Algebraic geometry from an $\mathbb{A}^1$-homotopic viewpoint

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Outline/Motivation

The goal of these notes is to provide a "modern" introduction to the theory of vector bundles on algebraic varieties. Officially, this course has a few prerequisites. I'll assume that you know something about

1. algebraic topology (as in our introductory algebraic topology course), i.e., that you know something about homology, homotopy equivalences, covering spaces, fundamental groups and
2. a little bit about differential geometry (as in our introductory differential geometry course), i.e., you know what a manifold is, you know something about Sard's theorem and degree and perhaps something about de Rham cohomology and
3. you have some familiarity with algebraic geometry (as in our introductory algebra sequence, in conjunction with the algebraic geometry class run last term), i.e., you know what a scheme is, some basic things about quasi-coherent sheaves and are at least familiar with the basic morphisms of schemes.

All that being said, the true prerequisite is willingness to learn on the fly. Given that background, the goal of this course is to teach you about vector bundles in algebraic geometry and algebraic topology, with the spectre of $\mathbb{A}^1$-homotopy theory lurking in the backgroung.

The theory of projective modules is by now a very classical subject: the formal notion of projective module goes back to the work of Cartan–Eilenberg in the foundations of homological algebra [**?**, Chapter I.2], but examples of projective modules arose much earlier (e.g., the theory of invertible fractional ideals in number theory). The notion of projective module becomes indispensible in cohomology, e.g., group cohomology may be computed using projective resolutions. One may look at the collection of projective modules over a ring as a certain invariant of the ring itself ("representations" of the ring).

The results of Serre showed that the language of algebraic geometry might provide a good language to study projective modules over commutative unital rings [**?**, §50 p. 242]. More precisely, Serre showed that finitely generated projective modules over commutative unital rings are precisely the same things as finite rank algebraic vector bundles over affine algebraic varieties. Serre furthermore showed that this dictionary was useful for providing a better understanding about projective modules because it allowed one to exploit an analogy between algebraic geometry and algebraic topology: projective modules over rings are analogous to vector bundles over topological spaces.

From this point of view, projective modules take on additional significance. For example, in differential topology, one may turn a non-linear problem (e.g., existence of an immersion of one manifold into another) into a linear problem by looking at associated bundles (a corresponding injection of tangent bundles in the case of immersions). In good situations, a solution to the linear problem can actually be promoted to a solution of the non-linear problem (in our parenthetical exmaple, this is an incarnation of the Hirsch-Smale theory of immersions).

Based on this analogy, Serre observed that if R is a Noetherian ring of dimension d, one could "simplify" projective modules P of rank $r > d$ greater than the dimension: any such module could be written as a sum of a projective module of rank $r' \leq d$ and a free module of rank $r' - r$ [**?**]. After the work of Bass [**?**], which furthermore amalgamated Grothendieck's ideas regarding K-theory with Serre's results, J.F. Adams wrote:

> "This leads to the following programme: take definitions, constructions and theorems

> from bundle-theory; express them as particular cases of definitions, constructions and statements about finitely-generated projective modules over a general ring; and finally, try to prove the statements under suitable assumptions".

My point of view in this class is that the Morel-Voevodsy $\mathbb{A}^1$-homotopy theory provides arguably the ultimate realization of this program.

Pontryagin and Steenrod observed that one could use techniques of homotopy theory to study vector bundles on spaces having the homotopy type of CW complexes. Indeed, the basic goal of the class will be to establish the analog in algebraic geometry of this result, at least for sufficiently nice (i.e., non-singular) affine varieties. Looking beyond this, just as the Weil conjectures provides a beautiful link between the arithmetic problem of counting the number of solutions of a system of equations over a finite field and a "topologically inspired" étale cohomology theory of algebraic varieties, $\mathbb{A}^1$-homotopy theory allows one to construct a link between the algebraic theory of projective modules over commutative rings, and an "algebro-geometric" analog of the homotopy groups of spheres!

After very quickly recalling some of the topological constructions that provide sources of inspiration (and which we will attempt to mirror), I will begin a brief study of the theory of affine algebraic varieties. While this will not suffice for our eventual applications, affine varieties are, arguably, intuitively appealing, and it seemed better not to require too much algebro-geometric sophistication at first.

Then, I will introduce a "naive" version of homotopy for algebraic varieties and, following the topological story, describe various "homotopy invariants" in algebraic geometry. Along the way, I will introduce a number of important invariants of algebraic varieties: projective modules, Picard groups, and K-theory. The ultimate goal is to prove Lindel's theorem that shows that the functor "isomorphism classes of projective modules" is homotopy invariant, in a suitable algebro-geometric sense, on suitably nice (i.e., non-singular, affine) algebraic varieties. Along the way, I will try to build things up in a way that motivates some of the tools used in the study of $\mathbb{A}^1$-homotopy theory over a field.

There are many texts that talk about cohomology theories in algebraic geometry and these notes are not intended to be another such text. Rather, there is a hope, supported by recent results, that $\mathbb{A}^1$-homotopy theory can give us information not just about cohomology of algebraic varieties, but actually about their geometry. We have attempted to illustrate this by focusing on projective modules and vector bundles on algebraic varieties.

What's next?

To proceed from the "naive" theory to the "true" theory, requires more sophistication: one needs to know some homotopy theory of simplicial sets, Grothendieck topologies, model categories etc. The syllabus listed the following plan, which I would argue is the "next step" beyond what I now want to cover in the class. The subsequent background is written with this plan in mind.

- Week 1. Some abstract algebraic geometry: the Nisnevich topology and basic properties.
- Week 2. Simplicial sets and simplicial (pre)sheaves.
- Week 3. Model categories in brief; the simplicial and $\mathbb{A}^1$-homotopy categories
- Week 4. Basic properties of the $\mathbb{A}^1$-homotopy category (e.g., homotopy purity)

- Week 5. Fibrancy, cd-structures and descent
- Week 6. Classifying spaces: simplicial homotopy classification of torsors
- Week 7. $\mathbb{A}^1$-homotopy classification results
- Week 8. Eilenberg-MacLane spaces and strong and strict $\mathbb{A}^1$-invariance
- Week 9. Postnikov towers
- Week 10. Homotopy sheaves and $\mathbb{A}^1$-connectivity
- Week 11. The unstable $\mathbb{A}^1$-connectivity property and applications
- Week 12. Loop spaces and relative connectivity
- Week 13. Gersten resolutions and strong/strict $\mathbb{A}^1$-invariance
- Week 14. $\mathbb{A}^1$-homology and $\mathbb{A}^1$-homotopy sheaves
- Week 15. $\mathbb{A}^1$-quasifibrations and some computations of homotopy sheaves

Background. While there isn't a specific textbook for the class, I will use a number of different sources for some of the background material. There is formally quite a lot of background for the subject of the class and I don't expect anyone to have digested all the prerequisites in any sort of linear fashion. Instead, there will be a lot of "on-the-fly" learning and going backwards to fill in details as necessary.

- To get started, I will expect that people know some basic things about commutative ring theory. A good introductory textbook is [**?**], but [**?**] is more comprehensive. For a discussion that is more algebro-geometric, you can look at [**?**]. We will also need more detailed results about modules, for which you consule [**?**].
- We will study affine varieties and eventually discuss sheaf cohomology on topological spaces. Beyond what I mention in the class, useful references for the theory of algebraic varieties will include [**?**, Chapters 1-2]. Useful background for the notions of sheaf cohomology we will need on topological spaces in general, and on schemes in particular, can be found in [**?**] or [**?**, Chapter 3]. Implicit here is a basic understanding of some ideas from homological algebra [**?**]. Furthermore, from the standpoint of references, I think there is now no better definitive source than Johan de Jong's Stacks Project [**?**].
- I will also expect some familiarity with basic concepts of algebraic and differential topology, e.g., topological spaces, smooth manifolds and maps, CW complexes, singular homology, covering spaces, vector bundles, and homotopy groups as can be found in [**?**] or [**?**]. The point of view exposed in [**?**] will also be useful.
- Finally, the course will, from the beginning, use category-theoretic terminology. Beyond the usual notions of categories, functors, and natural transformations, I will expect some familiarity with various kinds of universal properties, limits (and colimits) and adjoint functors and their properties, as can be picked up in [**?**] or [**?**]. As time goes on, we will need a bit of familiarity with "size" issues in category theory, so [**?**] is also a good reference.
- One theme throughout the course will be connections with the theory of projective modules and K-theory. For the topological story, [**?**] is a good reference, while [**?**] is a suitable reference for the theory of fiber bundles. In the algebraic setting, [**?**] is a good reference for K-theory, while [**?**] will provide excellent motivation.
- As we progress, it will also be useful to know some things about the theory of quadratic forms. The theory over fields is discussed in [**?**]; the theory over more general rings is developed in [**?**], and [**?**] has a nice discussion from a point of view that will be closely related with ours.

Other references. The following is simply a list of references regarding topics that will appear in the class; it is by no means complete.

- Grothendieck topologies, especially the Nisnevich topology: [**?**], [**?**], [**?**]
- Simplicial sets: [**?**], [**?**] or the original sources [**?**, **?**]
- Model Categories: [**?**], [**?**] for a survey, or [**?**], [**?**] for more detailed treatments.
- Sheaf theoretic homotopy theory: [**?**] or [**?**, **?**] for original sources.
- $\mathbb{A}^1$-homotopy theory: [**?**] for an overview, and [**?**] or [**?**] for (different) more detailed treatments

As is likely clear from this quick list of references, $\mathbb{A}^1$-algebraic topology has a number of prerequisites and a large collection of sources of inspiration.

Notation

We use the following standard categories. All the categories under consideration are *essentially small*, i.e., equivalent to small categories (see A.1 for more details about category theory as we will need it). As a consequence we will frequently abuse notation and use the same notation for a choice of an essentially small skeletal subcategory.

- Set - objects are sets and morphisms are functions
- Grp - objects are groups and morphisms are group homomorphisms
- Ab = the full subcategory of Grp with objects consisting of abelian groups
- Mod_R - objects are (left) R-modules, and morphisms are R-module homomorphisms
- Top - objects are topological spaces and morphisms are continuous maps
- Δ - objects are non-empty finite totally ordered sets and morphisms are order-preserving functions
- sSet - objects are functors $\mathrm{Fun}(\Delta^\circ, \mathrm{Set})$ and morphisms are natural transformations.
- Aff_k - objects are finitely generated, commutative, unital k-algebras, morphisms are k-algebra homomorphisms.
- Cat - the category of small categories.

Warning/Disclaimer: These notes are constantly being modified (especially while the class is going on). Moreover, all the material is in very rough form, especially that which appears in later sections. I will frequently be adding/revising material in earlier sections. Thus, in the off chance that you happen to be reading along and are not taking the class, use at your own risk! Furthermore, not everything that is discussed in the notes was mentioned in class. If you do see mistakes, or find things about which you are confused (and they aren't fixed in a later version), please do not hesitate to write me for clarification!

Chapter 1

Schemes and varieties

Contents

Our goal in this class is to study algebraic varieties. Classically, one studied algebraic subsets of affine or projective space (i.e., subsets defined by the vanishing of finitely many polynomial functions); this is analogous to studying manifolds as embedded subsets of Euclidean space. One problem with this definition is that it was hard to define and study algebraic maps between algebraic varieties. One goal of studying schemes was to streamline the definition of morphisms of algebraic varieties. Ever since Grothendieck, it has been standard to introduce the more general category of schemes. Since I assume some familiarity with algebraic varieties, what is written below is written with the goal of a) fixing notation, and b) introducing some important examples. I first define affine schemes without any mention of sheaves, which will allow us to get off the ground quickly.

1.1 Affine schemes

We begin by analyzing a special class of varieties: affine varieties. Loosely speaking, these are the varieties defined by finitely many polynomial equations in a polynomial ring of some number of variables. When the base k is an algebraically closed field, one can use intuition from usual calculus/analytic geometry to study such objects. We will augment this intuition by studying what happens when the base k is not an algebraically closed field. More strongly: every commutative unital ring is an algebra over the ring of integers $\mathbb{Z}$, and it will sometimes be convenient for us to take the base $k = \mathbb{Z}$ sometimes.

1.1.1 Affine varieties

We begin by studying affine varieties over a base k. Intuitively speaking affine varieties are very familiar objects: they are simultaneous vanishing loci of a finite collection of polynomials in finitely many variables. While this should always serve as important inspiration, this definition is only correct when one works over an algebraically closed base field. The basic premise of affine algebraic geometry is that an affine variety is equivalent to its ring of functions. We begin with a definition of affine schemes in general that takes this point of view seriously. You can think of our definition as adding several layers of complexity to the intuitive idea of affine variety above:

- the topological space underlying an affine variety can have points that are not closed;
- the ring of coordinate functions can have nilpotent elements; and
- the base k may not be a field.

The Zariski topology

If R is any commutative ring, we can associate with R a topological space called its *spectrum* as follows; the correspondence implicit in the construction is called the ideal-variety correspondence and is one justification for the choice of notation.

Definition 1.1.1.1. Suppose R is a commutative unital ring,

1. $\operatorname{Spec} R :=$ the set of prime ideals in R;
2. for a subset T of R (not necessarily an ideal!) $V_T :=$ prime ideals containing T;
3. given an element $f \in R$, $D_f :=$ prime ideals not containing f.

In the above definition, we allowed ourselves to consider varieties attached to subsets that are not ideals for convenience.

Exercise 1.1.1.2. *Suppose R is a commutative unital ring. Show that*

1. *Every non-zero ring has a maximal ideal.*
2. *The set $\operatorname{Spec} R$ is empty if and only if R is the zero ring.*

The following result elucidates key properties of the ideal/variety correspondence.

Exercise 1.1.1.3. *If I and J are ideals in a commutative unital ring R, then show that*

1. *if T is a subset of R, and (T) is the ideal generated by T, then $V_T = V_{(T)}$;*
2. *V_I is empty if and only if I is the unit ideal;*
3. *$V_I \cup V_J = V_{I \cap J}$;*
4. *if I is an ideal and $\sqrt{I}$ is its radical, then $V_I = V_{\sqrt{I}}$;*
5. *for any set of ideals $\{I_\alpha\}_{\alpha \in A}$, $\cap_{\alpha \in A} V(I_\alpha) = V(\cup_{\alpha \in A} I_\alpha)$;*
6. *if $f \in R$, then $D(f) \coprod V_f = \operatorname{Spec} R$;*
7. *if $f, g \in R$, then $D(fg) = D(f) \cap D(g)$;*
8. *if $\{f_i\}_{i \in I}$ is a set of elements in R, then $\cup_{i \in I} D(f_i)$ is the complement in $\operatorname{Spec} R$ of $V_{\{f_i\}_{i \in I}}$;*
9. *if $f = uf'$ for some unit $u \in R$, then $D(f) = D(f')$;*
10. *if $f \in R$ and $D_f = \operatorname{Spec} R$, then f is a unit.*

Remark 1.1.1.4. Given a ring R and an element $f \in R$, the sets D_f are called the basic (or principal) open sets of R.

Exercise 1.1.1.5. *If R is a commutative unital ring, defining closed sets to be sets of the form V_T equips $\operatorname{Spec} R$ with the structure of a topological space. The sets D_f for a basis for this topology.*

If $\varphi : R \to S$ is any ring homomorphism, then for any prime ideal $\mathfrak{p} \subset S$, $\varphi^{-1}(\mathfrak{p})$ is prime, so there is an induced function $\operatorname{Spec} S \to \operatorname{Spec} R$.

Exercise 1.1.1.6. *If $\varphi : R \to S$ is a ring homomorphism, then the induced function $\operatorname{Spec} S \to \operatorname{Spec} R$ is continuous. Moreover, Spec is a* contravariant *functor from the category of commutative unital rings to* Top.

Remark 1.1.1.7. We think of R as the ring of “regular functions” on the topological space $\operatorname{Spec} R$.

Suppose R is a commutative ring. The following two exercise provide a dictionary between certain open subsets of $\operatorname{Spec} R$ and localizations of R, and closed subsets of $\operatorname{Spec} R$ and ideals.

Exercise 1.1.1.8. *Let R be a commutative unital ring.*

1. *Suppose $S \subset R$ a multiplicative set. Show that the ring homomorphism $R \to R[S^{-1}]$ induces a homeomorphism*
$$\operatorname{Spec} R[S^{-1}] \longrightarrow \{\mathfrak{p} \in \operatorname{Spec} R | S \cap \mathfrak{p} = \emptyset\},$$
where the topology on the right hand side is the subspace topology induced from the Zariski topology on $\operatorname{Spec} R$.
2. *If $f \in R$, then the map $R \mapsto R_f$ induces a homeomorphism $\operatorname{Spec} R_f \to D_f \subset \operatorname{Spec} R$.*

Example 1.1.1.9. If R is a commutative unital ring and $I \subset R$ is an ideal, then the map $R \to R/I$ induces a homeomorphism
$$\operatorname{Spec} R/I \to V_I \subset \operatorname{Spec} R.$$
Indeed, this is a restatement of the correspondence theorem: the quotient homomorphism identifies ideals in R/I with ideals in R that contain I. By definition, V_I is the set of prime ideals that contain I, and so inverse image under the quotient homomorphism is bijective. It follows immediately that $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is a continuous bijection onto V_I. To conclude, it suffices to check that $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is closed (since a closed continuous bijection is a homeomorphism). The closed subsets of $\operatorname{Spec} R/I$ are precisely the sets of the form V_T where T is a subset of R/I; since $V_T = V_{(T)}$ as subsets, we may restrict our attention to ideals in R/I. In that case, the closedness of the map again follows from the correspondence theorem.

Important examples of spectra

Example 1.1.1.10. If k is a field, then $\operatorname{Spec} k$ is, as a topological space, a single point with the discrete topology.

Example 1.1.1.11. Suppose R is a domain that is not a field. In this case (0) is a prime ideal and therefore is a point of $\operatorname{Spec} R$. On the other hand, since (0) is contained in every ideal, it follows that this point is not closed and, in fact, contains every point of $\operatorname{Spec} R$ in its closure. The point (0) is the generic point of $\operatorname{Spec} R$.

Example 1.1.1.12. Suppose R is a commutative ring and $\mathfrak{p} \subset R$ is a prime ideal. In that case, $\mathfrak{p}$ is a point x of $\operatorname{Spec} R$. There is an induced ring homomorphism $R \to R/\mathfrak{p}$ and thus a morphism $\operatorname{Spec} R/\mathfrak{p} \to \operatorname{Spec} R$. By the correspondence theorem, the ideals of $\operatorname{Spec} R/\mathfrak{p}$ are precisely the ideals of R that contain $\mathfrak{p}$. In particular, the prime ideals of $\operatorname{Spec} R/\mathfrak{p}$ are the prime ideals of R containing $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, we know that $R/\mathfrak{p}$ is a domain, and it follows that (0) is a prime ideal of $R/\mathfrak{p}$; this corresponds to the ideal $\mathfrak{p}$ in R. Thus, the point x in $\operatorname{Spec} R$ can be thought of as the generic point of $\operatorname{Spec} R/\mathfrak{p}$. The points in the closure of x correspond to the prime ideals in $R/\mathfrak{p}$.

Example 1.1.1.13. Suppose R is a commutative ring. A point $x \in \operatorname{Spec} R$ is closed if and only if it is not properly contained in any prime ideal. In other words, closed points of $\operatorname{Spec} R$ correspond to maximal ideals $\mathfrak{m} \subset R$.

Example 1.1.1.14. Take $R = k[\epsilon]/\epsilon^2$. In this case, R has ideals (ϵ) and (0). The ideal (ϵ) is prime and determines a closed point of $\operatorname{Spec} R$. Note that (0) is not a prime ideal since $\epsilon \notin (0)$ but $\epsilon^2 \in (0)$. The inclusion $k \subset k[\epsilon]/(\epsilon^2)$ is split by the projection $R \to R/(\epsilon)$. Thus, the homomoprhism $\operatorname{Spec} R \to \operatorname{Spec} k$ has a splitting. One evocative image for $\operatorname{Spec} R$ in this case is a closed point together with "nilpotent fuzz". If $R = k[x]$, then (x^2) is an ideal of $k[x]$, and $k[x]/(x^2)$ is a quotient of $k[x]$. In particular, we can view $\operatorname{Spec} k[x]/(x^2)$ as a subscheme of $\operatorname{Spec} k[x]$.

Exercise 1.1.1.15. *Draw a picture of* $\operatorname{Spec} k[x]$. *In particular, observe that* $\operatorname{Spec} k[x]$ *is not a Hausdorff topological space.*

Example 1.1.1.16. More generally, a polynomial ring in n-variables $k[x_1, \ldots, x_n]$ is a reduced, integral k-algebra and $\operatorname{Spec} k[x_1, \ldots, x_n]$ is denoted $\mathbb{A}^n_k$ (affine k-space).

Example 1.1.1.17. Suppose R is a commutative ring and $\mathfrak{p} \subset R$ is a prime ideal corresponding to a point in $\operatorname{Spec} R$. In that case, $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{m} := \mathfrak{p}R_{\mathfrak{p}}$. The homomorphism $R \to R_{\mathfrak{p}}$ induces an identification of $\operatorname{Spec} R_{\mathfrak{p}}$ with an open subset of $\operatorname{Spec} R$. Since $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{m}$, it follows that $\operatorname{Spec} R_{\mathfrak{p}}$ has one closed point. We write $\kappa(\mathfrak{p})$ for field $R_{\mathfrak{p}}/\mathfrak{m}$; we call this the residue field at the maximal ideal $\mathfrak{m}$. Note that the homomorphism $R \to R_{\mathfrak{p}}$ induces a function $R/\mathfrak{p} \to R_{\mathfrak{p}}/\mathfrak{m} = \kappa(\mathfrak{p})$. The former ring is an integral domain and this homomorphism identfies $\kappa(\mathfrak{p})$ with the field of fractions of $R/\mathfrak{p}$.

If $\varphi : R \to S$ is a ring homomorphism, then we have the continuous map $\operatorname{Spec} S \to \operatorname{Spec} R$. We can ask when $\mathfrak{p}$ lies in the image of this map. To this end, suppose $\mathfrak{q} \subset S$ is a prime ideal and $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. In that case, the image of $R \setminus \mathfrak{p}$ in S under φ is again a multiplicative set and we write $S_{\mathfrak{p}}$ for the localization of S at $\varphi(R \setminus \mathfrak{p})$. Then, there is an induced homomorphism $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ (though the latter need not be a local ring!) and thus an induced homomorphism $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \to S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$. This homomorphism identifies $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ with the tensor product $\kappa(\mathfrak{p}) \otimes_R S$. The ring $\kappa(\mathfrak{p}) \otimes_R S$ is a $\kappa(\mathfrak{p})$-algebra that we will call the scheme-theoretic fiber of $\varphi : R \to S$ over $\mathfrak{p}$.

