A “homotopic” view of affine lines on varieties

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Outline

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The topological story

Notation/Definitions

- M^n will denote an n-dimensional manifold, which we take to mean either *topological* or *smooth* (we'll specify).

- The term *open manifold* means non-compact manifold, without boundary.

- We'll write I for the unit interval $[0, 1]$.

A *homotopy* between two continuous maps $f, g : X \longrightarrow Y$ is a continuous map $H : X \times I \longrightarrow Y$ such that $H(x, 0) = f$ and $H(x, 1) = g$.

A continuous map $f : X \longrightarrow Y$ is called a *homotopy equivalence* if there exists a map $g : Y \longrightarrow X$ such that $g \circ f \sim Id_X$ and $f \circ g \sim Id_Y$.

The *homotopy category* is the category whose objects are (nice) topological spaces and whose morphisms are continuous maps up to homotopy equivalence.

A topological space X is *contractible* if $f : X \longrightarrow pt$ is a homotopy equivalence, i.e., if X is equivalent to a point in the homotopy category. Note that Euclidean space $\mathbb{R}^n$ is an open contractible manifold: radially contract to the origin.

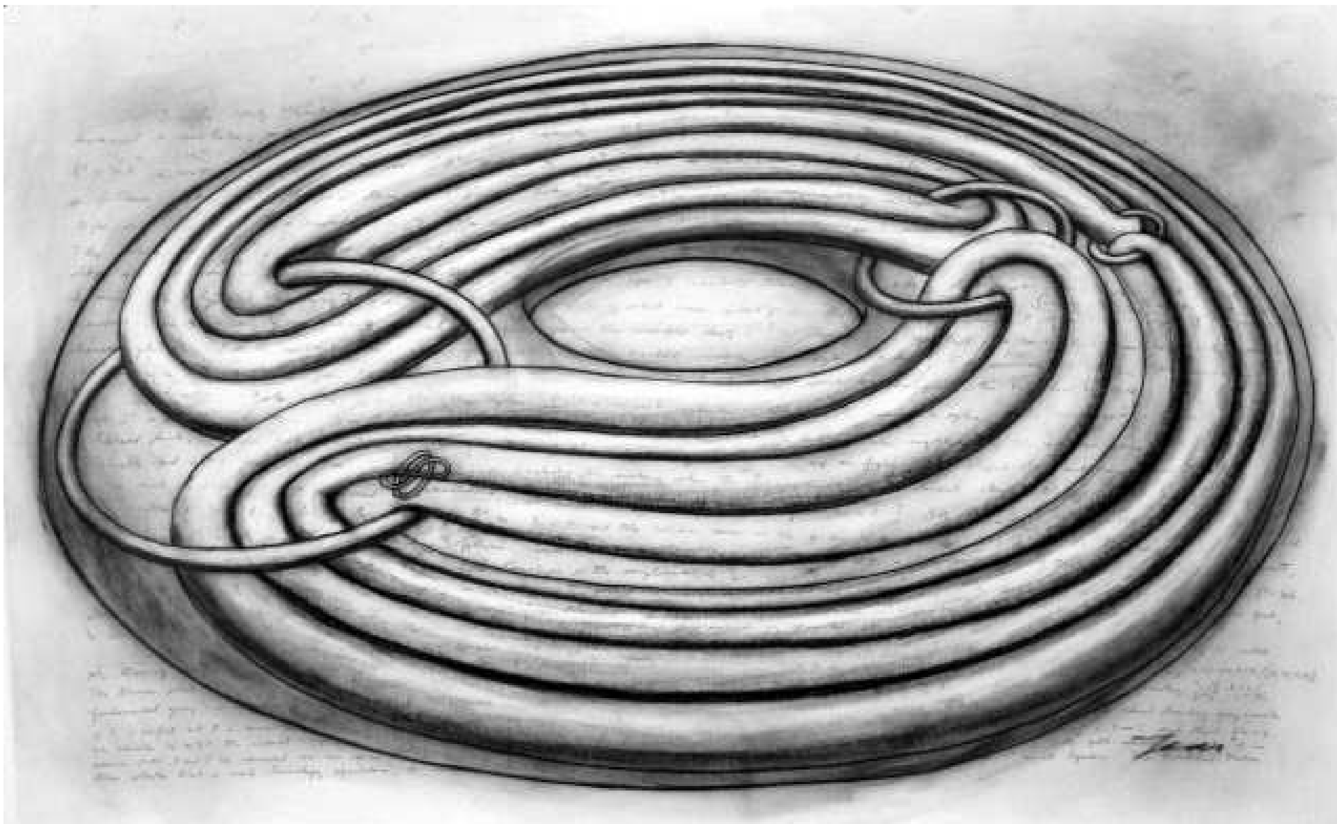
The *Poincaré conjecture* asks whether every manifold M^n homotopy equivalent to S^n is actually homeomorphic to S^n.

