Unipotent groups and some $\mathbb{A}^1$-contractible smooth schemes

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Outline

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History/Historical Motivation

- 1935 - Whitehead - Example of an open contractible 3-manifold not isomorphic to $\mathbb{R}^3$
- 1960-61 - Mazur, Poenaru - Examples of (smooth) open contractible 4-manifolds not isomorphic to $\mathbb{R}^4$

Key tool: "Fundamental group at infinity"

Definition 1. A manifold M is simply-connected at infinity if for every compact subset $C \subset M$, there exists a compact subset D such that $C \subset D \subset M$ and $\pi_1(M \setminus D) = 1$.

Theorem 2 (McMillan '62, Curtis-Kwun '65, Glaser '67)**.** *For every $n \geq 3$, there exist uncountably many pairwise non-isomorphic, open contractible n-manifolds.*

This theorem might lead one to believe that these objects are pathological and lacking significant structure. All of the above examples have the property that they are *not* simply connected at infinity.

Theorem 3 (Stallings '62)**.**

1. *If M is an open contractible n-manifold, $n \geq 5$, M is isomorphic to affine space if and only if M is simply-connected at infinity.*
2. *If M is an open contractible n-manifold, $n \geq 2$, $M \times \mathbb{R}$ is simply connected at infinity.*

Corollary 4. *Every open contractible n-manifold, $n \geq 4$ is a quotient of $\mathbb{R}^{n+1}$ by a free action of $\mathbb{R}$.*

Moral: There exist many open contractible n-manifolds, $n \geq 3$, all of which can be *constructed* as quotients of Euclidean space. We can *recognize* Euclidean space among these by computing the fundamental group at infinity.

All of the above constructions are inherently "topological." Might there be algebro-geometric versions?

We will say that a smooth complex algebraic variety X is (topologically) contractible if $X(\mathbb{C})$, equipped with the usual structure of a smooth manifold, is contractible.

Theorem 5 (Ramanujam '74)**.** *There exists a contractible smooth complex algebraic surface not homeomorphic to* $\mathbb{R}^4$*. A smooth contractible complex algebraic surface* X *is algebraically isomorphic to* $\mathbb{A}^2$ *if and only if* X *is simply connected at infinity.*

Key problem: Characterize $\mathbb{A}^n$ among all n-dimensional algebraic varieties.

Unfortunately, purely topological invariants are probably not enough to solve this problem for $n > 2$.

Theorem 6 (Dimca, Ramanujam)**.** *Any* n*-dimensional contractible smooth complex* affine *variety,* $n \geq 3$ *is diffeomorphic to* $\mathbb{R}^{2n}$*.*

Problems with this picture:

- There is no coherent picture describing the structure of contractible smooth complex algebraic varieties.

- The notion of contractiblity for algebraic varieties defined so far only makes sense for varieties defined over fields embeddable in $\mathbb{C}$.

To rectify these problems we consider...

The $\mathbb{A}^1$-homotopy black box

Let k be an arbitrary field of characteristic 0. Let $\mathcal{S}m_k$ denote the category of separated, finite type, schemes smooth over k; we will refer to these objects as smooth schemes in all that follows.

Mantra:(Morel-Voevodsky) There is a homotopy theory for smooth schemes over k where the affine line $\mathbb{A}^1$ plays a role analogous to that played by the unit interval in classical topology.

In particular, there is a notion of "$\mathbb{A}^1$-weak equivalence" of smooth schemes. Reasonable algebraic cohomology theories can not see the difference between $\mathbb{A}^1$-weakly equivalent smooth schemes. We will only use one example:

Example 7. Suppose X and Y are smooth schemes and $f : X \longrightarrow Y$ is a Zariski locally trivial morphism with fibers isomorphic to affine spaces. Then f is an $\mathbb{A}^1$-weak equivalence. Observe that

- if U is a connected unipotent k-group and $f : X \longrightarrow Y$ is a U-torsor, then f is an $\mathbb{A}^1$-weak equivalence.

Definition 8. A smooth scheme X is $\mathbb{A}^1$-contractible if the structure morphism $X \longrightarrow \operatorname{Spec} k$ is an $\mathbb{A}^1$-weak equivalence.