There is also an induced homomorphism $S_{\mathfrak{p}} \to S_{\mathfrak{q}}$. This induces a homomorphism $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \longrightarrow S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} = \kappa(\mathfrak{q})$, i.e., we get a homomorphism $\kappa(p) \otimes_R S \to \kappa(\mathfrak{q})$. Therefore, we conclude that $\mathfrak{p}$ lies in the image of $\operatorname{Spec}\varphi$ if and only if $\kappa(\mathfrak{p}) \otimes_R S$ is non-zero.

Example 1.1.1.18. Consider the ring map $k[x] \to k[x]$ given by $x \mapsto x^2$. Assume first that k is algebraically closed. In that case, maximal ideals in $k[x]$ are of the form $(x-a)$, $a \in k$, and these ideals exhaust all non-zero prime ideals. Let us look at the scheme-theoretic fibers of the induced ring map. First, let us look at fibers over closed points. In that case, the residue field at the point corresponding to a is simply k itself. The scheme-theoretic fiber is $k[x]/(x-a) \otimes_{k[x]} k[x]$. Since the map in the tensor product is induced by the ring homomorphism $x \mapsto x^2$, you can check that this may be identified as the k-algebra $k[x]/(x^2-a)$. If $a \neq 0$, then since k is algebraically closed, (x^2-a) splits as $(x-\sqrt{a})(x+\sqrt{a})$. Then, $k[x]/(x-\sqrt{a})(x+\sqrt{a})$ is a reduced k-algebra and its spectrum is simply two points. If $a = 0$, then the scheme-theoretic fiber is $k[x]/(x^2)$. This is 2-dimensional as a k-vector space, just like the scheme-theoretic fibers corresponding to $a \neq 0$, in contrast to the topological picture where the preimage consists of a single point. If we look at the generic point, then the corresponding residue field is the field of Laurent polynomials $k(x)$ (i.e., invert all irreducible polynomials in $k[x]$). In that case, we are forming $k[x] \otimes_{k[x]} Frac(k[x]_{(0)})$. In this case, the fiber identifies with $k(x)[t]/(x^2-t)$. It's a good exercise to work out what happens over $k = \mathbb{R}$, say.

Affine schemes

We now proceed to give the general definition of an affine k-scheme.

Definition 1.1.1.19. Fix a base k (e.g., $\mathbb{Z}$ or a field). The category of (finite type) *affine k-schemes* is the opposite of the category of (finitely generated) commutative, unital k-algebras. If k is a field, a finite type k-algebra will be called an *affine k-algebra*. We write Aff_k for the category of affine k-schemes and ring homomorphisms.

We first begin with some statements about the general topology of spectra.

Lemma 1.1.1.20. *If R is a commutative ring, then* $\operatorname{Spec} R$ *is quasi-compact and quasi-separated as a topological space, i.e., every open cover has a finite subcover and the intersection of two quasi-compact opens is again quasi-compact.*

Proof. First, we establish quasi-compactness. Since we know what the open sets of the form D_f form a basis for the topology on $\operatorname{Spec} R$, it suffices to prove that any cover of $\operatorname{Spec} R$ by basic open sets can be refined to a finite open cover. In other words, suppose $\operatorname{Spec} R = \cup_i D_{f_i}$. By definition, this means that $\cap_i V(f_i) = \emptyset$. Since $\cap_i V(f_i) = V(\{f_i\})$, we conclude that $\{f_i\}_{i \in I}$ generates the unit ideal in I. In other words, $1 = \sum_i a_i f_i$ for some finite subset $J \subset I$. The finitely many elements $\{f_j\}_{j \in J}$ provide the required refinement.

For quasi-separatedness assertion, first observe that by means of the identification $D_f = \operatorname{Spec} R_f$, it follows that $\operatorname{Spec} R$ has a basis consisting of quasi-compact open subsets. Now, suppose U and V are quasi-compact open subsets of $\operatorname{Spec} R$. We may write $U = \cup_i D_{f_i}$ and $V = \cup_j D_{g_j}$. In that case, since $D_f \cap D_g = D_{fg}$ we conclude that $U \cap V = \cup_{i,j} D_{f_i g_j}$ and is thus evidently quasi-compact. $\square$

Since the definitions we're using have not made any assumptions about the structure of the ring R (e.g., about existence of zero-divisors or nilpotent elements in R), we will now analyze those things a bit.

Definition 1.1.1.21. If R is a commutative unital ring, then say that

1. R is *reduced* if R has no nilpotent elements;
2. R is *integral* if R is an integral domain.

Example 1.1.1.22. If R is a commutative unital ring, then the nilpotent elements form an ideal $N(R)$ called the nilradical of R. As a consequence, we obtain a ring homomorphism $R \to R/N(R)$ and thus a continuous map $\operatorname{Spec} R/N(R) \to \operatorname{Spec} R$, which identifies the former as a closed subset of $\operatorname{Spec} R$. The quotient $R/N(R)$ is a reduced ring by construction. The nilradical is known to be equal to the intersection of all prime ideals in R and in fact the map $\operatorname{Spec} R/N(R) \to \operatorname{Spec} R$ is a homeomorphism. Since it is already a continuous closed map, it suffices to check that it is bijective, but this follows from the correspondence theorem. If $\mathfrak{p}$ is a prime ideal of R, then $N(R) \subset \mathfrak{p}$, and thus there is a bijection between prime ideals of R and prime ideals of $R/N(R)$. If $X = \operatorname{Spec} R$, then we will write X_{red} for $\operatorname{Spec} R/N(R)$.

Next, we discuss the implications of R being a domain.

Definition 1.1.1.23. A topological space X is *reducible* if it can be written as the union of two non-empty proper closed subsets (and *irreducible* if it is not reducible).

Proposition 1.1.1.24. *Suppose R is a commutative unital ring.*

1. *For a prime $\mathfrak{p} \subset R$, the closure of $\{\mathfrak{p}\}$ in the Zariski topology is $V(\mathfrak{p})$.*
2. *The irreducible closed subsets of* $\operatorname{Spec} R$ *are precisely those of the form $V(\mathfrak{p})$ for $\mathfrak{p}$ a prime ideal.*
3. *Under the correspondence described in Point (2), the irreducible components of* $\operatorname{Spec} R$ *correspond precisely with the* minimal *prime ideals.*

Proof. Exercise. □

Example 1.1.1.25. If R is an integral domain, then $\operatorname{Spec} R$ is irreducible.

Exercise 1.1.1.26. *Show that* $\operatorname{Spec} R$ *is irreducible if and only if $\sqrt{(0)}$ is a prime ideal.*

Definition 1.1.1.27. If k is a field, by an *affine k-algebra* we will mean a finitely generated reduced k-algebra. The category of affine k-varieties is the opposite of the category of reduced, affine k-algebras; we write Var_k^{aff} for the category of affine k-varieties.

Remark 1.1.1.28. According to our definition, affine varieties can be *reducible*.

Examples

There are a number of affine schemes we will routinely consider, in addition to affine space $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$.

Example 1.1.1.29. Fix a base ring k that we will suppress from the notation. Let X be a symbolic $n\times n$-matrix with elements x_{ij}. The determinant $\det X$ is a polynomial in x_{ij} (e.g., by any cofactor expansion). Define
$$GL_n := \operatorname{Spec} k[x_{11},\ldots,x_{nn},\frac{1}{\det X}]$$
, and
$$SL_n := \operatorname{Spec} k[x_{11},\ldots,x_{nn}]/(\det X - 1).$$
In this case, the formula for matrix multiplication is also evidently polynomial in the entries, and the multiplicativity of the determinant then implies that matrix multiplication determines a morphism $GL_n \times GL_n \to GL_n$. Likewise, if X is an invertible $n\times n$-matrix, then the fact that $X^{-1} := \frac{1}{\det X} adj X$ implies that the assignment $X \mapsto X^{-1}$ determines a morphism of varieties $GL_n \to GL_n$. The identity $n\times n$-matrix determines a distinguished homomorphism $\operatorname{Spec} k \to GL_n$. The variety GL_n thus has the structure of an algebraic group. Similar statements hold for SL_n. One can define orthogonal and symplectic groups in an analogous fashion.

Dimension

Definition 1.1.1.30. If R is a commutative ring, recall that a chain of prime ideals of length n in R is a sequence $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$, where each inclusion is proper. The *Krull dimension* of R is the supremum of the lengths of chains of prime ideals. We will say that an affine scheme $X = \operatorname{Spec} R$ has dimension d if R has Krull dimension d.

Remark 1.1.1.31. For an arbitrary ring, this number need not be finite. The Krull dimension of $\operatorname{Spec} R$ coincides with the dimension of $\operatorname{Spec} R$ as a topological space (the topological definition is involves the lengths of chains of irreducible subspaces).

Example 1.1.1.32 (Hyperbolic quadrics). Fix a base field k. Consider the subvariety of $\mathbb{A}^{2n}$, with coordinates $x_1,\ldots,x_n,y_1,\ldots,y_n$ defined by the equation $\sum_i x_iy_i = 1$. The expression $\sum_i x_iy_i$ is called the hyperbolic quadratic form and $Q_{2n-1} := \operatorname{Spec} k[x_1,\ldots,x_n,y_1,\ldots,y_n]/(\sum_i x_iy_i - 1)$; the subscript labels the Krull dimension $2n-1$ of the coordinate ring. Likewise, in $\mathbb{A}^{2n+1}$ (with additional coordinate z), we set $Q_{2n} := \operatorname{Spec} k[x_1,\ldots,x_n,y_1,\ldots,y_n,z]/(\sum_{i=1}^n x_iy_i - z(z+1))$. One obtains an isomorphic variety by replacing $z(1+z)$ by $z(1-z)$. Once again, the subscript $2n$ is the Krull dimension of coordinate ring.

Exercise 1.1.1.33. *Show that if* 2 *is a unit in k, then the ring* $k[x_1,\ldots,x_n,y_1,\ldots,y_n,z]/(\sum_{i=1}^n x_iy_i - z^2 - 1)$ *is isomorphic to* $k[x_1,\ldots,x_n,y_1,\ldots,y_n,z]/(\sum_{i=1}^n x_iy_i - z(z+1))$.

Exercise 1.1.1.34. *Show that if* -1 *is a square in k and* 2 *is invertible in k, then* Q_{2n-1} *is isomorphic to the "usual" sphere defined by* $\sum_{i=1}^{2n-1} w_i^2 - 1$ *and* Q_{2n} *is isomorphic to the variety defined by the equation* $\sum_{i=1}^{2n} w_i^2 - 1$.

Abstract vs. embedded varieties

To connect the above definitions more closely with geometric intuition, fix an affine k-algebra A. Just as in topology, there are an "abstract" and "embedded" point of view on affine k-varieties. By assumption A is finitely generated as a k-algebra, so we can choose a surjection $k[x_1,\ldots,x_n]$. Such a surjection corresponds to a map $\operatorname{Spec} A \to \mathbb{A}^n_k$; this map identifies $\operatorname{Spec} A$ as a closed subset of $\mathbb{A}^n_k$. Now, since $k[x_1, \ldots, x_n]$ is a Noetherian k-algebra, any ideal $I \subset k[x_1, \ldots, x_n]$ is finitely generated. Thus the kernel of $k[x_1, \ldots, x_n] \to A$ is a finitely generated ideal $I \subset k[x_1, \ldots, x_n]$. By picking generators $f_1, \ldots, f_r$ of I, we see that $\operatorname{Spec} A$ can be identified as the closed subset of $\mathbb{A}^n$ defined by the equations $f_1, \ldots, f_r$. Thus, we conclude that every affine k-variety is a closed subset of affine space. Here are some basic questions that one might ask, in parallel with questions from topology.

Question 1.1.1.35. *If A is an affine k-algebra, what is the minimal dimension of an affine space $\mathbb{A}^n_k$ into which* $\operatorname{Spec} A$ *embeds? In other words, what is it the smallest n for which there is a surjection $k[x_1, \ldots, x_n] \to A$?*

If A is an affine k-algebra, then we can look at the group $Aut_k(A)$ of k-algebra automorphisms of A. The group $Aut_k(k[x_1, \ldots, x_n])$ of automorphisms of a polynomial ring is quite large in general. There are two subgroups that are straightforward to write down: the subgroup k^n acting by translations, and the subgroup $GL_n(k)$ acting by $(x_1, \ldots, x_n)^t$ to $M(x_1, \ldots, x_n)^t$ and substitution. For $n = 1$, these two subgroups exhaust the automorphism group: $Aut_k(k[x])$ is a semidirect product of $k^* = GL_1(k)$ and k acting by translations. For $n \geq 2$, the evident semi-direct product is quite far from $Aut_k(k[x_1, \ldots, x_n])$. For example, $Aut_k(k[x_1, \ldots, x_n])$ contains the so-called *tame* subgroup generated, i.e., the subgroup generated by automorphisms of the form

$$x_i \longmapsto x_i + f_i(x_{i+1}, \ldots, x_n).$$

For $n = 2$, it is known that all automorphisms of $k[x_1, x_2]$ are tame, but for $n \geq 3$, this is false.

Question 1.1.1.36. *If A is an affine k-algebra, and we have two surjections $\varphi_1, \varphi_2 : k[x_1, \ldots, x_n] \to A$, then when can we find an element of $\psi \in Aut_k(k[x_1, \ldots, x_n])$ such that $\varphi_1 = \varphi_2 \circ \psi$.*

Remark 1.1.1.37. There are analogs of the (weak) Whitney embedding theorem in the algebro-geometric setting [**?**], which we'll return to later.

General topology of spectra

It is possible to characterize those topological spaces that are homeomorphic to prime ideal spectra of rings, but to do so requires a bit of general topology. We mention this here for the sake of curiosity, but also to explain how far the topological spaces that are spectra of rings are from the "standard" topological spaces one studies in algebraic topology (e.g., Hausdorff). This discussion is not something that we will have use for, but it's interesting in its own right.

Definition 1.1.1.38. A topological space X is called:

1. T_0 if given any two points $x, x' \in X$, there exists an open neighborhood U of x not containing x';
2. *quasi-compact* if every open cover of X admits a finite open subcover;
3. *quasi-separated* if the intersection of two quasi-compact subsets is again quasi-compact;

Exercise 1.1.1.39. *If R is a commutative unital ring, characterize the quasi-compact open subsets of* $\operatorname{Spec} R$ *as finite unions of basic open sets. Conclude that* $\operatorname{Spec} R$ *is both quasi-compact and quasi-separated. Show that* $\operatorname{Spec} R$ *is T_0.*

An irreducible component of a topological space X is a maximal irreducible subset of X. A point $x \in X$ is a generic point if the closure $\bar{x} = X$. We saw above that integral affine k-schemes have unique generic points (corresponding to the zero ideal (0)) and are therefore irreducible. In fact, Hochster characterized topological spaces that can appear as $\operatorname{Spec} R$ for some ring R; for more details, we refer the reader to [**?**, Tag 08YF].

Theorem 1.1.1.40 ([**?**, p. 43])**.** *A topological space X is $\operatorname{Spec} R$ for a commutative ring R if and only if X is quasi-compact, T_0, the quasi-compact open subsets of X form a basis for the open subsets of X, are closed under finite intersections, and every non-empty irreducible component of X has a unique generic point.*

1.1.2 The functor of points

We would like to think of the variety $\operatorname{Spec} A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ as the simultaneous "vanishing locus" of $f_1, \ldots, f_r$, but we have to take care in doing this. Indeed, if we look at the ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$, then the "vanishing locus" of $x^2 + y^2 - 1$ over $\mathbb{R}$ is simply a circle. However, there are other maximal ideals besides those corresponding to points on the graph. Indeed, there are maximal ideals corresponding to complex solutions of the equations.

To explain this more clearly, suppose we are given another k-algebra T (for test). A homomorphism $A \to T$ corresponds, using the description above, to specifying elements $x_1, \ldots, x_n$ in T such that $f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n) = 0$ in T. In other words, a map $\operatorname{Spec} T \to \operatorname{Spec} A$ corresponds to a "solution of the equations defining A with coefficients in T." The variety $\operatorname{Spec} A$ is not "determined" by its vanishing locus over k, but it *is* determined by looking at solutions in all possible ring extensions. Here is a precise statement.

Lemma 1.1.2.1. *The functor $A \mapsto \operatorname{Hom}_{\mathrm{Aff}_k}(A, -)$ from the category of affine k-algebras to the category of set-valued functors on the category of affine k-algebras is fully-faithful and we can identify Aff_k as the full-subcategory consisting of (co-)representable functors.*

Proof. This is a special case of the Yoneda lemma. □

Example 1.1.2.2. Suppose given a morphism $\varphi : A \to B$ of k-algebras and suppose we fix presentations $A = k[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ and $B = k[y_1, \ldots, y_n]/(g_1, \ldots, g_s)$. We claim that a morphism as above is essentially the restriction of a polynomial map. Indeed, the composite morphism $k[x_1, \ldots, x_m] \to B$ corresponds to specifying polynomials $\varphi(x_1), \ldots, \varphi(x_m)$ in B satisfying the equations $f_1, \ldots, f_r$. Moreover, because there is a surjection $k[y_1, \ldots, y_n] \to B$, these elements can all be lifted to $k[y_1, \ldots, y_n]$. A choice of such lifts then determines a homomorphism $k[x_1, \ldots, x_m] \to k[y_1, \ldots, y_n]$, which is precisely a morphism between affine spaces.

Example 1.1.2.3. It is even useful to consider "solutions" in non-reduced rings. E.g., suppose $T = k[\epsilon]/\epsilon^2$. Take $A = k[x, y]/(xy - 1)$. Suppose we would like to construct a homomorphism $k[x, y]/(xy - 1) \to k[\epsilon]/\epsilon^2$. First, we need to specify two elements x and y of $k[\epsilon]/\epsilon^2$; any element can be written as $a + b\epsilon$. So suppose we have two elements $x = a + b\epsilon$ and $y = a' + b'\epsilon$. Now, the equation $xy - 1$ imposes the relation $(a + b\epsilon)(a' + b'\epsilon) - 1 = 0$ in $k[\epsilon]/\epsilon^2$. In other words, $(aa' + (ab' + ba')\epsilon) - 1 = 0$. This means $aa' = 1$ and $ab' + ba' = 0$ or equivalently, $ab' = -ba'$. The first condition, corresponds simply to a solution of $xy = 1$ in k, i.e., a k-point on the graph.

The second condition can be interpreted as picking out the tangent space at (a, a'), i.e., we can think of a $k[\epsilon]/\epsilon^2$-valued point as a k-point together with a tangent vector at that point.

Given two k-algbras A and B, we can form their tensor product $A \otimes_k B$. The k-algebras A and B are k-modules, and as a k-module, the tensor product is the usual tensor product. We give $A \otimes_k B$ a k-algebra structure by defining $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ then extending to $A \otimes_k B$ by linearity. Note that $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n] \cong k[x_1, \ldots, x_m, y_1, \ldots, y_n]$. More generally, given presentations $A = k[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ and $B = k[y_1, \ldots, y_n]/(g_1, \ldots, g_s)$, the tensor product $A \otimes_k B$ can be identified with $k[x_1, \ldots, x_m, y_1, \ldots, y_n]/(f_1, \ldots, f_r, g_1, \ldots, g_s)$.

Remark 1.1.2.4. Note that $A \otimes_k B$ is a coproduct in the category of rings. More precisely, there are maps $A \to A \otimes_k B$ and $B \to A \otimes_k B$ such that if C is any k-algebra equipped with homomorphisms $A \to C$ and $B \to C$, then there exists a unique map $A \otimes_k B \to C$ making the relevant diagrams commute. Since Spec is a contravariant functor, it follows that $\operatorname{Spec} A \otimes_k B$ is a product in the category of affine k-schemes: i.e., it is the product of $\operatorname{Spec} A$ and $\operatorname{Spec} B$ in the category of k-schemes. Note that the topology on $\operatorname{Spec} A \otimes_k B$ is *not* the product topology in general. This can be seen already with $A = k[x]$ and $B = k[y]$! Nevertheless, we will still write $\operatorname{Spec} A \times_{\operatorname{Spec} k} \operatorname{Spec} B$ for the product variety. If it is clear from context, we will drop the subscript $\operatorname{Spec} k$ in the product. Thus, the functor Spec does not preserve products.

Fibers of a map

If $\varphi : A \to B$ is a ring homomorphism, then φ corresponds to a morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Given a point of $\operatorname{Spec} A$, we may therefore consider the fiber of f over that point. There are several things we could mean by this idea. Generally, a T-point of A corresponds to a ring homomorphism $A \to T$. In that case we can form the tensor product $B \otimes_A T$; this comes equipped with a morphism $B \to B \otimes_A T$. A useful case to consider is when T is reduction modulo a maximal ideal $\mathfrak{m} \subset A$. In that case, $\mathfrak{m}B$ is an ideal in B, which no longer needs to be maximal. The scheme-theoretic fiber of f over the closed point corresponding to $\mathfrak{m}$ coincides with the ring $B/\mathfrak{m}B$. Observe that $A/\mathfrak{m}$ is a field κ by assumption, and thus $B/\mathfrak{m}B$ is automatically a κ-algebra.

1.2 Presheaves and sheaves

Just as manifolds have local models that are open subsets of Euclidean space, general schemes are obtained by gluing together affine schemes. The gluing process is mediated by the theory of sheaves. We quickly review some facts about sheaves on topological spaces.

1.2.1 Sheaves on topological spaces

The notion of a sheaf on a topological space is useful for studying locally defined properties. Here is a motivating problem. Suppose X is a topological space, and $\{U_i\}_{i \in I}$ is an open cover of X. Given continuous functions $f_i : U_i \to \mathbb{C}^\times$, can we find a function $f : X \to \mathbb{C}^\times$ whose restriction to U_i coincides with f_i? Some compatibility amongst the f_i is necessary: if U_i and U_j are open sets that intersect, then we can restrict f_i and f_j to $U_i \cap U_j = U_{ij}$ and they must coincide there. On the other hand, if $f_i|_{U_i j} = f_j|_{U_{ij}}$, then we can define a function f on $U_i \cup U_j$ whose values at

$x \in X$ are given by $f_i(x)$ if $x \in U_i$ and $f_j(x)$ if $x \in U_j$. This extended function is continuous and by induction, assuming compatibility we can build a function f. Note that the function f that we have built is necessarily unique. The notion of a sheaf abstracts this gluing procedure, which has appeared repeatedly in previous sections.

Presheaves on a topological space

Definition 1.2.1.1. If X is a topological space, then define a category $\mathrm{Op}(X)$ as follows: objects are open sets of X for the given topology and given two open sets U and V, there is a unique morphism $U \to V$ if $U \subset V$.

Remark 1.2.1.2. Note that $\mathrm{Op}(X)$ has an initial object, corresponding to the empty set, and a final object, given by X itself.