History

- 1934 - Whitehead - Purported proof of 3-D Poincaré conjecture

  A lesson in how *not* to prove the Poincaré conjecture: start with a homotopy equivalence $f : M^3 \longrightarrow S^3$. Removing a point produces an open contractible manifold $M^3 \backslash pt$ and continuous map $f : M^3 \setminus pt \longrightarrow \mathbb{R}^3$. Whitehead claimed: All open contractible 3-manifolds are homeomorphic to $\mathbb{R}^3$...such a homeo. extends to a homeo. $M^3 \rightarrow S^3$....but...

- 1935 - Whitehead produced an example of an open contractible 3-manifold not homeomorphic to $\mathbb{R}^3$. We call this space the *Whitehead manifold* and denote it by W^3. The construction is very "topological" and uses the Whitehead link. Here's a picture:



The Whitehead Manifold

Lun-Yi Tsai

How do we distinguish this manifold from $\mathbb{R}^3$? Whitehead had a "geometric" argument. We will use another approach: try to use properties "at infinity."

Homotopy at infinity

Definition 1. An open manifold M^n is *connected at* ∞ or *has one end* if given any compact subset $C \subset M^n$, there is a larger compact subset $C \subset D \subset M^n$ such that $M^n \setminus D$ is connected (i.e., any two points can be connected by the image of path $I \longrightarrow M^n \setminus D$).

A *neighborhood of* ∞ in an open manifold M is a subset N such that the closure of $X \setminus N$ is compact. The manifold M is said to be *simply connected at infinity*, if for any neighborhood U of ∞, there is a smaller neighborhood of ∞, call it V, such that any continuous map of the circle into V is contractible in U.

The following condition (**C**) has been considered by many authors: given any compact subset $C \subset M^n$, there is a larger compact subset $C \subset D \subset M^n$ such $M^n \setminus D$ is simply connected (i.e., any map $S^1 \longrightarrow M^n \setminus D$ is homotopy equivalent to a trivial map). This condition implies simple-connectivity at infinity in the above sense, but fails to be a proper homotopy invariant.

- Simple-connectivity at infinity is a topological property.

- $\mathbb{R}$ is *not* connected at infinity, $\mathbb{R}^2$ *is* connected at infinity, but is *not* simply connected at infinity.

- $\mathbb{R}^n$, $n \geq 3$ is simply connected at infinity.

- The Whitehead manifold is *not* simply connected at infinity and thus not homeomorphic to $\mathbb{R}^3$.

Stability property

- 1960 - Glimm and Shapiro showed that (i) $W^3 \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$, and (ii) $W^3 \times W^3$ is homeomorphic to $\mathbb{R}^6$.

Key problems:

- (Existence) Are there other open contractible manifolds? (or "is the Whitehead manifold an isolated pathology?")

- (Characterization) If there do exist such, can one characterize $\mathbb{R}^n$ among open contractible manifolds.

- (Structure) What structural results, analogous to the above stability property, exist for open contractible manifolds.

The 1960s saw *most* of these problems solved and a beautiful framework emerge. The techniques developed in their solution was very important for the so-called "surgery" classification of manifolds. We'll just summarize the results as they stand today.

Theorem 2 (Existence results: 1934-1967)).

- *Every open contractible topological n-manifold, $n \leq 2$ is homeomorphic to $\mathbb{R}^n$.*
- *For every $n \geq 3$, there exist* uncountably many *pairwise non-homeomorphic open contractible topological n-manifolds*

Dimension ≤ 2, follows from classification theory. Dimension 3, McMillan generalized Whitehead's construction (1961). Dimension ≥ 5 proved by Curtis-Kwun (1965) using tricky group theoretical arguments. Dimension 4 Glaser (1967).

