# $\mathbb{A}^1$-contractibility and topological contractibility

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**Abstract**

These notes are an extended transcript of two lectures given at the University of Ottawa during the workshop "Group actions, generalized cohomology theories, and affine algebraic geometry." The first lecture was a general introduction to $\mathbb{A}^1$-homotopy theory focusing on motivating the choice of definitions in the construction, together with a definition of $\mathbb{A}^1$-contractible spaces. The second lecture attempted to understand $\mathbb{A}^1$-connectedness in greater detail, and to use this to understand better the relationship between $\mathbb{A}^1$-contractibility and topological contractibility. I have taken the liberty of including some material in these notes that I hoped to (but was unable) to discuss in the lectures. In particular, I have included some problems which I felt might be interesting to different groups of participants. Disclaimer: these notes were not carefully proofread; please e-mail me with any comments, corrections or questions.

## Contents



# 1 Lecture 1

We began by stating a slogan, loosely paraphrasing the first section of [MV99]:

> there should be a homotopy theory for algebraic varieties over a base where the affine line plays the role assigned to the unit interval in topology.

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The path to motivate the construction of the $\mathbb{A}^1$-homotopy category that I have followed is loosely based on the work of Dugger [Dug01]. The original constructions of the $\mathbb{A}^1$-homotopy category rely on [Jar87] and are to be found in [MV99]. A general overview of $\mathbb{A}^1$-homotopy theory can be found in [Voe98], and [Mor04] provides a more recent introductory text. Summaries of some more recent developments can be found in [Mor06].

## 1.1 Brief topological motivation

The jumping off point for the discussion was the classical Brown-representability theorem in unstable homotopy theory. First, recall that the ordinary homotopy category, denoted here $\mathscr{H}$, has as objects "sufficiently nice" topological spaces $\mathcal{T}op$ (including, for example, all CW complexes), and morphisms given by homotopy classes of continuous maps between spaces.

We consider contravariant functors $\mathscr{F}$ on $\mathcal{T}op$ that satisfy the following properties:

i) (Homotopy invariance axiom) If $X$ is a topological space, and $I = [0,1]$ is the unit interval, then the map $\mathscr{F}(X) \to \mathscr{F}(X \times I)$ is a bijection.

ii) (Mayer-Vietoris axiom) If $X$ is a CW complex, and $U$ and $V$ are subcomplexes with intersection $U \cap V$, then we have a diagram of the form

$$\mathscr{F}(X) \to \mathscr{F}(U) \times \mathscr{F}(V) \to \mathscr{F}(U \cap V),$$

and given $u \in \mathscr{F}(U)$ and $v \in \mathscr{F}(V)$, such that the images of $u$ and $v$ under the right hand map coincide, then there is an element of $\mathscr{F}(X)$ whose image under the first map is the pair $(u, v)$.

iii) (Wedge axiom) The functor $\mathscr{F}$ takes sums to products.

The first condition on a functor $\mathscr{F}$ as above implies that the functor gives rise to a functor on the homotopy category. The Brown representability theorem says that the second and third conditions imply that this functor is of the form $[-, Y]$ for some CW complex $Y$.

Note: the category of CW complexes is not "categorically good" there are various constructions one wants to perform in topology (quotients, loop spaces, mapping spaces, suspensions) that do not stay in the category of CW complexes (they only stay in the category up to homotopy).

*Remark* 1.1.1. In general, "cohomology theories" satisfy more properties than just those mentioned. The second condition will eventually imply the "usual" Mayer-Vietoris property for cohomology theories on a topological space.

*Remark* 1.1.2. If a "cohomology theory" is represented on CW complexes by a space $Z$, and the cohomology theory is geometrically defined (e.g., topological K-theory with $Z$ the infinite grassmannian), then the "representable" cohomology theory extended to all topological spaces need not coincide with the geometric definition for spaces that are not CW complexes.

## 1.2 Homotopy functors in algebraic geometry

We would like to guess what properties the "homotopy category" will have based on the known invariance properties of cohomology theories in algebraic geometry. To this end, we must think of some actual “cohomology theories." The examples we want to use are the theory of Chow groups [Ful98] and (higher) algebraic K-theory [Qui73]. Defining these quickly seems insensible, so let me just pick a concrete example: the Picard group, which is an example of a Chow (cohomology) group for sufficiently nice varieties.

It is known that the functor $Pic(X)$ is not homotopy invariant on schemes with singularities that are sufficiently bad [Tra70]. On the other hand, Quillen established a homotopy invariance property for algebraic K-theory of regular schemes (this is one of the fundamental properties of higher algebraic K-theory proven in [Qui73]). Thus, in order to speak of homotopy invariance, we first restrict to considering the category $\mathcal{S}m_k$ of schemes that are separated, smooth and have finite type over $k$ (we use smooth schemes rather than regular schemes since smoothness is more functorially well-behaved than regularity; if we assume $k$ is perfect, then there is no need to distinguish between the two notions). Our restriction to smooth schemes will be analogous to the restriction to (finite) CW complexes performed above.