A presheaf on X is a rule that assigns some structure to each open set, together with suitable "restriction" maps connecting the structures associated to different open sets. As before, it will be convenient to think of "algebraic structures" as simply the objects of a category $\mathbf{C}$. In practice, the category will be taken to be Set, Ab, Grp, or something similar; and until further notice, we will assume that $\mathbf{C}$ is a category of algebraic structures in this sense.

Definition 1.2.1.3. Suppose $\mathbf{C}$ is a category and X is a topological space. A $\mathbf{C}$-*valued presheaf* on X is a functor

$$\mathscr{F} : \mathrm{Op}(X)^{\circ} \longrightarrow \mathbf{C}.$$

A *morphism of* $\mathbf{C}$-*valued presheaves* on X is a natural transformation of functors. Write $\mathrm{PShv}(X, \mathbf{C})$ for the category of $\mathbf{C}$-valued presheaves on X.

Remark 1.2.1.4. While having a definition this general affords us considerable flexibility, it does come with some drawbacks. For example, we need to be a bit careful with terminology: if U is an open subset of a topological space X, then $\mathscr{F}(U)$ is just an object of the category $\mathbf{C}$ and need not have any "internal" structure: in particular, it does not make any sense to speak of elements of $\mathscr{F}(U)$. Often we will take $\mathbf{C} = \mathrm{Set}$ or Ab. In either of these cases, elements of $\mathscr{F}(U)$ are themselves sets or abelian groups and it makes sense to talk about their elements (more generally, this makes sense in any "concrete category", i.e., a category equipped with a faithful functor to the category of sets). In that case, elements of $\mathscr{F}(U)$ will be called *sections of* $\mathscr{F}$ *over* U. While it may not be immediately apparent, we will later want to work with categories that are not necessarily concrete, so the flexibility of the definition will become essential. Freyd showed that the homotopy category of pointed topological spaces $\mathscr{H}_*$ cannot be equipped with a faithful functor to Set and is therefore not concrete [**?**], so even "down to earth" categories may fail to be concrete.

Remark 1.2.1.5. Furthermore, note that we have imposed no restriction on the functor $\mathscr{F}$. Hartshorne restricts attention to $\mathbf{C} = \mathrm{Ab}$ and requires that $\mathscr{F}(\emptyset) = 0$ (the final object of Ab). Since a general category $\mathbf{C}$ as above need not have a final object, Hartshorne's definition does not even make sense in this generality.

Example 1.2.1.6. If $\mathbf{C}$ is a category and $A \in \mathbf{C}$ is an object, the constant presheaf on a topological space X is the presheaf assigning to each $U \in \mathrm{Op}(X)$ the object A and to each morphism the identity morphism.

Sheaves of sets on a topological space

We now define sheaves of sets by imposing the condition that sections are "locally" determined. More precisely, suppose U is an open subset of a topological space X and $\{U_i\}_{i\in I}$ is a open cover of U. In this case, if $\mathscr{F}$ is a presheaf on X, there are restriction maps $\mathscr{F}(U) \to \mathscr{F}(U_i)$ and we can take the product of these to obtain a function

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i).$$

If $s \in \mathscr{F}(U)$ is a section, this function sends s to $\{s_i\}_{i\in I}$, where s_i is, intuitively speaking, the restriction of s to U_i. Similarly, there are a pair of restriction maps of the form:

$$\textstyle\prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i_0,i_1\in I\times I} \mathscr{F}(U_{i_0} \times_U U_{i_1}).$$

Now, the locality condition can be phrased in two steps: (i) any section $s \in \mathscr{F}(U)$ is determined by its restriction to U_i, i.e., the first map above is injective, and (ii), given a family of sections $\{s_i\}_{i\in I}$ whose restrictions to two-fold intersections agree, there exists a (necessarily unique) section $s \in \mathscr{F}(U)$ whose restriction to U_i coincides with s_i. These two conditions can be phrased more categorically as follows.

Definition 1.2.1.7. If X is a topological space, $\mathscr{F}$ is a Set-valued presheaf on X, then say $\mathscr{F}$ is a Set-*valued sheaf on* X if for any open set U and any open cover $\{U_i\}_{i\in I}$ of U, the sequence

$$\mathscr{F}(U) \longrightarrow \textstyle\prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i_0,i_1\in I\times I} \mathscr{F}(U_{i_0} \times_U U_{i_1})$$

is an equalizer diagram.

Remark 1.2.1.8. As observed above, the empty set is the initial object of $\mathrm{Op}(X)$. The emtpy set also has a distinguished cover given by the empty cover. The indexing set for the empty cover of the empty set is the empty set as well. The empty product in a category is simply the final object. Therefore, implicit in our definition of a sheaf is the condition that $\mathscr{F}(\emptyset) = *$ (where $*$ is the singleton set).

Remark 1.2.1.9. If $\mathscr{F}$ is a presheaf of abelian groups, then the sheaf condition can be stated in terms that might be more familiar. Indeed, in that case, the injectivity condition is the condition that the map $\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$ has trivial kernel. Likewise, the locality condition can be rephrased as follows. Write $U_{ij} := U_i \times_U U_j$. Given a family of sections $s_i \in \mathscr{F}(U_i)$, the condition that the restriction to 2-fold intersections $\mathscr{F}(U_i \times_U U_j)$ is equivalent to requiring that $s_i|_{U_{ij}} - s_j|_{U_{ij}} = 0$. In other words, the sheaf condition is equivalent to *exactness* of the sequence of abelian groups

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_{i=1}^{n} \mathscr{F}(U_i) \longrightarrow \prod_{i,j} \mathscr{F}(U_{ij}),$$

where the first homomorphism is simply the sum of the restriction maps, and the second homomorphism sends $(s_1,\ldots,s_n)$ to $(\ldots, s_i|_{U_{ij}} - s_j|_{U_{ij}}, \ldots)$. We will frequently use this translation to check the sheaf condition.

Exercise 1.2.1.10. *Show that if X is a topological space and $\mathscr{F}$ is a Set-valued sheaf on X, and U and V are disjoint open subsets of X, then $\mathscr{F}(U \cup V) = \mathscr{F}(U) \times \mathscr{F}(V)$.*

Example 1.2.1.11. The fundamental example of a presheaf (of sets) that is not a sheaf (of sets) is the constant presheaf assigning to $U \in X$ a non-singleton set S. Indeed, $\mathscr{F}(\emptyset) = S$, rather than $*$.

Example 1.2.1.12. Suppose S is a set and view S as a topological space with the discrete topology. If X is a topological space, define the constant sheaf S_X to be the sheaf $\mathrm{Hom}_{\mathrm{Top}}(U, S)$ (i.e., continuous maps from U to S). If U is connected, such functions are constant, but if U is disconnected, then $\mathrm{Hom}(U, S)$ is only constant on connected components. Thus, S_X consists of "locally constant functions." We will refer to S_X as the *constant sheaf* associated with S.

Example 1.2.1.13. One standard class of sheaves arises by considering functions of various sorts. For example if $U \subset \mathbb{R}^n$ is an open subset, and we assign to an open subset $V \subset U$ the ring $C(V, \mathbb{R})$ or $C^\infty(V, \mathbb{R})$ of continuous or smooth real-valued functions $V \to \mathbb{R}$, then this assignment defines a sheaf on U; we will write $\mathscr{C}_U$ or $\mathscr{C}_U^{sm}$ for the resulting sheaf.

Sheaves valued in a general category

Now that we have a reasonable notion of sheaves of sets, there are several ways we can talk about $\mathbf{C}$-valued sheaves where $\mathbf{C}$ is a more general category. A fundamental problem is that if $\mathbf{C}$ is a general category, then the constructions being used to define "restriction" need not even make sense. For example, if $\prod_{i \in I} \mathscr{F}(U_i)$ may not exist, and even if it does, equalizers may not exist in the given category. Rather than necessitating the existence of all such products and equalizers in $\mathbf{C}$, we use the Yoneda embedding to allow us "reduce our problem" to only considering Set-valued sheaves. Indeed, we can identify $\mathbf{C}$ as the full subcategory of Set-valued contravariant functors on $\mathbf{C}$ of the form $\mathrm{Hom}_{\mathbf{C}}(-, Y)$ for Y an object in $\mathbf{C}$.

Exercise 1.2.1.14. *Show that, given an object $A \in \mathbf{C}$, the assignment $\mathscr{F}_A(U) := \mathrm{Hom}_{\mathbf{C}}(A, \mathscr{F}(U))$ defines a presheaf of sets $\mathscr{F}_A$ on X.*

Definition 1.2.1.15. Suppose X is a topological space, $\mathbf{C}$ is a category and $\mathscr{F}$ is a $\mathbf{C}$-valued presheaf on X. We will say that $\mathscr{F}$ is a $\mathbf{C}$-valued sheaf on X if for every object $A \in \mathbf{C}$, $\mathscr{F}_A$ is a Set-valued sheaf on X. A morphism of sheaves is simply a morphism of the underlying presheaves, i.e., a natural transformation of functors. Write $\mathrm{Shv}(X, \mathbf{C})$ for the category of $\mathbf{C}$-valued sheaves on X.

Exercise 1.2.1.16. *Show that if all necessary products and equalizers exist in $\mathbf{C}$, the definition above is equivalent to requiring that the diagram from the definition of a sheaf is an equalizer diagram in $\mathbf{C}$.*

Example 1.2.1.17. If X is any topological space, then X determines a Set-valued presheaf on X, i.e., $\mathrm{Hom}_{\mathrm{Op}(X)}(-, X)$. This presheaf is a sheaf. More generally, if Y is any topological space, then we can consider the presheaf that assigns to $U \subset X$ the set of continuous maps $U \to Y$. You can check that this presheaf is necessarily a sheaf as well. In particular, taking $Y = \mathbb{R}$ or $\mathbb{C}$ equipped with its usual topology, one can speak of the sheaf of real or complex valued continuous functions on X. We write $\mathbb{C}_X$ for this sheaf. If X happens to be a differentiable manifold, then we may also speak of the sheaf of smooth functions on X.

Example 1.2.1.18. If X is a topological space, and $\pi : \mathscr{E} \to X$ is a vector bundle on X, then assigning to $U \subset X$ the set of sections of $\mathscr{E}|_U$ defines a sheaf of modules over the sheaf of continuous functions on X.

Example 1.2.1.19. If X is a topological space, $x \in X$ and S is a set, then the skyscraper sheaf associated with x is defined as follows: $x_*S(U) = S$ if $x \in U$ and $\emptyset$ if $x \notin U$.

Example 1.2.1.20. Assume X is a topological space, and suppose we have a family of sets A_x parameterized by the points $x \in X$. We define a pre-sheaf $\mathscr{A}$ on X by the assignment $U \mapsto \prod_{x \in U} A_x$; the restriction maps attached to $V \subset U$ are given by the projection maps $\prod_{x \in U} A_x \to \prod_{x \in V} A_x$ (project away from the points in $U \setminus V$). Of course, this construction works more generally in any category $\mathbf{C}$ that has arbitrary products (e.g., groups, abelian groups, rings, etc.). It is straightforward to check that the $\mathscr{A}$ is sheaf on X.

1.2.2 Isomorphism, epimorphism and monomorphisms of sheaves

Since morphisms in $\mathrm{PShv}(X, \mathbf{C})$ are simply natural transformations of functors, it follows immediately that monomorphisms, epimorphims and isomorphisms are determined sectionwise. Detecting epimorphisms and isomorphisms of sheaves is more subtle as we now discuss.

Lemma 1.2.2.1. *If $\mathscr{F}_1, \mathscr{F}_2$ are $\mathbf{C}$-valued presheaves on X, then a morphism $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is a monomorphism if and only if the induced maps $\mathscr{F}_1(U) \to \mathscr{F}_2(U)$ are monomorphisms for every $U \in \mathrm{Ob}(X)$.*

Proof. Unwind the definitions. □

Detecting epimorphicity of sheaf maps is more complicated because of the "local" nature of the definition of sheaves. If X is a topological space and $x \in X$ is a point, then a neighborhood of x in X is an open set $x \in U \subset X$. Note that every point $x \in X$ has a neighborhood, namely X itself. If U_1 and U_2 are neighborhoods of U, then $U_1 \cap U_2$ is also a neighborhood of U. It follows that the subcategory of $\mathrm{Op}(X)$ consisting of neighborhoods of x is a partially ordered set, viewed as a category.

Definition 1.2.2.2. If $\mathscr{F}$ is a presheaf on a topological space and $x \in X$ is a point, then the *stalk of $\mathscr{F}$ at x*, denoted $\mathscr{F}_x$ is defined by the colimit

$$\mathscr{F}_x := \mathrm{colim}_{x \in U \subset X} \mathscr{F}(U),$$

assuming this colimit exists in $\mathbf{C}$.

Remark 1.2.2.3. Because the category indexing the colimit is filtered, we can give a very direct definition of the colimit for a presheaf of sets. Namely, $\mathscr{F}_x$ consists of pairs (U, s) where $x \in U$ and $s \in \mathscr{F}(U)$ modulo the equivalence relation given by $(U, s) \sim (U', s')$ if the sections s and s' coincide after a suitable refinement, i.e., there exists an open set $U'' \subset U \cap U'$ such that s and s' coincide upon restriction to $\mathscr{F}(U'')$. The same thing holds for presheaves of (abelian) groups.

Remark 1.2.2.4. The construction of the stalk is functorial in the input presheaf. More precisely, if $f : \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on a topological space, and $x \in X$ is a point, then there is an induced function $f_x : \mathscr{F}_x \to \mathscr{G}_x$ of stalks. If the presheaves have additional structure, i.e., they are presheaves of groups or rings, then f_x respects that structure as well, i.e., f_x will be a group or ring homomorphism.

Example 1.2.2.5. The object $\mathscr{F}_x$ is a generalization of the the notion of a "germ of a function" at a point. More precisely, let $X = \mathbb{R}^n$ and consider the sheaf $\mathscr{F}$ of real valued continuous functions on X. The stalk of $\mathscr{F}$ at x consists precisely of germs of continuous functions at x.

Proposition 1.2.2.6. *If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then φ is an epimorphism (resp. isomorphism) if and only if the induced map on stalks is an epimorphism (resp. isomorphism).*

Proof. By the Yoneda lemma, we reduce attention to set valued presheaves.

In that case, if φ is an epimorphism, then φ is surjective on stalks by unwinding the definitions.

Conversely, suppose $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is an epimorphism for each $x \in X$. Let $\psi_i : \mathscr{G} \to \mathscr{H}$, $i = 1, 2$ be two further morphisms of sheaves and assume $\psi_1 \circ \varphi = \psi_2 \circ \varphi$. We want to show that $\psi_1 = \psi_2$. Since taking stalks is functorial, it follows that

$$(\psi_1)_x \circ \varphi_x = (\psi_1 \circ \varphi)_x = (\psi_2 \circ \varphi)_x = (\psi_2)_x \circ \varphi_x.$$

By assumption, the induced maps on stalks are epimorphisms, and therefore $(\psi_1)_x = (\psi_2)_x$ for every $x \in X$.

Now, suppose $s \in \mathscr{G}(U)$ and consider $(\psi_1)_U(s)$ and $(\psi_2)_U(s)$. At each point $x \in U$, we can find a neighborhood V of x such that $(\psi_1)_U(s)$ and $(\psi_2)_U(s)$ coincide upon restriction to V. Doing this for every point $x \in U$, we obtain a cover of U on which the two sections agree after restriction and therefore, they must agree. Thus $\psi_1 = \psi_2$. $\square$

Exercise 1.2.2.7. *Describe the stalks of a skyscraper sheaf.*

Example 1.2.2.8. A surjective map of sheaves need not be surjective on sections. Here is a rather small example. Take $X = P, Q, R$ with open sets X, $\emptyset$, $\{P, R\}$, $\{Q, R\}$ and $\{R\}$. Consider first the constant sheaf $\mathbb{Z}_X$ on X. Define another sheaf on X by taking the sum of the skyscraper sheaves $P_*\mathbb{Z} \oplus Q_*\mathbb{Z}$. Restriction defines a map $\mathbb{Z}_X \to P_*\mathbb{Z} \oplus Q_*\mathbb{Z}$, but this morphism is not surjective on sections. Indeed, the map on sections sends $\mathbb{Z}$ to the diagonal in $\mathbb{Z} \oplus \mathbb{Z}$, which is evidently not surjective. However, this map is an epimorphism of sheaves.

1.2.3 Sheafification

If $\mathscr{F}$ is a $\mathbf{C}$-valued sheaf on a topological space X, then by simply forgetting that the sheaf condition holds one obtains a forgetful functor

$$\mathrm{Sh}(X, \mathbf{C}) \longrightarrow \mathrm{PSh}(X, \mathbf{C}).$$

This morphism is fully-faithful by definition (sheaves form a subcategory of presheaves). One can then ask: given a presheaf on X, is there a "best-approximation" of X by a sheaf? We can give the notion of "best-approximation" a precise meaning using universal properties, but for example, we would like that if $\mathscr{F}$ is already a sheaf, then the best-approximation is $\mathscr{F}$ itself. In order to motivate the construction of sheafification, we begin with the following example.

Example 1.2.3.1. Suppose $\mathscr{F}$ is a sheaf of sets on X. If $U \in X$ is an open set, then restriction of sections induces a map $\mathscr{F}(U) \to \mathscr{F}_x$ for any $x \in U$. Therefore, there is an induced map

$$\mathscr{F}(U) \longrightarrow \prod_{x \in U} \mathscr{F}_x.$$

The uniqueness statement in the sheaf condition guarantees that this map is injective. Of course, the image of the above map consists of $\{s_x\}_{x \in X}$ for which there exists a section $s \in \mathscr{F}(U)$ such that $s_x = s|_x$.

In conjunction with Example 1.2.1.20 we will use preceding example to build sheafification. If $\mathscr{F}$ is a presheaf on a topological space, then set

$$\Pi(\mathscr{F})(U) := \prod_{x \in X} \mathscr{F}_x;$$

this is a sheaf by the conclusion of Example 1.2.1.20. By construction, restriction of sections defines a map

$$\mathscr{F} \to \Pi(\mathscr{F});$$

thus, we see that $\mathscr{F}$ maps to a sheaf. However, this map is not an isomorphism even if $\mathscr{F}$ is a sheaf since the sections of $\Pi(\mathscr{F})$ correspond to elements in $\mathscr{F}_x$ where the elements at nearby points need *not* be related. Following Example 1.2.3.1, we define a sub pre-sheaf of $\Pi(\mathscr{F})(U)$ as follows. For any open $U \subset X$, define

$$\mathscr{F}^+(U) \subset \Pi(\mathscr{F})(U)$$

to be the subset consisting of sections

$$\{\}\{s_x\}_{x \in U} | \forall u \in U \exists V \subset U \text{ open, and } s \in \mathscr{F}(V) s.t. s_v = s|_v\}.$$

The condition in the statement is compatible with restriction maps, so we conclude that $\mathscr{F}^+$ is actually a sub-presheaf of $\Pi(\mathscr{F})$. Note also that the map $\mathscr{F} \to \Pi(\mathscr{F})$ has image in $\mathscr{F}^+$, so there is a factorization

$$\mathscr{F} \longrightarrow \mathscr{F}^+ \longrightarrow \Pi(\mathscr{F}).$$

Theorem 1.2.3.2 (Sheafification)**.** *If $\mathscr{F}$ is a presheaf of sets on a topological space X, then the pre-sheaf $\mathscr{F}^+$ is a sheaf that we will call the sheaf associated with $\mathscr{F}$; if $\mathscr{F}$ is already a sheaf, then the map $\mathscr{F} \to \mathscr{F}^+$ is an isomorphism. The assignment $\mathscr{F} \to \mathscr{F}^+$ is functorial. If $\mathscr{G}$ is any sheaf, and $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of pre-sheaves, then φ factors uniquely through a morphism of sheaves $\mathscr{F}^+ \to \mathscr{G}$; in other words there is a functorial bijection*

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathscr{F}^+, \mathscr{G}),$$

i.e., the functor of associated sheaf is left adjoint to the forgetful functor.

1.2.4 Building sheaves from a basis

Suppose X is a topological space and $\mathcal{B}$ is a basis of open sets for the topology on X. (Recall this means that we provide a set of open sets of X such that the elements cover X and given any two open sets in the base, the intersection can be covered by elements of the base). Often, it is convenient to specify some construction on the basis and show that it extends to all of X. We will do this now for sheaves on X. We abuse notation and write $\mathcal{B}$ for the full subcategory of $\mathrm{Op}(X)$ spanned by elements of the basis.

Example 1.2.4.1. The example to keep in mind for our later use is the case where $X = \operatorname{Spec} R$ for R a commutative unital ring. In this case, we have a good handle on a basis for the Zariski topology on $\operatorname{Spec} R$ (arising from principal open sets). The subset $\mathcal{B}$ defines a subcategory of $\mathrm{Op}(X)$ consisting of those open sets that are contained in $\mathcal{B}$.

Definition 1.2.4.2. If X is a topological space and $\mathcal{B}$ is a basis for the topology on X, then a $\mathbf{C}$-valued presheaf on $\mathcal{B}$ is a contravariant functor from $\mathcal{B}$ to $\mathbf{C}$.

Remark 1.2.4.3. Every presheaf on X determines a presheaf on $\mathcal{B}$, but there is no reason that a presheaf on $\mathcal{B}$ should determine a presheaf on X. Nevertheless, we will see now that sheaves on X are determined by their restriction to $\mathcal{B}$.

Definition 1.2.4.4. If X is a topological space and $\mathcal{B}$ is a basis for the topology of X, then a a presheaf of sets $\mathscr{F}$ on $\mathcal{B}$ is a sheaf on $\mathcal{B}$ if it satisfies the following additional property: for any $U \in \mathcal{B}$ and any covering $U = \cup_{i \in I} U_i$ with $U_i \in \mathcal{B}$ and any coverings $U_i \cap U_j = \cup_{k \in I_{ij}} U_{ijk}$ with $U_{ijk} \in \mathcal{B}$ the sheaf condition holds, i.e., for any collection of sections $s_i \in \mathscr{F}(U_i), i \in I$ such that for all $i, j \in I$ and for all $k \in I_{ij}$, $s_i|_{U_{ijk}} = s_j|_{U_{ijk}}$, there exists a unique section $s \in \mathscr{F}(U)$ such that $s_i = s|_{U_i}$ for all i.

Remark 1.2.4.5. If $\mathscr{F}$ is a sheaf of sets on X, then $\mathscr{F}$ determines a sheaf on the basis by restriction to $\mathscr{B}$. We now show that, conversely, there exists a unique extension of a sheaf on a basis to a sheaf on X. Given a sheaf on a basis, we begin by explaining how to describe sections of the extended sheaf over an arbitrary open set.

