism*

$$f^* : K_i^M(U') \longrightarrow K_i^M(U).$$

This homomorphism is functorial.

Proof. We first define a map when f is dominant. We then define the map when f is a closed immersion. Then we can define it for any map that can be factored as a composite of such maps. Every map admits such a factorization: closed immersion (graph) followed by a projection. Then we check that the map so-defined is independent of the choice of factorization. \square

Definition 11.3.1.3. *The unramified Milnor K-theory sheaf...*

11.3.2 Functoriality properties

Functoriality of unramified Milnor K-theory sheaf,

11.3.3 Homotopy invariance

Bass-Tate results:

Chapter 12

Milnor–Witt K-theory

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12.1 Milnor–Witt K-theory: definitions...

12.1.1 Definitions, basic properties and examples

Definition 12.1.1.1. If R is a ring, then the *Milnor–Witt K-theory ring* of R , denoted $K_*^{MW}(R)$ is the quotient of the free, graded, associative algebra on generators $[u], u \in R^\times$ of degree $+1$ and an element η of degree -1 subject to the following relations:

1. if $u \in R^\times$, then $[u]\eta = \eta[u]$;
2. if $u, v \in R^\times$, then $[uv] = [u] + [v] + \eta[u][v]$;
3. if $u \in R^\times \setminus 1$, then $[u][1 - u] = 0$; and
4. if $h = 2 + [-1]\eta$, then $\eta \cdot h = 0$.

We write $\mathbf{K}_n^{MW}(R)$ for the n -th graded piece of $K_*^{MW}(R)$.

The following result is immediate from the definitions.
Description of degree n part of Milnor–Witt K-theory.
Basic symbol manipulations.

Lemma 12.1.1.2. *The quotient $\mathbf{K}_*^{MW}(R)/\langle\eta\rangle \cong \mathbf{K}_*^M(R)$.*

12.2 Low-dimensional results

12.2.1 Milnor–Witt K-theory in degrees ≤ 0 and the Grothendieck–Witt ring

A presentation of the Grothendieck–Witt ring

The Grothendieck–Witt ring of a field F has a presentation. If F has characteristic unequal to 2, this follows from the fact that every form may be diagonalized.

Theorem 12.2.1.1. *The Grothendieck–Witt group of a field F is generated by the isometry classes of 1-dimensional symmetric bilinear forms subject to the following relation:*

- $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$, $a, b, a + b \in k^\times$.

Our goal will be to interpret this in terms of generators and relations and show that Milnor–Witt K_0 satisfies these generators and relations.

Definition 12.2.1.2. If R is a ring, and $u \in R^\times$, set

- $\langle u \rangle = 1 + \eta[u]$, and
- $\epsilon := -\langle -1 \rangle$

Observe that $h = 1 + \langle -1 \rangle = 1 - \epsilon$. The relation in Milnor–Witt K-theory $\eta \cdot h = 0$ can thus be rephrased as $\eta\epsilon = \eta$. We now deduce a number of properties of relations in Milnor–Witt K-theory.

Proposition 12.2.1.3. *Suppose F is a field, and $a, b \in F^\times$. The following relations hold in Milnor–Witt K-theory:*

1. $\langle ab \rangle = \langle a \rangle \langle b \rangle$;
2. $\langle 1 \rangle = 1$ and $[1] = 0$;
3. $\langle a \rangle$ is a unit in $K_0^{MW}(F)$ with inverse $\langle a^{-1} \rangle$;
4. $[\frac{a}{b}] = [a] - \langle \frac{a}{b} \rangle [b]$, in particular $[a^{-1}] = -\langle a^{-1} \rangle [a]$
5. $[a] [-a] = 0$;
6. $\langle a \rangle + \langle -a \rangle = h$;
7. $[a][a] = [a] [-1] = \epsilon [a] [-1] = [-1][a] = \epsilon [-1][a]$;
8. $[a][b] = [b][a]$;
9. $\langle ab^2 \rangle = \langle a \rangle$;
10. if $a + b = 1$, then $(\langle a \rangle - 1)(\langle b \rangle - 1) = 0$;
11. $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle (a + b)ab \rangle$;

Proof. We compute

$$\begin{aligned} \langle ab \rangle &:= 1 + \eta[ab] = 1 + \eta([a] + [b] + \eta[a][b]) \\ &= 1 + \eta[a] + \eta[b] + \eta^2[a][b] \\ &= (1 + \eta[a])(1 + \eta[b]) \\ &= \langle a \rangle \langle b \rangle. \end{aligned}$$

Since $\eta h = 0$, we conclude $\eta[1]h = 0$. Observe that $\eta[1] = -1 + \langle 1 \rangle$, while $h = 1 + \langle -1 \rangle$. Then, we manipulate:

$$0 = (-1 + \langle 1 \rangle)(1 + \langle -1 \rangle) = -1 + \langle 1 \rangle + \langle -1 \rangle + \langle 1 \rangle \langle -1 \rangle.$$

By the previous point, $\langle 1 \rangle \langle -1 \rangle = \langle -1 \rangle$, and thus we conclude that $0 = -1 + \langle 1 \rangle$, which is precisely what we wanted to show.

