Algebraic vs. topological vector bundles

joint with Jean Fasel and Mike Hopkins

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Question

Can we characterize the image of Φ_r?

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the morphism π is *not* a vector bundle projection; *is* topologically trivial.

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Example (Projective space)

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We call $\widetilde{\mathbb{P}(V)}$ *the* standard Jouanolou device of projective space. *If* $X \hookrightarrow \mathbb{P}^n$ *is a closed subvariety, then we get a Jouanolou device for* X *by restricting the standard Jounolou device for* $\mathbb{P}^n$.

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Representability:

Theorem (Pontryagin–Steenrod)

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Sub-question: are there restrictions on the possible Chern classes of algebraic vector bundles?

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Sub question: are there restrictions on these “further” cohomological invariants for algebraic vector bundles?

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Criticism: BGL_r is not equivalent to Gr_r in $\mathscr{H}_{alg}(k)$.

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Warning: if we enlarge Sm_k by including sufficiently singular varieties, then the functor $U \mapsto Pic(U)$ *fails* to be homotopy invariant.

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Theorem (Quillen–Suslin '76)

If k is a PID, then every vector bundle on $\mathbb{A}^n_k$ is trivial.

Conjecture (Bass–Quillen '72)

If k is a regular ring of finite Krull dimension, then for any $r \geq 0$

$$\mathscr{V}_r(\operatorname{Spec} k) \longrightarrow \mathscr{V}_r(\mathbb{A}^1_k)$$

is a bijection.

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Still open in completely generality!

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Unfortunately $\mathscr{V}_r^{alg}(-)$ fails to be homotopy invariant for $r \geq 2$.

Example (A., B. Doran '08)

Set $Q_4 = \operatorname{Spec} k[x_1, x_2, x_3, x_4, z]/\langle x_1x_2 - x_3x_4 = z(z+1)\rangle$

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this bundle restricts non-trivially to X_4*, i.e., contractible varieties may carry non-trivial vector bundles!*

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Example (A., B. Doran '08)

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$E_2 \subset Q_4$ *defined by* $x_1 = x_3 = z + 1 = 0$ *is isomorphic to* $\mathbb{A}^2$

If $k = \mathbb{C}$, $X_4 = Q_4 \setminus E_2$ *is* contractible*: in fact, there is an explicit morphism* $\mathbb{A}^5 \to X_4$ *that is Zariski locally trivial with fibers* $\mathbb{A}^1$

Q_4 *carries an explicit non-trivial rank* 2 *bundle (the Hopf bundle);*

this bundle restricts non-trivially to X_4*, i.e., contractible varieties may carry non-trivial vector bundles!*

Moral: homotopy invariance fails badly for non-affine varieties (even $\mathbb{P}^1$)!

The motivic homotopy category

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- $\mathscr{H}_{alg}(k)$ - invert Nisnevich local weak equivalences and
- $\mathscr{H}_{mot}(k)$ - inverting both Nisnevich local and $\mathbb{A}^1$-weak equivalences (this is the Morel–Voevodsky $\mathbb{A}^1$-homotopy category)

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- *More generally,* $Gr_n \to BGL_n$ *classifying the tautological vector bundle is an* $\mathbb{A}^1$*-weak equivalence*

Cannot expect representability of $\mathscr{V}_r^{alg}(X)$ in $\mathscr{H}_{mot}(k)$ for all smooth X, however:

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If k is a field or $\mathbb{Z}$, then for any smooth affine k-scheme X,

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A.–M. Hoyois–M. Wendt '15 (essentially self-contained: in essence, representability is equivalent to the Bass–Quillen conjecture for all smooth k-algebras)

Comparing algebraic and topological vb

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Does this factorization get us anything new?

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- if $k = \mathbb{C}$, cycle class map $cl : H^{2i,i}(X, \mathbb{Z}) \to H^{2i}(X, \mathbb{Z})$ sends c_i to c_i^{top}

Fundamental difference: $H^{2i,i}(X, \mathbb{Z})$ more complicated than $H^{2i}(X, \mathbb{Z})$ (e.g., X affine, $H^{2i}(X, \mathbb{Z})$ vanishes for $2i > \dim X$; false for motivic cohomology!)

Further cohomological invariants

Just like in topology, further invariants are torsion:

Theorem (A., J. Fasel, M. Hopkins)

If k a field and -1 is a sum of squares in k, then

$$Gr_r \longrightarrow \prod_{i=1}^{r} K(\mathbb{Z}(i), 2i)$$

is a rational $\mathbb{A}^1$-weak equivalence.