Theorem 3 (Characterization/Structure results: 1960-2005!).

- *Euclidean space $\mathbb{R}^n$ ($n \geq 3$) is the unique open contractible topological n-manifold that is simply connected at infinity.*
- *For any open contractible n-manifold M^n, the product $M^n \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{n+1}$.*

A smooth (/PL) version for $n \geq 5$ was given by Stallings (1962), though dimension ≥ 6 can be deduced from work of Smale.

The stated topological version has a more interesting history. Dimension ≥ 5 is due to Siebenmann (1968) following work of M.H.A. Newman, though proofs were claimed slightly earlier (Luft).

In any dimension the result is a consequence of the Poincaré conjecture. Thus, dimension ≥ 4 follows from Freedman's work (1980s) and dimension 3 is now known due to Perelman's recent results (2005?). The smooth version in dimension 4 is "very false" by work of Freedman, Donaldson, etc. on fake $\mathbb{R}^4$s.

Moral of the story: There exist many open contractible n-manifolds, $n \geq 3$, all of which can be *constructed* as quotients of Euclidean space by free "translation" actions of $\mathbb{R}$. We can *recognize* Euclidean space among these by checking simple connectivity at infinity.

Standard homotopy invariants such as singular homology, homotopy groups, vector bundles, etc. on such manifolds are all *trivial*! It is *difficult* to distinguish open contractible manifolds. In some sense these manifolds "measure" the difference between homotopy theory and homeomorphism.

Now to a new thread...

Some algebro-geometric problems

Notation/Conventions:

- We'll work only with smooth algebraic varieties that, for simplicity, are assumed defined over algebraically closed fields (take $\mathbb{C}$ for example).

- Smooth complex algebraic varieties X, while topologized with the Zariski topology, can also be viewed as smooth manifolds. We write $X(\mathbb{C})$ for this associated manifold.

- We'll work mainly with *affine* algebraic varieties or *quasi-affine* algebraic varieties, the latter are just Zariski open subsets of affine algebraic varieties.

- We write $\mathbb{A}^n$ for n-dimensional affine space; over $\mathbb{C}$ one has $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ with its *usual* structure of a complex manifold.

With definitions analogous to topology, we can consider *vector bundles* (or more general fiber bundles) though now we assume that local trivializations exist in the Zariski topology, i.e., the variety in question can be covered by Zariski open sets over which the restricted bundle is isomorphic to a product $U \times \mathbb{A}^n$, and transition functions are algebraic.

Serre Problem (1955): Are all algebraic vector bundles on affine space trivial? (Presumably this problem was motivated by the corresponding topological result...Serre was first a topologist.)

Cancellation Problem: Suppose $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$, are X and Y necessarily isomorphic?

The first problem has a positive solution due to the celebrated work of Quillen and Suslin (1976). The second problem however, has a negative solution due to the existence of stably trivial but non-trivial vector bundles, thus further restrictions were added.

Zariski cancellation problem (~1970): Suppose $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$, is X isomorphic to $\mathbb{A}^n$?

In this form, the problem is not actually due to Zariski (in 1949 he asked a different, but related, question about fields). We'll say that *Zariski cancellation holds in dimension* n if a positive answer to the question holds for algebraic varieties X of dimension n. Note that over $\mathbb{C}$, any variety X satisfying the hypotheses of the Zariski cancellation has the property that $X(\mathbb{C})$ is contractible. It was quickly shown that Zariski cancellation holds in dimension 1, and Fujita, Miyanishi and Sugie (1980s) proved Zariski cancellation holds in dimension 2. For later use, we make the following definition.

Definition 4. A complex algebraic variety is *topologically contractible* if $X(\mathbb{C})$ is contractible in the sense defined previously.