**Definition 1.2.1.** A (set-valued) contravariant functor $\mathscr{F}$ on $\mathcal{S}m_k$ is $\mathbb{A}^1$*-homotopy invariant* if the morphism $\mathscr{F}(U) \to \mathscr{F}(U \times \mathbb{A}^1)$ is a bijection.

The classical cohomology theories one studies (Bloch's higher Chow groups [Blo86, Blo94] and algebraic K-theory) satisfy “localization." In the world of Chow groups, if $X$ is a smooth variety, and $U \subset X$ is an open subvariety with closed complement $Z$ (say equi-dimensional of codimension $d$), there is an exact sequence of the form

$$CH^{*-d}(Z) \longrightarrow CH^*(X) \longrightarrow CH^*(U) \longrightarrow 0;$$

to extend this sequence further to the left, one needs to introduce Bloch higher Chow groups; but I will not do this here. From this sequence one can formally deduce that Chow groups have a Mayer-Vietoris property for Zariski open covers by two open sets. However, they actually have a more refined Mayer-Vietoris property.

One often considers the étale topology in algebraic geometry, and one might ask whether there is an appropriate Mayer-Vietoris sequence for étale covers. In this direction, consider the following situation. Suppose given an open immersion $j : U \hookrightarrow X$ and an étale morphism $\varphi : V \to X$ such that the pair $(j, \varphi)$ are jointly surjective and such that the induced map $\varphi^{-1}(X \setminus U) \to X \setminus U$ is an isomorphism, diagrammatically this is a picture of the form:

$$\begin{array}{ccc} U \times_X V & \xrightarrow{j'} & V \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ U & \xrightarrow{j} & X \end{array} .$$

We will refer to such diagrams as *Nisnevich distinguished squares*. One can show that $X$ is the *colimit* in the category of smooth schemes of the diagram $U \longleftarrow U \times_X V \longrightarrow V$.

For the purposes of our discussion, it is useful to know that, by chasing diagrams, one can show that the sequence

$$CH^*(X) \to CH^*(U) \oplus CH^*(V) \to CH^*(U \times_X V)$$

is exact: given an element $(u, v)$ in $CH^*(U) \oplus CH^*(V)$, if the restriction of $(u, v)$ to $CH^*(U \times_X V)$ is zero, then there is an element $x$ in $CH^*(X)$ whose restriction to $CH^*(U) \oplus CH^*(V)$ is $(u, v)$.

*Example* 1.2.2. Suppose $k$ is a field of characteristic unequal to 2. Consider the diagram where $X = \mathbb{A}^1$, $U = \mathbb{A}^1 \setminus \{1\}$, $V = \mathbb{A}^1 \setminus \{0, -1\}$. Let $j$ be the usual open immersion of $\mathbb{A}^1 \setminus \{1\}$ into $\mathbb{A}^1$, and let $\varphi$ be the étale map given by the composite $\mathbb{A}^1 \setminus \{0, -1\} \hookrightarrow \mathbf{G}_m \to \mathbf{G}_m \hookrightarrow \mathbb{A}^1$, where the map $\mathbf{G}_m \to \mathbf{G}_m$ is $z \mapsto z^2$. It is easily checked that this diagram provides a square as above.

This version of "covering" will inform our definition of Mayer-Vietoris in the sought-for homotopy category. Roughly speaking, we want to consider the universal category where $\mathbb{A}^1$-homotopy invariance and Mayer-Vietoris in the Nisnevich sense hold. In order to impose Mayer-Vietoris universally, we go back to topology again. Suppose $u : U \to X$ with $U = \coprod_i U_i$ is an open cover of a topological space $X$. We can form the simplicial topological space $\check{C}(u)$ whose $n$-simplices correspond to the $n+1$-fold fiber product $U \times_X \cdots \times_X U$. Such a simplicial topological space has a geometric realization, and the geometric realization is a topological space that is weakly equivalent to $X$ itself. In a sense, this fact is a "universal" form of the Mayer-Vietoris sequence.

In the algebro-geometric context, we will impose Mayer-Vietoris "universally" in two stages. First, we want to enlarge our category of smooth schemes to a slightly larger category so that many constructions necessary in homotopy theory can be performed. E.g., we would like to talk about mapping spaces, quotients by subspaces, etc., and not all of these constructions can be performed in the category of smooth schemes. More precisely, we would like to embed $\mathcal{S}m_k$ in a category that contains all small limits and colimits (i.e., is complete and cocomplete). There is a universal procedure to do this, but we would like to perform this enlargement in a fashion that respects certain colimits that already exist in smooth schemes (e.g., the Nisnevich distinguished squares).

**Definition 1.2.3.** A Nisnevich sheaf of sets on $\mathcal{S}m_k$ is a contravariant functor $\mathscr{F}$ from the category $\mathcal{S}m_k$ to the category $\mathcal{S}ets$ such that given any Nisnevich distinguished square (as above) the induced diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \longrightarrow & \mathscr{F}(U) \\ \downarrow & & \downarrow \\ \mathscr{F}(V) & \longrightarrow & \mathscr{F}(U \times_X V) \end{array}$$

is cartesian. We write $\mathcal{S}hv_k$ for the category of Nisnevich sheaves on $\mathcal{S}m_k$ (morphisms are natural transformations of functors).