Combining the above two points, we conclude that $\langle a \rangle$ is a unit in $K_0^{MW}(F)$ with inverse $\langle a^{-1} \rangle$. Since $\langle ab^2 \rangle = \langle a \rangle \langle b^2 \rangle$ and since $\langle a \rangle$ is a unit, it suffices to prove that $\langle b^2 \rangle = \langle 1 \rangle$. Suppose $a + b = 1$. Since $[a][b] = 0$, it follows that $\eta^2[a][b] = 0$. However,

$$\eta^2[a][b] = (\eta[a])(\eta[b]) = (\langle a \rangle - 1)(\langle b \rangle - 1).$$

Suppose a and b are arbitrary units such that $a + b$ is a unit. In that case, $a' = \frac{a}{a+b}$ and $b' = \frac{b}{a+b}$ have the property that $a' + b' = 1$ and $a'b' = ab(a+b)^2$. Since $a' + b' = 1$, we conclude that $(\langle a' \rangle - 1)(\langle b' \rangle - 1) = 0$ by the previous point. Expanding, we conclude that

$$\langle a' \rangle \langle b' \rangle + 1 = \langle a' \rangle + \langle b' \rangle.$$

Since $a'b' = ab(a+b)^2$, we conclude that

$$\langle a' \rangle \langle b' \rangle = \langle a'b' \rangle = \langle ab(a+b)^2 \rangle = \langle ab \rangle.$$

where the final point is deduced from the relation above. Multiplying through by the unit $\langle a+b \rangle$ and observing that $\langle a' \rangle \langle a+b \rangle = \langle a \rangle$ and $\langle b' \rangle \langle a+b \rangle = \langle b \rangle$, we conclude that

$$\langle ab(a+b) \rangle + \langle a+b \rangle = \langle a \rangle + \langle b \rangle,$$

which is precisely what we wanted to show. \square

If k is a field, then every symmetric bilinear form can be diagonalized. Therefore, there is a homomorphism $\mathbb{Z}[k^\times] \rightarrow GW(k)$.

Theorem 12.2.1.4. *Suppose k is a field. The assignment $\mathbb{Z}[k^\times] \rightarrow K_0^{MW}(k)$ defined by $u \mapsto \langle u \rangle$ yields a surjective ring map. This map factors through an isomorphism $GW(k) \xrightarrow{\sim} K_0^{MW}(k)$.*

12.2.2 Negative degrees

12.2.3 Degree 1 and Barge–Lannes theory

$K_i^{MW}(k) = GW_i^i(k)$ for $i \leq 3$. Give a proof of the results of Karoubi–Barge–Lannes for $i = 1$.

12.2.4 Degree 2 and Suslin’s theorem

Suslin on symplectic K_2 .

12.3 Fiber product presentation of Milnor–Witt K-theory

Milnor’s homomorphism $K_n^M/2(k) \rightarrow I^n(k)/I^{n+1}(k)$.

Appendix A

Sets, Categories and Functors

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A.1 Categories

For the most part, we use naive set theory, though we will differentiate between sets and classes. We will assume the axiom of choice. We will use category theoretic language to attempt to keep track of “structure” present in the objects under consideration. Nevertheless, it is undoubtedly the case that there can come a point where “structure” becomes so refined as to be unwieldy.

A.1.1 Sets

We will not pay too much attention to set theory, but for the most part it will suffice to think “intuitively” about such things (though perhaps even saying this is unintuitive). As most people have probably heard, we should not talk about the set of all sets, since one runs into paradoxical constructions like “the set of all sets that do not contain themselves” (Russell’s paradox). We require that the following constructions can be performed with sets; I hope you agree that all these constructions are reasonable.

1. For each set X and each “property” P , we can form the set $\{x \in X | P(x)\}$ of all members of X that have the property P ;
2. For each set X , the collection $\{x | x \in X\}$, is a set; this set is sometimes denoted 2^X or $\mathcal{P}(X)$ and referred to as the *power set* of X .
3. Given any pair of sets X and Y , we can form the following sets:
 - a) the set $\{X, Y\}$ whose members are exactly X and Y ;
 - b) the (ordered) pair (X, Y) with first coefficient X and second coefficient Y ; more generally for any natural number n and sets X_1, \dots, X_n we may form the ordered n -tuple (X_1, X_2, \dots, X_n) ;
 - c) the union $X \cup Y := \{x | x \in X \text{ or } x \in Y\}$;
 - d) the intersection $X \cap Y := \{x | x \in X \text{ and } x \in Y\}$;
 - e) the Cartesian product $X \times Y := \{(x, y) | x \in X \text{ and } y \in Y\}$;
 - f) the relative complement $X \setminus Y := \{x | x \in X \text{ and } x \notin Y\}$;
 - g) a *function* $f : X \rightarrow Y$ is a triple (X, Y, f) consisting of a subset of $f \subset X \times Y$ with the property that for each $x \in X$, there is a unique y such that $(x, y) \in f$; the set Y^X of all functions $X \rightarrow Y$ is a set.
4. For any set I and any family of sets X_i indexed by I (write $\{X_i\}_{i \in I}$), we can form the following sets:
 - a) the image $\{X_i | i \in I\}$ of the indexing function;
 - b) the union $\cup_{i \in I} X_i := \{x | x \in X_i \text{ for some } i \in I\}$;
 - c) the intersection $\cap_{i \in I} X_i := \{x | x \in X_i \text{ for all } i \in I\}$, provided $I \neq \emptyset$;
 - d) the Cartesian product $\prod_{i \in I} X_i := \{f : I \rightarrow \cup_{i \in I} X_i | f(i) \in X_i \text{ for each } i \in I\}$;
 - e) the disjoint union $\coprod_{i \in I} X_i := \cup_{i \in I} (X_i \times \{i\})$.
5. We can form the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of all natural numbers, integers, rational numbers, real numbers, complex numbers.

