Toward a meta-stable range in $\mathbb{A}^1$-homotopy theory of punctured affine spaces

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Aravind Asok, Jean Fasel

Suppose k is a perfect field having characteristic unequal to 2. Write $\mathcal{S}_k$ for the category of schemes that are separated, smooth and of finite type over k. Write $\mathcal{H}_\cdot(k)$ for the Morel-Voevodsky unstable pointed $\mathbb{A}^1$-homotopy category [MoVo99]. A (pointed) k-space $\mathcal{X}$ (resp. $(\mathcal{X}, x)$) is a (pointed) simplicial Nisnevich sheaf on $\mathcal{S}_k$. Given two pointed k-spaces $(\mathcal{X}, x)$ and $(\mathcal{Y}, y)$, we write $[(\mathcal{X}, x), (\mathcal{Y}, y)]_{\mathbb{A}^1}$ for $\mathrm{Hom}_{\mathcal{H}_\cdot(k)}(\mathcal{X}, \mathcal{Y})$. If $(\mathcal{X}, x)$ is a pointed k-space, write $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ for the Nisnevich sheaf associated with the presheaf $U \mapsto [S^i_s \wedge U_+, (\mathcal{X}, x)]_{\mathbb{A}^1}$.

Point $\mathbb{A}^n \setminus 0$ by $(1, 0, \ldots, 0)$, and suppress this base-point from notation. Results of Morel yield a description of $\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$, $n \geq 2$, as the sheaf $\mathbf{K}^{MW}_n$ of "unramified Milnor-Witt K-theory." In previous work, the authors provided a description of $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ [AsFa12a, AsFa12b]. The goal of the talk was to provide a conjectural description of $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$ for $n \geq 4$. The proposed description is in two parts.

Suslin matrices and the degree map

Schlichting and Tripathi constructed an orthogonal Grassmannian OGr and showed that $\mathbb{Z} \times OGr$ represents Hermitian K-theory in the unstable $\mathbb{A}^1$-homotopy category [ScTr12]. They also establish a geometric form of Bott periodicity in Hermitian K-theory that identifies various loop spaces of $\mathbb{Z} \times OGr$ in terms of other natural spaces; we summarize this result as follows.

Proposition 1. *There are weak equivalences of the form*

$$\Omega^1_s \Omega^i_{\mathbb{P}^1}(\mathbb{Z} \times OGr) \xrightarrow{\sim} \begin{cases} O & \textit{if } i \equiv 0 \mod 4 \\ GL/Sp & \textit{if } i \equiv 1 \mod 4 \\ Sp & \textit{if } i \equiv 2 \mod 4, \textit{ and} \\ GL/O & \textit{if } i \equiv 3 \mod 4; \end{cases}$$

Here $O := \mathrm{colim}_n\, O(q_{2n})$, where q_{2n} is the standard hyperbolic form, $Sp := \mathrm{colim}_n\, Sp_{2n}$, $GL/Sp := \mathrm{colim}_n\, GL_{2n}/Sp_{2n}$, and $GL/O := \mathrm{colim}_n\, GL_{2n}/O(q_{2n})$.

The class of $\langle 1 \rangle \in GW(k)$ yields a distinguished element in $GW(k) = [\mathrm{Spec}\, k_+, \mathbb{Z} \times OGr]_{\mathbb{A}^1}$. An adjunction argument can be used to show that this element corresponds to a distinguished class in $[\mathbb{A}^n \setminus 0, P_n]_{\mathbb{A}^1}$, where P_n is either O, GL/O, Sp or GL/Sp depending on whether n is congruent to 0, 1, 2 or 3 modulo 4.

Let Q_{2n-1} be the smooth affine quadric defined as a hypersurface in $\mathbb{A}^{2n}$ given by the equation $\sum_i x_i x_{n+i} = 1$. There is an $\mathbb{A}^1$-weak equivalence $Q_{2n-1} \to \mathbb{A}^n \setminus 0$ defined by projecting onto the first n variables. Each variety P_n is an ind-algebraic variety, and Suslin inductively defined certain matrices S_n that correspond to morphisms $s_n : Q_{2n-1} \to P_n$ [Su77].

Proposition 2. *The distinguished homotopy classes $[\mathbb{A}^n \setminus 0, P_n]_{\mathbb{A}^1}$ described in the previous paragraph is represented by the morphism $s_n : Q_{2n-1} \to P_n$ given by the matrix S_n.*

The $\mathbb{A}^1$-homotopy sheaves of $O, GL/O, Sp$ and GL/Sp can be identified in terms of the Nisnevich sheafification of Schlichting's higher Grothendieck-Witt groups. Indeed, $\pi_i^{\mathbb{A}^1}(O) \cong \mathbf{GW}^0_{i+1}$, $\pi_i^{\mathbb{A}^1}(GL/O) \cong \mathbf{GW}^1_{i+1}$, $\pi_i^{\mathbb{A}^1}(Sp) \cong \mathbf{GW}^2_0$ and $\pi_i^{\mathbb{A}^1}(GL/Sp) \cong \mathbf{GW}^3_{i+1}$. In general, the sheaves $\mathbf{GW}^j_i$ are viewed as 4 periodic in j. Therefore, the morphism s_n yields, upon applying the functor $\pi_n^{\mathbb{A}^1}(\cdot)$, a morphism

$$s_{n*} : \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}^n_{n+1}.$$

This morphism is not surjective for $n \geq 4$, but it does coincide with a corresponding morphism constructed in the computations of $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$.