*Remark* 1.2.4. Of course, Nisnevich sheaves are sheaves for a Grothendieck topology (the Nisnevich topology) on $\mathcal{S}m_k$. As a consequence, we will be able to speak of sheafification. A morphism $f : U \to X$ is a Nisnevich cover if it is an étale cover and for every point $x$ of $X$, there exists a point $u$ in $U$ such that the induced map on residue fields $\kappa(u) \to \kappa(x)$ is an isomorphism.

If $X$ is a smooth $k$-scheme, one can check that the contravariant representable functor $\mathrm{Hom}_{\mathcal{S}m_k}(\cdot, X)$ is a Nisnevich sheaf in the sense just defined. The Yoneda embedding lemma says that the induced functor $\mathcal{S}m_k \to \mathcal{S}hv_k$ is fully-faithful. On the other hand, in order for the "universal" Mayer-Vietoris, we would like to work with simplicial sheaves, and therefore we define $\mathcal{S}pc_k$ to be the category of simplicial Nisnevich sheaves on $\mathcal{S}m_k$. To distinguish general spaces from schemes, we will use calligraphic letters for spaces, and roman letters for schemes.

The $\mathbb{A}^1$-homotopy category can be defined by means of a universal property. The $\mathbb{A}^1$-homotopy category is the universal category constructed from $\mathcal{S}pc_k$ in which the following two classes of maps are formally inverted:

i) if $u : U \to X$ is a Nisnevich cover, then we consider the morphism $\check{C}(u) \to X$, and

ii) if $\mathscr{X}$ is a space, then we consider the projection morphism $\mathscr{X} \times \mathbb{A}^1 \to \mathscr{X}$.

Showing that such a category exists is difficult, but there are numerous constructions now (the "universal" point-of-view espoused here is due to Dan Dugger; see [Dug01] for more details).

**Notation 1.2.5.** We write $\mathscr{H}(k)$ for the category obtained from $\mathcal{S}pc_k$ by formally inverting all of the above classes of morphisms. An isomorphism in $\mathscr{H}(k)$ will be called an $\mathbb{A}^1$-weak equivalence. To emphasize the analogy with topology, we write $[\mathscr{X}, \mathscr{Y}]_{\mathbb{A}^1}$ for the morphisms $\mathrm{Hom}_{\mathscr{H}(k)}(\mathscr{X}, \mathscr{Y})$; we will read this as the set of $\mathbb{A}^1$-homotopy classes of maps between $\mathscr{X}$ and $\mathscr{Y}$.

Just like in topology, there is a Brown representability theorem characterizing homotopy functors in algebraic geometry. In addition to homotopy invariance, one wants functors that turn Nisnevich distinguished squares into "homotopy" fiber products; for more details, see the work of Jardine [Jar11].

## 1.3 Basic constructions

There are a number of basic constructions that one imports from topology. First, a pointed $k$-space is a pair $(\mathscr{X}, x)$ consisting of a space $\mathscr{X}$ together with a morphism $\mathrm{Spec}\, k \to \mathscr{X}$. We will often denote $\mathrm{Spec}\, k$ simply by $*$. Using the fact that $\mathcal{S}pc_k$ has all small limits and colimits, the following definitions make sense. It is important to emphasize that these constructions are being made in the category of *spaces* and NOT in the category of schemes. However, moving outside of the category of spaces has a number of tangible benefits.

- If $(\mathscr{X}, x)$ and $(\mathscr{Y}, y)$ are pointed $k$-spaces, then the wedge sum $\mathscr{X} \vee \mathscr{Y}$ is the pushout (colimit) of the diagram $\mathscr{X} \xleftarrow{x} * \xrightarrow{y} \mathscr{Y}$.
- If $(\mathscr{X}, x)$ and $(\mathscr{Y}, y)$ are pointed $k$-spaces, then the smash product $\mathscr{X} \wedge \mathscr{Y}$ is the quotient of $\mathscr{X} \times \mathscr{Y} / \mathscr{X} \vee \mathscr{Y}$.
- We write $S^1_s := \mathbb{A}^1 / \{0, 1\rangle$ (remember: we are not thinking of this as a scheme!).
- We set $S^i_s := S^1_s \wedge \cdots \wedge S^1_s$ ($i$-times).

Having given a definition of sphere, we can define homotopy groups. Because our category of spaces already consists of sheaves, it will be necessary for us to consider homotopy *sheaves.*

**Definition 1.3.1.** Given a space $\mathscr{X}$, the sheaf of $\mathbb{A}^1$-connected components, denoted $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathscr{X})$, is the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [U, \mathscr{X}]_{\mathbb{A}^1}$ (where $U \in \mathcal{S}m_k$).

It makes sense to talk about pointed $\mathbb{A}^1$-homotopy classes of maps between pointed spaces. Using this observation, we can make the following definition.

