Connectedness in the homotopy theory of algebraic varieties

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Outline

1 Conventions, definitions and basic examples

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- We'll write I for the unit interval $[0, 1]$.

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- If K/k is an extension, we'll write $X(K)$ for the set of K-valued solutions to the (local) equations defining X.

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- If $k = \mathbb{Q}$, or a finite field, then the set of solutions looks like a discrete set of points.

Invariants

- A *homotopy* between two continuous maps $f, g : M \longrightarrow M'$ is a continuous map $H : M \times I \longrightarrow M'$ such that $H(x, 0) = f$ and $H(x, 1) = g$.
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- The *homotopy category* is the category whose objects are (nice) topological spaces and whose morphisms are homotopy classes of continuous maps.

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- Homology groups $H_i(M, \mathbb{Z})$, start w/ free abelian group on the cts. maps $\Delta^n \to M$, define boundary operator, take cycles modulo boundaries.. (Harder to define, easier to compute)

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  - Dimension 1. Only connected closed manifold is S^1.
  - Dimension 2. Connected closed manifolds are either S^2, connected sums of $S^1 \times S^1$, connected sums of $\mathbb{RP}^2$.

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  - Any finitely presented group can appear as the fundamental group of a manifold of dimension ≥ 4.
- Dimensions ≥ 5. Better problem: classify all manifolds having a fixed homotopy type.

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- Problem: (Over $\mathbb{R}$) Might be the empty manifold, e.g., $x^2 + y^2 = -1$. More generally, we lose arithmetic information about solutions over the ground field.
- Problem: Even if the field k can be embedded into $\mathbb{C}$, the topological invariants can depend on the choice of embedding $k \hookrightarrow \mathbb{C}$ (Serre).

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- In topology, essentially all invariants one needs for the classification problem are homotopy invariants, and two properties are distinguished: "gluing" and "homotopy invariance."
    - *Gluing* means invariants can be computed locally and then glued together (e.g., Mayer-Vietoris sequence for homology or van Kampen theorem for fundamental group).
    - *Homotopy invariance* means invariant takes the same value on M and $M \times I$.

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- Gluing makes sense if we use Zariski open sets.
- Even better than trying to define invariants, why not try to define a good "homotopy category"?

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- Show this gives a good theory (e.g., recovers old invariants, proves old conjectures).

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- Detects much arithmetic information, including information regarding rational points.

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- We can form wedge sums of pointed spaces (one point unions)
- We can form smash products of pointed spaces (take Cartesian product and collapse one point union).
- We define $S_s^i \wedge \mathbb{G}_m^{\wedge j}$, and call this a motivic sphere.

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- We have many of the same computational tools

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- The space $\mathbb{A}^n$ is contractible in $\mathbb{A}^1$-homotopy theory, i.e., $\mathbb{A}^1$-weakly equivalent to a point.
- Are there other (non-isomorphic) spaces like this?

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Example

Take the variety Q_4 defined by the equation $x_1x_3 + x_2x_4 = x_5(x_5 + 1)$ and remove the locus of points where $x_1 = x_2 = 0, x_5 = -1$; this is $\mathbb{A}^1$-contractible.

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Idea of proof.

Take $\mathbb{A}^n$, equip it with a translation action of $\mathbb{G}_a$ (additive group of the affine line) and construct a quotient. □

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If X is a smooth proper variety over a field k, then $\pi_0^{\mathbb{A}^1}(X) = X(k)/\sim_{\mathbb{A}^1}$ where $\sim_{\mathbb{A}^1}$ is the equivalence relation on k-points generated by connecting points by the images of a map from $\mathbb{A}^1$.

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- This theorem provides a homotopical characterization of separably rationally connected varieties.

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- More generally, the zeroth homology controls a number of important invariants, e.g., the Brauer group.

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- The elements of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ admit an interpretation in terms of the theory of quadratic forms.

Classification in dimension 1

Theorem

The only $\mathbb{A}^1$-connected smooth proper algebraic curve (up to isomorphism) is $\mathbb{P}^1$.

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- The elements of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ admit an interpretation in terms of the theory of quadratic forms.
- We do not have a "geometric" interpretation of elements of elements of the $\mathbb{A}^1$-fundamental group in general, but they are large.

Classification in dimension 2

Theorem (A., Morel '09)

Suppose k is an algebraically closed field. Every $\mathbb{A}^1$-connected smooth proper algebraic surface is $\mathbb{A}^1$-weakly equivalent to either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at a fixed (possibly empty) finite set of distinct points.

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- The $\mathbb{A}^1$-fundamental group distinguishes $\mathbb{A}^1$-homotopy types of $\mathbb{A}^1$-connected surfaces; in fact, $X \setminus *$ is $\mathbb{A}^1$-weakly equivalent to a wedge of copies of $\mathbb{P}^1$.

Classification in dimension ≥ 3

Theorem (A. '11)

In every dimension $d \geq 3$, there exist $\mathbb{A}^1$-connected smooth proper varieties X and X' such that the all $\mathbb{A}^1$-homotopy groups of X and X' are abstractly isomorphic, yet which are not $\mathbb{A}^1$-weakly equivalent

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- The $\mathbb{A}^1$-connected varieties of dimension ≥ 3 can fail to be "cellular"
- We do not know whether $\mathbb{A}^1$-homotopy classification is impossible in higher dimensions (though we strongly suspect this is true).

A surgical approach to classification

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- However, smooth proper $\mathbb{A}^1$-connected varieties always have non-trivial $\mathbb{A}^1$-fundamental group.
- Many other invariants of surgery theory can be defined, but we do not know if they have reasonable geometric interpretations.

Thank you!

See http://www-bcf.usc.edu/~asok for more information