Remark A.1.1.1. With the above requirements, each topological space is a set, i.e., it is a pair (X, τ) consisting of a set X and a topology τ on X : the topology τ , which is given by the set of open sets in X , is a subset of $\mathcal{P}(\mathcal{P}(X))$. Likewise, each group is a set, each ring is a set, etc.. While spelling everything out in terms of sets is possible in principle, in practice, it would be extremely cumbersome.

While I hope you agree that whatever notion of set one takes one should be able to perform the above constructions, requiring that one can perform such operations is closely related with the notion of a Grothendieck universe, whose definition we now recall.

Definition A.1.1.2. A *Grothendieck universe* is a set \mathcal{U} with the following properties:

1. If $X \in \mathcal{U}$ and if $y \in X$, then $y \in \mathcal{U}$;
2. If $X, Y \in \mathcal{U}$, then $\{X, Y\} \in \mathcal{U}$;
3. If $X \in \mathcal{U}$, then $\mathcal{P}(X) \in \mathcal{U}$;
4. If $\{X_i\}_{i \in I}$ is a family of elements of \mathcal{U} and if $I \in \mathcal{U}$, then $\cap_{i \in I} X_i \in \mathcal{U}$.

Example A.1.1.3. Grothendieck universes are difficult to construct in general: the empty set gives an example. There is another example of a countable universe (that of hereditarily finite sets). If we want, as we do, to work in a universe that contains an uncountable set, then this amounts to a “largeness hypothesis” on our universe. In any case, it turns out that positing the existence of such

a universe requires adjoining an axiom to the usual axioms of set theory, and for this reason some people prefer to avoid using universes.

If we fix a Grothendieck universe \mathcal{U} , an element $X \in \mathcal{U}$ is called a \mathcal{U} -small set, or simply a set. We will also need to consider “larger” constructions, and for this one introduces the notion of a *class*. We require that (1) the members of each class are sets, and (2) for any property “P”, one can form the class of all sets with property P , (3) every set is a class. Classes that are not sets are called *proper classes*. Thus, one speaks of the class of all sets, or the class of all topological spaces.

A.1.2 Categories and Functors

Loosely speaking, categories are structures we introduce to keep track of mathematical structures (objects) and the relations between them (morphisms). One can compose morphisms, and there is an identity morphism from any object to itself. More formally, one makes the following definition.

Definition A.1.2.1. A category \mathcal{C} is a quadruple $(\text{Ob}, \text{Hom}, id, \circ)$ consisting of

1. A class $\text{Ob}_{\mathcal{C}}$ of objects;
2. For each pair $X, Y \in \text{Ob}_{\mathcal{C}}$, a set $\text{Hom}_{\mathcal{C}}(X, Y)$;
3. For each object X , a morphism $id_X \in \text{Hom}_{\mathcal{C}}(X, X)$;
4. For each triple of objects X, Y, Z , a function

$$\circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{C}}(X, Z);$$

these data are subject to the following axioms:

1. composition is associative, i.e., given four objects W, X, Y, Z , and morphism $f : W \rightarrow X$, $g : X \rightarrow Y$ and $h : Y \rightarrow Z$, $h \circ (g \circ f) = (h \circ g) \circ f$;
2. id_X is an identity, i.e., for $f \in \text{Hom}_{\mathcal{C}}(W, X)$ and $g \in \text{Hom}_{\mathcal{C}}(X, Y)$, $id_X \circ f = f$ and $g \circ id_X = g$;
3. the sets $\text{Hom}_{\mathcal{C}}(X, Y)$ are pairwise disjoint.

If \mathcal{U} is a universe, and if $\text{Ob}_{\mathcal{C}}$ is a \mathcal{U} -small set, then \mathcal{C} will be called a \mathcal{U} -small category.

Remark A.1.2.2. What we are calling *categories* are often called *locally small* categories in the literature.

Definition A.1.2.3. If \mathcal{C} is any category, then we can define the opposite category \mathcal{C}° to be the category where objects are those of \mathcal{C} and the direction of morphisms is reversed.

Definition A.1.2.4. If \mathcal{C} and \mathcal{D} are categories, then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of a function that assigns to each object X in \mathcal{C} an object $F(X)$ in \mathcal{D} , and to each pair of objects X, Y , assigns a function $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ (also denoted F) such that

1. F preserves composition, i.e., given $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, $F(g \circ f) = F(g) \circ F(f)$;
2. F preserves identities, i.e., $F(id_X) = id_{F(X)}$.

Example A.1.2.5. If \mathcal{C} is any category, we write $id_{\mathcal{C}}$ for the functor $\mathcal{C} \rightarrow \mathcal{C}$ that is the identity on objects and morphisms. The composite of two functors is a functor.

Definition A.1.2.6. If $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors, then a *natural transformation* $\theta : F \rightarrow G$ (or a morphism of functors) is a rule that assigns to each object X of \mathcal{C} a morphism $\theta_X : F(X) \rightarrow G(X)$ such that, if $f : X \rightarrow Y$ is any morphism in \mathcal{C} , then the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\theta_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\theta_Y} & G(Y) \end{array}$$

If for every object X of \mathcal{C} the morphism θ_X is an isomorphism in \mathcal{D} , then θ is called a *natural equivalence* (or *natural isomorphism* or *isomorphism of functors*).

