The unstable n-$\mathbb{A}^1$-connectivity theorem

Aravind Asok

April 1, 2009

1 Introduction

Fix k a commutative unital ring having finite Krull dimension. Let $\mathcal{S}m_k$ denote the category of $\operatorname{Spec} k$-schemes that are separated, smooth and have finite type over $\operatorname{Spec} k$. Endow $\mathcal{S}m_k$ with the structure of a site by equipping this category with the Nisnevich topology. Recall the following definition.

Definition 1.1. A sheaf of groups $\mathcal{G}$ is said to be *strongly* $\mathbb{A}^1$*-invariant* if for any $X \in \mathcal{S}m_k$, and for i either 0 or 1, the maps

$$H^i_{Nis}(X, \mathcal{G}) \to H^i_{Nis}(X \times \mathbb{A}^1, \mathcal{G})$$

induced by pullback along the projection $X \times \mathbb{A}^1 \to X$ are bijections.

We will use "standard" terminology from $\mathbb{A}^1$-homotopy theory.

Definition 1.2. Let k be a commutative unital ring having finite Krull dimension. We say that the *strong* $\mathbb{A}^1$*-invariance property* holds for k if, for any pointed space $(\mathcal{X}, x)$, the sheaf $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant.

Definition 1.3. Let k be a commutative unital ring. We say that the *unstable* $\mathbb{A}^1$*-n-connectivity property holds over* k if any simplicially n-connected space $\mathcal{X}$ is $\mathbb{A}^1$-n-connected.

The goal of this note is to prove the following result.

Theorem 1.4. *If k is a commutative unital ring for which the strong $\mathbb{A}^1$-invariance property holds, then, for every integer $n \geq 0$, the unstable $\mathbb{A}^1$-n-connectivity property holds over k.*

Theorem 3.1 of [Mor06] establishes the strong $\mathbb{A}^1$-invariance property over any field k. Combining this with Theorem 1.4, one immediately deduces the following result, which is sometimes called the *unstable* $\mathbb{A}^1$*-n-connectivity theorem.*

Corollary 1.5 (*cf.* [Mor06, Theorem 3.38])**.** *If k is a field, then for every integer $n \geq 0$, the unstable $\mathbb{A}^1$-n-connectivity property holds over k.*

Remark 1.6. In [Mor06], this result is stated without proof. Our goal here is to indicate that the result follows without any of the machinations appearing in §3.2 of [Mor06] (where the result is stated). Ayoub has shown that if the Krull dimension of k is ≥ 2, then the unstable $\mathbb{A}^1$-connectivity property will fail, thus the strong $\mathbb{A}^1$-invariance property cannot hold over 2-dimensional rings. However, Morel has indicated the hope (*cf.* [Mor06, Remark 3.32]) that if k is a Dedekind ring, the strong $\mathbb{A}^1$-invariance property holds for k.

2 Preliminary results

In this section, we recall some general facts about the Postnikov towers in the setting of simplicial homotopy theory. Then, we recall some facts about $\mathbb{A}^1$-local objects. The general reference for this material is [MV99].

Postnikov towers

We begin by summarizing some consequences of existence and functoriality of the Postnikov tower; these statements hold quite generally. To this end, let $\mathcal{G}$ be an arbitrary sheaf of groups. By replacing $\mathcal{BG}$ by a simplicially fibrant model, we can assume it is fibrant. Furthermore, by its very definition, $\mathcal{BG}$ is simplicially 0-connected. Any pointed morphism of simplicial sheaves $\mathcal{X} \to \mathcal{BG}$ induces a group homomorphism $\pi_1^s(\mathcal{X}, x) \to \pi_1^s(\mathcal{BG})$. Thus, we obtain a map

$$\hom_{\Delta^\circ Shv_{Nis}(\mathcal{S}m/k)_\bullet}((\mathcal{X}, x), (\mathcal{BG}, *)) \to \hom_{\mathcal{G}r_k}(\pi_1^s(\mathcal{X}), \mathcal{G}).$$

Since π_1^s is a simplicial homotopy invariant, two simplicially homotopy equivalent maps determine the same group homomorphism. Since $\mathcal{BG}$ is assumed fibrant, the observation of the last sentence shows that we obtain a well-defined map

$$[(\mathcal{X}, x), (\mathcal{BG}, *)]_s \to \hom_{\mathcal{G}r_k}(\pi_1^s(\mathcal{X}), \mathcal{G}).$$

We will now show that under reasonable hypotheses, this map is a bijection.