Lemma 1.2.4.6. *Suppose X is a topological space and $\mathcal{B}$ is a basis for the topology of X. Let $\mathscr{F}$ be a sheaf of sets on $\mathcal{B}$. Given $U \in \mathcal{B}$, the map (see* Example 1.2.3.1*)*

$$\mathscr{F}(U) \longrightarrow \prod_{x \in U} \mathscr{F}_x$$

identifies $\mathscr{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property that for any $x \in U$ there exists a $V \in \mathcal{B}$ with $x \in V$ and a section $\sigma \in \mathscr{F}(V)$ such that for all $y \in V$ the equality $s_y = (V, \sigma) \in \mathscr{F}_y$.

Proof. As observed in Example 1.2.3.1 the map $\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$ is injective. To establish surjectivity, take any element $(s_x)_{x \in U}$ on the right hand side satisfying the condition of the statement. We can find an open cover $\{U_i\}_{i \in I}$ of U with $U_i \in \mathcal{B}$ such that $(s_x)_{x \in U_i}$ comes from a section $s_i \in \mathscr{F}(U_i)$. For every $y \in U_i \cap U_j$, the sections s_i and s_j agree in $\mathscr{F}_y$. Therefore, we can find an open set $y \in V_{ijy} \in \mathcal{B}$ such that s_i and s_j restricted to this open set agree. The sheaf condition then guarantees that the sections s_i can be patched to obtain a section of $\mathscr{F}(U)$. $\square$

Using this observation, one may extend sheaves defined on a base of open sets for the topology on X to sheaves on all of X.

Theorem 1.2.4.7. *Suppose X is a topological space and $\mathcal{B}$ is a base for the topology of X.*

1. *If $\mathscr{F}$ is a sheaf of sets on $\mathcal{B}$, then there exists a unique sheaf $\mathscr{F}^{ex}$ on X such that $\mathscr{F}^{ex}(U) = \mathscr{F}(U)$ for all $U \in \mathcal{B}$ compatibly with restriction mappings.*
2. *The assignment $\mathscr{F} \to \mathscr{F}^{ex}$ provides a quasi-inverse to the restriction functor from sheaves on X to sheaves on $\mathcal{B}$, i.e., restriction determines an equivalence between the category of sheaves on X and the category of sheaves on $\mathcal{B}$.*

Proof. For an open subset U of X, define $\mathscr{F}^{ex}(U)$ to be the subset of $\prod_{x \in U} \mathscr{F}_x$ consisting of sections such that for any $x \in U$, there exists a $V \in \mathcal{B}$ containing x and a section $\sigma \in \mathscr{F}(V)$ such that for all $y \in V$, $s_y = (V, \sigma)$ in $\mathscr{F}_y$. Restriction equips this assignment with the structure of a presheaf of sets on X. By Lemma 1.2.4.6, we conclude that $\mathscr{F}^{ex}(U)$ coincides with $\mathscr{F}(U)$ for any $U \in \mathcal{B}$.

To see that $\mathscr{F}^{ex}$ is a sheaf on X is a direct check. Suppose U is an open set and $\{U_i\}_{i \in I}$ is an open cover of U. It is immediate from the definitions that $\mathscr{F}^{ex}(U) \to \prod_{i \in I} \mathscr{F}^{ex}(U_i)$ is injective. Suppose we are given sections $s_i \in \mathscr{F}^{ex}(U_i)$. By definition, each s_i consists of $(s_i)_x, x \in U_i$. If these sections agree upon restriction to $\mathscr{F}^{ex}(U_i \cap U_j)$, we claim they patch together as required. We leave this as an exercise. □

Example 1.2.4.8. Take $X = \mathbb{C}^n$. We know how to speak about holomorphic functions on X. An open disc in $\mathbb{C}$, centered at x is an open subset of the form $D_\epsilon(x)$ consisting of all points of distance at most ϵ from x. A polydisc in $\mathbb{C}^n$ centered a point $x = (x_1, \ldots, x_n)$ is a subset isomorphic to $D_{\epsilon_1}(x_1) \times \cdots D_{\epsilon_n}(x_n)$. Polydiscs provide a basis for the topology on X. Moreover, it makes sense to speak of holomorphic functions on a polydisc. Using the procedure above, one can define a sheaf $\mathscr{O}^{hol}_{\mathbb{C}^n}$ of holomorphic functions on X. More generally, the same procedure works for any complex manifold X to produce a sheaf $\mathscr{O}^{hol}_X$ of holomorphic functions on X.

1.2.5 Basic functoriality

Suppose $f : X \to Y$ is a continuous map of topological spaces. In that case, the definition of continuity implies that f induces a functor $f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$. In particular, if $\mathscr{F}$ is a $\mathbf{C}$-valued pre-sheaf on X, then there is an induced $\mathbf{C}$-valued presheaf $f_*\mathscr{F}$ on X. In other words, there is an induced functor

$$f_* : PSh(X, \mathbf{C}) \longrightarrow PSh(Y, \mathbf{C}).$$

Explicitly, if $U \subset Y$ is a subset, then $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$. The functor f_* is called the *push-forward* functor.

Lemma 1.2.5.1. *If $f : X \to Y$ is a continuous map, and $\mathscr{F}$ is a $\mathbf{C}$-valued sheaf on X, then $f_*\mathscr{F}$ is a $\mathbf{C}$-valued presheaf on Y.*

Example 1.2.5.2. If X is a topological space, $x \in X$ is a point, $i : x \to X$ is the inclusion, and $\mathscr{F}$ is a $\mathbf{C}$-valued presheaf on x, then $i_*\mathscr{F}$ is the skyscraper sheaf we described above. More generally, if $i : Z \hookrightarrow X$ is the inclusion of a closed subset, then i_* allows one to extend sheaves from Z to X.

Henceforth, we will assume that $\mathbf{C}$ is a category of algebraic structures. If X is a topological space, $\mathscr{F}$ is a (pre)sheaf on X, and $U \subset X$ is an open subset, then we may always restrict $\mathscr{F}$ to a sheaf on U, for which we will write $\mathscr{F}|_U$. However, if $Z \subset X$ is a subset that is not necessarily open, then restricting (pre)-sheaves on X to (pre-sheaves) on X is more complicated since open subsets of Z are typically not open subsets of X. Instead, suppose $f : X \to Y$ is a continuous map of topological spaces, and $\mathscr{F}$ is a pre-sheaf on Y. If U is an open set in X, then the collection of open subsets of Y containing $f(U)$ is partially ordered by inclusion and filtered because the intersection of any two open subsets of Y that contain $f(U)$ again contains $f(U)$. We then define the "pullback" presheaf $f^{-1}\mathscr{F}$ on X by defining sections over an open $U \subset X$ via the formula:

$$f^{-}\mathscr{F}(U) := \operatorname{colim}_{V \supset f(U) \subset Y} \mathscr{F}(V).$$

If $\mathscr{F}$ is a sheaf, there is, in general, no reason to expect that $f^{-}\mathscr{F}$ is again a sheaf, we define $f^{-1}\mathscr{F}$ on sheaves by sheafifying the above $f^{-1}\mathscr{F} := f^{-}\mathscr{F}^{+}$.

Assume $f : X \to Y$ is a continuous map as above and suppose $\mathscr{F}$ is a sheaf on Y and $\mathscr{G}$ is a sheaf on X. In that case, suppose we have a morphism of sheaves $\mathscr{F} \to f_*\mathscr{G}$. In that case, for any open U in Y, we have $\mathscr{F}(U) \to f_*\mathscr{G}(U) = \mathscr{G}(f^{-1}U)$. In particular, if we fix an open $W \subset X$ and consider opens U in Y that contain $f(W)$ we get such a map. It follows that there is an induced map

$$\operatorname{colim}_{U \supset f(W)} \mathscr{F}(U) \longrightarrow \mathscr{G}(W),$$

that is functorial in W in the sense that there is an induced morphism of presheaves $f^{-}\mathscr{F} \to \mathscr{G}$. By the universal propety of sheafification, this morphism of presheaves factors uniquely through a morphism of sheaves $f^{-1}\mathscr{F} \to \mathscr{G}$. The assignment just described defines a function

$$\operatorname{Hom}_{\operatorname{Shv}(Y,\mathbf{C})}(\mathscr{F}, f_*\mathscr{G}) \longrightarrow \operatorname{Hom}_{\operatorname{Shv}(X,\mathbf{C})}(f^{-1}\mathscr{F}, \mathscr{G}).$$

In fact, unwinding the definitions, one can construct an explicit inverse function. The following exercise summarizes the properties of the above function.

Exercise 1.2.5.3. *If $f : X \to Y$ is a continuous map, $\mathscr{F}$ is a $\mathbf{C}$-valued sheaf on Y and $\mathscr{G}$ is a $\mathbf{C}$-valued sheaf on X, then the function*

$$\operatorname{Hom}_{\operatorname{Shv}(Y,\mathbf{C})}(\mathscr{F}, f_*\mathscr{G}) \longrightarrow \operatorname{Hom}_{\operatorname{Shv}(X,\mathbf{C})}(f^{-1}\mathscr{F}, \mathscr{G}).$$

is a bijection, functorial in both input sheaves.

There are many natural questions to ask here. Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous maps. In that case we obtain functors f_*, g_* and $(g \circ f)_*$. The composite $g_* \circ f_*$ has the same source and target as $(g \circ f)_*$, and comparing definitions, one sees that if $\mathscr{F}$ is a sheaf on X, then for any open U in Z, the identity map defines a bijection

$$(g_* \circ f_*)\mathscr{F}(U) \to (g \circ f)_*\mathscr{F}(U),$$

i.e., there is an equality of functors $g_* \circ f_* = (g \circ f)_*$. You can analyze pullbacks similarly.

1.3 Schemes in general

Our goal is to specify a formalism for gluing sheaves. There are several ways to do this, each with its own benefits and complications. Previously, given a commutative ring R, we defined a topological space $\operatorname{Spec} R$, and we defined the category of affine schemes as the opposite category of the category of commutative rings. We'd now like to think of R as "functions" on $\operatorname{Spec} R$ and define a sheaf of rings on $\operatorname{Spec} R$ whose global sections are R itself.

1.3.1 The structure sheaf of an affine scheme

Since the basic open set $D_f \subset \operatorname{Spec} R$ corresponds to the ring R_f under the ideal variety correspondence, the sheaf we would like to build should have sections over D_f equal to R_f for consistency with the principle just described. Since the opens D_f form a basis for the topological space $\operatorname{Spec} R$, *any* sheaf we would like to build on $\operatorname{Spec} R$ is uniquely specified by its values on D_f. Thus, to build a sheaf on $\operatorname{Spec} R$, it suffices to show that the assignment $D_f \to R_f$ is a presheaf on the basis D_f and then to check the sheaf condition. We begin by checking the presheaf condition: for this, we need to show that an inclusion of basic open sets $D_g \subset D_f$ corresponds to a ring homomorphism $R_f \to R_g$. The following lemma establishes this in slightly greater generality.

Lemma 1.3.1.1. *Suppose R is a commutative ring, and $f \in R$.*

1. *If $g \in R$ is such that $D_g \subset D_f$, then f is invertible in R_g, there exists an integer $e \geq 1$ and $a \in R$ such that $g^e = af$, there is a unique ring map $R_f \to R_g$ inducing the inclusion $D_g \subset D_f$, and for any R-module M, there is an induced morphism $M_f \to M_g$.*
2. *Any open covering of D_f can be refined to a finite open covering of the form $D_f = \cup_i D_{g_i}$.*
3. *If $g_1, \ldots, g_n \in R$ then $D_f \subset \cup_i D_{g_i}$ if and only if $g_1, \ldots, g_n$ generate the unit ideal in R_f.*

Proof. Suppose $g \in R$ and $D_g \subset D_f$. We know that $D_g = \operatorname{Spec} R_g$. Now, $\operatorname{Spec} R_g$ corresponds to those prime ideals that do not contain g, so it follows that f is not contained in any prime ideal containing g; this means that f is invertible in R_g. In that case, we may write the inverse of f in R_g as $\frac{a}{g^d}$ for some integer $d \geq 1$. Then, $g^d - af$ is annihilated by some power of g, and we may write $g^e = af$ as claimed. The morphism $R_f \to R_g$ is that arising by the universal property of localization; explicitly, an element of the form $\frac{b}{f^n}$ is sent to $\frac{a^n b}{g^{ne}}$. Likewise, if M is an R-module, then $M_f \to M_g$ is induced by the same formula.

The second assertion follows from the fact that D_f is quasi-compact and the basic open sets form a basis for the Zariski topology. The final statement is an exercise. □

Corollary 1.3.1.2. *If R is a commutative ring and M is an R-module, then the assignment $M \mapsto M_f$ determines a presheaf $\tilde{M}$ on the basis D_f of $\operatorname{Spec} R$. If $x \in \operatorname{Spec} R$ corresponds to the prime ideal $\mathfrak{p}$, then the stalk $\tilde{M}_x$ coincides with the localization $M_{\mathfrak{p}}$.*

Proof. The first statement is a consequence of Lemma 1.3.1.1. For the second statement, it suffices to observe that the stalk can be computed by restricting attention to basic open sets containing a given point. Now, if f_1 and f_2 are basic open sets, then $D_{f_1 f_2} = D_{f_1} \cup D_{f_2}$. Now, let us order the collection of elements $f \in R, f \notin \mathfrak{p}$ as follows: we will say that $f \geq g$ if $D_f \subset D_g$. With respect to this ordering, we see that

$$\tilde{M}_x := \operatorname{colim}_{f \in R | f \notin \mathfrak{p}} R_f$$

Note also that $f_1 f_2 \geq f_1$ with respect to this ordering. Now, one just has to observe that if S is a multiplicative set in a ring R, then the localization $M[S^{-1}]$ can be realized as the colimit $\operatorname{colim}_{f \in S} M_f$ where the ordering on $f \in S$ is that $f \geq f'$ if $f = f'f''$ for some $f'' \in R$. $\square$

Now, we check that $\tilde{M}$ is actually a sheaf on the basis of open sets of $\operatorname{Spec} R$. By what we learned above, suppose we have an open cover of D_f by $\cup_i D_{g_i}$ for some $g_1, \ldots, g_n$. In that case, the intersections are the open sets of the form $D(g_i g_j)$, and we thus want to check exactness of the sequence

$$M_f \longrightarrow \bigoplus_{i=1}^{n} M_{g_i} \longrightarrow \bigoplus_{i,j=1}^{n} M_{g_i g_j}.$$

Since $D_f = \cup_i D_{g_i}$, we saw above that $g_1, \ldots, g_n$ generate the unit ideal in R_f. Moreover, without loss of generality, we may replace g_i by $f g_i$ and $g_i g_j$ by $f g_i g_j$. Thus, to establish $\tilde{M}$ is a sheaf, we need the following result.

Lemma 1.3.1.3. *Assume R is a commutative ring, and $g_1, \ldots, g_n$ are elements in R that generate the unit ideal and M is an R-module. The sequence*

$$0 \longrightarrow M \longrightarrow \bigoplus_{i=1}^{n} M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

where the first map sends $m \in M$ to $(m, \ldots, m)$ and the second map sends $(\frac{m_1}{g^{a_1}}, \ldots, \frac{m_n}{g^{a_n}})$to the difference of the restrictions.

Proof. It suffices to show that the localization of the sequence at any maximal ideal $\mathfrak{m} \subset R$ is exact (see Appendix). Since $g_1, \ldots, g_n$ generate the unit ideal in R, there is an integer i such that $g_i \notin \mathfrak{m}$. Renumbering the g_i if necessary, we may assume $i = 1$. Since localizations commute, we see that $(M_{g_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{g_i}$ and likewise that $(M_{g_i g_j})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{g_i g_j}$. In particular, $(M_{g_1})_{\mathfrak{m}} = M_{\mathfrak{m}}$ and $(M_{g_1 g_i})_{\mathfrak{m}} = (M_{\mathfrak{m}})_{g_i}$, because g_1 is a unit. Note that the maps in the sequence are the canonical ones coming from Lemma 10.9.7 and the identity map on M. Having said all of this, after replacing R by $R_{\mathfrak{m}}$, M by $M_{\mathfrak{m}}$, and g_i by their image in $R_{\mathfrak{m}}$, and g_1 by $1 \in R_{\mathfrak{m}}$, we reduce to the case where $g_1 = 1$.

Assume $g_1 = 1$. Injectivity of the first map is now immediate. Let $m = (m_1, \ldots, m_n)$ lie in the kernel of the second map. Then $m_1 \in M_{g_1} = M$. The assertion that m is sent to zero by the second map implies that $m_1 = m_i$ for $i = 1, \ldots, n$. In that case, the image of m_1 under the first map is m and we're done. $\square$

If R is a commutative ring, then we will write $\mathscr{O}_{\operatorname{Spec} R}$ for the sheaf on $\operatorname{Spec} R$ whose existence is established by the preceding results. We now summarize what we know about this sheaf of rings.

Theorem 1.3.1.4. *Let R be a ring. Let M be an R-module. Let $\tilde{M}$ be the sheaf of $O_{\operatorname{Spec} R}$-modules attached to M. The following statements hold:*

1. *$\Gamma(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R} = R$; and*
2. *$\Gamma(\operatorname{Spec} R, \tilde{M}) = M$ as an R-module.*
3. *For every $f \in R$, $\Gamma(D_f, \mathscr{O}_{\operatorname{Spec} R}) = R_f$.*
4. *For every $f \in R$, $\Gamma(D_f, \tilde{M}) = M_f$ as an R_f-module.*

5. *Whenever $D_g \subset D_f$ the restriction mappings on $\mathscr{O}_{\operatorname{Spec} R}$ and $\tilde{M}$ are the maps $R_f \to R_g$ and $M_f \to M_g$ above.*
6. *If $\mathfrak{p} \subset R$ is a prime ideal and $x \in \operatorname{Spec} R$ is the corresponding point, then $\mathscr{O}_{\operatorname{Spec} R, x} = R_{\mathfrak{p}}$.*
7. *If $\mathfrak{p} \subset R$ is a prime ideal and $x \in \operatorname{Spec} R$ is the corresponding point, then $\tilde{M}_x = M_{\mathfrak{p}}$.*

Moreover, all these identifications are functorial in the R module M. In particular, the assignment $M \to \tilde{M}$ is an exact functor from the category of R-modules to the category of $\mathscr{O}_{\operatorname{Spec} R}$-modules.

1.3.2 Ringed and locally ringed spaces

We now want to build more general schemes by gluing together affine schemes. Here is a simple example that shows that in performing gluing constructions we cannot expect to stay within the category of affine schemes. In topology, one defines S^2 as glued from two copies of $\mathbb{C}$ over the intersection, $\mathbb{C}^{\times}$. The maps defining the gluing are algebraic functions: if z is a coordinate on the first copy of $\mathbb{C}$ and z^{-1} is a coordinate on the other, then the gluing map is defined on the intersection by $z \mapsto z^{-1}$. We can try to perform this construction in the category of rings, but to do so we reverse all the arrows. Namely, we want to obtain the coproduct of rings $k[z]$ and $k[z^{-1}]$ over $k[z, z^{-1}]$. However, the collection of functions (f_1, f_2) such that $f_1(z) = f_2(z^{-1})$ consists only of elements of k. Thus, the "gluing" in the category of affine schemes is $\operatorname{Spec} k$, which is evidently not what we have in mind when we think of $\mathbb{P}^1$. The "problem" is that we are only thinking about functions that are globally defined and in complex analysis one learns that a polynomial function on the Riemann sphere is constant. Thus, we must expand our view beyond the world of rings to obtain a reasonable notion of quotient.

Definition 1.3.2.1. A ringed space is a pair $(X, \mathscr{O}_X)$ consisting of a topological space and a sheaf of commutative rings on X.

Example 1.3.2.2. The examples to keep in mind are: M a topological manifold and $\mathscr{C}_M$ the sheaf of (say, real-valued) continuous functions on M, M a smooth manifold and $\mathscr{C}^{\infty}_M$ the sheaf of (say, real-valued) smooth functions on M.

A map of manifolds induces a corresponding pullback of functions; this corresponds to a suitable map of sheaves, albeit on different topological spaces. We now introduce a notion to compare sheaves on different topological spaces. The following notion is a formalization of what happens to functions under pullback along a morphism of smooth or topological manifolds.

Definition 1.3.2.3. Suppose $f : X \to Y$ is a continuous map of topological spaces. If $\mathscr{F}$ is a sheaf on X and $\mathscr{G}$ is a sheaf on $\mathscr{G}$, then an f-map $\xi : \mathscr{G} \to \mathscr{F}$ is a collection of maps $\xi_V : \mathscr{G}(V) \to \mathscr{F}(f^{-1}(V))$ indexed by open sets $V \subset Y$ that commutes with restriction in a suitable sense.

Definition 1.3.2.4. If $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces, a morphism of ringed spaces is a continuous map $f : X \to Y$ and an f-map of sheaves of rings $\mathscr{O}_Y \to \mathscr{O}_X$.

In all the geometric situations we consider (e.g., topological and smooth manifolds, schemes), the sheaves of function rings on our topological spaces have stalks that are *local rings*. E.g., the stalk of $\mathscr{C}_M$ at a point $x \in M$ is the ring of germs of continuous functions at x; this ring is a local ring with maximal ideal those continuous functions vanishing at x. Moreover, a map of smooth manifolds sends points to points and therefore induces corresponding maps of stalks (by functoriality of stalks);

the resulting maps of stalks are local homomorphisms of local rings. Generalizing this observation, one makes the following definition.

Definition 1.3.2.5. A ringed space $(X, \mathscr{O}_X)$ is *locally ringed* if for each point $x \in X$, the stalks $\mathscr{O}_{X,x}$ are local rings. A morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that, for any $x \in X$, the map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is a local homomorphism of local rings.

Example 1.3.2.6. If R is a commutative unital ring, then $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a locally ringed space. Indeed, if $x \in \operatorname{Spec} R$ is a point corresponding to a prime ideal $\mathfrak{p}$, we saw that $\mathscr{O}_{\operatorname{Spec} R,x} = R_{\mathfrak{p}}$, which is a local ring.

1.3.3 Schemes

Earlier, we defined the category of affine schemes over a base ring k to be the opposite of the category of commutative k-algebras. Above, we showed how to associate a locally ringed space with any commutative unital k-algebra. We now show that this assignment identifies the category of affine k-schemes with its image.

Proposition 1.3.3.1. *Sending a commutative unital k-algebra R to the locally ringed space* $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ *extends to a fully-faithful functor from the category of affine schemes to the category of locally ringed spaces.*

Proof. See [**?**, Lemma 25.6.4]. □

Definition 1.3.3.2. We write Aff for the full subcategory of locally ringed spaces spanned by affine schemes. If k is a commutative unital ring, we write Aff_k for comma category of Aff consisting of affine schemes equipped with a morphism to $(\operatorname{Spec} k, \mathscr{O}_{\operatorname{Spec} k})$ (in particular, $\mathrm{Aff} = \mathrm{Aff}_{\mathbb{Z}}$).