Example A.1.2.7. Given any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is an identity natural transformation $id : F \rightarrow F$, which is simply the identity map $id_X : F(X) \rightarrow F(X)$ for every object X in \mathcal{C} . If F, G are two functors, and θ and θ' are natural transformations, it makes sense to compose $\theta \circ \theta'$, with composition given by objectwise composition.

Example A.1.2.8. If \mathcal{C} and \mathcal{D} are categories, then we can form a new category $F(\mathcal{C}, \mathcal{D})$ where objects are functors from \mathcal{C} to \mathcal{D} and morphisms are natural transformations of functors; the identity is given by the identity functor, and composition is composition of natural transformations as described in Example A.1.2.7. Even if \mathcal{C} and \mathcal{D} are small categories, the functor category $F(\mathcal{C}, \mathcal{D})$ is typically not small.

Definition A.1.2.9. Suppose \mathcal{C} and \mathcal{D} are categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor. We say that F is

- *faithful* if for any pair of objects $X, Y \in \mathcal{C}$, the function $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective;
- *reflects isomorphisms* (or *conservative*) if for any arrow $f \in \mathcal{C}$, $F(f)$ is an isomorphism implies f is an isomorphism.
- *an embedding* if it is faithful and injective on objects;
- *full* if for any pair of objects $X, Y \in \mathcal{C}$, the function $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective;
- *fully faithful* if it is both full and faithful;
- *essentially surjective* (or *isomorphism dense*) if for any object $D \in \mathcal{D}$, there exists an object $C \in \mathcal{C}$ and an isomorphism $F(C) \xrightarrow{\sim} D$;
- *an equivalence of categories* if there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural equivalences $FG \xrightarrow{\sim} id_{\mathcal{C}}$ and $GF \xrightarrow{\sim} id_{\mathcal{D}}$.

Proposition A.1.2.10. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor that is fully faithful and essentially surjective, then there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and isomorphisms of functors $FG \xrightarrow{\sim} id_{\mathcal{C}}$ and $GF \xrightarrow{\sim} id_{\mathcal{D}}$. In other words, F is an equivalence of categories.*

A.1.3 Indexing categories

When we speak about limits and colimits, we will use “indexing categories”. Sometimes indexing categories are drawn as diagrams. For example, if we want to speak about pullbacks, we can think of the category pictured as follows:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet;$$

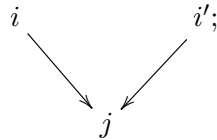
this category has three objects, and we have drawn the non-identity morphisms. Similarly, any directed graph defines a category: one has an object for each vertex and one non-identity morphism for each arrow pictured.

Example A.1.3.1. If (P, \leq) is a category, we can define a directed graph by creating one vertex for each element of P and where there is a unique non-identity morphism $a \rightarrow b$ if $a \leq b$.

We now describe further diagram categories.

Definition A.1.3.2. A category \mathbf{I} is *filtered* if:

1. \mathbf{I} is non-empty;
2. for every pair of objects $i, i' \in I$, there exists an object j and two maps $i \rightarrow j$ and $i' \rightarrow j$; pictorially:



3. for every pair of morphisms $\alpha, \beta : i \rightarrow j$, there exists an object k and an arrow $\gamma : j \rightarrow k$, pictorially:

$$i \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} j \xrightarrow{\exists \gamma} k,$$

such that $\gamma\alpha = \gamma\beta$.

Analogously, a category \mathbf{I} is *cofiltered* if \mathbf{I}^{op} is filtered.

Notation A.1.3.3. Typically, indexing categories take the form described above, but formally, any category can be viewed as an indexing category.

Definition A.1.3.4. If \mathbf{C} is a category, and \mathbf{I} is a category, then an *\mathbf{I} -diagram* is a functor $\mathbf{I} \rightarrow \mathbf{C}$. The category $\text{Fun}(\mathbf{I}, \mathbf{C})$ is called the category of \mathbf{I} -diagrams in \mathbf{C} (i.e., morphisms are natural transformations of functors).

Example A.1.3.5. If \mathbf{C} is a category, $A \in \mathbf{C}$ is an object, and \mathbf{I} is a category, then the *constant \mathbf{I} -diagram* (with value A) is the functor that assigns to each object $i \in \mathbf{I}$ the object A and to each morphism $i \rightarrow i' \in \mathbf{I}$ the identity morphism. Sending an object A to the constant \mathbf{I} -diagram defines a functor

$$\Delta : \mathbf{C} \longrightarrow \text{Fun}(\mathbf{I}, \mathbf{C});$$

this functor is typically called the diagonal.

Remark A.1.3.6. One point of view on limits and colimits is that an \mathbf{I} -indexed limit is simply a right adjoint to the diagonal functor while an \mathbf{I} -indexed colimit is a left adjoint to the diagonal functor.