All the previous constructions of topological manifolds were very "topological" so it's not even clear that topologically contractible smooth complex algebraic varieties different from affine space exist! However, the following groundbreaking result appeared:

Theorem 5 (Ramanujam '74)**.** *There exists a topologically contractible smooth complex algebraic surface X such that $X(\mathbb{C})$ is not homeomorphic to $\mathbb{R}^4$. A smooth contractible complex algebraic surface X is algebraically isomorphic to $\mathbb{A}^2$ if and only if $X(\mathbb{C})$ is simply connected at infinity.*

Constructed by removing a certain curve from a Hirzebruch surface. Recently this example has been used by Seidel and Smith in symplectic topology!

This *did not* solve the Zariski cancellation problem in dimension 2, though it did suggest the following problem.

Characterization problem: Characterize $\mathbb{A}^n$ among all n-dimensional algebraic varieties.

Unfortunately, purely topological invariants are probably not enough to solve this problem for $n > 2$ because of the following result.

Theorem 6 (Dimca, Ramanujam)**.** *Any topologically contractible n-dimensional smooth complex* affine *variety X, $n \geq 3$, has $X(\mathbb{C})$ diffeomorphic to $\mathbb{R}^{2n}$.*

Later attempts at studying Zariski cancellation proceeded by constructing other examples of topologically contractible smooth algebraic varieties–many examples of the latter are now known; Zaidenberg has a nice survey of these results. Furthermore, there was some interest in a generalization of Serre's problem.

Generalized Serre problem: Are all algebraic vector bundles on topologically contractible complex varieties trivial?

Problems with this picture:

- There is no coherent structure theory for contractible smooth complex algebraic varieties (of higher dimension).

- Furthermore, we have only defined a notion of contractibility for smooth *complex* algebraic varieties, what about varieties defined over other fields?

To rectify these problems one might try to introduce some ideas from homotopy theory into algebraic geometry to parallel the topological story.

Homotopy theory for varieties (a black box)

Basic idea: the affine line is analogous to the unit interval. Introduce a homotopy theory for algebraic varieties where homotopies are "parameterized" by the affine line.

Unfortunately this naive idea runs into various technical complications and one has to work harder. Morel and Voevodsky considered a slightly less naive notion that leads to a theory with good technical properties. We will avoid a technical discussion and consider instead the following (rough) dictionary.

Topology	Algebraic Geometry
Unit interval	Affine line $\mathbb{A}^1$
(finite) CW complex	Smooth algebraic variety
category of topological spaces	category of k-spaces $\mathcal{S}pc_k$
homotopy equivalence	$\mathbb{A}^1$-weak equivalence
homotopy category	$\mathbb{A}^1$-homotopy category
singular homology	Voevodsky's "motivic" homology
singular chain complex of Dold-Thom construction	Voevodsky motive

The precise definition of an $\mathbb{A}^1$-weak equivalence is technical, but it will suffice for us to consider an example.

Example 7. If X and Y are smooth varieties and $f : X \longrightarrow Y$ is a Zariski locally trivial fiber bundle with fibers isomorphic to affine spaces, then f is an $\mathbb{A}^1$-weak equivalence.

Remark 8. It is non-obvious that there exist pairwise non-isomorphic smooth proper (compact) varieties that are $\mathbb{A}^1$-weakly equivalent.

$\mathbb{A}^1$-contractibility

Definition 9. A smooth variety X is $\mathbb{A}^1$-contractible if the structure morphism $X \longrightarrow pt$ is an $\mathbb{A}^1$-weak equivalence.

- The smooth variety $\mathbb{A}^n$ is $\mathbb{A}^1$-contractible (radial scaling again!).
- Any variety that is $\mathbb{A}^1$-weakly equivalent to affine space is $\mathbb{A}^1$-contractible.
- Any variety that satisfies the hypotheses of Zariski cancellation is $\mathbb{A}^1$-contractible.
- Any $\mathbb{A}^1$-contractible smooth complex variety is topologically contractible.

...but why is making this definition a good thing?

Naive question: do there even exist $\mathbb{A}^1$-contractible smooth varieties that are not isomorphic to affine space?

