In what sense are algebraic varieties like manifolds?

Aravind Asok
UCLA

January 20, 2009

Manifolds

- Throughout, M^n will denote an n-dimensional manifold.
  - The manifold M^n is constructed by *gluing*
  - *locally* looks an open subset of $\mathbb{R}^n$
  - *global* conditions, i.e., Hausdorff (separation), and paracompactness (finiteness).
- The term *open manifold* means non-compact manifold without boundary.
- The term *closed manifold* means compact manifold without boundary.
- We'll write I for the unit interval $[0, 1]$.

Algebraic varieties

- *Algebraic varieties* will be denoted by the letter X.
  - A variety X over a field k is constructed by *gluing*
  - *locally* looks like the the simultaneous vanishing locus of a finite collection of polynomials in an affine space with coefficients in k.
  - *global* conditions (separation and finiteness).
  - topologize with Zariski topology
- If L/k is an extension, we'll write $X(L)$ for the set of L-valued solutions to the (local) equations defining X.
- We'll write $\mathbb{A}^n$ for affine space of dimension n.
- We'll write $\mathbb{P}^n$ for projective space of dimension n.

Comparisons

- Depends on the field in question, e.g., consider $x^2 + y^2 = -1$ and $x^2 + y^2 = 1$ over $\mathbb{R}$, and over $\mathbb{C}$.
- More "local models"
- If coefficients are in k to begin, one can also consider them to have coefficients in any extension L/k.
  - If $k = \mathbb{C}$, one can consider $\mathbb{C}(t)$: a $\mathbb{C}(t)$ solution is a 1-parameter family of solutions over $\mathbb{C}$.
- If $k = \mathbb{R}$ or $\mathbb{C}$, the set of real or complex solutions is a manifold.
- If $k = \mathbb{Q}$, or a finite field, then the set of solutions looks like a discrete set of points!

Basic problem: Classify manifolds

- How? Construct invariants.

- A *homotopy* between two continuous maps $f, g : M \longrightarrow M'$ is a continuous map $H : M \times I \longrightarrow M'$ such that $H(x, 0) = f$ and $H(x, 1) = g$.
(Think: continuous deformation)

- A cts. map $f : M \longrightarrow M'$ is a *homotopy equivalence* if there exists cts. $g : M' \longrightarrow M$ s.t. $g \circ f \sim Id_M$ and $f \circ g \sim Id_{M'}$.

- The *homotopy category* is the category whose objects are (nice) topological spaces and whose morphisms are homotopy classes of continuous maps.

- Basic invariants
  - Set of path connected components $\pi_0(M)$: maps $* \rightarrow M$ up to homotopy equivalence.
  - Homotopy groups $\pi_i(M, *)$ $(i \geq 1)$ are cts. maps $S^n \rightarrow M$ (preserving a chosen base-point) up to homotopy. (Easy to define, hard to compute)
  - Homology groups $H_i(M, \mathbb{Z})$, start w/ free abelian group generated by maps $\Delta^n \rightarrow M$, define boundary operator, take cycles modulo boundaries. (Harder to define, easier to compute)

Topological Observations

- In low dimensions classification is possible using only invariants (in fact π_1 is enough):
  - Dimension 1. Only connected closed manifold is S^1. Only open manifold is $\mathbb{R}$.
  - Dimension 2. Connected closed manifolds are either S^2, connected sums of $S^1 \times S^1$, connected sums of $\mathbb{RP}^2$.
- In dimension 3 invariants are not enough: there exist homotopy equivalent non-diffeomorphic manifolds.
- In dimensions ≥ 4 there are "too many" invariants, e.g., any finitely presented group can appear as the fundamental group of a manifold of dimension ≥ 4.

Better problem:

Fix invariants (homotopy type) and try to classify!

Invariants in algebraic geometry

- Idea 1: Work over $\mathbb{R}$ or $\mathbb{C}$ and use topological invariants of the associated manifold.
  - Problem: (Over $\mathbb{C}$) Take a smooth cubic curve in $\mathbb{P}^2$. Underlying topological space is a torus, though algebraically there are many models. Picture: quotient $\mathbb{C}/\Lambda$; lattice can vary.
  - Problem: (Over $\mathbb{R}$) Might be the empty manifold, e.g., $x^2 + y^2 = -1$.
- Idea 2: Try to imitate topological constructions in algebraic geometry in less formal way.
  - E.g., fundamental group classifies regular covering spaces $\rightarrow$ étale fundamental group.
- Idea 3: Do neither, i.e., use intrinsic structure of the ground field.
  - Smooth cubic curve in $\mathbb{P}^2$ with $\mathbb{Q}$-coefficients gives rise to a finitely generated abelian group (Mordell-Weil theorem). Take the rank of this group.
  - Use field of rational functions.