**Definition 1.3.2.** Given a pointed space $(\mathscr{X}, x)$, the $i$-th $\mathbb{A}^1$-homotopy sheaf, denoted $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{X}, x)$, is the Nisnevich sheaf on $\mathcal{S}m_k$ associated with the presheaf $U \mapsto [S_s^i \wedge U_+, (\mathscr{X}, x)]_{\mathbb{A}^1}$. Here, we write $U_+$ for $U \coprod \operatorname{Spec} k$ pointed by $\operatorname{Spec} k$.

*Remark* 1.3.3. One can formally show that $\boldsymbol{\pi}_1^{\mathbb{A}^1}(\mathscr{X}, x)$ is a Nisnevich sheaf of groups, and $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{X}, x)$ is a Nisnevich sheaf of abelian groups for $i > 1$. In fact, results of Morel show that, just like in topology, these sheaves of groups are “discrete" in an appropriate sense; see [Mor06] for an introduction to these ideas and [Mor11] for details.

The following result, called the $\mathbb{A}^1$-Whitehead theorem for its formal similarity to the ordinary Whitehead theorem for CW complexes, is a formal consequence of the definitions.

**Proposition 1.3.4** ([MV99, §3 Proposition 2.14])**.** *A morphism $f : \mathscr{X} \to \mathscr{Y}$ of $\mathbb{A}^1$-connected spaces is an $\mathbb{A}^1$-weak equivalence if and only if for any choice of base-point $x$ for $\mathscr{X}$, setting $y = f(x)$ the induced morphism*

$$\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{X}, x) \to \boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{Y}, y)$$

*is an isomorphism.*

### 1.4 Examples of $\mathbb{A}^1$-weak equivalences

We now return to concrete geometry. By the very definitions, the morphism $\mathbb{A}^1 \to \operatorname{Spec} k$ is an $\mathbb{A}^1$-weak equivalence. Similarly, one shows that $\mathbb{A}^n \to \operatorname{Spec} k$ is an $\mathbb{A}^1$-weak equivalence. Combining this with the “universal" Mayer-Vietoris property, one can produce the following general example of an $\mathbb{A}^1$-weak equivalence.

*Example* 1.4.1. If $f : X \to Y$ is a Zariski locally trivial morphism of smooth schemes with fibers isomorphic to $\mathbb{A}^n$, then $f$ is an $\mathbb{A}^1$-weak equivalence. In particular, if $\xi : E \to X$ is a (geometric) vector bundle, then $\xi$ is an $\mathbb{A}^1$-weak equivalence.

We also make the following definition.

**Definition 1.4.2.** A space $\mathscr{X}$ is $\mathbb{A}^1$-contractible if the structure morphism $\mathscr{X} \to \operatorname{Spec} k$ is an $\mathbb{A}^1$-weak equivalence.

Combining the example above with the definition, we have the following.

*Example* 1.4.3. If $X$ is a smooth scheme, and $f : \mathbb{A}^N \to X$ is a Zariski locally trivial morphism with affine space fibers, then $X$ is $\mathbb{A}^1$-contractible. In particular, if $X$ is any smooth scheme that is stably isomorphic to $\mathbb{A}^n$, then $X$ is $\mathbb{A}^1$-contractible.

For the purposes of the next lecture, we note that the Whitehead theorem shows that a space $\mathscr{X}$ is $\mathbb{A}^1$-contractible if and only if it is $\mathbb{A}^1$-connected and $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{X}, x)$ is trivial.

## 2 Lecture 2

### 2.1 A brief review of Lecture 1 and some complements

Last time, we introduced $\mathbb{A}^1$-homotopy theory. We fix a field $k$ (no restrictions were imposed on the characteristic of $k$), and we consider the category $\mathcal{S}m_k$ of schemes that were separated, smooth and had finite type over $k$. We enlarged this category to a category $\mathcal{S}pc_k$; in other words, we described a category $\mathcal{S}pc_k$ together with a fully-faithful functor $\mathcal{S}m_k \hookrightarrow \mathcal{S}pc_k$ such that $\mathcal{S}pc_k$ had all small limits and colimits and such that the inclusion preserved the so-called Nisnevich distinguished squares. In practice, $\mathcal{S}pc_k$ can be taken either to be the category of Nisnevich sheaves of topological spaces, or the category of Nisnevich sheaves of simplicial sets, but there are other "reasonable" choices of the category of spaces.

The Morel-Voevodsky $\mathbb{A}^1$-homotopy category has as objects the objects of $\mathcal{S}pc_k$, but the morphisms in $\mathcal{S}pc_k$ are "$\mathbb{A}^1$-homotopy classes of maps;" providing a precise definition of the latter required the theory of model categories, but we swept this under the rug.

*Remark* 2.1.1. The main difficulty in computing the set of $\mathbb{A}^1$-homotopy classes of maps between two spaces is analogous to a fundamental problem in homological algebra: one has to take something like an "injective resolution of $\mathscr{Y}$" (a fibrant replacement for the chosen model structure). While we know that such "resolutions" exist, and we can even provide an explicit model, in practice the model is not sufficiently explicit that one can actually perform computations. Instead, one usually finds a convenient work-around.

## 2.2 $\mathbb{A}^1$-connectedness and geometry

Last time, given a space $\mathscr{X}$, we defined a sheaf $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathscr{X})$; this was the sheaf associated with the presheaf $U \mapsto [U, \mathscr{X}]_{\mathbb{A}^1}$.