Example 9. The smooth scheme $\mathbb{A}^n$ is $\mathbb{A}^1$-contractible. Any scheme that is $\mathbb{A}^1$-weakly equivalent to affine space is $\mathbb{A}^1$-contractible.

Main Results

Idea: Construct $\mathbb{A}^1$-contractible smooth schemes by taking quotients of $\mathbb{A}^n$ by free actions of unipotent groups.

Problems: 1) Quotient of a smooth scheme by the free action of an algebraic group doesn't exist as a scheme in general, only as an algebraic space.

2) Even if a quotient exists as a scheme, how does one tell whether or not it is isomorphic to affine space?

Furthermore, unipotent groups acting on affine varieties can have non-finitely generated rings of invariants (Nagata's famous counter-examples) so understanding the quotients *may* be hard. The interested listener can peruse the paper by Doran-Kirwan

(see http://arxiv.org/math.ag/0703131)

for more details.

To deal with problem (1), we use a version of geometric invariant theory (GIT) for unipotent group actions.

Rough Idea. "Pass to reductive GIT:" Suppose X is a (quasi-) affine scheme equipped with an action of a connected unipotent group U.

1) Fix an embedding $U \hookrightarrow SL_n$ (such an embedding always exists for n sufficiently large).

2) Consider the contracted product scheme $(SL_n \times X)/U$ (quotient of $SL_n \times X$ by the right action of U on SL_n and left action of U on X). This quotient exists as a scheme by descent theory.

3) The trivial bundle on $SL_n \times X/U$ has a unique structure of SL_n-equivariant line bundle. One can then consider the usual notion of stability for *this* linearized SL_n-action.

4) Check this notion of stability is "independent of the embedding" (actually, one makes an intrinsic defintion, and checks that one may compute with a fixed embedding)

Definition 10. Given an affine scheme X equipped with an action of a connected unipotent group U, we say that the action is *everywhere stable* if every geometric point of $X \hookrightarrow (SL_n \times X)/U$ is stable for the induced SL_n-action linearized by the trivial bundle.

5) In this case, the quotient of the contracted product by the SL_n-action (which exists by usual GIT) is the quotient of X by U.

Theorem 11 (A., Doran)**.** *Suppose X is an affine scheme equipped with an action of a connected unipotent group U. Then a quotient $q : X \longrightarrow X/U$ exists as a U-torsor with quasi-affine base X/U if and only if U acts everywhere stably on X.* If furthermore X is smooth, then X/U is smooth.*

To deal with Problem (2), we proceed as follows. Suppose we now take $\mathbb{A}^n$ equipped with the action of a connected unipotent group U. Suppose further that U acts everywhere stably on $\mathbb{A}^n$. Then either the quotient is affine or it is strictly quasi-affine (i.e. quasi-affine but not affine). If the quotient is strictly quasi-affine, then it can not be isomorphic to affine space. We then (re-)prove the following result (due essentially to Greuel-Pfister, Kambayashi-Miyanishi-Takeuchi):

Theorem 12. *Suppose U is a connected unipotent group acting on an affine scheme X. Then the following are equivalent:*

- *The quotient $q : X \longrightarrow X/U$ exists as a U-torsor with X/U affine (and is thus trivial).*
- *The Lie algebra cohomlogy group $H^1(\mathfrak{u}, k[X]) = 0$, where $\mathfrak{u} = Lie(U)$.*

Key Point: These theorems are *computationally effective* in the sense that their exists algorithms to check the relevant hypotheses are satisfied.

*A connected unipotent group has no non-trivial finite subgroups.

Upshot: To construct $\mathbb{A}^1$-contractible smooth schemes not isomorphic to affine space, it suffices to write down everywhere stable actions of unipotent groups on a fixed $\mathbb{A}^n$ with quotient a strictly quasi-affine variety.

Example 13. Take the affine quadric $Q \subset \mathbb{A}^5$ defined by

$$x_1x_4 - x_2x_3 - x_5(x_5 + 1) = 0.$$

Let $V = \{x_1 = x_2 = 0, x_5 = -1\} \subset Q$. Then $V \cong \mathbb{A}^2$ and $Q \setminus V$ is an $\mathbb{A}^1$-contractible smooth scheme. Over $\mathbb{C}$, this is diffeomorphic to the complement of a cotangent space in T^*S^4.