A homotopic view of affine lines

Example 10. Take the smooth quadric hypersurface Q_4 defined by $x_1x_4 - x_2x_3 = x_5(x_5+1)$ in $\mathbb{A}^5$, and remove the subvariety E_2 (isomorphic to a 2-dimensional affine space) defined by $x_1 = x_2 = 0$, $x_5 = -1$; set

$$Q_4 \setminus E_2 := X_4$$

How? Construct an explicit fiber bundle $\mathbb{A}^5 \longrightarrow X_4$ that is Zariski locally trivial with $\mathbb{A}^1$-fibers. Can this be generalized? Yes (in two ways!).

<u>"Existence results" for $\mathbb{A}^1$-contractibles</u>

Theorem 11 (A., B. Doran)**.** *For every integer $n \geq 4$, there exist infinitely many pairwise non-isomorphic $\mathbb{A}^1$-contractible smooth varieties of dimension n. For every pair of integers $n \geq 6$, $m > 0$, there exist m-dimensional families of $\mathbb{A}^1$-contractible smooth varieties of dimension n, all of whose fibers are pairwise non-isomorphic.*

Idea of proof: Construct $\mathbb{A}^1$-contractible smooth varieties by using a version of geometric invariant theory to construct quotients of affine spaces by unipotent ("translation") group actions. We study the particular case of $\mathbb{G}_a$-actions on affine space, motivated by the topological constructions! Can give explicit defining equations and the theory is computationally effective (implemented in Singular).

Note: The $\mathbb{A}^1$-contractible varieties appearing in our construction that are *affine* cannot provably be distinguished from affine space. Thus, potential counterexamples to Zariski cancellation *may* exist here! We'll return to this...

Theorem 12 (A.). *Every $\mathbb{A}^1$-contractible smooth complex variety of dimension $d \leq 2$ is isomorphic to $\mathbb{A}^2$.*

Corollary 13 (A.). *There exist arbitrary dimensional families of topologically contractible, not $\mathbb{A}^1$-contractible complex varieties of every dimension $d \geq 2$.*

Dimension 1 is easy, so we only need to consider dimension 2. Idea of proof: as with topological proof, start with the beautiful classification of topologically contractible smooth varieties of dimension 2 due to Gurjar, Shastri, Miyanishi, Kaliman, Zaidenberg, Makar-Limanov, etc. However, two important "topological" ideas appear. Basically, we study connectedness from the standpoint of $\mathbb{A}^1$-homotopy theory.

- Recall that a variety is *path connected* if any two points can be joined by a map from the unit interval. Similarly, one can hope that a variety is connected from the standpoint of $\mathbb{A}^1$-homotopy theory if every pair of points can be connected by a morphism from $\mathbb{A}^1$.

- Prove an "excision" result stating that if X is $\mathbb{A}^1$-contractible any open subvariety whose complement has codimension $d \geq 2$ is $\mathbb{A}^1$-connected; this uses recent work of Morel quite significantly.

- Observe that every topologically contractible smooth complex surface not isomorphic to $\mathbb{A}^2$ has an open subset U whose complement that is of codimension 2 that is *not* $\mathbb{A}^1$-connected.

- We hope that $\mathbb{A}^2$ is the only $\mathbb{A}^1$-connected topologically contractible smooth complex surface!

Moral: Checking that a variety is $\mathbb{A}^1$-connected is a *non-trivial* condition; it should mean that the variety is covered by affine lines! This should fail for "most" topologically contractible smooth complex varieties! Thus, $\mathbb{A}^1$-contractibility of a variety is a significantly stronger restriction than topological contractibility on possible counter-examples to Zariski cancellation.

Generalized Serre problems

Related to "bundle theory" for algebraic varieties, which doesn't work in the manner suggested by topology. The answer to the generalized Serre problem is a resounding no. An interesting dichotomy appears.

Theorem 14 (A., B. Doran)**.**

(Essentially) Every known $\mathbb{A}^1$-contractible smooth variety that is quasi-affine yet not affine has non-trivial vector bundles.

In fact, one can construct $\mathbb{A}^1$-contractible smooth varieties with arbitrary dimensional moduli of vector bundles.

However, $\mathbb{A}^1$-contractible smooth affine varieties all have trivial vector bundles (of rank $\neq 2$).

The last statement follows from recent work of Morel. The first two statements are proven by explicit constructions.