Recall the contraction of a sheaf $\mathcal{F}$ is defined by the formula $\mathcal{F}_{-1}(U) := \ker((id \times e)^* : \mathcal{F}(\mathbf{G}_m \times U) \to \mathcal{F}(U))$, where $e : \operatorname{Spec} k \to \mathbf{G}_m$ is the unit section. One defines $\mathcal{F}_{-i}$ inductively as $(\mathcal{F}_{-(i-1)})_{-1}$.

Theorem 3. *The morphism s_{n*} becomes surjective after $(n-3)$-fold contraction and split surjective after n-fold contraction.*

Motivic Hopf maps and the kernel of the degree map

In [AsFa12b], we introduced the geometric Hopf map $\nu : \mathbb{A}^4 \setminus 0 \to \mathbb{P}^{1\wedge 2}$ and showed that it was $\mathbb{P}^1$-stably essential (i.e., is not null $\mathbb{A}^1$-homotopic after repeated $\mathbb{P}^1$-suspension). For any integer $n \geq 2$, set

$$\nu_n := \Sigma^{n-2}_{\mathbb{P}^1} \nu : \mathbb{A}^{n+2} \setminus 0 \longrightarrow \mathbb{P}^{1\wedge n}.$$

Applying $\pi_n^{\mathbb{A}^1}(\cdot)$ to the above morphism yields a map

$$(\nu_n)_* : \mathbf{K}^{MW}_{n+2} \longrightarrow \pi^{\mathbb{A}^1}_{n+1}(\mathbb{P}^{1\wedge n}).$$

For $n \geq 4$, Morel's Freudenthal suspension theorem yields isomorphisms

$$\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \pi^{\mathbb{A}^1}_{n+1}(\mathbb{P}^{1\wedge n}),$$

so in this range, we can view $(\nu_n)_*$ as giving a map $\mathbf{K}^{MW}_{n+2} \to \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$.

For $n = 3$, Morel's Freudenthal suspension theorem only yields an epimorphism. We can refine this result to provide an analog of the beginning of the EHP sequence in $\mathbb{A}^1$-homotopy theory. A particular case of the general result we can establish can be stated as follows.

Theorem 4. *There is an exact sequence of the form*

$$\pi_5^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3}) \xrightarrow{H} \pi_5^{\mathbb{A}^1}(\Sigma_s^1(\mathbb{A}^3 \setminus 0)^{\wedge 2}) \xrightarrow{P} \pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \xrightarrow{E} \pi_4^{\mathbb{A}^1}(\mathbb{P}^{1\wedge 3}) \longrightarrow 0.$$

The morphism H in the above exact sequence *conjecturally* admits a description as a variant of the Hopf invariant in Chow-Witt theory. Assuming this, the results we have proven on $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0)$ show that ν_{3*} factors through an explicit quotient of $\mathbf{K}_5^{MW}$. In turn, this (conjectural) computation suggests the following conjecture.

Conjecture 5. *For any integer $n \geq 3$, the morphism ν_{n*} factors through a morphism $\mathbf{K}_{n+2}^{M}/24 \to \pi_{n+1}^{\mathbb{A}^1}(\mathbb{P}^{1\wedge n})$.*

The structure of $\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)$

We now study the relationship between the two morphisms constructed above. Using an obstruction theory argument, one can demonstrate the following result.

Proposition 6. *For any integer $n \geq 4$, the composite map*

$$\mathbf{K}_{n+2}^{MW} \longrightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}_{n+1}^{n}$$

is zero.

Combining everything discussed so far, one is led to the following conjecture.

Conjecture 7. *For any integer $n \geq 4$, there is an exact sequence of sheaves of the form*

$$\mathbf{K}_{n+2}^{M}/24 \longrightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \longrightarrow \mathbf{GW}_{n+1}^{n}.$$

The sequence becomes short exact after n-fold contraction.

Remark 8. The conjecture above stabilizes to an unpublished conjecture of F. Morel on the stable motivic π_1 of the motivic sphere spectrum. Using the motivic Adams(-Novikov) spectral sequence, K. Ormsby and P.-A. Østvær have checked that after taking sections over fields having 2-cohomological dimension ≤ 2, the 2-primary part of the stable conjecture is true. Nevertheless, the stable conjecture does not imply the conjecture above (even for large n) because of a lack of a Freudenthal suspension theorem for $\mathbb{P}^1$-suspension. On the other hand, the conjecture above for every n sufficiently large implies the stable conjecture.

Remark 9. By the results of [AsFa12b], the above conjecture immediately implies "Murthy's conjecture:" if X is a smooth affine $(d+1)$-fold over an algebraically closed field k, and $\mathcal{E}$ is a rank d vector bundle on X, then $\mathcal{E}$ splits off a free rank 1 summand if and only if $0 = c_d(\mathcal{E}) \in CH^d(X)$. However, the conjecture is much stronger: it gives the complete secondary obstruction to splitting a free rank 1 summand of a vector bundle on a smooth affine scheme.

References

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