Lemma 2.1. *For any pointed simplicially* 0*-connected space* $(\mathcal{X}, x)$ *the map*

$$[(\mathcal{X}, x), (\mathcal{BG}, *)]_s \to \hom_{\mathcal{G}r_k}(\pi_1^s(\mathcal{X}), \mathcal{G})$$

is a bijection.

Proof. Any homomorphism of sheaves of groups $\mathcal{H} \to \mathcal{G}$ induces by functoriality of classifying spaces a pointed morphism of simplicial sheaves $\mathcal{BH} \to \mathcal{BG}$. Since the higher simplicial homotopy group sheaves of $\mathcal{BH}$ and $\mathcal{BG}$ both vanish, the observations of the previous paragraph show that this map induces a bijection

$$\hom_{\mathcal{G}r_k}(\mathcal{H}, \mathcal{G}) \to [(\mathcal{BH}, *), (\mathcal{BG}, *)]_s;$$

the inverse is provided by the functor π_1^s. The first stage in the simplicial Postnikov tower for any pointed connected space $\mathcal{X}$ can be identified with $\mathcal{B}\pi_1^s(\mathcal{X}, x)$, the classifying space of the first *simplicial* homotopy group of $\mathcal{X}$. By functoriality of the Postnikov tower, any pointed morphism $\mathcal{X} \to \mathcal{BG}$ induces a morphism $\mathcal{B}\pi_1^s(\mathcal{X}) \to \mathcal{BG}$. Thus, any pointed morphism $\mathcal{B}\pi_1^s(\mathcal{X}) \to \mathcal{BG}$ can be lifted to a pointed morphism $\mathcal{X} \to \mathcal{BG}$. This construction provides an inverse to the construction of the previous paragraph. □

Preservation of $\mathbb{A}^1$-locality

We begin by stating the following well-known lemma.

Lemma 2.2. *A sheaf of groups* $\mathcal{G}$ *is strongly* $\mathbb{A}^1$*-invariant if and only if the simplicial classifying space* $\mathcal{BG}$ *is* $\mathbb{A}^1$*-local.*

Notation 2.3. Let $\mathcal{G}r_k^{\mathbb{A}^1}$ denote the full subcategory of $\mathcal{G}r_k$ consisting of strongly $\mathbb{A}^1$-invariant sheaves of groups.

Lemma 2.4. *If $\mathcal{X}$ is a pointed, fibrant and $\mathbb{A}^1$-local space, then $\Omega^1_s\mathcal{X}$ is again $\mathbb{A}^1$-local.*

Proof. We have to show that for any space $\mathcal{Y}$, the map

$$\hom_{\mathcal{H}_s((\mathcal{S}m/k)_{Nis})}(\mathcal{Y},\Omega^1_s\mathcal{X}) \to \hom_{\mathcal{H}_s((\mathcal{S}m/k)_{Nis})}(\mathcal{Y}\times\mathbb{A}^1,\Omega^1_s\mathcal{X})$$

is a bijection. We can rewrite $\mathcal{Y}\times\mathbb{A}^1$ as $\mathcal{Y}_+\wedge\mathbb{A}^1$. Then, by adjunction of suspension and looping, we have

$$\hom_{\mathcal{H}_s((\mathcal{S}m/k)_{Nis})}(\Sigma^1_s(\mathcal{Y}_+\wedge\mathbb{A}^1),\mathcal{X}) \xrightarrow{\sim} \hom_{\mathcal{H}_s((\mathcal{S}m/k)_{Nis})}(\mathcal{Y}\times\mathbb{A}^1,\Omega^1_s\mathcal{X}).$$