Given our identification of affine schemes above, we may now give the general definition of a scheme: a scheme is a locally ringed space obtained by gluing together ringed spaces of the form $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ for a commutative unital ring R. We formalize this in two steps.

Definition 1.3.3.3. A scheme is a locally ringed space $(X, \mathscr{O}_X)$ that is locally isomorphic to an affine scheme, i.e., given any point $x \in X$ there is an open neighborhood U of $x \in X$ such that $(U, \mathscr{O}_X|_U)$ is an affine scheme. We write Sch for the full subcategory of the category of locally ringed spaces consisting of schemes (i.e., a morphism of schemes is morphism of locally ringed spaces). If S is a base-scheme, we write Sch_S for the full subcategory of Sch consisting of schemes admitting a morphism to S.

Example 1.3.3.4. If $(X, \mathscr{O}_X)$ is a scheme, and $U \subset X$ is an open subset of the topological space X, then U carries the structure of scheme by defining $\mathscr{O}_U = \mathscr{O}_X|_U$. We refer to this as the induced open subscheme structure on U. A morphism of schemes $f : X \to Y$ is called an *open immersion* if it induces an isomorphism of X with an open subscheme of Y.

Example 1.3.3.5 (Reduced schemes). We defined affine k-varieties by restricting attention to reduced, finite-type k-algebras. We can globalize the notion of reducedness in the following way: a scheme $(X, \mathscr{O}_X)$ is reduced if $\mathscr{O}_{X,x}$ is reduced for each $x \in X$. One checks immediately that

$(X, \mathscr{O}_X)$ is reduced if and only if for each open $U \subset X$, the ring $\mathscr{O}_X(U)$ is reduced. Indeed, the map $\mathscr{O}_X(U) \to \prod_{x \in U} \mathscr{O}_{X,x}$ is injective since $\mathscr{O}_X$ is a sheaf. Thus, if $f \in \mathscr{O}_X(U)$ is an element such that $f^n = 0$. In that case, the image of f in $\mathscr{O}_{X,x}$ is zero for every $x \in X$ by the assumption that X is reduced, and thus f must be zero to begin with. Conversely, if $\mathscr{O}_X(U)$ is reduced for all $U \subset X$, then any non-zero element $f \in \mathscr{O}_{X,x}$ can be represented by a section over some open neighborhood of x. Since that element is non-zero, it is necessarily not nilpotent either. From this one checks that an affine scheme $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is reduced if and only if R is a reduced k-algebra.

Just as in differential geometry, we may construct schemes by gluing.

Example 1.3.3.6. Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are schemes and $U \subset X$ and $V \subset Y$ are open subsets. The subset U inherits the structure of a locally ringed space by setting $\mathscr{O}_U = \mathscr{O}_X|_U$. Suppose we are given an isomorphism of locally ringed spaces $\varphi : (U, \mathscr{O}_U) \xrightarrow{\sim} (V, \mathscr{O}_V)$. In that case, we define a new scheme with underlying topological space $W := X \coprod Y/(x \sim \varphi(x))$ (equipped with the quotient topology). Note that there are continuous maps $i_X : X \to W$ and $i_Y : Y \to W$ by the definition of W. Using these maps, we can define a structure sheaf $\mathscr{O}_W$ by gluing: for $Z \subset W$, define $\mathscr{O}_W(Z)$ to consist of pairs (s_1, s_2) such that $s_1 \in \mathscr{O}_X(i_X^{-1}(Z))$ and $s_2 \in \mathscr{O}_Y(i_Y^{-1}(Z))$ such that $\varphi(s_1|_{i_X^{-1}(Z) \cap U}) = s_2|_{i_Y^{-1}(Z) \cap V}$. The pair $(W, \mathscr{O}_W)$ is still a scheme because every point in W has a neighborhood isomorphic to an affine scheme.

Example 1.3.3.7. In an analogous fashion, we may glue morphisms of schemes.

Example 1.3.3.8. The category Sch has a terminal object, namely $(\operatorname{Spec} \mathbb{Z}, \mathscr{O}_{\operatorname{Spec} \mathbb{Z}})$. Indeed, this is clear if X is an affine scheme and we may glue morphisms to obtain the morphism for a general scheme from this one.

If X and Y are topological spaces, then we can equip $X \times Y$ with the structure of a topological space making it a product in the category of topological spaces. We would like to have a similar construction in the category of schemes, but this requires more work. Since the the category of affine schemes is the opposite of the category of commutative rings, the universal property of the product is dualized in the category of rings. Recall the universal property of a (fibered) product in a category: if X and Y are objects, equipped with morphisms $\varphi : X \to Z$ and $\psi : Y \to Z$, then a fibered product consists of an object $X \times_Z Y$ in $\mathbf{C}$ together with morphisms $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ such that given any other object T and morphisms $T \to X$ and $T \to Y$ whose composites to Z via φ and ψ agree, there exists a unique morphism $T \to X \times_Z Y$ whose composites with the projections p_X and p_Y agree with φ and ψ respectively.

Using the fact that the category of affine schemes is the opposite of the category of commutative rings, if we restrict attention to affine schemes, then the above universal property of a fibered product becomes the universal property of a coproduct. In particular, if A and B are C-algebras, then the pushout $A \otimes_C B$ exists in the category of rings. Thus, we claim that $\operatorname{Spec} A \otimes_C B$ realizes the fibered product $\operatorname{Spec} A \otimes_{\operatorname{Spec} C} \operatorname{Spec} B$. Using gluing, one then can build fibered products of arbitrary schemes as sketched in the following exercise.

Exercise 1.3.3.9. *Show that the category of schemes has finite products.*

1. *Show that the tensor product of rings equips the the category of affine schemes with a product.*
2. *Assuming X and Y are schemes, show that $X \times Y$ can be equipped with a natural scheme structure by gluing (first, assume Y is affine, and inductively use the gluing construction*

for a suitable open cover of X by affine schemes, then use gluing again to obtain a scheme structure on $X \times Y$).

3. *Show that the scheme structure you obtained on $X \times Y$ in the previous part makes it a product.*

Example 1.3.3.10 (Punctured affine space). Suppose n is an integer ≥ 1. We define a scheme $\mathbb{A}^n \setminus 0$ inductively as follows. For $n = 1$, we set $\mathbb{A}^1 \setminus 0$ to be $\operatorname{Spec} \mathbb{Z}[t, t^{-1}], \mathscr{O}_{\operatorname{Spec} \mathbb{Z}[t,t^{-1}]}$. For $n = 2$, we glue the affine schemes $\mathbb{A}^1 \times \mathbb{A}^1 \setminus 0$ and $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$ along $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \setminus 0$ via the identity map. More generally we may define $\mathbb{A}^2 \setminus 0 \times \mathbb{A}^m$ for any $m \geq 0$ by gluing $\mathbb{A}^1 \times \mathbb{A}^1 \setminus 0 \times \mathbb{A}^m$ and $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \times \mathbb{A}^m$ along $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1 \setminus 0$. Inductively we define $\mathbb{A}^n \setminus 0$ by gluing $\mathbb{A}^{n-1} \setminus 0 \times \mathbb{A}^1$ and $\mathbb{A}^{n-1} \times \mathbb{A}^1 \setminus 0$ over $\mathbb{A}^{n-1} \setminus 0 \times \mathbb{A}^1 \setminus 0$.

Example 1.3.3.11 (The diagonal morphism). One important example of fiber products and gluing morphisms can be realized as follows. Suppose $f : X \to Y$ is a morphism of schemes. In that case, we may form the fiber product $X \times_Y X$. The universal property of the fibered product applied to the identity map $X \to X$ then yields a unique morphism $\Delta_{X/Y} : X \to X \times_Y X$ that we will call the relative diagonal. By the exercise above, the general case is reduced to the case where X and Y are affine by means of gluing, so let's assume that f is obtained from a ring homomorphism $B \to A$. In that case, the fibered product is obtained by taking the spectrum of $A \otimes_B A$. The universal property of the coproduct applied to the identity map $A \to A$ thus corresponds to a ring homomorphism $A \otimes_B A \to A$. Since the restriction of this map to each factor is the identity, one checks that the product map sending $a_1 \otimes a_2 \to a_1 a_2$ is the required ring homomorphism. Thus, for affine schemes, the diagonal morphism corresponds at the level of rings to the product homomorphism.

1.3.4 Properties of morphisms

In general, we are not interested in arbitrary topological spaces, but certain classes thereof. For the purposes of this course, we will be interested in schemes that are closely related to manifolds. As such, we will begin by introducing various "finiteness" properties of schemes. Recall that we established above that affine schemes were quasi-compact and quasi-separated. Let us begin by rephrasing those definitions more generally. Quasi-compactness is a statement about the underlying topological space of a scheme, so the definition is perhaps easiest to understand, however we will make a relative version of this statement.

Definition 1.3.4.1. A scheme X is *quasi-compact* if the underlying topological space of X is quasi-compact, i.e., every open cover has a finite subcover. A morphism $f : X \to S$ of schemes is quasi-compact if the pre-image of every quasi-compact open in S is quasi-compact in X.

Quasi-separatedness was the condition that intersections of any pair of quasi-compact opens was again a quasi-compact open. We can rephrase this condition in terms of quasi-compactness of the diagonal, and as with quasi-compactness we will also make a relative definition.

Definition 1.3.4.2. If $f : X \to S$ is a morphism of schemes, then we will say that f is quasi-separated if the diagonal morphism $\Delta_{X/S}$ is quasi-compact.

Exercise 1.3.4.3. *Check that if X is an affine scheme, then the structure morphism $f : X \to \operatorname{Spec} Z$ is quasi-separated in this sense.*

Now, a manifold is typically a paracompact Hausdorff topological space that is locally Euclidean. So far, our schemes are simply spaces that are "locally affine" and we have introduced neither finiteness nor separation properties. As a first step towards isolating a "nice" class of schemes we begin by introducing various finiteness properties. The first two generalize the natural finiteness properties we analyzed in the context of affine schemes.

Definition 1.3.4.4. A morphism of schemes $f : X \to S$ has finite type at $x \in X$, if there exists an affine open neighborhood $\operatorname{Spec} A$ of x and an affine open neighborhood $\operatorname{Spec} R$ of $f(x)$ such that f maps A a finite type R-algebra. A morphism of schemes has locally finite type if it has finite type at every point $x \in X$ and has finite type if f has locally finite type and f is quasi-compact.

Definition 1.3.4.5. A morphism of schemes $f : X \to S$ is finitely presented at $x \in X$, if there exists an affine open neighborhood $\operatorname{Spec} A$ of x and an affine open neighborhood $\operatorname{Spec} R$ of $f(x)$ such that f maps $\operatorname{Spec} A$ to $\operatorname{Spec} R$ and makes A into a finitely presented R-algebra. A morphism of schemes is locally of finite presentation, if it is finitely presented at $x \in X$ for every point $x \in X$. A morphism of schemes is finitely presented if it is locally of finite presentation, quasi-compact *and* quasi-separated.

1.4 Constructions of schemes

1.4.1 Projective space

Example 1.4.1.1 (The projective line). We can define projective space by gluing as well by mimicking the construction in topology: to obtain $\mathbb{P}^1$ simply glue two copies of the affine line over $\mathbf{G}_m$ by means of the isomorphism $z \mapsto z^{-1}$. More precisely, consider the affine scheme associated with $\mathbb{Z}[t]$ and with $\mathbb{Z}[t^{-1}]$. Consider the isomorphism of affine schemes $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]$ given by $t \mapsto t^{-1}$. To say this a bit more systematically, with the goal of generalizing the construction to higher dimensional projective spaces, let us think instead of presenting $\mathbb{P}^1$ in terms of lines through the origin in a 2-dimensional space. If we choose coordinates x_0, x_1 in 2-dimensions, then a line is specified by a vector up to rescaling. In that case, $x_0 \neq 0$ (i.e., the line is not "vertical"), the line is uniquely determined by its slope $\frac{x_1}{x_0}$. Likewise, if $x_1 \neq 0$ (i.e., the line is not "horizontal"), the line is uniquely determined by $\frac{x_0}{x_1}$. We can think of $\frac{x_1}{x_0}$ as a coordinate on one copy of $\mathbb{A}^1_{\mathbb{Z}}$ and $\frac{x_0}{x_1}$ as a coordinate on another copy of $\mathbb{A}^1_{\mathbb{Z}}$. On the intersection where both x_0 and x_1 are non-zero, the two descriptions of the slope are related by the formula $(\frac{x_1}{x_0})^{-1} = \frac{x_0}{x_1}$. Thus, if we write $x = \frac{x_1}{x_0}$ and x^{-1} for $\frac{x_0}{x_1}$, we recover the gluing description above.

Example 1.4.1.2 (Projective spaces by gluing). More generally, we can define $\mathbb{P}^n$ inductively by gluing $n + 1$ copies of $\mathbb{A}^n$. We can keep track of the gluing as we did for $\mathbb{P}^1$. Consider coordinates $x_0, \ldots, x_n$ on an $(n + 1)$-dimensional affine space. As above, a line is determined up to scaling. If $x_i \neq 0$, then we consider the affine space with coordinates $\mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \ldots, \frac{x_n}{x_i}$. If x_i and x_j are simultaneously non-zero, then we can write down gluing maps in a fashion analogous to those above and inductively realize $\mathbb{P}^n_{\mathbb{Z}}$ by gluing.

Example 1.4.1.3. A slight modification of the construction of the projective line produces an example that strays from our geometric intuition: glue two copies of the affine line over the identity map $\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1 \setminus 0$. The result is a scheme that one usually draws as an affine line with "doubled" origin. From one point of view, this kind of example is pathological (and in gluing manifolds, one usually eliminates this kind of example!), but from another point of view it gives us flexibility in the constructions we can make.

Another useful way to think about the construction above is as follows. Recall that the classical way to construct projective space over a field k is via lines through the origin in an $n+1$-dimensional k-vector space V. If we fix a basis of the line, then we simply get a non-zero vector in V. Two non-zero vectors determine the same line if one can be obtained from the other up to scaling. Thus, alternatively, we can think in terms of elements of $V \setminus 0$ invariant under scaling: the scaling action determines an action on functions via pullback. If x_i is a coordinate function on V, $i = 0, \ldots, n$, then $x_i(v)$ is the i-th coordinate of the vector v in terms of the standard basis $e_0, \ldots, e_n$. Thus, $x_i(\lambda v) = \lambda x_i(v)$. This action induces a grading on $k[x_0, \ldots, x_n]$.

If f is a homogeneous degree d polynomial, then f determines a graded ideal in the graded ring $k[x_0, \ldots, x_n]$. By homogeneity, the vanishing locus of f determines a scaling invariant subset of V and therefore passes to a corresponding subset of projective space. More generally, if we have an ideal defined by homogeneous polynomials, then its vanishing locus is again a subset of V that is invariant under scaling and therefore passes to a subset of projective space.

1.4.2 The Proj construction

We now formalize the discussion just made by attaching a scheme to any graded ring. Following the analysis of Spec, we first attach a topological space to a graded ring, and then define an associated structure sheaf. For the most part, when we write "graded ring" we will mean positively graded (i.e., graded by the natural numbers). At a few points, we will need to consider $\mathbb{Z}$-graded rings, in which case we will make that clear by explicitly saying $\mathbb{Z}$-graded ring. We fix the following notation: if S is a graded ring, we write S_+ for the subset of positively graded elements.

Definition 1.4.2.1. If S is a graded ring, define $\mathrm{Proj}\, S$ to be the set of homogeneous prime ideals $\mathfrak{p}$ of S such that $S_+ \not\subset \mathfrak{p}$. We view $\mathrm{Proj}\, S \subset \mathrm{Spec}\, S$ and equip it with the structure of a topological space via the induced topology.

Remark 1.4.2.2. The assignment $S \to \mathrm{Proj}\, S$ is not as "well-behaved" as the assignment $S \to \mathrm{Spec}\, S$ in a number of ways. First, Proj does not yield a functor from graded rings. Indeed, if we think classically, and consider a vector space map $V \to W$, then there is no induced map $\mathbb{P}(V) \to \mathbb{P}(W)$ in general. Indeed if $\varphi : V \to W$ is a surjective map, then any line L contained in the kernel of φ is sent to $0 \subset W$, which does not correspond to a point in $\mathbb{P}(W)$. In ring-theoretic terms, given a graded ring map $\varphi : A \to B$, the inverse image $\varphi^{-1}(\mathfrak{q})$ of a homogeneous prime $\mathfrak{q} \subset B$ may still contain A_+. On the other hand, if $V \to W$ is an injective ring map, then there is an induced ring map $\mathbb{P}(V) \to \mathbb{P}(W)$. See Remark 1.4.2.9 below for more details.

In another direction, while $\mathrm{Spec}\, R$ is always a quasi-compact topological space, there are graded rings S for which $\mathrm{Proj}\, S$ is *not* a quasi-compact topological space.

If S is concentrated in degree 0, then $\mathrm{Proj}\, S$ coincides with $\mathrm{Spec}\, S$. If S is a graded ring, write S_0 for the subring of elements of degree 0. In this case, the inclusion map induces a continuous map $\mathrm{Proj}\, S \to \mathrm{Spec}\, S_0$. In some instances, this map is not very interesting (e.g., if $S = \mathbb{Z}[x_0, \ldots, x_n]$ as above).

However, in projective space as described above, if we look at the complement of the vanishing locus of a homogeneous polynomial of positive degree, then we obtain a set that has many functions. To make this precise, let $S = \bigoplus_{d \geq 0} S_d$ be a positively graded ring. If $f \in S_d$ is a homogeneous degree d element, then set $S_{(f)}$ to be the subring of the localization S_f consisting of elements of the form $\frac{r}{f^n}$ with r homogeneous and where the degree of r is nd. Likewise, if M is a graded module, we define an $S_{(f)}$-module $M_{(f)}$ as the submodule of M_f consisting of elements of the form $\frac{x}{f^n}$ with x homogeneous of degree nd.

Example 1.4.2.3. Consider the complement of the vanishing locus of x_i; this corresponds to looking at $\mathbb{Z}[x_0, \ldots, x_n]_{(x_i)}$ as just described. The elements $\frac{x_j}{x_i}$, $j \neq i$ are degree 0. Geometrically, the complement of $x_i = 0$ is an affine space with precisely the coordinates described via projection.

The following result generalizes this observation to the situation where we invert a homogeneous element f of positive degree.

Lemma 1.4.2.4. *If S is a $\mathbb{Z}$-graded ring containing a homogeneous invertible element of positive degree, then the set $G \subset \operatorname{Spec} S$ of $\mathbb{Z}$-graded primes of S (with the induced topology) maps homeomorphically to* $\operatorname{Spec} S_0$.

Proof. We show that the map is a bijection by constructing an inverse: given a prime $\mathfrak{p}_0$ of S_0, we want to associate with it a $\mathbb{Z}$-graded prime of S. By assumption, we can find an invertible $f \in S_d$, $d > 0$. If $\mathfrak{p}_0$ is a prime of S_0, then $\mathfrak{p}_0 S$ is a $\mathbb{Z}$-graded ideal of S such that $\mathfrak{p}_0 S \cap S_0 = \mathfrak{p}_0$, If $ab \in \mathfrak{p}_0 S$ with a, b homogeneous, then $\frac{a^d b^d}{f^{deg(a)deg(b)}} \in \mathfrak{p}_0$. Therefore, either $\frac{a^d}{f^{dega}} \in \mathfrak{p}_0$ or $\frac{b^d}{f^{degb}} \in \mathfrak{p}_0$, i.e., either $a^d \in \mathfrak{p}_0 S$ or $b^d \in \mathfrak{p}_0 S$. Therefore, $\sqrt{\mathfrak{p}_0 S}$ is a $\mathbb{Z}$-graded prime ideal of S whose intersection with S_0 is $\mathfrak{p}_0$. $\square$

Given this observation, we now define principal open sets in $\operatorname{Proj} S$.

Definition 1.4.2.5. If $f \in S$ is a homogeneous element of degree > 0, define $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} S | f \notin \mathfrak{p}\}$. If $I \subset S$ is a homogeneous ideal, define $V_+(I) = \{\mathfrak{p} \in \operatorname{Proj} S | I \subset \mathfrak{p}\}$. More generally, if E is any set of homogeneous elements, then we define $V_+(E) = \{\mathfrak{p} \in \operatorname{Proj} S | E \subset \mathfrak{p}\}$.

Proposition 1.4.2.6. *Suppose $S = \bigoplus_{d \geq 0} S_d$, is a graded ring and $f \in S$ is a homogeneous element of positive degree.*

1. *The sets $D_+(f)$ are open subsets of* $\operatorname{Proj} S$.
2. *The equality $D_+(ff') = D_+(f) \cap D_+(f')$ holds.*
3. *The sets $D_+(f)$ form a basis for the topology on* $\operatorname{Proj} S$.
4. *The localized ring S_f has a natural $\mathbb{Z}$-grading.*
   The ring maps $S \to S_f \leftarrow S_{(f)}$ induce homeomorphisms
$$D_+(f) \longleftarrow \{\mathbb{Z} - \textit{graded primes of } S_f\} \longrightarrow \operatorname{Spec}(S_{(f)}).$$
5. *The sets $V_+(I)$ for I a homogeneous ideal are closed subsets of* $\operatorname{Proj} S$ *and any closed subset of* $\operatorname{Proj} S$ *is of the form $V_+(I)$ for some homogeneous ideal $I \subset S$.*

We can define a structure sheaf on $\operatorname{Proj} S$ using the sets $D_+(f)$ by assigning to $D_+(f)$ the ring $S_{(f)}$. Likewise, if M is a graded S-module, then we will define a sheaf of modules $\widetilde{M}$ by assigning to $D_+(f)$ the module $M_{(f)}$ defined in a fashion exactly analogous to $S_{(f)}$ as the degree 0 part of the localization M_f thought of as a graded ring in the same way as we did with S_f.

Proposition 1.4.2.7. *Suppose S is a graded ring and M is a graded S-module.*

1. *The assignment $D_+(f) \mapsto S_{(f)}$ determines a sheaf of rings on the basis $D_+(f)$ of $\operatorname{Proj} S$ and therefore extends uniquely to a sheaf of rings $\mathscr{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$.*
2. *The assignment $D_+(f) \mapsto M_{(f)}$ determines a sheaf of on $\operatorname{Proj} S$ that is a sheaf of $\mathscr{O}_{\operatorname{Proj} S}$-modules.*
3. *The ringed space $(\operatorname{Proj} S, \mathscr{O}_{\operatorname{Proj} S})$ is a scheme.*

Definition 1.4.2.8. We define $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_n]$, where x_i has degree $+1$.