A.2 Monoidal categories

A.2.1 Monoidal categories

Definition A.2.1.1. A *monoidal category* $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ consists of

- a category \mathcal{C} ,
 - a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$,
 - a distinguished *unit* object $\mathbb{1} \in \mathcal{C}$,
 - natural isomorphisms $l_X : \mathbb{1} \otimes X \rightarrow X$, $r_X : X \otimes \mathbb{1} \rightarrow X$, and
 - natural associativity isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$;
- these data are supposed to satisfy two *coherence* axioms:

1. given a pair of objects $X, Y \in \mathcal{C}$, the diagram

$$\begin{array}{ccc}
 (X \otimes \mathbb{1}) \otimes Y & \xrightarrow{a_{X\mathbb{1}Y}} & X \otimes (\mathbb{1} \otimes Y) \\
 \searrow^{r_X \otimes \text{id}} & & \swarrow^{\text{id} \otimes l_Y} \\
 & X \otimes Y &
 \end{array}$$

commutes;

2. given four objects W, X, Y, Z , the diagram

$$\begin{array}{ccc}
 ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a_{W \otimes X, Y, Z}} & (W \otimes X) \otimes (Y \otimes Z) \xrightarrow{a_{W, X, Y \otimes Z}} W \otimes (X \otimes (Y \otimes Z)) \\
 \downarrow^{a_{W, X, Y} \otimes \mathbb{1}_Z} & & \uparrow^{\mathbb{1}_W \otimes a_{X, Y, Z}} \\
 (W \otimes (X \otimes Y)) \otimes Z & \xrightarrow{a_{W, X \otimes Y, Z}} & W \otimes ((X \otimes Y) \otimes Z)
 \end{array}$$

commutes.

Remark A.2.1.2. A priori, there are infinitely many more diagrams whose commutativity we could request (e.g., the associativity relations for 5 or greater arrows). There is a “coherence theorem” that shows that requesting commutativity of the above diagrams guarantees commutativity of various more complicated diagrams...

Example A.2.1.3. The category *Set* of sets with Cartesian product is monoidal. The category *Grp* of groups with the usual product of groups is monoidal. Likewise, the category *Ab* is a monoidal subcategory of *Grp*. The category *Cat* of categories with product of categories is monoidal. the category *Top* of topological spaces with the Cartesian product (equipped with the product topology) is monoidal.

A.2.2 Enriched categories

Given a category \mathcal{C} and three objects X, Y, Z , a priori one has a set of homomorphisms $\text{Hom}_{\mathcal{C}}(X, Y)$ and composition determines a function $\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$ via the formula $(f, g) \mapsto f \circ g$. In many cases of interest, $\text{Hom}_{\mathcal{C}}(X, Y)$ has additional structure, e.g., it is an abelian group or a vector space over a field, and the composition operation respects this additional structure. We now introduce some the standard terminology one uses to keep track of all of the compatibilities inherent in such a structure.

Definition A.2.2.1. Suppose $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ is a symmetric monoidal category. We will say that a (locally small) category \mathcal{E} is \mathcal{C} -enriched (or simply an \mathcal{C} -category) if

- for every pair of objects X, Y , the set $\text{Hom}_{\mathcal{E}}(X, Y)$ is an object of \mathcal{E} ;

- for each object $X \in \mathcal{C}$, there is an identity element $i_X : \mathbf{1} \rightarrow \text{Hom}_{\mathcal{C}}(X, X)$;
- for every triple of objects X, Y, Z , there is a composition law $M_{X,Y,Z} : \text{Hom}_{\mathcal{C}}(Y, Z) \otimes \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$;

the following axioms hold:

1. composition is associative, i.e., for any four objects W, X, Y, Z the diagram

$$\begin{array}{ccc}
 (\text{Hom}_{\mathcal{C}}(Y, Z) \otimes \text{Hom}_{\mathcal{C}}(X, Y)) \otimes \text{Hom}_{\mathcal{C}}(W, X) & \xrightarrow{a} & \text{Hom}_{\mathcal{C}}(Y, Z) \otimes (\text{Hom}_{\mathcal{C}}(X, Y) \otimes \text{Hom}_{\mathcal{C}}(W, X)) \\
 \downarrow M \otimes id & & \downarrow id \otimes M \\
 \text{Hom}_{\mathcal{C}}(X, Z) \otimes \text{Hom}_{\mathcal{C}}(W, X) & \xrightarrow{M} & \text{Hom}_{\mathcal{C}}(W, Z) \xleftarrow{M} \text{Hom}_{\mathcal{C}}(Y, Z) \otimes \text{Hom}_{\mathcal{C}}(W, X)
 \end{array}$$

commutes;

2. composition is compatible with units, i.e., for any pair of objects X, Y in \mathcal{C} the diagram

$$\begin{array}{ccccc}
 \text{Hom}_{\mathcal{C}}(Y, Y) \otimes \text{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{M} & \text{Hom}_{\mathcal{C}}(X, Y) & \xleftarrow{M} & \text{Hom}_{\mathcal{C}}(X, Y) \otimes \text{Hom}_{\mathcal{C}}(X, X) \\
 i \otimes id \uparrow & & l \nearrow & & \nwarrow r & id \otimes i \uparrow \\
 \mathbf{1} \otimes \text{Hom}_{\mathcal{C}}(X, Y) & & & & & \text{Hom}_{\mathcal{C}}(X, Y) \otimes \mathbf{1}
 \end{array}$$

commutes.

Example A.2.2.2. Every (locally small) category is a Set-enriched category.