**Definition 2.2.1.** Say $\mathscr{X}$ is $\mathbb{A}^1$-connected if the canonical morphism $\boldsymbol{\pi}_0^{\mathbb{A}^1}(\mathscr{X}) \to \operatorname{Spec} k$ is an isomorphism (and $\mathbb{A}^1$-disconnected otherwise).

Any $\mathbb{A}^1$-contractible space is $\mathbb{A}^1$-connected, by the very definition, but it would be nice to have a "geometric" definition. For this, we recall how connectedness is studied in topology: a topological space is *path* connected if any two points can be connected by a map from the unit interval. Replacing the unit interval by the affine line, we could define a notion of $\mathbb{A}^1$-path connectedness. For flexibility, we will use a slightly more general definition.

**Definition 2.2.2.** If $X$ is a smooth $k$-scheme, say that $X$ is $\mathbb{A}^1$*-chain connected* if for every separable, finitely generated extension $K/k$, $X(K)$ is non-empty, and for any pair $x, y \in X(K)$, there exist an integer $N$ and a sequence $x = x_0, x_1, \ldots, x_N = y \in X(K)$ together with morphisms $f_1, \ldots, f_N : \mathbb{A}^1_K \to X$ with the property that $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$; loosely speaking: any two points can be connected by the images of a chain of maps from the affine line.

*Remark* 2.2.3. Note: $K$ is not necessarily a finite extension, so this definition is non-trivial even when $k = \mathbb{C}$. Indeed, in that case, we ask, e.g., that $\mathbb{C}(t)$-points, $\mathbb{C}(t_1, t_2)$-points, etc. can all be connected by the images of chains of affine lines. If $k$ has characteristic 0, and $X$ is a smooth proper $k$-scheme, then stably $k$-rational varieties are $\mathbb{A}^1$-chain connected by this definition (one way to see this is to check that the property is stable under blow-ups and to use weak factorization, but it can also be seen directly with an application of Hironaka's resolution theorem). On the other hand, there are rationally connected varieties over $\mathbb{C}$ that do NOT satisfy this property (this can be tested indirectly, e.g., by studying cohomological consequences of this definition).

From the definitions given, it is not clear that either $\mathbb{A}^1$-connectedness implies $\mathbb{A}^1$-chain connectedness or vice versa. In one direction, the problem is that $\mathbb{A}^1$-chain connectedness only imposes conditions over fields: while fields are examples of stalks in the Nisnevich topology, they do not exhaust all examples of stalks.

**Proposition 2.2.4** ([Mor04, Lemma 3.3.6] and [Mor05, Lemma 6.1.3])**.** *If $X$ is an $\mathbb{A}^1$-chain connected smooth variety, then $X$ is $\mathbb{A}^1$-connected.*

*Idea of proof.* The proof uses the fact that we are working with the Nisnevich topology in a fairly crucial way. To check triviality of all stalks, it suffices to show that $\boldsymbol{\pi}_0^{\mathbb{A}^1}(X)(S)$ is trivial for $S$ a henselian local scheme. Chain connectedness implies that the sections over the generic point of $S$ are trivial and also that the sections over the closed point are trivial. We can then try to use a sandwiching argument to establish that sections over $S$ are also trivial: we use the Thom isomorphism theorem! □

Conversely, it is not clear that $\mathbb{A}^1$-connectedness implies $\mathbb{A}^1$-chain connectedness. However, we prove the following result.

**Theorem 2.2.5** ([AM11, Theorem 6.2.1])**.** *If $X$ is a smooth proper $k$-variety, and $K/k$ is any separable finitely generated extension, then $\boldsymbol{\pi}_0^{\mathbb{A}^1}(X)(K) = X(K)/\sim$. In particular, if $X$ is $\mathbb{A}^1$-chain connected, then $X$ is $\mathbb{A}^1$-connected.*

*Remarks on the proof.* The proof uses properness (via the valuative criterion) in an essential way. Again, we sandwich $\boldsymbol{\pi}_0^{\mathbb{A}^1}(X)(K)$ between two sets that are easier to describe. □

**Proposition 2.2.6.** *If $U$ is an $\mathbb{A}^1$-connected smooth variety and $X$ is any smooth proper compactification of $U$, then $X$ is also $\mathbb{A}^1$-connected.*

The basic problem with $\mathbb{A}^1$-connectedness for non-proper varieties $U$ is that it is not clear that a rational curve connecting two points can always be moved to a chain of affine lines lying wholly in $U$. Nevertheless, these results give a fairly good handle on $\mathbb{A}^1$-connectedness, but to finish the story it would be nice to answer the following question.

**Question 2.2.7.** *Is it true that for an arbitrary smooth $k$-scheme $X$ that $\mathbb{A}^1$-connectedness is equivalent to $\mathbb{A}^1$-chain connectedness?*

## 2.3 $\mathbb{A}^1$-contractibility and topological contractibility

Recall from the last lecture that the basic example of an $\mathbb{A}^1$-contractible variety is provided by the following fact: if there exists a Zariski locally trivial morphism $f : \mathbb{A}^N \to X$ with fibers isomorphic to affine spaces, then $X$ is $\mathbb{A}^1$-contractible. In particular, if $X$ is a quotient of $\mathbb{A}^N$ by a scheme-theoretically free action of a (split) unipotent group, then $X$ is $\mathbb{A}^1$-contractible. This point of view is explored in great detail in [AD07].