Remark 1.4.2.9. If $\varphi : A \to B$ is a homomorphism of graded rings, then we can define an open subscheme of $\operatorname{Proj} B$ that maps to $\operatorname{Proj} A$ as follows: take $U(\varphi)$ to be the union of $D(\varphi(f))$ as f ranges over the homogeneous elements of A_+. There is a canonical map of schemes $U(\varphi) \to \operatorname{Proj} A$.

1.5 Interlude: naive $\mathbb{A}^1$-invariants

We now attempt to transpose some of the ideas about homotopies between maps of topological spaces to the category of affine schemes. First, we need an analog of the unit interval and to do this, we isolate some of the formal properties of I. The properties we use are as follows: (i) there are two distinguished points $0, 1 \in I$, (ii) there is a multiplication map $I \times I \to I$ that makes I into a topological monoid.

1.5.1 $\mathbb{A}^1$-invariants

We claim that $\mathbb{A}^1$ is an analog of I in topology. We could work with the affine line over the integers, but since the only changes that occur working relative to a fixed commutative base ring k are notational, we will work more generally in that context and suppress k from the notation.

There are two maps $0, 1 : \operatorname{Spec} k \to \operatorname{Spec} k[x]$, which at the level rings are given by the evaluation maps $ev_0, ev_1 : k[x] \to k$. The k-algebra structure map $k \to k[x]$ splits either of the above ring homomorphisms.

The monoid operation $\operatorname{Spec} k[x] \times_{\operatorname{Spec} k} \operatorname{Spec} k[x] \to \operatorname{Spec} k[x]$ corresponds to a ring map $k[x] \to k[x_1, x_2]$. At the level of coordinate functions, the product map sends $(x_1, x_2) \mapsto x_1 x_2$, and the map we want is $x \mapsto x_1 x_2$. The element 1 is the identity for multiplication. If we evaluation $f(x_1 x_2)$ at either $x_1 = 1$ or $x_2 = 1$, then we get the identity function. Thus the multiplication map just described can be thought of a providing a homotopy parameterized by $\mathbb{A}^1$ between the identity map $\mathbb{A}^1 \to \mathbb{A}^1$ and the map induced by the constant map to 0 followed by inclusion. With this notation, we can now formally introduce the notion of an $\mathbb{A}^1$-homotopy invariant.

Definition 1.5.1.1. Suppose $\mathscr{C}$ is some category of abstract algebraic structures (e.g., groups, rings, etc.). Suppose $\mathscr{S} \subset \mathrm{Sch}_k$ is some sub-category of k-schemes (we will assume this subcategory is closed under formation of fiber product with $\mathbb{A}^1$). A *$\mathscr{C}$-valued invariant on $\mathscr{S}_k$* is a contravariant functor $\mathscr{F} : \mathscr{S}_k \to \mathscr{C}$. A $\mathscr{C}$-valued invariant $\mathscr{F}$ on $\mathscr{S}_k$ is called $\mathbb{A}^1$-*invariant* if, for any $X \in \mathscr{S}_k$, the pullback along the projection $p_X : X \times \mathbb{A}^1_k \to X$ induces a $\mathscr{C}$-isomorphism $\mathscr{F}(X) \to \mathscr{F}(X \times \mathbb{A}^1)$.

If $k = \mathbb{R}$ or $\mathbb{C}$, then one way to produce invariants as above is to use invariants from topology. For concreteness, fix $k = \mathbb{R}$ and take $\mathscr{S} = \mathrm{Aff}_{\mathbb{R}}$. If $X \in \mathrm{Aff}_{\mathbb{R}}$, then the set $X(\mathbb{R}) := \mathrm{Hom}(\mathrm{Spec}\,\mathbb{R}, X)$ can be equipped with the structure of a topological space in the usual sense. Indeed, if $X = \mathrm{Spec}\,A$, and we fix a presentation $A = k[x_1, \ldots, x_m]/(f_1, \ldots, f_r)$ for A then we realize $X(\mathbb{R})$ as a closed subset of $\mathbb{A}^n(\mathbb{R}) = \mathbb{R}^n$ and we can view it as a topological space with the induced topology.

If we fix a different presentation $A = k[x_1, \ldots, x_n]/(g_1, \ldots, g_s)$ then we get an *a priori* different topological space $X'(\mathbb{R}) \to \mathbb{A}^n(\mathbb{R})$. However, since the two coordinate rings are abstractly isomorphic, we can fix an isomorphism $k[x_1, \ldots, x_m]/(f_1, \ldots, f_r) \cong k[x_1, \ldots, x_n]/(g_1, \ldots, g_s)$. By Example 1.1.2.2, this isomorphism corresponds to a pair of polynomial maps that restrict to real solutions of the respective systems of equations that are mutually inverse. Since polynomial maps are continuous, we conclude that $X(\mathbb{R})$ and $X'(\mathbb{R})$ are actually homeomorphic. In a similar vein, if $f : X \to Y$ is a morphism of affine algebraic varieties, then we conclude that the induced maps $X(\mathbb{R}) \to Y(\mathbb{R})$ are continuous. Therefore, we conclude that the assignment $X \mapsto X(\mathbb{R})$ yields a functor $\mathrm{Var}_{\mathbb{R}}^{aff} \to \mathrm{Top}$.

Remark 1.5.1.2. The category of affine algebraic varieties over the real numbers is very rich. There is a famous theorem of Nash-Tognoli: given a compact differentiable manifold M, there exists an integral $X \in \mathrm{Var}_{\mathbb{R}}^{aff}$ and a diffeomorphism M and $X(\mathbb{R})$ [**?**] (such an X is called an *algebraic model* of M. In fact, the situation is even more interesting: such representations are very far from unique. Since X is integral, it has a well-defined fraction field $k(X)$: this field is a finitely generated extension of k. Say that two integral affine schemes X and X' are birationally equivalent if $k(X) \cong k(X')$ as fields. One can even show that there are infinitely many birationally inequivalent models of M. In dimension 1 this is a fun exercise: if $n > 0$, the equation $x^{2n} + y^{2n} = 1$ has real points diffeomorphic to S^1 for every n, but the function fields of each of these varieties differs as n varies. More generally, any dimension 1 manifold is a disjoint union of circles. For every $n > 0$, the variety given by the equation $y^{2n} = -(x^2-1)(x^2-2)\cdots(x^2-m)$ has graph consisting of m disjoint circles and the resulting varieties can be shown to be birationally inequivalent for different values of n.

Exercise 1.5.1.3. *Show that* $X \mapsto X(\mathbb{C})$ *determines a functor* $\mathrm{Aff}_{\mathbb{C}} \to \mathrm{Top}$.

With this in mind, one way to produce $\mathbb{A}^1$-invariants is simply to take a homotopy invariant on Top and compose with the "realization" functors just described.

Example 1.5.1.4. Suppose k is a field, and assume we have an embedding $\iota : k \hookrightarrow \mathbb{R}$ (or similarly with $\mathbb{R}$ replaced by $\mathbb{C}$). For example, we could take $k = \mathbb{Q}$. The choice of ι defines a functor $\mathrm{Aff}_k \to \mathrm{Top}$ that we will call a *realization* functor. If $\mathscr{F}$ is any $\mathscr{C}$-valued invariant of Top, one obtains a corresponding $\mathscr{C}$-valued invariant of Var_k. If $\mathscr{F}$ is a $\mathscr{C}$-valued homotopy invariant, then since $\mathbb{A}^1(\mathbb{R})$ is a contractible topological space, it follows that the composite functor $\mathrm{Var}_k \to \mathscr{C}$ is $\mathbb{A}^1$-invariant. Note that, in contrast to the situation explained above, inequivalent embeddings ι can yield varieties that are not just topologically inequivalent, but in fact *homotopy inequivalent*! The first example of this goes back to J.P.-Serre.

Remark 1.5.1.5. Take $k = \mathbb{C}$ and X an affine $\mathbb{C}$-variety. One natural question to ask is: what can you say about the homotopy type of $X(\mathbb{C})$? For example, when does $X(\mathbb{C})$ have the homotopy type of a finite CW complex. A classical result of Andreotti-Frankel [**?**], generalized independently by

Karchyauskas [**?**] and Hamm [**?**] shows that this is always the case. Moreover, they show that if X is a dimension d complex affine variety, then $X(\mathbb{C})$ has the homotopy type of a CW complex of dimension $\leq d$.

While the above examples are restricted to work over subfields of the real or complex numbers, it is also possible to produce $\mathbb{A}^1$-invariants that are purely algebraic. We now describe an example that arises in elementary algebra.

Homotopy invariance of units

Set $\mathbf{G}_m = \operatorname{Spec} k[t, t^{-1}]$. Suppose $X = \operatorname{Spec} A \in \mathrm{Aff}_k$ and suppose we are given a map $X \to \mathbf{G}_m$. Such an element corresponds to a homomorphism $\varphi : k[t, t^{-1}] \to A$. Such a homomorphism corresponds to an element $\varphi(t) \in A$ such that $\varphi(t)^{-1} \in A$ as well. In other words, $\varphi(t)$ is a unit. Conversely, given a unit $u \in A$, define a homomorphism $k[t, t^{-1}] \to A$ by sending $t \mapsto u$ and extending by linearity. We write $A^\times$ for the set of units in A. This description of units shows that the assignment $A \mapsto A^\times$ is actually a functor. As a consequence there is always a group homomorphism $A^\times \mapsto A[x]^\times$. On the other hand, the evaluation at 0 homomophism provides a section of $A \to A[x]$, i.e., a homomorphism $A[x] \to A$ such that the composite $A \to A[x] \to A$ is the identity. It follows that the map $A^\times \to A[x]^\times$ is injective.

We can analyze surjectivity of this map. Indeed, if $f \in A[x]$ is a unit, then we can write $f = a_0 + a_1 x + \cdots + a_n x$ and what we just said shows that a_0 must be a unit. In that case, we write $a_0^{-1} f = 1 + \alpha_1 x + \cdots + \alpha_n x$ where $\alpha_i = a_0^{-1} a_i$. This takes the form $1 + z$ where $z = \alpha_1 x + \cdots + \alpha_n x$. In that case, an inverse is given by $\frac{1}{1+z} = \sum_{j \geq 0} (-1)^n z^n$. In order for this element to lie in $A[x]$, we require that $z^n = 0$ for all n sufficiently large, i.e., z^n is nilpotent. The following result characterizes the units in $A[x]$.

Proposition 1.5.1.6. *An element $f = a_0 + a_1 x + \cdots a_n x^n \in A[x]$ is a unit if and only if $a_0 \in A^\times$ and a_i is nilpotent for $i > 0$. In particular, the functor $X \mapsto \mathbf{G}_m(X)$ is $\mathbb{A}^1$-invariant on Var_k^{aff} (in particular, there are no non-constant morphisms $\mathbb{A}^1 \to \mathbf{G}_m$).*

Exercise 1.5.1.7. *Prove* Proposition 1.5.1.6.

1. *Show that if A is a ring, and x is a nilpotent element of A then $1 + x$ is a unit in A.*
2. *Show that if $\alpha_0, \ldots, \alpha_n$ are nilpotent elements of A, then $\sum_{i=0}^n \alpha_i x^i$ is a nilpotent element of $A[x]$.*
3. *If f is a unit in $A[x]$ and $g = \sum_{i=1}^m b_i x^i$ is an inverse of f, prove by induction on r that $a_n^{r+1} b_{m-r} = 0$ and conclude that a_n is nilpotent.*

Remark 1.5.1.8. In fact, the results stated above globalize via gluing, and we conclude that $\mathrm{Hom}_{\mathrm{Sch}_k}(-, \mathbf{G}_m)$ is actually $\mathbb{A}^1$-invariant on the category of all *reduced* k-schemes.

Definition 1.5.1.9. If X is a reduced k-scheme, we set

$$H^{1,1}(X, \mathbb{Z}) := \mathrm{Hom}_{\mathrm{Sch}_k}(X, \mathbf{G}_m).$$

Remark 1.5.1.10. This group can be thought of as analogous to the group $[M, S^1]$ of homotopy classes of maps from a CW-complex to S^1, which is naturally isomorphic to $H^1(X, \mathbb{Z})$. The above notation is that used in motivic cohomology.

Remark 1.5.1.11. It is important to note that, in the above, homotopy invariance did not hold for the functor $\mathbf{G}_m$ on *all* k-schemes, only for reduced k-schemes. Indeed, this is a phenomenon of which we must be aware: even for "natural" functors, homotopy invariance need not hold for all k-schemes.

Remark 1.5.1.12. The proposition above highlights one difference between the algebraic and the continuous category. Indeed, $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$, while $\mathbf{G}_m(\mathbb{C}) = \mathbb{C}^\times$. While there are no non-trivial algebraic maps $\mathbb{A}^1 \to \mathbf{G}_m$, by evaluation on $\mathbb{C}$, observe that there are *continuous* maps $\mathbb{C} \to \mathbb{C}^\times$, e.g., the exponential map.

Algebraic singular homology

We now produce a "purely algebraic" version of singular homology for an arbitrary affine scheme. We begin by recalling a construction of "simplices" in algebraic geometry by naively transplanting the definitions from topology.

Example 1.5.1.13. If k is a field, then define $\Delta^n_k = \operatorname{Spec} k[x_0, \ldots, x_n]/(\sum_{i=0}^n x_i - 1)$. If $n = 0$, then Δ^0_k is isomorphic to $\operatorname{Spec} k$. If $n = 1$, then Δ^1_k is the line $x_0 + x_1 = 1$ in $\mathbb{A}^2_k$. More generally, Δ^n_k is isomorphic to $\mathbb{A}^n_k$ (though the isomorphism with a polynomial ring depends on a choice). As in the topological setting, there are face morphisms $\Delta^n_k \to \Delta^{n-1}_k$ and degeneracy morphisms $\Delta^{n-1}_k \to \Delta^n_k$. These morphisms are defined by projection away from x_i and the inclusion of $x_i = 0$.

Definition 1.5.1.14. If X is an affine scheme over a base k, then the *algebraic singular simplicial set* attached to X, denoted $\operatorname{Sing}^{\mathbb{A}^1} X(k)$, is the simplicial set whose n-simplices are the k-morphisms $\operatorname{Hom}(\Delta^n_k, X)$ and where the face and degeneracy maps are induced by the structures just defined.

Remark 1.5.1.15. The affine n-simplex described above seems to have originally been considered by D. Rector in the 1970s [**?**, Remark 2.5].

For another purely algebraic $\mathbb{A}^1$-invariant, we can appeal to constructions involving $\operatorname{Sing}^{\mathbb{A}^1} X$ for X a smooth affine k-variety.

Exercise 1.5.1.16. *Show that there is an isomorphism* $\operatorname{Sing}^{\mathbb{A}^1} X \times_k Y \cong \operatorname{Sing}^{\mathbb{A}^1} X \times \operatorname{Sing}^{\mathbb{A}^1} Y$.

Indeed, mimicking the definition of ordinary singular homology, we observed that we define a chain complex as follows.

Definition 1.5.1.17. The *algebraic singular chain complex* of an affine scheme X is the chain complex $C^{alg}_*(X, \mathbb{Z})$ with $C^{alg}_n(X, \mathbb{Z}) := \mathbb{Z}(\operatorname{Sing}^{\mathbb{A}^1}{}_n X)$ and with differential $d_i := \sum_{i=0}^n (-1)^i s^*_{n,i}$. The *algebraic singular homology* of X is the homology of the chain complex $H^{alg}_i(X, \mathbb{Z}) := H_i(C^{alg}_*(X, \mathbb{Z}))$.

Lemma 1.5.1.18. *The functor* $X \mapsto H^{alg}_i(X, \mathbb{Z})$ *is* $\mathbb{A}^1$*-invariant.*

Example 1.5.1.19. These groups are often not that interesting. For example, if $X = \mathbf{G}_m$ is considered over a base ring k, then $\operatorname{Sing}^{\mathbb{A}^1}{}_n \mathbf{G}_m = \mathbf{G}_m(k)$ for every integer n. In particular, one computes directly that $H^{alg}_0(\mathbf{G}_m, \mathbb{Z}) = \mathbb{Z}(\mathbf{G}_m(k))$, while the higher groups $H^{alg}_i(\mathbf{G}_m, \mathbb{Z})$ are all trivial. A similar statement holds for any (non-empty) proper open subset of $\mathbb{A}^1$. This kind of example that suggests the construction of "interesting" $\mathbb{A}^1$-invariants will require real work.

Remark 1.5.1.20. Just as in the topological situation, the functor sending X to its ring of functions is not $\mathbb{A}^1$-invariant. This functor also has a nice description. Indeed if $X = \operatorname{Spec} A$, then an element $a \in A$ determines a homomorphism $k[t] \to A$ by sending t to a. Conversely, given a homomorphism $k[t] \to A$ the homomorphism is uniquely specified by the image of t, i.e., an element of A. Therefore, $\operatorname{Hom}_{\mathrm{Var}_k^{aff}}(-, \mathbb{A}^1)$ represents the functor "functions on X".

1.5.2 Naive $\mathbb{A}^1$-homotopies

We now transport the definitions from classical homotopy theory to the algebro-geometric setting.

Definition 1.5.2.1. If $f, g : X \to Y$ are two morphisms of affine k-schemes, then a *naive* $\mathbb{A}^1$-*homotopy* between f and g is a morphism $H : X \times \mathbb{A}^1 \to Y$ such that $H(x, 0) = f$ and $H(x, 1) = g$; in this case we will say that f and g are connected by a naive $\mathbb{A}^1$-homotopy.

Lemma 1.5.2.2. *If $F : X \times \mathbb{A}^1 \to Y$ is a naive $\mathbb{A}^1$-homotopy between morphisms f and f', and if $G : Y \times \mathbb{A}^1 \to Z$ is a naive $\mathbb{A}^1$-homotopy between g and g', then $g \circ f$ and $g' \circ f'$ are connected by a naive $\mathbb{A}^1$-homotopy.*

Proof. If X and Y are affine k-schemes, and $F : X \times \mathbb{A}^1 \to Y$ is a naive $\mathbb{A}^1$-homotopy between f and f' and $G : Y \times \mathbb{A}^1 \to Z$ is a naive $\mathbb{A}^1$-homotopy between g and g', then we can define a naive $\mathbb{A}^1$-homotopy between $g \circ f$ and $g' \circ f'$ by taking $H(x, t) := G(F(x, t), t)$. $\square$

Notice that f is always connected to f by a naive $\mathbb{A}^1$-homotopy (namely the composite of the projection map $X \times \mathbb{A}^1 \to X$ and the map $f : X \to Y$). Likewise, if f and g are connected by a naive $\mathbb{A}^1$-homotopy, then g and f are connected by a naive $\mathbb{A}^1$-homotopy. Indeed, consider the map $\mathbb{A}^1 \to \mathbb{A}^1$ given by $\phi(x) = 1 - x$: this map is an isomorphism that sends 0 to 1 and 1 to 0. Therefore given $H : X \times \mathbb{A}^1 \to Y$, we can pre-compose with $id \times \phi : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ to obtain a new map $H' : X \times \mathbb{A}^1 \to Y$ with $H'(x, 0) = g$ and $H'(x, 1) = f$.

In topology, the fact that homotopy equivalence is transitive stems from the fact that setting copies of the unit interval "end-to-end", i.e., taking $I \coprod I$ where we identity 1 in the first factor with 0 in the second factor, is homeomorphic to I itself. An explicit homeomorphism is gotten by identifying the first copy of I with $[0, \frac{1}{2}]$ and then the second copy of I with $[\frac{1}{2}, 1]$. Unfortunately, the rigidity inherent in algebraic varieties manifests itself in the fact that the relation that two maps are naively $\mathbb{A}^1$-homotopic is not an equivalence relation. Even though the relation f is connected to g by a naive $\mathbb{A}^1$-homotopy is reflexive and symmetric, the following example shows that it need not be transitive in general.

Example 1.5.2.3. Take X to be the affine scheme $\operatorname{Spec} k[x, y]/(xy)$. Geometrically, this scheme consists of two copies of the affine line glued at the origin; as usual, refer to the line $y = 0$ as the x-axis and the line $x = 0$ as the y-axis. Consider the points $(1, 0)$, $(0, 0)$ and $(0, 1)$. While each consecutive pair of points are naively $\mathbb{A}^1$-homotopic, there is no morphism $\mathbb{A}^1 \to X$ that connects $(1, 0)$ and $(0, 1)$. We claim any non-constant morphism $\mathbb{A}^1 \to X$ factors through one of the components. Indeed, given a non-constant morphism $k[x, y]/(xy) \to k[t]$, the relation $xy = 0$ shows that at most one of x and y is sent to a non-constant element of $k[t]$.

The essential point of this example is that X is a reducible algebraic variety and one can envision more complicated examples. One can also envision situations in which naive $\mathbb{A}^1$-homotopy is well-behaved: for example, if we consider a target variety Y, and any morphism from X to Y is naively

$\mathbb{A}^1$-homotopic to a morphism $\mathbb{A}^1$ to Y, then we can sequentially replace "chains" of maps from $\mathbb{A}^1$ to a single morphism from $\mathbb{A}^1$. For example, we can identify X with a closed subscheme of $\mathbb{A}^2$ and then identify $\mathbb{A}^1$ with the closed subvariety of $\mathbb{A}^2$ given by the equation $x + y = 1$. In that case, $\mathbb{A}^2$ itself can be thought of as a deformation of X to $\mathbb{A}^1$. Then, given a morphism $f : X \to Y$, we can extend f to a morphism $\mathbb{A}^2 \to Y$, then we will obtain a suitable condition.

As a consequence of the example, we have to consider the equivalence relation generated by "f is connected to g by a naive $\mathbb{A}^1$-homotopy" (his seemingly innocuous distinction between the algebro-geometric and the topological categories is the source of many of the complications that will arise in our setting). We make the following definition.

Definition 1.5.2.4. Suppose $f, g : X \to Y$ are two morphisms of affine k-schemes. We will say that f and g are *naively* $\mathbb{A}^1$*-homotopic* if they are equivalent for the equivalence relation generated by naive $\mathbb{A}^1$-homotopy. Likewise, two affine k-schemes X and Y are *naively* $\mathbb{A}^1$*-weakly equivalent* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that the two composites are naively $\mathbb{A}^1$-homotopic to the respective identity maps. We write $[X, Y]_N$ for the set of naive $\mathbb{A}^1$-homotopy classes of maps from X to Y.

Exercise 1.5.2.5. *If $\mathscr{F}$ is a $\mathscr{C}$-valued $\mathbb{A}^1$-homotopy invariant, and if f and g are naively $\mathbb{A}^1$-homotopic maps, then $\mathscr{F}(f) = \mathscr{F}(g)$.*

Example 1.5.2.6. If X is an affine k-scheme, then X and $\mathbb{A}^n \times X$ are naively $\mathbb{A}^1$-weakly equivalent for any X. The composite $X \to \mathbb{A}^n \times X \to X$ is equal to the identity. To see that the other composite is naively $\mathbb{A}^1$-homotopic to the identity, we use the map $\mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$ given by $(t, x) \mapsto tx$.

