Rational points vs. 0-cycles of degree 1 in stable $\mathbb{A}^1$-homotopy

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Suppose k is a field, and X is a smooth variety over k. Let $\mathcal{H}(k)$ denote the $\mathbb{A}^1$-homotopy category of smooth schemes over k [MV99]; abusing notation, we write X for the isomorphism class of a smooth scheme in $\mathcal{H}(k)$. Let $\mathcal{SH}(k)$ denote the stable $\mathbb{A}^1$-homotopy category of smooth schemes over k, i.e., the category of $\mathbb{P}^1$-spectra over k [Mor05]. The suspension spectrum $\Sigma^{\infty}_{\mathbb{P}^1} \operatorname{Spec} k_+$, denoted $\mathbb{S}^0$ for notational convenience, is called the motivic sphere spectrum.

If U is another smooth variety, write $[U,X]_{\mathbb{A}^1}$ for the set $\operatorname{Hom}_{\mathcal{H}(k)}(U,X)$ and write $[U,X]_{st}$ for the abelian group $\operatorname{Hom}_{\mathcal{SH}(k)}(\Sigma^{\infty}_{\mathbb{P}^1} U_+, \Sigma^{\infty}_{\mathbb{P}^1} X_+)$. Define $\pi_0^{\mathbb{A}^1}(X)$ to be the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [U,X]_{\mathbb{A}^1}$ and $\pi_0^s(X)$ to be the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [U,X]_{st}$. Each of these sheaves determines "by restriction" a functor on the category of finitely generated separable extensions L/k.

Stable homotopy theory and rational points

If $\pi_0^{\mathbb{A}^1}(X)(k)$ is non-empty, we say that X has a rational point up to unstable $\mathbb{A}^1$-homotopy. It is known that if X has a rational point up to unstable $\mathbb{A}^1$-homotopy, then X has a rational point [MV99]. Thus, existence of a rational point is an unstable $\mathbb{A}^1$-homotopy invariant.

Similarly, say that X has a rational point up to stable $\mathbb{A}^1$-homotopy if the canonical map $\pi_0^s(X) \to \pi_0^s(\mathbb{S}^0)$ is a split epimorphism; a choice of a splitting will be called a rational point up to stable $\mathbb{A}^1$-homotopy. Any rational point up to unstable $\mathbb{A}^1$-homotopy determines a rational point up to stable $\mathbb{A}^1$-homotopy by taking iterated $\mathbb{P}^1$-suspensions. If X is smooth and proper, there is a group homomorphism from $\pi_0^s(X)(k)$ to the group of 0-cycles of degree 1; *a priori* it is not clear that this map is either surjective or injective.

Theorem 1. *Assume k is a field having characteristic 0. If X is a smooth proper k-variety, then X has a 0-cycle of degree 1 if and only if X has a rational point up to stable $\mathbb{A}^1$-homotopy.*

Sheaves of connected components

We deduce the above result from a description of the sheaf $\pi_0^s(X)$ for any smooth proper variety. The description is motivated by foundational work of Morel describing the sheaf $\pi_0^s(\mathbb{S}^0)$ in terms of the Grothendieck-Witt ring [Mor04]. There is a "Hurewicz" functor from the stable $\mathbb{A}^1$-homotopy category to Voevodsky's derived category of motives. The analog of the stable π_0 computed in Voevodsky's derived category of motives is the 0-th Suslin homology sheaf. For a smooth proper variety X, the sections of this sheaf over fields coincide with the Chow group of 0-cycles on X_L (*cf.* [Dég08, §3.4]).

We use the theory of oriented Chow groups, or Chow-Witt groups, as invented by J. Barge and F. Morel [BM00], and developed in detail by J. Fasel [Fas08, Fas07]. For any n-dimensional smooth proper k-scheme X, one can define the oriented Chow group $\widetilde{CH}_0(X)$ by means of a certain "oriented Chow cohomology group" $\widetilde{CH}^n(X, \omega_X)$ (see [Fas08, Definition 10.2.17] for details). This latter group is defined by means of an explicit Gersten resolution, and has functorial pushforwards for proper morphisms.

Theorem 2. *If X is a smooth proper k-variety over a field k having characteristic 0, then there is an isomorphism (natural with respect to X) between the functor $L \mapsto \pi_0^s(X)(L)$ and the functor $L \mapsto \widetilde{CH}_0(X_L)$.*

Sketch of proof of Theorem 2. One first reduces to the case where X is projective, and deals with an associated "abelianized" problem using a version of $\mathbb{A}^1$-homology that has been stabilized with respect to $\mathbb{G}_m$. When X is projective, the idea of the proof is to use Spanier-Whitehead duality: the Spanier-Whitehead dual of a smooth scheme X is the Thom space of the negative tangent bundle (see, e.g., [Hu05, Theorem A.1] or [Rio05, Théorème 2.2]).

When X has trivial tangent bundle, one can prove the result by proving a $\mathbb{P}^1$-bundle formula for the oriented Chow group of 0-cycles—this involves some facts about contractions of the sheaf $\mathbf{K}_n^{MW}$ as discussed at the end of [Mor06, §2.3]. In the general case, one has to show that the twist arising from non-triviality of the negative tangent bundle only appears through the canonical bundle ω_X of X. Locally the tangent bundle is trivial, and a careful patching argument (using the fact that any element of GL_n is $\mathbb{A}^1$-homotopic to its determinant) can be used to finish the proof; this involves an "unstable" construction of the map inducing duality as given by Voevodsky in [Voe03]. □

Sketch of proof of Theorem 1. The "only if" direction is straightforward. For the "if" direction, it suffices to show that the "forgetful" morphism $\widetilde{CH}_0(X_L) \to CH_0(X_L)$—functorial in L and X—is always a surjection. For any field F, the canonical map $GW(F) \to \mathbb{Z}$ given by the rank homomorphism is always surjective. Each of these groups is computed by means of a Gersten resolution. One then just uses the fact that X has Nisnevich cohomological dimension n. □

Remark 3. In fact, we prove a more precise result. The sheaf $\pi_0^s(X)$ is a strictly $\mathbb{A}^1$-invariant sheaf of groups by [Mor05, Theorem 6.2.7] and therefore "unramified" in an appropriate sense; one can then describe the sections of the sheaf $\pi_0^s(X)$ over a smooth scheme U in terms of sections over $k(U)$ together with information coming from discrete valuations associated with codimension 1 points of U.

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