Motivic vector bundles and algebraicity

Theorem (A., J. Fasel, M. Hopkins)

Suppose X is a smooth complex affine variety of dimension 4, and $\mathcal{E}^{an} \to X^{an}$ is a rank 2 complex analytic vector bundle with Chern classes $c_i^{top} \in H^{2i}(X^{an}, \mathbb{Z})$. Assume the Chern classes c_i^{top} of $\mathcal{E}^{an}$ are algebraic, i.e., lie in the image of the cycle class map cl. The bundle $\mathcal{E}^{an}$ is algebraizable if and only if we may find $(c_1, c_2) \in H^{2,1}(X) \times H^{4,2}(X)$ with $(cl(c_1), cl(c_2)) = (c_1^{top}, c_2^{top})$ such that $Sq^2 c_2 + c_1 \cup c_2 = 0 \in H^{6,3}(X, \mathbb{Z}/2)$.

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Conjecture (A., J. Fasel, M. Hopkins)

For "cellular" smooth $\mathbb{C}$-varieties X, $[X, Gr_r]_{\mathbb{A}^1} \to [X, Gr_r]$ is a bijection.

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which is the problem of *descent* for vector bundles.

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What happens in low dimensions?

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Theorem (A., J. Fasel, M. Hopkins)

If X is a smooth projective variety of dimension ≤ 2 over $\mathbb{C}$, and $(\tilde{X}, \pi)$ is a Jouanolou device for X, then $\pi^* : \mathscr{V}_r^{alg}(X) \to \mathscr{V}_r^{alg}(\tilde{X})$ is surjective.

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- Use the Hartshorne–Serre correspondence (between codimension 2 lci schemes and rank 2 vector bundles) to construct the required vb on X.

□

Obstruction theory in $\mathbb{A}^1$-homotopy

Classical homotopy theory gives techniques for providing a "cohomological" description of homotopy classes: one factors a space into homotopically simple spaces (Eilenberg–Mac Lane spaces). F. Morel developed these ideas in algebraic geometry.

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- $\mathbb{A}^1$-Postnikov tower: given a pointed $\mathbb{A}^1$-connected space, we can build $\mathscr{X}$ inductively out of Eilenberg-Mac Lane spaces $K(\boldsymbol{\pi}, n)$; these have exactly 1 non-trivial $\mathbb{A}^1$-homotopy sheaf in degree n

Obstruction theory in $\mathbb{A}^1$-homotopy

Classical homotopy theory gives techniques for providing a "cohomological" description of homotopy classes: one factors a space into homotopically simple spaces (Eilenberg–Mac Lane spaces). F. Morel developed these ideas in algebraic geometry.

- If $(\mathscr{X}, x)$ is a pointed space, we may define $\mathbb{A}^1$-homotopy sheaves $\boldsymbol{\pi}_i^{\mathbb{A}^1}(\mathscr{X}, x)$.
- $\mathbb{A}^1$-Postnikov tower: given a pointed $\mathbb{A}^1$-connected space, we can build $\mathscr{X}$ inductively out of Eilenberg-Mac Lane spaces $K(\boldsymbol{\pi}, n)$; these have exactly 1 non-trivial $\mathbb{A}^1$-homotopy sheaf in degree n
- We can inductively describe the set of maps $[U, \mathscr{X}]_{\mathbb{A}^1}$ using sheaf cohomology with coefficients in $\mathbb{A}^1$-homotopy sheaves

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For any $n \geq 2$, the map $BGL_n \to BGL_1$ coming from $\det : GL_n \to GL_1$ induces an isomorphism $\pi_1^{\mathbb{A}^1}(BGL_n) = GL_1$.

Example (F. Morel)

There are isomorphisms

$$\pi_2^{\mathbb{A}^1}(BSL_n) \xrightarrow{\sim} \begin{cases} \mathbf{K}_2^{MW} & \text{if } n = 2 \\ \mathbf{K}_2^{M} & \text{if } n \geq 3. \end{cases}$$

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the map $BSp_\infty \to BGL_\infty$ yields a map $\mathbf{K}_2^{MW} \to \mathbf{K}_2^M$; this map is an epimorphism of sheaves and its kernel may be described via the "fundamental ideal" in the Witt ring (A. Suslin)

Motivic vector bundles on algebraic surfaces

Theorem

If k is algebraically closed, and $\tilde{X}$ is the Jouanolou device of a smooth projective surface X, then for $r \geq 2$ the map

$$(c_1, c_2) : \mathscr{V}_r^{alg}(\tilde{X}) \longrightarrow Pic(\tilde{X}) \times H^{4,2}(\tilde{X}, \mathbb{Z})$$

is an isomorphism.

Proof.

Obstruction theory! Case of trivial determinant: there is a canonical "Euler class" map

$$BSL_2 \longrightarrow K(\mathbf{K}_2^{MW}, 2);$$

if $\tilde{X}$ is as in the statement, then $H^2(\tilde{X}, \mathbf{K}_2^{MW}) \to H^2(\tilde{X}, \mathbf{K}_2^{M}) \cong H^{4,2}(\tilde{X}, \mathbb{Z})$ is an isomorphism; any class in $H^2(\tilde{X}, \mathbf{K}_2^{MW})$ lifts uniquely to $[\tilde{X}, BSL_2]_{\mathbb{A}^1}$. □

Thank you!