1.5.3 The naive $\mathbb{A}^1$-homotopy category

We can try to formally define a "universal" $\mathbb{A}^1$-homotopy invariant by constructing a new category whose objects are objects in Var_k^{aff} and whose morphisms are naive $\mathbb{A}^1$-homotopy classes of morphisms between k-varieties. The following definition makes sense because composites of naively $\mathbb{A}^1$-homotopic maps are naively $\mathbb{A}^1$-homotopic.

Definition 1.5.3.1 (Naive $\mathbb{A}^1$-homotopy category)**.** The naive $\mathbb{A}^1$-homotopy category over a field k is the category $\mathscr{N}(k)$ whose objects are those of Var_k^{aff} and whose morphisms are the sets of naive $\mathbb{A}^1$-homotopy classes of maps between affine sk-varieties.

Lemma 1.5.3.2. *If $X, Y \in \mathrm{Var}_k^{aff}$, then the projection map $[X, Y]_N \to [X \times \mathbb{A}^1, Y]_N$ is a bijection.*

Proof. The map in question is evidently split (via any inclusion $X \hookrightarrow X \times \mathbb{A}^1$) and therefore injective. Thus, it suffices to demonstrate surjectivity. Suppose $f : X \times \mathbb{A}^1 \to Y$ is a morphism. We want to show that f is $\mathbb{A}^1$-homotopic to $f(x, 0)$. To this end, consider the product map $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. Note that $\mu(t, 0) = 0$, while $\mu(t, 1) = 1$. Then, define a naive $\mathbb{A}^1$-homotopy between f and $f(x, 0)$ by considering the map $X \times \mathbb{A}^1 \times \mathbb{A}^1 \to Y$ given by $f(x, \mu(t, s))$. $\square$

Example 1.5.3.3. The functor sending X to $\mathbf{G}_m(X)$ is representable on Var_k^{aff} by $[-, \mathbf{G}_m]_N$.

1.5.4 Naive $\mathbb{A}^1$-homotopy calculations

We now give some examples to show that naive $\mathbb{A}^1$-homotopy classes of maps can sometimes be determined in practice; we begin with a few exercises.

Exercise 1.5.4.1. *Suppose $Z \subset \mathbb{A}^1$ is a closed subset defined by a polynomial f. Let $U \subset \mathbb{A}^1$ be the complement of Z, with coordinate ring $k[U] = k[x, \frac{1}{f}]$.*
1. *Show that $[\operatorname{Spec} k, U]_N = U(k)$.*
2. *More generally, show that if X is any smooth affine scheme, then $U(X) = [X, U]_N$.*

Example 1.5.4.2. Consider the variety $\mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$; this can be identified as the spectrum of the ring $k[x, x^{-1}, y]$. Even though there is a copy of the affine line passing through every point, there are no morphisms $\mathbb{A}^2 \to \mathbb{A}^1 \setminus 0 \times \mathbb{A}^1$.

Exercise 1.5.4.3. *Suppose G is an affine algebraic k-group (i.e., a group object in the category Var_k^{aff}). Show that for any $X \in \mathrm{Var}_k^{aff}$, the set $[X, G]_N$ inherits a group structure making the map $G(X) \to [X, G]_N$ a homomorphism. Moreover, this group structure is functorial in both X and G.*

Pick coordinates x_{ij} on the n^2-dimensional affine space M_n of $n \times n$-matrices. With this choice, if $X \in M_n$ is an $n \times n$-matrix, then $\det X$ is a polynomial of degree n in the variables x_{ij}. In particular, we can define $GL_n = \operatorname{Spec} k[x_{ij}, \det X^{-1}]$ and $SL_n = \operatorname{Spec} k[x_{ij}]/(\det X = 1)$. The explicit formulas for matrix multiplication and matrix inversion show that GL_n and SL_n are affine algebraic groups (the identity is given by the $n \times n$-identity matrix). A T-point of GL_n, for some test k-algebra T, is precisely an invertible $n \times n$-matrix with coefficients in T. Likewise, a T-point of SL_n is an invertible $n \times n$-matrix with coefficients in T and whose determinant is equal to 1.

Proposition 1.5.4.4. *If k is any field, then $[\operatorname{Spec} k, SL_n]_N = Id_n$.*

Proof. Any "elementary matrix" gives rise to a matrix naively $\mathbb{A}^1$-homotopic to the identity. Indeed, let e_{ij} be a matrix unit (i.e., an $n \times n$-matrix such that $(e_{ij})_{kl} = 1$ if $i = k$ and $j = l$ and 0 otherwise). If $i \neq j$, then consider the matrix $E_{ij}(\alpha) := Id_n + \alpha e_{ij}$; this matrix is called an elementary matrix (or an elementary shearing matrix). Observe that $E_{ij}(t\alpha)$ provides a naive homotopy between $E_ij(\alpha)$ and Id_n. The statement follows from the following observation: any element of $SL_n(k)$ can be written as a product of elementary matrices.

To begin, recall that any element of $GL_n(k)$ can be written as a product of an elementary matrix as in the previous paragraph, matrices of the form $Id_n + (\alpha - 1)e_{ii}$ for $1 \leq 1 \leq n, \alpha \in k^*$, and permutation matrices. We claim that every permutation matrix is a product of elementary matrices and matrices of the form $Id_n + (\alpha - 1)e_{ii}$. Indeed, any element of the symmetric group can be written as a product of transpositions. For $GL_2(k)$, we simply perform row operations to transform $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ into such a matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, by fixing different embeddings $GL_2(k) \to GL_n(k)$, we conclude that similar formulas hold for arbitrary elements. As a consequence, we see that every matrix $X \in GL_n(k)$ can be written as a product $\prod_i E_i$ where each E_i is either elementary or of the form $Id_n + (\alpha - 1)e_{ii}$ for $1 \leq i \leq n, \alpha \in k^*$. We now study the possible commutators of elements. Since elementary matrices act as row operations, and matrices of the form $Id_n + (\alpha - 1)e_{ii}$ act by multiplying a row by α, we conclude that the commutator of an elementary matrix and one of the form $Id_n + (\alpha - 1)e_{ii}$ is the identity unless i is one of the rows being acted upon by the elementary matrix. In the remaining case, one immediately verifies the following identities

$$\begin{pmatrix}1 & \alpha \\ 0 & 1\end{pmatrix}\begin{pmatrix}\beta & 0 \\ 0 & 1\end{pmatrix} = \begin{pmatrix}\beta & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{\alpha}{\beta} \\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & \alpha \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & \beta\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & \alpha\beta \\ 0 & 1\end{pmatrix}.$$

By taking transposes, we obtain similar formulas for lower triangular matrices. Using these observations, we conclude that we can write any element of $GL_n(k)$ as a product of a diagonal matrix D and a product of elementary matrices.

Now, suppose $X \in SL_n(k)$ and write $X = DE_1 \cdots E_n$. Observe that $\det D = 1$ (though we cannot necessarily assume that D is the identity matrix based on the way in which we formulated our algorithm above). However, if $D = diag(\alpha_1, \ldots, \alpha_n)$, then we can write $D = diag(\alpha_1, \alpha_1^{-1}, 1, \ldots, 1)(1, \alpha_1\alpha_2, \alpha_3, \ldots, \alpha_n)$, i.e., any diagonal matrix with determinant 1 can be written as the product of a diagonal matrix in $SL_2(k)$ embedded in $SL_n(k)$ and a diagonal matrix in $SL_{n-1}(k)$ embedded in $SL_n(k)$. Thus, proceeding recursively, we see that any diagonal matrix of determinant 1 can be written as a product of diagonal matrices that are in the image of $SL_2(k)$. Now, it is straightforward to show that a diagonal matrix in $SL_2(k)$ can be written as a product of elementary matrices. For example, one can write:

$$\begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix} = \begin{pmatrix}1 & \alpha - 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & \frac{1-\alpha}{\alpha} \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ -\alpha & 1\end{pmatrix}.$$

Taken together, we conclude that every element of $SL_n(k)$ can be written as a product of elementary matrices. Since every element elementary matrix is naively $\mathbb{A}^1$-homotopic to the identity, we conclude that $[\operatorname{Spec} k, SL_n]_N = \{Id_n\}$ (i.e., as a group it is the trivial group). $\square$

Exercise 1.5.4.5. *Show that the determinant homomorphism* $\det : GL_n \to \mathbf{G}_m$ *induces an isomorphism* $[\operatorname{Spec} k, GL_n] \to k^\times$.

Remark 1.5.4.6. One possible generalization of the units functor is the functor sending a ring R to the group $GL_n(R)$ of invertible $n \times n$-matrices over R and we can investigate the $\mathbb{A}^1$-homotopy invariance of this functor. As before since $GL_n(-)$ is a functor, there is an evident map $GL_n(R) \to GL_n(R[x])$ and evaluation at zero shows that this map is injective. Note that, if $n \geq 2$, this map is never surjective. Indeed, take any element f of $R[x]$ that is not in R and consider the elementary matrix $Id_n + fe_{ij}$ (with $i \neq j$): this is an element of $GL_n(R[x])$ that does not lie in the image of $GL_n(R)$.

Example 1.5.4.7. Suppose $X \in GL_n(R[x])$. The matrix $X(0)$ is in $GL_n(R[x])$. Setting $Y := X(0)^{-1}X$ produces a matrix such that $ev_0(X(0)^{-1}X) = Id_n$. Now, we can also consider the evaluation map $M_n(R[x]) \to M_n(R)$ and note that $ev_0(X(0)^{-1}X - Id_n) = 0$. Consider the matrix $Y(tx)$. For $t = 0$, this matrix is $Y(0) = Id_n$, while for $t = 1$ it is simply $Y(x)$. Therefore, the matrix Y is naively $\mathbb{A}^1$-homotopic to the identity. It follows that the matrix X is naively $\mathbb{A}^1$-homotopic to $X(0)$.

Further generalizations of these computations

Closely related naive $\mathbb{A}^1$-homotopy classes to those studied above can be very interesting. Fix an integer n, and work over a field whose characteristic does not divide n. The center of SL_n is a affine algebraic group $\mu_n := \operatorname{Spec} k[t, t^{-1}]/(t^n - 1)$; this affine algebraic group is also a subgroup of $\mathbf{G}_m$. Now, the group scheme μ_n acts by left multiplication on SL_n, i.e., there is a morphism $\mu_n \times SL_n \to SL_n$. We can form the quotient by this group action. More precisely, define $PGL_n := \operatorname{Spec} k[SL_n]^{\mu_n}$, i.e., the spectrum of the ring of invariant functions. The elements of $PGL_n(k)$ are precisely the invertible $n \times n$-matrices over k *up to scaling*.

Exercise 1.5.4.8. *Show that* $[\operatorname{Spec} k, PGL_n]_N = k^*/(k^*)^n$*, i.e., the quotient of the group* k^* *by the subgroup of* n*-th powers.*

Chapter 2

Projective modules, gluing and vector bundles

Contents

The goal of this section is to introduce a dictionary between algebraic geometry/commutative ring theory and topology. Here, we will introduce and analyze projective modules over rings and relate these notions with vector bundles. We will begin by introducing modules over a ring As mentioned in the introduction, by following parallels with topology, it is natural to study vector bundles on affine varieties. We develop in this chapter the basic properties of such objects, i.e., projective modules over a ring.

2.1 Modules over a ring

2.1.1 Finiteness properties

We begin by developing the basic ideas in a slightly more general context than we studied before. Suppose R is an arbitrary commutative unital ring and write Mod_R for the category of *all* R-modules.

Definition 2.1.1.1. If R is a commutative ring, an R-module M is called

1. *finitely generated* if there is an epimorphism $R^{\oplus n} \to M$;
2. *finitely presented* if M is the cokernel of a map $R^{\oplus n} \to R^{\oplus m}$ (equivalently, M is finitely generated, and for some surjection $\varphi : R^{\oplus m} \to M$, the kernel $\ker \varphi$ is finitely generated as well).
3. *coherent* if M is finitely generated and any finitely generated (not necessarily proper) submodule is itself finitely presented.

It follows from the definitions that for any ring R that M coherent implies M finitely presented, and M finitely presented implies M finitely generated. For general rings R, the reverse implications need not hold. We write Mod_R^{fg}, Mod_R^{fp} and Mod_R^{coh} for the full subcategories of Mod_R consisting of finitely generated, finitely presented or coherent R-modules.

Example 2.1.1.2. Beware: over "big" rings strange things can happen. For example: a submodule of a finitely generated R-module need not be finitely generated. For instance, take $R = k[x_1, x_2, \ldots]$ be a polynomial ring in infinitely many variables. You can check that the ideal $\langle x_1, x_2, \ldots \rangle$ is an R-submodule of a free R-module of rank 1, yet fails to be finitely generated.

Lemma 2.1.1.3. *Let R be a commutative unital ring and suppose*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence of R-modules. The following statements hold:

1. *Any extension of finitely generated R-modules is finitely generated, i.e., if M' and M'' are finitely generated, then so is M.*
2. *Any extension of finitely presented R-modules is finitely presented, i.e., if M' and M'' are finitely presented, so is M.*
3. *Any quotient of a finitely generated module is finite, i.e., if M is finitely generated, so is M''.*
4. *Any quotient of a finitely presented module by a finitely generated submodule is finitely presented, i.e., if M is a finitely presented R-module, and M' is finitely generated, then M'' is finitely presented as well.*
5. *If M'' is finitely presented, and M is finitely generated, then M' is finitely generated as well.*

Proof. Exercise $\square$

Remark 2.1.1.4. Our desire to work with arbitrary commutative rings is not generality for its own sake. If M is a compact manifold, then the rings $C(M)$ or $C^\infty(M)$ of complex-valued continuous functions on M or complex valued smooth functions on M *need not* be Noetherian rings.

The following fact is fundamental.

Theorem 2.1.1.5. *The category* Mod_R*, equipped with the usual structures of direct sum and tensor product is abelian and symmetric monoidal (see* Appendices A.3.1 and A.2.3*).*

Remark 2.1.1.6. In general, the categories Mod_R^{fg} and Mod_R^{fp} *need not* be abelian categories (the example above shows what can go wrong for finitely generated R-modules), but Mod_R^{coh} turns out to be an abelian category [**?**, Tag 05CU]. When R is a Noetherian ring, all three notions are equivalent (reference?).

Remark 2.1.1.7. A ring R is coherent if it is coherent as a module over itself (i.e., all finitely generated ideals in R are finitely presented). Rings appearing in topology are rarely coherent. For example a result due to Neville [**?**] characterizes those topological spaces for which the ring of continuous functions is coherent: such spaces are called *basically disconnected*. More precisely, this means that for any continuous function f, the closure of the open set $\{x \in X | f(x) \neq 0\}$ is again open.

One of our eventual goal is the problem of classification of modules over a ring. The model for classification results is provided by the structure theorem for finitely generated modules over a PID: every finitely generated module can be written as a direct sum of a free module and a torsion module, and we can give a nice classification of the torsion modules in terms of the (non-zero) prime ideals of the ring. There is a straightforward generalization of torsion modules to arbitrary commutative rings.

Definition 2.1.1.8. A module M over a commutative unital ring R is called a *torsion* R-module, if there exists a regular element $r \in R$ (i.e., a non-zero divisor) such that $rM = 0$. An R-module M that is not a torsion R-module is called *torsion-free*.

As the "classification" of prime ideals for rings that are not PIDs is more complicated (the spectrum of the ring is a precise measure of the complexity), the structure of torsion modules for more general rings becomes more complicated. Moreover, even the "easy" part of the structure theorem is more complicated: if R is not a principal ideal domain, it is not necessarily the case that torsion free R-modules are themselves free.

Functoriality

If $\varphi : R \to S$ is a ring homomorphism, then there are two functors that we will frequently consider on categories of modules. If M is an S-module, then the ring homomorphism $R \to S$ equips M with the structure of an R-module as well thus defining a functor

$$\varphi_* : \mathrm{Mod}_S \longrightarrow \mathrm{Mod}_R.$$

Our choice of notation here reflects the fact that φ induces a morphism of affine schemes $\operatorname{Spec} S \to \operatorname{Spec} R$, and if $\tilde{M}$ is the sheaf of $\mathscr{O}_{\operatorname{Spec} S}$-modules corresponding to M, then $\varphi_* M$ coincides with the global sections of the sheaf $\varphi_* \tilde{M}$ by construction.

Remark 2.1.1.9. Note that this functor can easily fail to preserve finiteness properties studied above. E.g., take R to be a field and S to be a polynomial ring in 1 variable over a k. If M is the free S-module of rank 1, which is even coherent as an S-module, viewing M as a k-vector space yields a countably generated k-module.

The other functor we consider is the "extension of scalars" functor. If φ is as above, and M is an R-module, then we may consider the S-module $M \otimes_R S$. Functoriality of tensor product then yields a functor

$$(-) \otimes_R S : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S.$$

Note that, by construction, this functor does preserve finite generation, finite presentation and coherence of modules. Unlike the case of the functor φ_* described above, this functor is less straightforward to relate to the functor φ^{-1} at the level of sheaves for a simple reason: the sheaf $\varphi^{-1}\tilde{M}$ on $\operatorname{Spec} S$ does not have the structure of an S-module.

2.1.2 Sheaves of modules over sheaves of rings

Suppose $(X, \mathscr{O}_X)$ is a ringed space. The discussion of modules we just gave can be globalized to talk about sheaves of $\mathscr{O}_X$-modules. We write $\mathrm{Mod}(\mathscr{O}_X)$ for the category of sheaves of $\mathscr{O}_X$-modules. We'd like to analyze the structures described on the ordinary categories of modules above in this context as well.

First, let us observe that $\mathrm{Mod}(\mathscr{O}_X)$ can be equipped with the structure of an additive category. First, we need to equip the set of homomorphisms between two objecs with an abelian group structure. For this, if $f, g : \mathscr{F} \to \mathscr{G}$ are two morphisms of sheaves of $\mathscr{O}_X$-modules, then we may define $f+g : \mathscr{F} \to \mathscr{G}$ sectionwise, i.e., for any open $U \subset X$, $\mathscr{F}(U)$ and $\mathscr{G}(U)$ are $\mathscr{X}_X(U)$-modules, and $f+g$ on U is the sum defined sectionwise. Analogously, we define the trivial sheaf of $\mathscr{O}_X$-modules 0, which is the constant sheaf with value 0 for every open $U \subset X$. A morphism $f : \mathscr{F} \to \mathscr{G}$ is trivial if and only if it factors through the zero morphism if and only if it is the zero map on sections. The fact that these structures equip $\mathrm{Hom}_{\mathrm{Mod}(\mathscr{O}_x)}(\mathscr{F}, \mathscr{G})$ with a structure of abelian group can then be checked section wise.

We define direct sum of $\mathscr{O}_X$-modules sectionwise, i.e., if $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of $\mathscr{O}_X$-modules, then for any open $U \subset X$, we set

$$(\mathscr{F} \oplus \mathscr{G})(U) := \mathscr{F}(U) \oplus \mathscr{G}(U).$$

The section-wise canonical inclusions $\mathscr{F} \to \mathscr{F} \oplus \mathscr{G}$ and $\mathscr{G} \mapsto \mathscr{F} \oplus \mathscr{G}$ equip the direct sum with the structure of a coproduct in the category of $\mathscr{O}_X$-modules. This formula works for *finite* direct sums of $\mathscr{O}_X$-modules. If we want to define infinite direct sums, then the sectionwise definition can fail to be a sheaf, so we must sheafify.

If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathscr{O}_X$-modules, we may define $\ker(\varphi)$ sectionwise; a priori this is a presheaf, but you may check immediately that it is in fact a sheaf. The situation with the cokernel is slightly different for exactly the same reason that surjectivity of a morphism of sheaves needs to be checked stalkwise. Indeed, if $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of $\mathscr{O}_X$-modules, then the assignment

$$U \mapsto \mathrm{coker}(\mathscr{F}(U) \to \mathscr{G}(U))$$

is not a priori a sheaf. Nevertheless, if we define $\mathrm{coker}(\varphi)$ to be the sheafification of the above presheaf, then this is a cokernel in the usual sense. Since taking stalks commutes with sheafification we conclude that $\mathrm{coker}(\varphi)_x = \mathrm{coker}(\varphi_x)$, and thus a morphism of sheaves is surjective if and only if the cokernel sheaf is trivial. Equipped with these definitions, the category $\mathrm{Mod}(\mathscr{O}_X)$ has the structure of an abelian category. A sequence of sheaves of $\mathscr{O}_X$-modules is exact if and only if it is exact stalkwise.

We can also equip the category $\mathrm{Mod}(\mathscr{O}_X)$ with a symmetric monoidal structure. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of $\mathscr{O}_X$-modules, then we may consider the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U).$$

Unfortunately, this presheaf may fail to be a sheaf. In order to have the correct universal property for *sheaves* of $\mathscr{O}_X$-modules, we must therefore sheafify the above presheaf. We will write $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ for the sheaf associated with the presheaf displayed above. One may check that tensor product of $\mathscr{O}_X$-modules is again right exact in either variable, and distributes over direct sum. The sheaf $\mathscr{O}_X$ provides a unit for $\otimes$. The following result generalizes the corresponding statement for modules (since one may take $X = *$ and $\mathscr{O}_X = R$ for some ring R).

Theorem 2.1.2.1. *If $(X, \mathscr{O}_X)$ is a ringed space, then $\mathrm{Mod}(\mathscr{O}_X)$ equipped with the direct sum and tensor product just mentioned has the structure of an abelian symmetric monoidal category.*

Finiteness conditions on sheaves of modules

If M is a module over a ring R, then M is finitely generated if there is a surjection $R^{\oplus n} \to M$. We make corresponding definitions in categories of sheaves of modules, except now we only impose finiteness locally. Now, any module over a ring has some presentation (i.e., it can be written as the cokernel of a map of free R-modules), but in defining infinite direct sums of sheaves, sheafification was necessary.

Definition 2.1.2.2. If $(X, \mathscr{O}_X)$ is a ringed space, then a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is said to have finite type if for every $x \in X$ if there exists a neighborhood U of x, an integer n and a surjection $\mathscr{O}_U^{\oplus n} \to \mathscr{F}|_U$. Likewise, $\mathscr{F}$ is said to have finite presentation if for every $x \in X$ there exists a neighborhood U of x, and a morphism $\varphi : \mathscr{O}_U^{\oplus m} \to \mathscr{O}_U^{\oplus n}$ such that $\mathscr{F}|_U$ is isomorphic to $\mathrm{coker}(\varphi)$.