*Remark* 2.3.1. Here is another upshot of working in our category of spaces: we don't actually care whether the quotient exists as a scheme! Indeed, in the situation above, the algebraic space quotient of $\mathbb{A}^N$ by the free action of $U$ exists by a result of Artin, but since $U$-torsors are Zariski locally trivial by vanishing of coherent cohomology, this “algebraic space" quotient admits a rational section and coincides with the categorical quotient of the $U$-action in our category of spaces (repeat the definition of algebraic space with the Nisnevich topology to get a notion of “Nisnevich algebraic spaces"). If fact, there exist examples of Nisnevich algebraic spaces of this form that are not actually schemes, and likely this phenomenon is quite common.

*Remark* 2.3.2. We do not know whether all $\mathbb{A}^1$-contractible schemes arise in this fashion. For example, the axioms of model categories allow one to conclude that any retract of an $\mathbb{A}^1$-contractible space is itself $\mathbb{A}^1$-contractible. However, if $X$ is a retract of affine space, it is by no means clear that $X$ is stably isomorphic to affine space.

Using the results established above, the following arithmetic/geometric result is established in a “purely homotopic" way by means of the proposition we stated above.

**Corollary 2.3.3.** *If $X$ is an $\mathbb{A}^1$-contractible smooth variety, then $X(K)/R$ is trivial for every finitely generated separable $K/k$. Thus, if $k$ admits resolution of singularities, then any compactification $\bar{X}$ of $X$ is $\mathbb{A}^1$-connected and thus “nearly rational."*

To get a better idea of the strength of the condition of $\mathbb{A}^1$-contractibility, it is useful to compare our purely algebraic definition of contractibility with the classical notion of contractibility of a topological space. To this end, suppose $k$ is a field that can be embedded in $\mathbb{C}$. The choice of embedding gives rise to a functor from the category $\mathcal{S}m_k$ to the category of complex manifolds: take a smooth $k$-scheme $X$, extend scalars to $\mathbb{C}$ by means of the chosen embedding, take the resulting set of complex points with its usual analytic topology. Morel and Voevodsky [MV99, §3.3] show how to extend this functor to a functor

$$\mathcal{H}(k) \longrightarrow \mathcal{H}$$

which we refer to as a (topological) realization functor; see [DI04] for more discussion of topological realization functors.

*Remark* 2.3.4. The choice of embedding of $k$ into $\mathbb{C}$ is important: Serre showed that it is possible to find smooth algebraic varieties over a number field together with two embeddings of $k$ into $\mathbb{C}$ such that the resulting complex manifolds are homotopy inequivalent. In fact, more recently examples were provided by F. Charles of two smooth algebraic varieties over a number field $k$ together with two embeddings of $k$ into $\mathbb{C}$ such that the real cohomology algebras of the resulting complex manifolds are not isomorphic [Cha09]. Said differently, the real homotopy type of a smooth $k$-scheme depends on the choice of embedding of $k$ into $\mathbb{C}$.

Because of the caveat mentioned in the previous remark, we make the following definition (we restrict ourselves to considering only smooth schemes).

**Definition 2.3.5.** A smooth $k$-scheme $X$ is *topologically contractible* if for some embedding $k \hookrightarrow \mathbb{C}$, the space $X(\mathbb{C})$ is contractible in the usual homotopy category.

**Problem 2.3.6.** *Prove or give a counterexample demonstrating that topological contractibility is independent of the choice of embedding of a field into $\mathbb{C}$.*

*Remark* 2.3.7. Recall that a connected topological space $X$ is $\mathbb{Z}$-acyclic if $H^i(X,\mathbb{Z}) = 0$ for all $i > 0$. Of course, contractible topological spaces are $\mathbb{Z}$-acyclic. One can show that the property of being $\mathbb{Z}$-acyclic is independent of the choice of embedding using étale cohomology. By the Artin-Grothendieck comparison theorem, the cohomology of $X(\mathbb{C})$ with $\mathbb{Z}/n$-coefficients is isomorphic to the étale cohomology of $X$ with $\mathbb{Z}/n$-coefficients, and étale cohomology of $X$ is independent of the choice of embedding of $k$ into $\mathbb{C}$. Furthermore, the Betti numbers of $X(\mathbb{C})$ are determined by the étale cohomology of $X$.

To my knowledge, all the examples where homotopy types change with the embedding involve a non-trivial fundamental group. If $X$ is a topologically contractible smooth $k$-scheme, then its étale fundamental group is independent of the choice of embedding. Furthermore, the étale fundamental group of $X$ is the profinite completion of the topological fundamental group of $X(\mathbb{C})$. Thus, if $X$ is topologically contractible, then any of the manifolds $X(\mathbb{C})$ has a fundamental group with trivial profinite completion. If one could prove that $X(\mathbb{C})$ has trivial fundamental group for any choice of embedding, the above problem would have a positive solution as a consequence of the usual Whitehead theorem. Let me also note that, working with étale homotopy types, one can deduce restrictions on the profinite completions of the other homotopy groups of $X(\mathbb{C})$.