Algebro-geometric observations

- There are many varieties that have very few non-trivial invariants: any smooth proper complex variety that can be rationally parameterized has trivial fundamental group (*cf.* Serre)!

- There are too many different kinds of invariants!
  - However, many of the invariants above are closely connected (though in highly non-obvious ways).
  - Can we unify the picture?

- Classification is possible using invariants in (complex) dimension ≤ 1.

- In dimension 2 already things are a bit obscure, e.g., over non-algebraically closed fields. In dimension ≥ 4, essentially nothing is known.

Isolating invariance properties

- In topology, good invariants are characterized by two properties: "gluing" and "homotopy invariance."
  - *Gluing* means invariants can be computed locally and then glued together (e.g., Mayer-Vietoris sequence for homology or van Kampen theorem for fundamental group).
  - *Homotopy invariance* means invariant takes the same value on M and $M \times I$.
- Many invariants have an algebro-geometric form of homotopy invariance, i.e., value of an invariant on X and $X \times \mathbb{A}^1$ often coincides. Thus, perhaps one can impose some kind of "gluing condition."
- Even better than trying to define invariants, why not try to define a good "homotopy category"?

Reorganize invariants then revisit classification!

$\mathbb{A}^1$-homotopy theory

- Start with $\mathcal{S}m_k$ (smooth algebraic varieties).

- Enlarge this category so that one can form quotients of varieties, or increasing unions of such things (+ some other categorical properties)

- Impose "gluing" and $\mathbb{A}^1$-homotopy invariance. Roughly speaking: invert homotopies parameterized by the affine line.

- The resulting category was first constructed by F. Morel and V. Voevodsky.

- Show this gives a good theory (e.g., recovers old invariants, proves old conjectures).

So what has it done for me lately?

- Most famously, it was a tool in Voevodsky's proof of the Milnor conjecture (which implies some classical conjectures about quadratic forms!)

- E.g., he showed (building on work of many others!) that algebraic K-theory could be constructed in this category.

- Organized many of the old cohomology theories.

- I claim it can be used to help organize geometric ideas as well.

Basic new objects of study

- Define homotopy groups.
  - Simplicial sphere $\mathbb{A}^1/\{0,1\}$ (think $I/\partial I$).
  - Tate sphere: $\mathbb{G}_m$, e.g., $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ (pointed by 1)
  - We can form wedge sums of pointed spaces (i.e., one point unions)
  - We can form smash products of pointed spaces.
  - We define $S_s^i \wedge \mathbb{G}_m^{\wedge j}$, and call this a motivic sphere.
  - We define homotopy groups of a space: $\pi_i^{\mathbb{A}^1}(X,x) = [S_s^i,(X,x)]_{\mathbb{A}^1}$ (homotopy classes computed in new category).
  - Can also define homology $H_i^{\mathbb{A}^1}(X)$.
- Try to study simple spaces and low dimensions to get a handle on the theory.
- Understand how these invariants differ from (or are similar to) the topological situation.

Warm-up

Theorem 1 (A., Doran '08). *The quadric hypersurface*

$$\sum_i x_i x_{n+i} = 1$$

has the $\mathbb{A}^1$-homotopy type of $S_s^{n-1} \wedge \mathbb{G}_m^{\wedge n}$. The quadric hypersurface

$$\sum_i x_i x_{n+i} = x_{2n+1}(1 + x_{2n+1})$$

has the $\mathbb{A}^1$-homotopy type of $S^n \wedge \mathbb{G}_m^{\wedge n}$. The spheres $S_s^i \wedge \mathbb{G}_m^{\wedge j}$, $i > j$ can't be realized by smooth schemes.