Generalized Hodge conjecture

Classically, the Hodge conjecture asks whether certain cohomology classes can be represented by algebraic cycles. However, for varieties over $\mathbb{C}$, there is a formulation due to A. Huber that can be viewed as a comparison between singular cohomology (enhanced in an appropriate way) and Voevodsky's motivic cohomology for arbitrary (not necessarily compact) smooth varieties. Applied to topologically contractible smooth varieties this general conjecture produces:

Conjecture 15. *The motive (recall the dictionary) with rational coefficients of a topologically contractible smooth complex variety is necessarily isomorphic to the motive with rational coefficients of a point.*

We can prove some portion of this conjecture. Note that the conclusion of the conjecture is essentially automatic for $\mathbb{A}^1$-contractible smooth varieties. Thus, the conjecture suggests that any topologically contractible smooth complex variety that is *not* $\mathbb{A}^1$-contractible is a potential counter-example to this general formulation of the Hodge conjecture.

Theorem 16 (A.). *If X is a topologically contractible smooth complex variety of dimension ≤ 2, then the motive with* integral *coefficients of X is isomorphic to that of a point.*

Conjectures, Problems in affine geometry,

and a little wild speculation...

A comparison of contractibles

	Topology	Algebraic geometry
Exist.	Only $\mathbb{R}^n$ in dim. $n \leq 2$	Only $\mathbb{A}^n$ in dim. $n \leq 2$
	∞ of dim. 3	???
	∞ of dim. 4-5	countably many of dim. 4-5
	∞ of dim ≥ 6	∞ of dim ≥ 6
Charact.	$\mathbb{R}^n$, $n \geq 3$ is unique s.c at ∞	???
Struct.	$M^n \times \mathbb{R} \cong \mathbb{R}^{n+1}$	See below...

How do we fill in the table? Given our constructions of $\mathbb{A}^1$-contractibles, a first question to ask is:

Question 17. *Can every $\mathbb{A}^1$-contractible smooth variety be realized as a quotient of affine space by the free action of a unipotent group?*

Unfortunately, the answer to this question is (almost certainly) no. We believe that we can construct explicit counterexamples using generalizations of the variety X_4 studied above involving higher even dimensional quadrics. A less-likely-to-be-wrong generalization would be:

Conjecture 18. *Is every $\mathbb{A}^1$-contractible smooth variety the base of a Zariski (really Nisnevich!) locally trivial fibration with total space and fibers isomorphic to affine spaces?*

Also, we didn't say anything about $\mathbb{A}^1$-contractibles in dimension 3. However, recent "topological" evidence suggests that smooth (strictly) quasi-affine $\mathbb{A}^1$-contractibles of dimension 3 do *not* exist.

Conjecture 19. *There exist no strictly quasi-affine smooth $\mathbb{A}^1$-contractible varieties of dimension 3.*

Furthermore, we have yet to construct a single *affine* $\mathbb{A}^1$-contractible smooth variety.

What about $x + x^2y + z^2 + t^3 = 0$ (Russell cubic)?

Nevertheless, we make the following (optimistic) conjecture.

Conjecture 20. *Zariski cancellation is* false *in every dimension ≥ 4.*

How about the problem of characterizing affine space? The topological results suggest trying to define an appropriate notion of $\mathbb{A}^1$-homotopy "at infinity." We have a tentative definition (at least "stably") of such a notion. Under this definition, affine space of dimension $n \geq 3$ is $\mathbb{A}^1$-simply connected at infinity. The following conjecture, which might be called an $\mathbb{A}^1$-Poincaré conjecture, is thus wild speculation at the moment.

Conjecture 21. *Affine n-space, $n \geq 3$, is the unique smooth n-dimensional variety that is $\mathbb{A}^1$-contractible, and $\mathbb{A}^1$-simply connected at infinity.*

Moral: Why is the Zariski cancellation problem so hard? 1) There are very few computable invariants available to distinguish non-isomorphic ($\mathbb{A}^1$-)contractible affine varieties (the $\mathbb{A}^1$-homotopy type at infinity would be such an invariant). 2) There are very few (computable) techniques to construct possible counterexamples (though various reductions of the problem exist, our methods give potential counter-examples).

For more information (e.g., preprint versions of papers) visit

http://www.math.washington.edu/~asok

Thank You!