Definition A.2.2.3. A category \mathcal{C} is called

- *pre-additive* if it is an Ab-enriched category;
- *pre- R -linear* if it is a Mod_R -enriched category, with R a commutative unital ring;
- *topological* if it is a Top-enriched category; and
- *simplicial* if it is an sSet-enriched category.

Given an enriched category, it will be important to consider functors that preserve the additional structure present on morphism sets. This notion is summarized in the next definition.

Definition A.2.2.4. Given a monoidal category $(\mathcal{E}, \otimes, \mathbf{1}, a, l, r)$ and two \mathcal{E} -enriched categories \mathcal{C} and \mathcal{D} , a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ will be called an \mathcal{E} -enriched functor, or simply an \mathcal{E} -functor if for any pair of objects $X, Y \in \mathcal{C}$, the map $F_{AB} : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a morphism in \mathcal{E} , and the following conditions are satisfied:

1. the functor is compatible with the monoidal structure, i.e., given three objects X, Y, Z the diagram

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{C}}(Y, Z) \otimes \text{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{M} & \text{Hom}_{\mathcal{C}}(X, Z) \\
 \downarrow F \otimes F & & \downarrow F \\
 \text{Hom}_{\mathcal{D}}(F(Y), F(Z)) \otimes \text{Hom}_{\mathcal{D}}(F(X), F(Y)) & \xrightarrow{M} & \text{Hom}_{\mathcal{D}}(F(X), F(Z))
 \end{array}$$

commutes;

2. the functor is compatible with units, i.e., given any object $X \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} & \mathbf{1} & \\ & \swarrow & \searrow \\ \text{Hom}_{\mathcal{C}}(X, X) & \xrightarrow{F} & \text{Hom}_{\mathcal{D}}(F(X), F(X)) \end{array}$$

commutes.

Example A.2.2.5. An \mathcal{E} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of \mathcal{E} -categories is called

- *pre-additive* if $\mathcal{E} = \text{Ab}$;
- *pre- R -linear* if $\mathcal{E} = \text{Mod}_R$ for R a commutative unital ring.

A.2.3 Symmetric monoidal categories

Definition A.2.3.1. If $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ is a monoidal category, a *symmetric* structure on \mathcal{C} is the data of a natural isomorphism $c_{XY} : X \otimes Y \rightarrow Y \otimes X$ (the commutativity isomorphism) satisfying the following coherence axioms:

1. $c^2 = id_{\mathcal{C}}$, i.e., for every pair of objects $X, Y \in \mathcal{C}$, the diagram

$$\begin{array}{ccccc} & & id & & \\ & & \curvearrowright & & \\ X \otimes Y & \xrightarrow{c} & Y \otimes X & \xrightarrow{c} & X \otimes Y \end{array}$$

commutes;

2. compatibility with the unit, i.e., for every object $X \in \mathcal{C}$ the diagram

$$\begin{array}{ccc} \mathbf{1} \otimes X & \xrightarrow{c} & X \otimes \mathbf{1} \\ & \searrow^{l_X} & \swarrow_{r_X} \\ & X & \end{array}$$

commutes;

3. compatibility between commutativity and associativity, i.e., for every triple X, Y, Z of objects in \mathcal{C} the diagram

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{c^{X,Y \otimes Z}} & (Y \otimes Z) \otimes X \\ c \otimes id \downarrow & & & & \downarrow a_{Y,Z,X} \\ (Y \otimes X) \otimes Z & \xrightarrow{a_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{id \otimes c} & Y \otimes (Z \otimes X) \end{array}$$

commutes.

A monoidal category equipped with a symmetric structure will be called a *symmetric monoidal category*.

Example A.2.3.2. The category of abelian groups equipped with the isomorphism $c_{A,B} : A \times B \rightarrow B \times A$ given by switching the two factors is a symmetric monoidal category. The same holds for the category of R -modules over a commutative unital ring R .

A.3 Other types of categories

A.3.1 Abelian categories

Definition A.3.1.1. An triple $(\mathcal{C}, 0)$ consisting of an Ab-enriched category \mathcal{C} and a distinguished object 0 is called *abelian* if

1. the object 0 is a zero object, i.e., it is both initial and final;
2. for any two objects X, Y , a biproduct $X \times Y$ exists in \mathcal{C} ;
3. every morphism in \mathcal{C} has a kernel and a cokernel;
4. every monomorphism is a kernel of its cokernel, and every epimorphism is a cokernel of its kernel.

A.3.2 Exact categories

Exact categories were initially defined by Quillen [Qui73, §2]. With time, simplifications of Quillen's axioms were observed (cf. [GR92, §9.1]). The following definition is due to Keller [Kel90, Appendix A]. See [Büh10] for a more detailed treatment.

Definition A.3.2.1. Given an additive category \mathcal{A} , a pair of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

is called *exact* if i is a kernel of d and d is a cokernel of i . We will refer to a diagram as a pair (i, p) as above as an *exact pair*; the morphism i will be called an *admissible monomorphism* and the morphism p will be called an *admissible epimorphism*.

Definition A.3.2.2. Given an additive category \mathcal{A} , an *exact structure* on \mathcal{A} consists of a class $\mathcal{E} \subset \mathcal{A}$ of exact pairs closed under isomorphisms and satisfying the following axioms:

- E0 the identity morphism on the zero object is an admissible epimorphism;
- E1 admissible epimorphisms are stable by composition;
- E2 admissible epimorphisms are stable by pullback;
- E2° admissible monomorphisms are stable by pushout;

A pair $(\mathcal{A}, \mathcal{E})$ consisting of an additive category and an exact structure will be called an *exact category*.