Corollary 2. *Over $\mathbb{C}$, the usual spheres are motivic spheres.*

Theorem 3 (Morel '05). *The first non-vanishing $\mathbb{A}^1$-homotopy group of a sphere can be computed explicitly. The answer is closely related to quadratic forms.*

High dimensional flexibility

- How "rigid" is this theory?

- Contractible manifolds measure the difference between homotopy theory and homeomorphism.
  - The space $\mathbb{A}^n$ is contractible in this world. Are there other spaces like this? First recall:

Theorem 4 (Whitehead '34, Mazur '61, McMillan '62, Curtis-Kwun '65, Glaser '67)**.** *There exist uncountably many open contractible manifolds M^n of every dimension $n \geq 3$.*

Theorem 5 (A., Doran '07)**.** *There exist arbitrary dimensional smooth families of $\mathbb{A}^1$-contractible smooth varieties (over any field) of dimension ≥ 6. Infinitely many in each dimension ≥ 4.*

Example 6. Take the variety Q_4 defined by the equation $x_1x_3 + x_2x_4 = x_5(x_5 + 1)$ and remove the locus of points where $x_1 = x_2 = 0, x_5 = -1$. This is $\mathbb{A}^1$-contractible.

Thus, the theory is very flexible.

Idea of proof: take $\mathbb{A}^n$, equip it with a translation action of $\mathbb{G}_a$ (additive group of the affine line) and construct a quotient.

Low dimensional results

- All the varieties constructed so far can be rationally parameterized.

- In fact, being connected from the standpoint of $\mathbb{A}^1$-homotopy theory means that one is "nearly" rationally parameterized.

Theorem 7. *The only $\mathbb{A}^1$-connected smooth proper algebraic curve (up to isomorphism) is $\mathbb{P}^1$.*

Theorem 8 (Morel '06)**.** *The group $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ is highly non-trivial and can be explicitly computed.*

Theorem 9 (A., Morel '08)**.** *Suppose k is an algebraically closed field. Every $\mathbb{A}^1$-connected smooth proper algebraic surface is $\mathbb{A}^1$-homotopy equivalent to either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at a fixed (possibly empty) finite set of distinct points.*

- Up to isomorphism, there are many more such surfaces! E.g., over $\mathbb{C}$, there are continuous families parameterizing non-isomorphic blow-ups of 5 points on $\mathbb{P}^2$.

- The $\mathbb{A}^1$-fundamental group is the only necessary invariant!

Topological interlude

- An h-cobordism between two closed manifolds M and M' of dim n is an $n+1$-dimensional compact manifold W whose boundary is a disjoint union of M and M' and such that both the inclusion of M into W and the inclusion of M' into W are homotopy equivalences.

- Smale proved that an h-cobordism between simply connected closed manifolds of dimension $n \geq 5$ is necessarily diffeomorphic to a product of the form $M \times I$.

- Thus, to classify simply connected manifolds in a given homotopy type, it suffices to identify h-cobordism classes.

- Milnor and Kervaire did this for spheres!

Ideas of proof and for the future

- Key idea of proof is to do something intermediate between $\mathbb{A}^1$-homotopy theory and geometry, and motivated by topology.

Definition 10. An *$\mathbb{A}^1$-h-cobordism* is a pair (W, f) consisting of a smooth variety W and a morphism $f : W \to \mathbb{A}^1$ such that the inclusions of the fibers over 0 and 1 are $\mathbb{A}^1$-homotopy equivalences.

Theorem 11. *Any* $\mathbb{A}^1$-h-cobordism *between smooth proper $\mathbb{A}^1$-connected and $\mathbb{A}^1$-simply connected varieties is trivial.*

Why? The $\mathbb{A}^1$-fundamental group of any smooth proper $\mathbb{A}^1$-connected variety of dimension ≥ 1 is non-trivial!

Question 12. *Can one classify smooth proper varieties up to $\mathbb{A}^1$-homotopy?*

Fundamental Problem 13. *Identify the $\mathbb{A}^1$-h-cobordism classes of smooth proper varieties in a given $\mathbb{A}^1$-homotopy type.*

Fundamental Problem 14. *Try to determine when two $\mathbb{A}^1$-h-cobordant smooth proper varieties are isomorphic.*

How? Look to topology! (Surgery theory)
Need: More computations and examples!

Thank you!

See **http://www.math.ucla.edu/~asok**
for more information.