On the other hand, $\mathbb{A}^1$-contractibility is a completely intrinsic notion. The following result is essentially a consequence of the existence of the realization functor mentioned above.

**Lemma 2.3.8.** *If $X$ is $\mathbb{A}^1$-contractible, then $X$ is topologically contractible.*

To compare the two notions in greater detail, it would be nice to know if the converse to the lemma is true. The best we can do at the moment is to proceed dimension by dimension. The only topologically contractible smooth curve is $\mathbb{A}^1$. However, already in dimension 2 problems appear to arise. With the exception of $\mathbb{A}^2$, most topologically contractible surfaces have very few affine lines. Thus, it seems reasonable to expect that they are disconnected from the standpoint of $\mathbb{A}^1$-homotopy theory. This leads us to suggest the following conjecture.

**Conjecture 2.3.9.** *A smooth topologically contractible surface $X$ is $\mathbb{A}^1$-contractible if and only if it is isomorphic to $\mathbb{A}^2$.*

*Remark* 2.3.10. Based on the classification results (see [Zaĭ99] and the references therein for more details), it suffices to treat the case of surfaces of logarithmic Kodaira dimensions 1 and 2 (there are no contractible surfaces of logarithmic Kodaira dimension 0). The topologically contractible surfaces of logarithmic Kodaira dimension 2 contain no contractible curves by work of Zaidenberg [Zaĭ87, Zaĭ91] and Miyanishi-Tsunoda [MT92]; what can one say about morphisms from the affine line to such a surface? For example, are such surfaces algebraically hyperbolic in the sense that there are no non-constant morphisms from the affine line? The surfaces of logarithmic Kodaira dimension 1 are all obtained from some special surfaces (the so-called tom Dieck-Petrie surfaces) by repeated application of a procedure called an affine modification (an affine variant of a blow-up). How does $\mathbb{A}^1$-chain connectedness behave with respect to affine modifications (we understand well how $\mathbb{A}^1$-chain connectedness behaves with respect to blow-ups of projective schemes with smooth centers). One could also try to use the rationality results of Gurjar-Shastri, i.e., that any smooth compactification of a topologically contractible surface is rational [GS89a, GS89b].

Note also: it is an open question whether all topologically contractible varieties are rational. For $\mathbb{A}^1$-contractible varieties, by “soft" methods, one can establish “near rationality" as we observed above. The upshot of this discussion is that $\mathbb{A}^1$-contractibility is a significantly stronger restriction on a space than topological contractibility.

## 2.4 Cancelation problems and the Russell cubic

One important problem in affine algebraic geometry is *the* cancelation problem: if $X \times \mathbb{A}^1 \cong \mathbb{A}^n$, then is $X$ isomorphic to $\mathbb{A}^{n-1}$? (The emphasis was cribbed from Peter Russell's talk at the workshop). Here are some observations regarding this question.

- Any counterexample to the cancelation problem is an affine $\mathbb{A}^1$-contractible smooth variety (as we saw above this is a much stronger restriction than the assertion that $X$ be topologically contractible).
- The corresponding question in topology has a negative answer. Namely, if $M$ is an open contractible manifold, and $M \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{n+1}$, then $M$ need not be homeomorphic to $\mathbb{R}^n$ if $n \geq 3$.
- Topologists have much stronger statements (Perelman, Freedman, Siebenmann-Stallings)
  - For every $n \geq 3$, there exist infinitely many pairwise non-homemorphic open contractible manifolds of dimension $n$.
  - If $M$ is an open contractible manifold of dimension $n \geq 3$, then $M \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{n+1}$.
  - If $M$ is an open contractible manifold of dimension $n \geq 3$, then $M$ is homeomorphic to $\mathbb{R}^n$ if and only if $M$ is simply connected at infinity.
- I think it is expected that counterexamples to **the** cancelation problem exist in dimensions $\geq 4$ (maybe 3?).

At the moment, $\mathbb{A}^1$-contractibility is not a very well developed notion if $X$ is affine. As a first step, it is useful to know that $\mathbb{A}^1$-contractibility can be established by computing only finitely many homotopy groups. For example, one has the following result.

**Theorem 2.4.1** ([AD07])**.** *In every dimension $d \geq 4$, there exist infinitely many pairwise non-isomorphic smooth quasi-affine (but not affine) $\mathbb{A}^1$-contractible varieties of dimension $d$.*

**Lemma 2.4.2.** *If $X$ is a smooth variety, then $X$ is $\mathbb{A}^1$-contractible if and only if $X$ is $\mathbb{A}^1$-connected and $\boldsymbol{\pi}_i^{\mathbb{A}^1}(X,x)$ is trivial for $1 \leq i \leq n$.*

*Idea of proof.* Schemes over a field have Nisnevich cohomological dimension equal to their Krull dimension. One can prove the result using obstruction theory via the Postnikov tower in $\mathbb{A}^1$-homotopy theory. □

At the moment it seems unreasonable to test $\mathbb{A}^1$-contractibility by computing $\mathbb{A}^1$-homotopy groups, it does have some other nice "geometric consequences." The following result is a consequence of a much more general $\mathbb{A}^1$-homotopy classification of vector bundles of rank $n \geq 3$ on a smooth affine variety due to F. Morel [Mor11, §7]. The results that (I believe) L.-F. Moser will discuss tomorrow imply that this result can be extended to give a homotopy classification of vector bundles of rank 2 as well [Mos11].