Lemma A.3.2.3. If $(\mathcal{A}, \mathcal{E})$ is an exact category, then for any pair of objects X, Y of \mathcal{C} , the pair

$$X \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} Y$$

is an exact pair.

Lemma A.3.2.4. If $(\mathcal{A}, \mathcal{E})$ is an exact category, then admissible monomorphisms are stable under composition.

Bibliography

- [AF59] A. Andreotti and T. Frankel. The Lefschetz theorem on hyperplane sections. *Ann. of Math. (2)*, 69:713–717, 1959. [25](#)
- [AGP02] M. Aguilar, S. Gitler, and C. Prieto. *Algebraic topology from a homotopical viewpoint*. Universitext. Springer-Verlag, New York, 2002. Translated from the Spanish by Stephen Bruce Sontz. [6](#)
- [AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. [6](#)
- [AR94] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. [6](#)
- [Ati89] M. F. Atiyah. *K-theory*. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, second edition, 1989. Notes by D. W. Anderson. [6](#), [80](#)
- [Bas64] H. Bass. *K*-theory and stable algebra. *Inst. Hautes Études Sci. Publ. Math.*, (22):5–60, 1964. [4](#)
- [BG73] K. S. Brown and S. M. Gersten. Algebraic *K*-theory as generalized sheaf cohomology. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 266–292. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973. [7](#)
- [BHS64] H. Bass, A. Heller, and R. G. Swan. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math.*, (22):61–79, 1964. [102](#)
- [Bir13] G. D. Birkhoff. A theorem on matrices of analytic functions. *Math. Ann.*, 74(1):122–133, 1913. [158](#)
- [BL08] J. Barge and J. Lannes. *Suites de Sturm, indice de Maslov et périodicité de Bott*, volume 267 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2008. [6](#), [175](#)
- [Bou98] N. Bourbaki. *Commutative algebra. Chapters 1–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation. [88](#)
- [Bro73] K. S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973. [7](#)
- [Büh10] T. Bühler. Exact categories. *Expo. Math.*, 28(1):1–69, 2010. [207](#)
- [Caz12] C. Cazanave. Algebraic homotopy classes of rational functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(4):511–534 (2013), 2012. [175](#)
- [CE99] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original. [4](#), [80](#)
- [Cla66] L. Claborn. Every abelian group is a class group. *Pacific J. Math.*, 18:219–222, 1966. [68](#)
- [Coq06] T. Coquand. On seminormality. *J. Algebra*, 305(1):577–584, 2006. [75](#)
- [Cur71] E. B. Curtis. Simplicial homotopy theory. *Advances in Math.*, 6:107–209 (1971), 1971. [7](#)
- [DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. [7](#)
- [DW82] R. Dedekind and H. Weber. Theorie der algebraischen Functionen einer Veränderlichen. *J. Reine Angew. Math.*, 92:181–290, 1882. [158](#)

- [Eis95] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry. 6
- [Fos73] R. M. Fossum. *The divisor class group of a Krull domain*. Springer-Verlag, New York-Heidelberg, 1973. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74*. 68
- [Fre04] P. Freyd. Homotopy is not concrete. *Repr. Theory Appl. Categ.*, (6):1–10, 2004. Reprinted from *The Steenrod Algebra and its Applications*, 25–34, *Lecture Notes in Math.*, 168, Springer, Berlin, 1970 [MR0276961]. 131
- [GJ60] L. Gillman and M. Jerison. *Rings of continuous functions*. The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960. 61
- [GJ09] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612]. 7
- [God73] R. Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973. Troisième édition revue et corrigée, *Publications de l'Institut de Mathématique de l'Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No. 1252*. 6
- [GR92] P. Gabriel and A. V. Roïter. Representations of finite-dimensional algebras. In *Algebra, VIII*, volume 73 of *Encyclopaedia Math. Sci.*, pages 1–177. Springer, Berlin, 1992. With a chapter by B. Keller. 207
- [Gro57] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957. 158
- [Ham95] H. A. Hamm. Affine varieties and Lefschetz theorems. In *Singularity theory (Trieste, 1991)*, pages 248–262. World Sci. Publ., River Edge, NJ, 1995. 25
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. *Graduate Texts in Mathematics, No. 52*. 6
- [Hat02] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. 6
- [Hir94] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original. 52
- [Hir03] P. S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003. 7
- [HM82] M. Hazewinkel and C. F. Martin. A short elementary proof of Grothendieck's theorem on algebraic vectorbundles over the projective line. *J. Pure Appl. Algebra*, 25(2):207–211, 1982. 157
- [Hoc69] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969. 22
- [Hor64] G. Horrocks. Projective modules over an extension of a local ring. *Proc. London Math. Soc. (3)*, 14:714–718, 1964. 161
- [Hov99] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. 7
- [Hus94] D. Husemoller. *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1994. 6
- [Jar15] J. F. Jardine. *Local homotopy theory*. Springer Monographs in Mathematics. Springer, New York, 2015. 7
- [Kan58a] D. M. Kan. A combinatorial definition of homotopy groups. *Ann. of Math. (2)*, 67:282–312, 1958. 7
- [Kan58b] D. M. Kan. On homotopy theory and c.s.s. groups. *Ann. of Math. (2)*, 68:38–53, 1958. 7
- [Kap58] I. Kaplansky. Projective modules. *Ann. of Math (2)*, 68:372–377, 1958. 43
- [Kar77] K. K. Karčauskas. A generalized Lefschetz theorem. *Funkcional. Anal. i Priložen.*, 11(4):80–81, 1977. 25
- [Kel90] B. Keller. Chain complexes and stable categories. *Manuscripta Math.*, 67(4):379–417, 1990. 207
- [Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714]. 6
- [Knu91] M.-A. Knus. *Quadratic and Hermitian forms over rings*, volume 294 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni. 6

