Vector bundles and $\mathbb{A}^1$-homotopy theory

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July 31, 2015

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Theorem (N. Steenrod)

The map

$$[M, BO(n)] \xrightarrow{\sim} \mathcal{V}_n(M),$$

given by pull-back of the universal vector bundle, is a bijection.

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is the primary obstruction to splitting off a trivial rank 1 *summand.If* dim $M \leq n$*, then the vanishing of* $e(\xi)$ *is the* only *obstruction to splitting off a trivial rank* 1 *summand.*

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- If $n \geq 4$ (and ξ is orientable), $\pi_n(S^{n-1}) \cong \mathbb{Z}/2$, and this obstruction (S. Liao '54, F. Peterson-N. Stein '62) can be shown to be an element of a coset

$$\mathcal{O}_{(2)}(\xi) \in H^{n+1}(M, \mathbb{Z}/2)/(Sq^2 + w_2 \cup)H^{n-1}(M, \mathbb{Z}/2).$$

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- If dim $M \leq n + 1$, the primary and secondary obstructions are the only obstructions to splitting.

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- If M is a manifold of dimension d, and $\xi : V \to M$ is a vector bundle of rank r, the situation is both theoretically and computationally satisfying for $d - r$ "small" relative to d (to guarantee homotopy groups of spheres are "regular," e.g., stable, meta-stable,...).

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Consider the "stabilization" function

$$s_{r,X} : \mathcal{V}_r(X) \longrightarrow \mathcal{V}_{r+1}(X)$$

given by $\mathcal{E} \mapsto \mathcal{E} \oplus \mathbf{1}_X$.

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Problem (H. Bass '64)

Characterize the image of s_r.

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- (J.-P. Serre '58) If $r \geq d$, then $s_{r,X}$ is surjective.
- If $r \leq d$ the problem is much harder.

Theorem (R. Swan-M.P. Murthy '76, N.M. Kumar-Murthy '82, Murthy '94)

If k is algebraically closed, and X is a smooth affine d-fold over k, then the image of $s_{d-1,X}$ consists of those $\mathcal{E}$ (of rank d) such that $0 = c_d(\mathcal{E}) \in CH^d(X)$.

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Conjecture (Murthy)

If k is algebraically closed, and X a smooth affine d-fold over k, then the image of $s_{d-2,X}$ consists of those $\mathcal{E}$ (of rank $d-1$) such that $0 = c_{d-1}(\mathcal{E}) \in CH^{d-1}(X)$.

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Thus, the Euler class is still the only obstruction to splitting!

Theorem (A., J. Fasel '12)

If k has characteristic unequal to 2, X is a smooth affine d-fold over k and $d \leq 4$, then the image of $s_{d-2,X}$ consists of $\mathcal{E}$ such that $0 = c_{d-1}(\mathcal{E}) \in CH^{d-1}(X)$.

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Remark

We expect Murthy's conjecture to be true in general (without restriction on d).

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- Necessary condition: topological Chern classes (in $H^{2i}(X^{an}, \mathbb{Z})$) must be algebraic, i.e., lie in the image of the cycle class map from Chow groups.

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- If $\dim X <= 2$, this condition is sufficient to guarantee algebraizability (Schwarzenberger, Murthy-Swan).

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Suppose X is a smooth complex affine 4-fold.

- *Given a topological rank 2 vector bundle $\mathcal{E}^{top}$ with topological Chern classes c_i^{top}, then $\mathcal{E}^{top}$ is algebraizable if and only if there exist $c_i \in CH^i(X)$ mapping to c_i^{top} under the cycle class map*

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- *Moreover, there exists a smooth complex affine 4-fold carrying a topological rank 2 vector bundle with algebraic Chern classes that is not algebraizable.*

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- The proofs of all the results above are connected.

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- Q_{2n+1} is like a sphere.
- Serre showed that there is a topologically non-trivial vector bundle on S^{2p+1} (using $\pi_{2p}(S^3)$ is non-trivial).

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- The functor $X \mapsto \mathcal{V}_1(X)$ is $\mathbb{A}^1$-invariant, i.e., $\mathcal{V}_1(X) \to \mathcal{V}_1(X \times \mathbb{A}^1)$ is a bijection.

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- The functor $X \mapsto \mathcal{V}_1(X)$ is $\mathbb{A}^1$-invariant, i.e., $\mathcal{V}_1(X) \to \mathcal{V}_1(X \times \mathbb{A}^1)$ is a bijection.
- Algebraic K-theory is representable in $\mathcal{H}(k)$.
- Various different models of BGL_n become equivalent in this category (e.g., the simplicial "bar" model and the infinite Grassmannian).