**Theorem 2.4.3.** *If $X$ is an affine $\mathbb{A}^1$-contractible variety, then all vector bundles on $X$ are trivial.*

*Remark* 2.4.4. The affineness assertion in the statement is definitely necessary. This point is discussed in great detail in [AD08].

For concreteness, it's useful to focus on one particular case, which has already been mentioned a few times. Let $X$ be the so-called Russell cubic, i.e., the smooth variety in $\mathbb{A}^4$ defined by the equation:

$$x + x^2 y + z^2 + t^3 = 0$$

As mentioned yesterday, $X$ is not isomorphic to $\mathbb{A}^3$, but it is presently unknown whether $X$ is stably isomorphic to $\mathbb{A}^3$. Furthermore, M.P. Murthy showed that all vector bundles on $X$ are trivial [Mur02] (it is also known that the chow groups of $X$ are trivial). However, $X$ admits a $\mathbf{G}_m$-action with an isolated fixed point. If $X$ is not stably isomorphic to $\mathbb{A}^3$, there might be an $\mathbb{A}^1$-homotopic obstruction to stable isomorphism.

**Question 2.4.5.** *Is the Russell cubic $\mathbb{A}^1$-contractible?*

First, one might try to compute the $\mathbb{A}^1$-homotopy groups; for this even to be sensible, we should make sure that the first obstruction to $\mathbb{A}^1$-contractibility vanishes.

**Proposition 2.4.6** (B. Antieau (unpublished))**.** *The Russell cubic is $\mathbb{A}^1$-chain connected.*

2.4.7 (Approach 1). Can one detect non-triviality of any of the higher $\mathbb{A}^1$-homotopy groups of $X$? One approach to this problem is to think "naively" of, e.g., the $\mathbb{A}^1$-fundamental group. Think of chains of maps from $\mathbb{A}^1$ that start and end at a fixed point up to "naive" homotopy equivalence (this naturally forms a monoid rather than a group). The resulting object maps to the actual $\mathbb{A}^1$-fundamental group, but what can one say about its image?

2.4.8 (Approach 2). Since to disprove $\mathbb{A}^1$-contractiblity, we only need one cohomology theory that is $\mathbb{A}^1$-representable that detects non-triviality, it is useful to look at invariants that are not as "universal" as $\mathbb{A}^1$-homotopy groups. For another approach, using group actions, let me mention that J. Bell showed that rational $\mathbf{G}_m$-equivariant $K_0$ of $X$ is actually non-trivial [Bel01]. Unfortunately, his computations together with the Atiyah-Segal completion theorem in equivariant algebraic K-theory also show that the "Borel style" equivariant $K_0$ is isomorphic to the Borel style equivariant $K_0$ of a point [AS69]. Nevertheless, $\mathbb{A}^1$-homotopy theory gives a wealth of new cohomology theories with which to study the Russell cubic. For example, it would be interested to know if one of the more "refined" Borel style equivariant theories is refined enough to detect failure of $\mathbb{A}^1$-contractibility.

If $\mu_n \subset \mathbf{G}_m$ is a sufficiently "large" subgroup then the $\mu_n$-equivariant $K_0$ of $X$ is also non-trivial. Moreover, the fixed-point loci for the $\mu_n$-actions are all affine spaces (indeed, if $n$ is prime, then the only non-trivial subgroup is the trivial subgroup which has the total space as fixed point locus). Thus, for many purposes, one might simply look at $\mu_n$-equivariant geometry.

In equivariant topology, a map is a "fine" equivariant weak equivalence if it induces a weak equivalence on fixed point loci for all subgroups. Transplanting this to $\mathbb{A}^1$-homotopy theory: if we knew that equivariant algebraic K-theory was representable on an appropriate equivariant $\mathbb{A}^1$-homotopy category (such categories have been constructed for finite groups by Voevodsky [Del09] and Hu-Kriz-Ormsby [HKO]), and we knew enough about the weak equivalences in the theory, Bell's result might formally imply that $X$ is not $\mathbb{A}^1$-contractible. (It is not at present known whether equivariant K-theory is representable in Voevodsky's equivariant category.)

Establishing $\mathbb{A}^1$-connectedness is a first step towards understanding $\mathbb{A}^1$-homotopy type.

**Problem 2.4.9.** *Which classes of topologically contractible varieties are known to be $\mathbb{A}^1$-chain connected?*

**Question 2.4.10.** *Looking further forward: can one characterize affine space among affine $\mathbb{A}^1$-contractible varieties (e.g., by defining some notion of $\mathbb{A}^1$-fundamental group at infinity)?*

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