- [Koh58] C. W. Kohls. Prime ideals in rings of continuous functions. *Illinois J. Math.*, 2:505–536, 1958. 61
- [Lam99] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. 6, 41
- [Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005. 6, 185
- [Lam06] T. Y. Lam. *Serre's problem on projective modules*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. 6, 97, 162, 164, 165, 166, 167
- [Lin82] H. Lindel. On the Bass-Quillen conjecture concerning projective modules over polynomial rings. *Invent. Math.*, 65(2):319–323, 1981/82. 151, 152
- [Mat89] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid. 6, 40, 63, 65, 69, 70
- [ML98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998. 6
- [Mor06a] F. Morel. *Homotopy theory of schemes*, volume 12 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2006. Translated from the 1999 French original by James D. Lewis. 7
- [Mor06b] F. Morel. \mathbb{A}^1 -algebraic topology. In *International Congress of Mathematicians. Vol. II*, pages 1035–1059. Eur. Math. Soc., Zürich, 2006. 4
- [Mor12] F. Morel. \mathbb{A}^1 -Algebraic Topology over a Field, volume 2052 of *Lecture Notes in Math*. Springer, New York, 2012. 4, 175
- [MV99] F. Morel and V. Voevodsky. \mathbb{A}^1 -homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, 90:45–143 (2001), 1999. 7
- [Nev90] C. W. Neville. When is $C(X)$ a coherent ring? *Proc. Amer. Math. Soc.*, 110(2):505–508, 1990. 35
- [Nis89] Ye. A. Nisnevich. The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K -theory. In *Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987)*, volume 279 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 241–342. Kluwer Acad. Publ., Dordrecht, 1989. 6
- [Oja90] M. Ojanguren. *The Witt group and the problem of Lüroth*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1990. With an introduction by Inta Bertuccioni. 175
- [Ple08] J. Plemelj. Riemannsche Funktionenscharen mit gegebener Monodromiegruppe. *Monatsh. Math. Phys.*, 19(1):211–245, 1908. 158
- [Pop89] D. Popescu. Polynomial rings and their projective modules. *Nagoya Math. J.*, 113:121–128, 1989. 170
- [Qui67] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967. 7
- [Qui73] D. G. Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973. 207
- [Rec71] D. L. Rector. K -theory of a space with coefficients in a (discrete) ring. *Bull. Amer. Math. Soc.*, 77:571–575, 1971. 26
- [Roi79] M. Roitman. On projective modules over polynomial rings. *J. Algebra*, 58(1):51–63, 1979. 167
- [Ser55] J.-P. Serre. Faisceaux algébriques cohérents. *Ann. of Math. (2)*, 61:197–278, 1955. 4, 36
- [Ser58a] J.-P. Serre. Espaces Fibrés Algébriques. In *Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications*, page 37. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958. 108
- [Ser58b] J.-P. Serre. Modules projectifs et espaces fibrés à fibre vectorielle. In *Séminaire P. Dubreil, M.-L. Dubreil-Jacotin et C. Pisot, 1957/58, Fasc. 2, Exposé 23*, page 18. Secrétariat mathématique, Paris, 1958. 4, 54

- [Spa81] E. H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint. 6
- [Sri91] V. Srinivas. On the embedding dimension of an affine variety. *Math. Ann.*, 289(1):125–132, 1991. 21
- [Sta15] The Stacks Project Authors. *Stacks Project*. <http://stacks.math.columbia.edu>, 2015. 6, 22, 35, 43, 70, 71, 86, 87, 96, 118, 137, 150, 153
- [Ste67] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967. 14
- [Swa68] R. G. Swan. *Algebraic K-theory*. Lecture Notes in Mathematics, No. 76. Springer-Verlag, Berlin-New York, 1968. 97
- [Swa80] R. G. Swan. On seminormality. *J. Algebra*, 67(1):210–229, 1980. 75
- [Tog73] A. Tognoli. Su una congettura di Nash. *Ann. Scuola Norm. Sup. Pisa (3)*, 27:167–185, 1973. 24
- [Tra70] C. Traverso. Seminormality and Picard group. *Ann. Scuola Norm. Sup. Pisa (3)*, 24:585–595, 1970. 75
- [TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990. 6
- [Ulm71] F. Ulmer. On the existence and exactness of the associated sheaf functor. *J. Pure Appl. Algebra*, 3:295–306, 1971. 147
- [Vas86] L. N. Vaserstein. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 294(2):749–755, 1986. 13
- [Voe98] V. Voevodsky. \mathbb{A}^1 -homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604 (electronic), 1998. 7
- [Voe10] V. Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. *J. Pure Appl. Algebra*, 214(8):1384–1398, 2010. 6
- [Wei94] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994. 6, 92
- [Wei13] C. A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic K -theory. 6

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