However, by associativity of smash products, we know that $\Sigma^1_s(\mathcal{Y}_+\wedge\mathbb{A}^1)\cong(\Sigma^1_s\mathcal{Y}_+)\wedge\mathbb{A}^1$. The result then follows immediately from the assumption that $\mathcal{X}$ is $\mathbb{A}^1$-local. □

Suppose $\mathcal{X}$ is a pointed, simplicially 0-connected fibrant space. By the previous lemma $\Omega^1_s L_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-local. Thus, the canonical map $\Omega^1_s\mathcal{X}\to\Omega^1_s L_{\mathbb{A}^1}(\mathcal{X})$ induced by functoriality of looping factors through a map

$$L_{\mathbb{A}^1}(\Omega^1_s(\mathcal{X})) \longrightarrow \Omega^1_s L_{\mathbb{A}^1}(\mathcal{X}).$$

We need the following somewhat difficult result regarding this map.

Theorem 2.5 ([MV99, §2 Theorem 2.34])**.** *If $(\mathcal{X},x)$ is a pointed simplicially 0-connected and simplicially fibrant space, the map*

$$L_{\mathbb{A}^1}(\Omega^1_s\mathcal{X}) \longrightarrow \Omega^1_s(L_{\mathbb{A}^1}(\mathcal{X}))$$

just constructed is a simplicial weak equivalence.

3 Proof of Theorem 1.4

Proof of Theorem 1.4. We prove the theorem by first establishing the result for $n=1$ (see Proposition 3.1), and then showing how to use homotopy theoretic results to deduce the result for $n>1$ from the case where $n=1$ (see Proposition 3.2). □

Proposition 3.1. *If $(\mathcal{X},x)$ is a pointed simplicially 1-connected space, then $\mathcal{X}$ is an $\mathbb{A}^1$-1-connected space.*

Proof. If $\mathcal{X}$ is $\mathbb{A}^1$-connected, its $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(\mathcal{X})$ is simplicially 0-connected by definition. Conversely, the unstable $\mathbb{A}^1$-0-connectivity theorem (i.e., [MV99, §2 Corollary 3.22]) shows that if $\mathcal{X}$ is simplicially 0-connected, $\mathcal{X}$ is $\mathbb{A}^1$-0-connected.[1] This establishes the result for $n=0$.

Thus, assume $n\geq 1$, $\mathcal{X}$ is a simplicially n-connected space, and the theorem holds for $j\leq n-1$. We need to show that $\pi_n^{\mathbb{A}^1}(\mathcal{X},x)$ is trivial.

If $n=1$, then by assumption $\pi_1^{\mathbb{A}^1}(\mathcal{X})$ is strongly $\mathbb{A}^1$-invariant. Furthermore, since $\mathcal{X}$ is $\mathbb{A}^1$-0-connected, the $\mathbb{A}^1$-localization $L_{\mathbb{A}^1}(\mathcal{X})$ is by definition simplicially 0-connected. Let $\mathcal{G}$ be an arbitrary strongly $\mathbb{A}^1$-invariant sheaf of groups. By Lemma 2.2, the canonical map $\mathcal{BG}\to L_{\mathbb{A}^1}(\mathcal{BG})$ is a simplicial weak equivalence. Thus, we deduce that the map

$$[\mathcal{X},\mathcal{BG}]_s \longrightarrow [\mathcal{X},L_{\mathbb{A}^1}\mathcal{BG}] := [\mathcal{X},\mathcal{BG}]_{\mathbb{A}^1}$$

is a bijection. We now relate the two sides of the above map to the connectivity properties of $\mathcal{X}$ using the Postnikov tower.

[1] In fact, this result is known to hold over a general base scheme, and even for the $\mathbb{A}^1$-homotopy category considered in other topologies.