In fact, this is obtained from the linear action of $\mathbb{G}_{\mathbf{a}}$ on the direct sum of three copies of the standard 2-dimensional representation of SL_2 by taking the quotient of an explicit $\mathbb{G}_{\mathbf{a}}$-stable hypersurface isomorphic to $\mathbb{A}^5$.

We can generalize this construction to prove the following theorem.

Theorem 14 (A., Doran)**.** *For every $n \geq 4$, there exist infinitely many pairwise non-isomorphic $\mathbb{A}^1$-contractible smooth schemes of dimension n.*

For every $n \geq 6$ and every $m > 0$, there exist m-dimensional families of $\mathbb{A}^1$-contractible smooth schemes of dimension n (all of whose fibers are non-isomorphic).

Conjecturally, we should be able to take $n \geq 3$ in the second statement. At the moment we have no examples. The situation seems quite close in spirit to the "existence" part of the discussion of topologically contractible varieties. What about the "recognition" part?

Conjectures, Problems in affine geometry,

and a little wild speculation...

In the above theorems, we avoided the case where the quotient was affine because it is very difficult to tell whether a given affine variety is isomorphic to affine space. For example, the algebraic automorphism group of $\mathbb{A}^n$, $n \geq 2$, is infinite dimensional and highly non-trivial. Suppose U is a k-dimensional connected unipotent group acting everywhere stably on affine space $\mathbb{A}^n$. If we have $H^1(\mathfrak{u}, k[\mathbb{A}^n]) = 0$, then we have an equality

$$\mathbb{A}^n/U \times \mathbb{A}^k \cong \mathbb{A}^n.$$

But recall:

Zariski Cancellation Problem: We say Zariski cancellation holds in dimension ℓ, if given X an ℓ-dimensional affine algebraic variety such that $X \times \mathbb{A}^k \cong \mathbb{A}^{\ell+k}$, there exists an isomorphism $X \cong \mathbb{A}^\ell$.

In fact, the Zariski cancellation problem is equivalent to the statement that every affine quotient of affine space is isomorphic to affine space. We can even reduce Zariski cancellation to the ("easier") question of whether for $\mathbb{G}_\mathbf{a}$-actions on hypersurfaces isomorphic to affine space in linear $\mathbb{G}_\mathbf{a}$-representations, any principal $\mathbb{G}_\mathbf{a}$-bundle quotient is isomorphic to affine space. Also, Zariski cancellation is known to hold in dimension ≤ 2.

We have a (conjectural) definition of an $\mathbb{A}^1$-fundamental group at infinity.

Questions: 1) Does Zariski cancellation fail in all dimensions ≥ 3?

2) Can one characterize affine space as being the unique $\mathbb{A}^1$-contractible variety having appropriate $\mathbb{A}^1$-fundamental group at infinity?

Questions are linked: it is extremely difficult to tell whether a variety is isomorphic to affine space. It is also generally difficult to compute the $\mathbb{A}^1$-fundamental group at infinity....

3) Is every $\mathbb{A}^1$-contractible variety a quotient of an affine space by the free action of a unipotent group?

Remarks: All of these examples also highlight the limits of $\mathbb{A}^1$-homotopy invariants.

1) The $\mathbb{A}^1$-homotopy category misses a tremendous amount of information about algebraic varieties (since it doesn't see $\mathbb{A}^1$-contractible varieties).

2) There is no bundle theory in $\mathbb{A}^1$-homotopy theory. In other words, isomorphism classes of algebraic vector bundles on a smooth scheme X are not in bijection with $\mathbb{A}^1$-homotopy classes of maps to BGL_n (appropriately defined). Indeed, (essentially) all $\mathbb{A}^1$-contractible strictly quasi-affine smooth varieties X have non-trivial algebraic vector bundles (but $K_0(X) \cong \mathbb{Z}$!), e.g., see Example 13.

3) There exist smooth algebraic varieties over $\mathbb{C}$ which are topologically contractible but *NOT* $\mathbb{A}^1$-contractible. In fact any contractible smooth algebraic surface of log general type has this property. Amazingly, $\mathbb{A}^1$-homotopy theory (conjecturally) *does* provide invariants to distinguish these from affine space!