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- the theorem of D. Quillen and A. Suslin ('76) solving Serre's problem, i.e., vector bundles on affine space are trivial, and
- H. Lindel's solution to the Bass-Quillen conjecture ('81), i.e., $\mathbb{A}^1$-invariance for X smooth affine.

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Varieties that are $\mathbb{A}^1$-contractible can have non-trivial vector bundles (but their algebraic K-theory must be that of the base field)!

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- Apply this with $\mathcal{X} = BGL_n$; descent for vector bundles $\Longrightarrow$ excision, Bass-Quillen conjecture $\Longrightarrow$ homotopy invariance.

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Thus, using a version of the Postnikov tower in $\mathbb{A}^1$-homotopy theory, the splitting problem is controlled by homotopy theory of $\mathbb{A}^n \setminus 0$.

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For any $n \geq 2$, there is an isomorphism $\pi^{\mathbb{A}^1}_{n-1}(\mathbb{A}^n \setminus 0) \cong \mathbf{K}^{MW}_n$.

Both of Morel's theorems admit "explanations"

- Set $\Delta^n = \operatorname{Spec} k[x_0, \ldots, x_n]/(\sum_i x_i - 1)$ and define $Sing^{\mathbb{A}^1}(X)(U) = \operatorname{Hom}(U \times \Delta^\bullet, X)$.

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- Connectivity theorem: move "algebraic" spheres to avoid 0 and then contract them.
- Computation: It follows from results of Suslin that Milnor K-theory measures the failure of stabilization for homology of GL_n, and Morel's result is essentially this result after "keeping track of the action of the fundamental group of BGL_n".

The sections of $\mathbf{K}_n^{MW}$ over finitely generated extensions of the base field can be explicitly described in terms of generators and relations (assuming Milnor's conjecture on quadratic forms, now a theorem of D. Orlov-A. Vishik-V. Voevodsky '07).

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Theorem (F. Morel '04)

There is a short exact sequence of the form

$$0 \longrightarrow I^{n+1}(F) \longrightarrow K_n^{MW}(F) \longrightarrow K_n^M(F) \longrightarrow 0,$$

where $I^{n+1}(F)$ *is the* $(n+1)$*st power of* $I(F) = \ker(GW(F) \to \mathbb{Z})$*, and* $K_n^M(F)$ *is Milnor K-theory.*

Theorem (A, J. Fasel)

The stabilization map $Sp_2 \hookrightarrow Sp_\infty$ gives a surjective map $\pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \to \mathbf{K}_3^{Sp} = \mathbf{GW}_3^2$ whose kernel is (almost) $\mathbf{K}_4^M/12$ (witnessing $\pi_6(S^3) = \mathbb{Z}/12$).

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The stabilization map $SL_4/Sp_4 \hookrightarrow GL_4/Sp_4 \hookrightarrow GL/Sp$ gives a surjective map $\pi_3^{\mathbb{A}^1}(\mathbb{A}^3 \setminus 0) \to \mathbf{GW}_4^3$ whose kernel is (almost) $\mathbf{K}_5^M/24$ (and witnesses $\pi_8(S^5) = \mathbb{Z}/24$).

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- In each case “almost” means that there are unstable factors coming from suitable powers of the fundamental ideal in the Witt ring.

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- Murthy's conjecture follows from this together with one other vanishing theorem.

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Construct a "p-local" splitting of punctured affine spaces using "Suslin matrices" and use this to lift Serre's generator to $\mathbb{A}^1$-homotopy theory.

For $n \geq 4$, we expect a kind of stabilization.

Conjecture (A., J. Fasel)

For every $n \geq 4$, there is an exact sequence of the form

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- Evidence in recent work with B. Williams and K. Wickelgren ("unstable" stuff is killed after one further suspension).

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For every $n \geq 4$, there is an exact sequence of the form

$$\mathbf{K}^M_{n+2}/24 \xrightarrow{(\nu_n)_*} \pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) \xrightarrow{(s_n)_*} \mathbf{GW}^n_{n+1}.$$

- Here the factor $\mathbf{GW}^n_{n+1}$ "incarnates" $\mathbb{Z}/2 = \pi_m(S^{m-1})$ (it is detected by **KO**-degree map),$\mathbb{Z}/2 = \pi_{m+1}(S^{m-1})$, and
- the factor $\mathbf{K}^M_{n+2}/24$ "incarnates" $\mathbb{Z}/24 = \pi_{n+2}(S^{n-1})$.
- Roughly, homotopy sheaves have "weights", which see classical homotopy groups of spheres of different degrees.
- Evidence in recent work with B. Williams and K. Wickelgren ("unstable" stuff is killed after one further suspension).
- Basic problem: there is no known Freudenthal suspension theorem for $\mathbb{P}^1$-suspension!

Thank you!