Consider $L_{\mathbb{A}^1}(\mathcal{X})$. By definition $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x) := \pi_1^s(L_{\mathbb{A}^1}(\mathcal{X}), x)$. By assumption, the latter sheaf of groups is strongly $\mathbb{A}^1$-invariant. Applying Lemma 2.1 to $L_{\mathbb{A}^1}(\mathcal{X})$, we deduce that the map

$$[(L_{\mathbb{A}^1}(\mathcal{X}), x), (\mathcal{BG}, *)]_s \longrightarrow \hom_{\mathcal{G}r_k}(\pi_1^{\mathbb{A}^1}(\mathcal{X}), \mathcal{G})$$

is a bijection. As $\mathcal{BG}$ is $\mathbb{A}^1$-local, we deduce that the first set can be identified with $[(L_{\mathbb{A}^1}(\mathcal{X}), x), (\mathcal{BG}, *)]_{\mathbb{A}^1}$. But since the map $\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$ is a pointed $\mathbb{A}^1$-weak equivalence, the induced map

$$[(L_{\mathbb{A}^1}(\mathcal{X}), x), (\mathcal{BG}, *)]_{\mathbb{A}^1} \longrightarrow [(\mathcal{X}), x), (\mathcal{BG}, *)]_{\mathbb{A}^1}$$

is a bijection. Thus, for any strongly $\mathbb{A}^1$-invariant sheaf of groups $\mathcal{G}$, we deduce the existence of a bijection

$$[(\mathcal{X}, x), (\mathcal{BG}, *)]_{\mathbb{A}^1} \longrightarrow \hom_{\mathcal{G}r_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(\mathcal{X}, x), \mathcal{G}).$$

Now, combining all of our observations, we deduce the existence of a bijection

$$\hom_{\mathcal{G}r_k}(\pi_1^s(\mathcal{X}, x), \mathcal{G}) \longrightarrow \hom_{\mathcal{G}r_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(\mathcal{X}), \mathcal{G}).$$

As $\mathcal{X}$ is assumed to be simplicially 1-connected, we deduce that the set $\hom_{\mathcal{G}r_k}(\pi_1^s(\mathcal{X}), \mathcal{G})$ is trivial for *any* strongly $\mathbb{A}^1$-invariant sheaf of groups $\mathcal{G}$. Consequently the set $\hom_{\mathcal{G}r_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(\mathcal{X}, x), \mathcal{G})$ is trivial in this situation as well. However, by the covariant form of the Yoneda lemma, we conclude that $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ must be trivial, and therefore that $\mathcal{X}$ is $\mathbb{A}^1$-1-connected. □

Proposition 3.2. *If for any pointed space* $(\mathcal{X}, x)$ *simplicial* 1*-connectedness implies* $\mathbb{A}^1$*-*1*-connectedness, then simplicial* n*-connectedness implies* $\mathbb{A}^1$*-*n*-connectedness.*

Proof. If $\mathcal{X}$ is a pointed, simplicially n-connected space, then without loss of generality we can assume $\mathcal{X}$ is simplicially fibrant. It follows immediately that $\Omega_s^{n-1}\mathcal{X}$ is pointed, simplicially 1-connected and fibrant. Applying Proposition 3.1, we deduce that $L_{\mathbb{A}^1}(\Omega_s^{n-1}\mathcal{X})$ is simplicially-1-connected.

For any integer $n > 1$, the map of Theorem 2.5 induces a map

$$L_{\mathbb{A}^1}\Omega_s^n\mathcal{X} \to \Omega_s^n L_{\mathbb{A}^1}\mathcal{X}.$$

that one may show is a simplicial weak equivalence by a straightforward induction argument. Combining this observation with the results of the previous paragraph, we deduce that $\Omega_s^{n-1}L_{\mathbb{A}^1}\mathcal{X}$ is simplicially 1-connected. Again, by definition of the $\mathbb{A}^1$-homotopy sheaves, it follows that $L_{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-n-connected. □

References

[Mor06] F. Morel. $\mathbb{A}^1$-algebraic topology over a field. 2006. *Preprint,* available at `http://www.mathematik.uni-muenchen.de/~morel/preprint.html`. 1

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999. 2, 3