Definition 2.1.2.3. If $(X, \mathscr{O}_X)$ is a ringed space, then a sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules is said to be *quasi-coherent* if for every point $x \in X$, there exists a neighborhood U of x such that $\mathscr{F}|_U$ is the cokernel of a morphism $\varphi : \bigoplus_{i \in I} \mathscr{O}_U \to \bigoplus_{j \in J} \mathscr{O}_U$. The sheaf $\mathscr{F}$ is said to be *coherent* if $\mathscr{F}$ has finite type, and for every open $U \subset X$, and any morphism $\varphi : \mathscr{O}_U^{\oplus n} \to \mathscr{F}|_U$, $\ker(\varphi)$ has finite type.

Note that taking $X = pt$ and $\mathscr{O}_X = R$ for a commutative ring R, the definition of coherence coincides with the notion of coherence for modules described above.

2.1.3 Projective and flat modules

Of course free modules are torsion-free. We now introduce some other properties that measure "torsion-freeness" of modules.

Definition 2.1.3.1. Suppose R is a commutative unital ring. An R-module M is called:

1. *flat* if $- \otimes_R M$ is an exact functor on Mod_R, i.e., preserves exact sequences;
2. *projective* if $\mathrm{Hom}_R(M, -)$ is an exact functor on Mod_R,

3. *invertible* if $- \otimes_R M$ is an auto-equivalence of Mod_R.

Remark 2.1.3.2. Note that $- \otimes_R M$ is an exact functor if and only if $M \otimes_R -$ is exact since included in the statement that Mod_R is symmetric monoidal is a natural isomorphism between these two functors.

Definition 2.1.3.3. If $f : R \to S$ is a homomorphism of commutative rings, then we say that f is a *flat ring homomorphism* if S is a flat R-module.

Example 2.1.3.4. Any free R-module is projective (or flat). A free R-module of rank 1 is invertible. Any finitely generated projective (or flat) module over a principal ideal domain is necessarily free (this follows from the structure theorem). In particular, if k is a field, any finitely generated projective $k[t]$-module is free.

Remark 2.1.3.5. In 1955, Serre asked whether finitely generated projective $k[t_1, \ldots, t_n]$-modules (k a field) are free [**?**]. This question stimulated much work in the theory of projective modules and was answered by Quillen and Suslin (independently) in 1976.

Example 2.1.3.6. If R is a ring, then we can consider $R \times 0$ as an $R \times R$-module. This module is evidently a direct summand of a free module (namely $R \times R$), but is not itself free. Thus, there exist examples of projective modules that are not free.

Lemma 2.1.3.7. *An arbitrarily indexed direct sum of R-modules is flat (resp. projective) if and only if each summand is flat (resp. projective).*

Proof. Since arbitrary direct sums commute with tensor products in the category of R-modules, there is an isomorphism of functors $(\bigoplus_{i \in I} M_i) \otimes_R - \cong \bigoplus_{i \in I} (M_i \otimes_R -)$. Thus, the first functor is exact if and only if the second functor is exact. Likewise, there is an isomorphism of functors $\mathrm{Hom}_R(\bigoplus_{i \in I} P_i, -) \cong \prod_i \mathrm{Hom}_R(P, -)$ and the left-hand-side is exact if and only if the right hand side is exact. $\square$

Lemma 2.1.3.8. *An R-module L is invertible if and only if there exists an R-module L' such that $L \otimes_R L' \xrightarrow{\sim} R$.*

Proof. If $L \otimes_R -$ is an auto-equivalence, then the existence of L' is an immediate consequence of the fact that equivalences of categories are essentially surjective. In the other direction, if L' exists as in the statement, then $- \otimes_R L'$ is a quasi-inverse to $- \otimes_R L$ since $(- \otimes_R L) \otimes_R L' \cong - \otimes_R (L \otimes_R L') \cong - \otimes_R R$, which is the identity functor. $\square$

Injective modules

There is a notion of injective module that is dual to that of projective module. More precisely, one makes the following definition.

Definition 2.1.3.9. If R is a commutative unital ring, then an R-module M is injective if $\mathrm{Hom}_R(-, M)$ is exact.

Concretely, an R-module M is injective if given any R-module map $j : N \to M$ and an injective R-module map $N \to N'$, there exists an R-module map $j' : N' \to M$ extending j, i.e., such that the composite $N \to N' \to M$ concides with j.

Exercise 2.1.3.10. *Show that any product of modules is injective if and only if each factor is injective.*

Localizations are flat ring homomorphisms

The following elementary fact about localization will be used repeatedly in what follows.

Theorem 2.1.3.11. *Suppose R is a commutative unital ring, and $S \subset R$ is a multiplicative subset.*
1. *If M is an R-module, then $M[S^{-1}] = M \otimes_R R[S^{-1}]$.*
2. *The assignment $M \mapsto M[S^{-1}]$ is an exact functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R[S^{-1}]}$.*
3. *In particular, $R \to R[S^{-1}]$ is a flat ring homomorphism.*

Proof. For Point (1). Consider the map $M \times R[S^{-1}] \to M[S^{-1}]$ given by $(m, \frac{r}{s}) \mapsto \frac{rm}{s}$. This map is R-bilinear by construction, and therefore there exists a map $M \otimes R[S^{-1}] \to M[S^{-1}]$ such that$m \otimes \frac{r}{s} \mapsto \frac{rm}{s}$. Define a map $M[S^{-1}] \to M \otimes R[S^{-1}]$ by the formula $\frac{m}{s} = m \otimes \frac{1}{s}$; we claim this is well-defined. Indeed, if $\frac{m'}{s'}$ presents the same element of $M[S^{-1}]$, then we can find t and $t' \in S$ such that $ms't = m'st'$. In that case, $m \otimes \frac{1}{s} = m \otimes \frac{s't}{ss't} = ms't \otimes \frac{1}{ss't} = m'st \otimes \frac{1}{ss't} = m' \otimes \frac{st}{ss't} = m' \otimes \frac{1}{s'}$. It is straightforward to check these two maps are inverses.

For Point (2), since tensoring is always right exact, it suffices to prove that if $M \to M'$ is an injective R-module map, $M[S^{-1}] \to M'[S^{-1}]$ remains injective. If we view M as a sub-module of M', an element $\frac{x}{s} \in M'[S^{-1}]$ is zero if and only if there exists $t \in S$ such that $tx = 0$. However, the latter happens if and only if $\frac{x}{s} = 0$ in M itself.

The third statement is a consequence of the first two. □

2.2 Projective modules and their properties

In this lecture we analyze further the basic properties of projective modules.

2.2.1 Properties of projective modules

Lemma 2.2.1.1. *Suppose R is a commutative unital ring. If P is an R-module, the following conditions on P are equivalent:*
1. *P is projective;*
2. *Any R-module epimorphism $M \to P$ is split.*
3. *P is a direct summand of a free R-module;*

Proof. (1) $\Longrightarrow$ (2). Suppose we are given a surjection $\varphi : A \to P$; we can complete this into an exact sequence $0 \to \ker(\varphi) \to A \to P \to 0$. Now, since P is projective, $\mathrm{Hom}_R(P, -)$ is an exact functor, and applying it to the previous short exact sequence yields a short exact sequence of the form

$$0 \longrightarrow \mathrm{Hom}_R(P, \ker(\varphi)) \longrightarrow \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, P) \longrightarrow 0.$$

In particular, we may lift the identity $1 \in \mathrm{Hom}_R(P, P)$ to a morphism $P \to A$ that, by construction, splits the given epimorphism.

(2) $\Longrightarrow$ (3). By choosing generators of P, we may build an epimorphism from a free module $\varphi : F \to P$. By (2), such an epimorphism is split, and we obtain a morphism $P \to F$. Then, we conclude that $F \cong P \oplus \ker(\varphi)$.

(3) $\Longrightarrow$ (1). Suppose $P \oplus Q \cong F$, where F is a free module. Example 2.1.3.4 shows that F is projective, and then appeal to Lemma 2.1.3.7 allows us to conclude that any summand of a free R-module is also projective. $\square$

Remark 2.2.1.2. Recall that a projection operator on a k-vector space V is an endomorphism P such that $P^2 = P$. Upon choice of a basis of V, such an endomorphism amounts to an idempotent matrix. The projection onto a summand is an example of a projection operator. Given a finitely generated projective module M over a ring R, we may always fix a direct sum decomposition $R^{\oplus n} \cong P \oplus Q$. In that case, the composite map $R^{\oplus n} \to P \hookrightarrow R^{\oplus n}$ is represented by an idempotent matrix on $R^{\oplus n}$, i.e., finitely generated projective modules correspond to projection operators.

Example 2.2.1.3. Lemma 2.2.1.1 shows that any module M such that $M \oplus R^{\oplus n} \cong R^{\oplus N}$ is projective; such modules are called *stably free*. We now explain how to construct stably free R-modules. Suppose R is a ring and $a_1, \ldots, a_n$ is a sequence of elements in R. We will say that a sequence $(a_1, \ldots, a_n)$ is a *unimodular row (of length n)* if there exist elements $b_1, \ldots, b_n$ such that $\sum_i a_i b_i = 1$. In other words, the row $(a_1, \ldots, a_n)$, viewed as a $1 \times n$-matrix, has a *right inverse*. Given unimodular row of length n, we can define an epimorphism $R^{\oplus n} \to R$ via multiplication by $\mathbf{a} := (a_1, \ldots, a_n)^t$. Unimodularity ensures that this homomorphism is surjective. In that case, the kernel $P_{\mathbf{a}} := \ker \mathbf{a}$ is a projective R-module.

In general, $P_{\mathbf{a}}$ need not be a free R-module, though it is sometimes difficult to prove this algebraically. One easy source of unimodular rows of length n is as follows. If A is an $n \times n$-invertible matrix of determinant 1, then its first row is a unimodular row of length n; this follows from the formula for $\det A$ by expansion along the first row. A unimodular row that is the first row of an invertible matrix will be called *completable*. If A is completeable, then we may pick a new basis of $R^{\oplus n}$ consisting of the rows of A. It follows immediately that the projective module $P_{\mathbf{a}}$ corresponding to a completeable unimodular row of length n is automatically free of rank $n-1$. The following exercise shows that the converse also holds.

Exercise 2.2.1.4. *Show that the projective module $P_{\mathbf{a}}$ associated with a unimodular row $(a_1, \ldots, a_n)$ is free if and only if $(a_1, \ldots, a_n)$ is the first row of an invertible $n \times n$-matrix with determinant* 1.

Example 2.2.1.5. Unimodular rows of length 2 are unfortunately not so interesting. In that case, in the previous construction, one obtains a module P such that $P \oplus R \cong R^{\oplus 2}$. However, any unimodular sequence of length 2 is completable, so $P \cong R$. Indeed, if (a_1, a_2) is our unimodular row, then by definition we can find (b_1, b_2) such that $a_1 b_1 + a_2 b_2 = 1$. In that case, the matrix

$$\begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix}$$

has determinant $a_1 b_1 + a_2 b_2 = 1$.

Example 2.2.1.6. If M is any R-module, then we can consider the R-module dual $M^\vee := \operatorname{Hom}_R(M, R)$; this has a natural R-module structure. Note that, if M is finitely generated free module, then so is

$\mathrm{Hom}_R(M,R)$. If P is a finitely generated projective module, then we claim $\mathrm{Hom}_R(P,R)$ is projective. Indeed, if $P \oplus Q \cong R^{\oplus n}$, then $\mathrm{Hom}_R(P \oplus Q, R) \cong \mathrm{Hom}_R(R^{\oplus n,R}) \cong R^{\oplus n}$. Since finite direct products of modules are also direct sums, it follows that $\mathrm{Hom}_R(P,R)$ is a summand of $R^{\oplus n}$ as well.

Remark 2.2.1.7. For $n \geq 3$, it is more difficult to determine whether a given unimodular row of length n is completable. On the other hand, with the technology developed so far, it is also not clear whether there are any non-trivial examples.

Corollary 2.2.1.8. *If R is a commutative unital ring, and M is an R-module, then following implications hold:*

$$M \textit{is invertible} \Longrightarrow M \textit{is projective} \Longrightarrow M \textit{ is flat}.$$

Proof. We leave the first implication as an exercise. For the second implication, since free R-modules are flat, direct summands of flat R-modules are flat, and since any projective R-module is a direct summand of a free R-module, it is necessarily flat as well. $\square$

Lemma 2.2.1.9. *Assume R is a commutative unital ring.*
1. *Any finitely generated projective R-module is finitely presented.*
2. *Any invertible R-module is finitely presented.*

Proof. Since projective modules are summands of free modules, it suffices to observe that any summand of a finitely generated free R-module is finitely presented. Indeed, suppose $R^{\oplus n} \cong P \oplus Q$ for two R-modules P and Q. In that case, we can view Q as the kernel of a surjection $R^{\oplus n} \to P$ and $R^{\oplus n}$ surjects onto Q as well.

For the second point, it suffices after the first point to show that invertible R-modules are finitely generated; this second statement is essentially a consequence of the definition of a tensor product in terms of finite sums of "pure" tensors. More precisely, suppose L is an invertible R-module and we are given an isomorphism $\varphi : L \otimes M \to R$. In that case, $\varphi^{-1}(1) = \sum_{i=1}^{s} x_i \otimes y_i$ for elements $x_i \in L$ and $y_i \in M$. Now, let $L' \subset L$ be the sub-module generated by $x_1, \ldots, x_s$. By construction L' is finitely generated and we will show that $L' \to L$ is an isomorphism.

To this end, note that the morphism $L' \otimes M \to L \otimes M \cong R$ is still surjective since $\varphi^{-1}(1) \in L' \otimes M$ by construction. Since $L' \to L$ is injective, we just want to show that the quotient L'/L is trivial. Consider the exact sequence $0 \to L' \to L \to L/L" \to 0$. Since M is invertible, we conclude that there is a short exact sequence of the form:

$$0 \longrightarrow L' \otimes_R M \longrightarrow L \otimes_R M \longrightarrow L/L' \otimes_R M \longrightarrow 0$$

Since the first map in this exact sequence is surjective by what we asserted before, we conclude that $L'' \otimes M = 0$. However, by associativity (and commutativity) of tensor product, $(L/L' \otimes_R M) \otimes_R L \cong L/L' \otimes_R (M \otimes_R L) \cong L/L' \otimes_R R \cong L/L'$. Thus, $L/L' = 0$, and we conclude. $\square$

Remark 2.2.1.10. Since finitely generated projective modules are automatically finitely presented, a necessary condition that a finitely generated flat module be projective is that it is also finitely presented. In fact, we will show later that finitely presented flat modules are exactly the finitely generated projective modules.

Over Noetherian rings, the link between projectivity and flatness is even more close: every finitely generated flat module is projective; this follows essentially from the equational criterion of flatness; see [**?**, Theorem 4.38]. Over non-Noetherian rings, there may be finitely generated flat modules that are not projective. Indeed, if $R = C^\infty(\mathbb{R})$ the ring of real valued smooth functions on the real line, and $\mathfrak{m}$ is the ideal of smooth functions vanishing at 0, then $R_\mathfrak{m}$ is the ring of germs of smooth functions at the origin. This module is flat because it is a localization. Set I to be the ideal of functions $f \in C^\infty(\mathbb{R})$ such that there exists $\epsilon > 0$ and $f(x) = 0$ for all $|x| < \epsilon$. One can check that, $R_\mathfrak{m} \cong R/I$, so $R_\mathfrak{m}$ is finitely generated as well. If R/I were projective, then the surjection $R \to R/I$ would split, and we could write $R \cong R/I \oplus I$. However, one can check that I is not even finitely generated.

Interlude: compact objects

In this interlude, we give a categorical interpretation of finitely presented modules and in doing so we give a simple proof of the fact that invertible R-modules are finitely presented.

Definition 2.2.1.11. If $\mathscr{C}$ is a category that admits filtered colimits, then an object $X \in \mathscr{C}$ is called *compact* if the functor $\mathrm{Hom}_\mathscr{C}(X, -) : \mathscr{C} \to \mathrm{Set}$ preserves filtered colimits.

The category Mod_R admits filtered colimits (inherited from the category of sets), so it makes sense to speak of compact objects in Mod_R. In Lemma 2.2.1.9 we observed that direct summands of finitely generated R-modules are necessarily finitely presented; we use this observation together with the following exercise to better understand compact objects in Mod_R.

Exercise 2.2.1.12. *Show that any R-module M can be written as a filtered colimit of finitely presented modules.*

Lemma 2.2.1.13. *Any compact object in Mod_R is finitely presented.*

Proof. Suppose M is compact. We can write M as a filtered colimit of finitely presented R-modules $M = \mathrm{colim}_i M_i$. Then, there are, by definition of compactness, a sequence of isomorphisms

$$\mathrm{Hom}_R(M, M) \cong \mathrm{Hom}_R(M, \mathrm{colim}_i M_i) \cong \mathrm{colim}_i \mathrm{Hom}_R(M, M_i).$$

In particular, the identity map $M \to M$ factors through a map $M \to M_i$ for some i sufficiently large. It follows that the inclusion map $M_i \to M$ can be split, so M is a direct summand of the finitely presented module M_i. To conclude, we appeal to the fact that direct summands of finitely presented R-modules are finitely presented. □

In fact, the converse to the above lemma holds. To see this, we will need some further information about filtered colimits in the category of R-modules. The following result is sometimes phrased as the assertion that "filtered colimits are exact in the category of R-modules."

Lemma 2.2.1.14. *Assume R is a commutative ring, I is a partially ordered set, viewed as a category. Assume given functors $M', M, M'' : I \to \mathrm{Mod}_R$, and natural transformations of functors $\varphi : M' \to M$, $\psi : M \to M''$ (we will write M_i for the value of M on $i \in I$ and $\mu_{ij} : M_i \to M_j$ for the corresponding morphism). Assume that for each $i \in I$, the sequence of R-modules*

$$M'_i \xrightarrow{\varphi_i} M_i \xrightarrow{\psi_i} M''_i$$

is a complex of R-modules (i.e., the composite $\psi_i \circ \varphi_i$ = 0) with homology H_i. In that case, there is an induced functor $H : I \to \mathrm{Mod}_R$ sending $i \in I$ to H_i, and the sequence

$$\mathrm{colim}_I\, M' \longrightarrow \mathrm{colim}_I\, M \longrightarrow \mathrm{colim}_I\, M''$$

is again a complex whose homology agrees with $\mathrm{colim}_I\, H$.

Proof. By naturality, there are induced maps $\ker(\psi_i) \to \ker(psi_j)$ whenever $i \leq j$ and likewise, there are maps $im(\varphi_i) \to im(\varphi_j)$ whenever $i \leq j$. Since $im(\varphi_i) \subset \ker(\psi_i)$ for all i by assumption, there are also induced maps $H_i \to H_j$ whenever $i \leq j$, which yields the functor $I \mapsto \mathrm{Mod}_R$.

Next, let us construct the map comparing the homology of the colimit with the colimit of the homology. There is a map $H_i \to \ker(\mathrm{colim}_I\, \psi)/im(\mathrm{colim}_I\, \varphi)$ for each i that simply sends a representative element in $\ker(\psi_i)$ to its image in the colimit. These morphism thus induce a morphism

$$\mathrm{colim}_I\, H \longrightarrow \ker(\mathrm{colim}_I\, \psi)/im(\mathrm{colim}_I\, \varphi).$$

We claim that this morphism is both injective and surjective, and this amounts to a careful analysis of the explicit construction of $\mathrm{colim}_I\, M$ as a suitable quotient of the coproduct of the M_i modulo some equivalence relation.

Take $h \in \ker(\mathrm{colim}_I\, \psi)/im(\mathrm{colim}_I\, \varphi)$. Choose a representative $[m] \in \ker(\mathrm{colim}_I\, \psi_i)$. Such an element comes from $m_i \in M_i$ for some i. The assumption that $[m]$ lies in $\ker(\mathrm{colim}_I\, \psi_i)$ means that the image of $\psi_i(m_i)$ in M''_j is zero for some $j \geq i$. After replacing i by j, it follows that h comes from H_i, which yields surjectivity.

For injectivity, suppose $h_i \in H_i$ has image zero in $\ker(\mathrm{colim}_I\, \psi)/im(\mathrm{colim}_I\, \varphi)$. We may represent h_i by an element $m_i \in \ker(\psi_i) \subset M_i$. Since the image of h_i is zero in $\ker(\mathrm{colim}_I\, \psi)/im(\mathrm{colim}_I\, \varphi)$, it follows that the image of m_i in $\mathrm{colim}_I\, M$ lies in $im(\mathrm{colim}_I\, \varphi)$. In other words, there exists $m'_j \in M'_j$ for some $j \geq i$ such that $\varphi_j(m'_j)$ coincides with the image of m_i in M_j. It follows that the image of h_i in H_j is necessarily zero, which yields the required injectivity. □

Proposition 2.2.1.15. *The compact objects in Mod_R are precisely the finitely presented R-modules.*

Proof. We already saw that compact objects are finitely presented in Lemma 2.2.1.13, so it remains to establish that finitely presented R-modules are compact objects. To see this, we use a series of reductions. First, suppose $N = \mathrm{colim}_i\, N_i$. Since $\mathrm{Hom}_R(R, -)$ is the identity functor on Mod_R, we conclude that $\mathrm{Hom}_R(R, \mathrm{colim}_i\, N_i) = \mathrm{colim}_i\, \mathrm{Hom}_R(R, N_i)$. Next, we can observe that both the functors $\mathrm{colim}_i\, \mathrm{Hom}_R(-, N_i)$ and $\mathrm{Hom}_R(-, N)$ commute with finite direct sums. Since filtered colimits are exact in the category of R-modules by Lemma 2.2.1.14, both of the functors just mentioned are actually right exact. Now, if we pick a presentation $R^{\oplus m} \to R^{\oplus n} \to M \to 0$ for a finitely presented R-module, then combining the observations just made with the 5-lemma allows us to conclude that finitely presented R-modules are compact. □

If $\mathscr{C}$ is any monoidal category, then we can speak of *invertible objects* in $\mathscr{C}$:

Definition 2.2.1.16. If $\mathscr{C}$ is any monoidal category, then an object $X \in \mathscr{C}$ is invertible if tensoring with X is an auto-equivalence of $\mathscr{C}$.

Remark 2.2.1.17. Invertible objects in Mod_R (with respect to $\otimes$) are precisely invertible R-modules.

Exercise 2.2.1.18. *An object X in a monoidal category $\mathscr{C}$ is invertible if and only if there exists an object $X^* \in \mathscr{C}$ such that $X \otimes X^*$ is isomorphic to the unit object in $\mathscr{C}$.*

Lemma 2.2.1.19. *If $\mathscr{C}$ is a monoidal category that admits filtered direct limits, then invertible objects are compact.*

Proof. Auto-equivalences of categories preserve filtered direct limits. □

2.2.2 Tensor products and extension of scalars

We now study the behavior of these various kinds of modules under tensor product.

Lemma 2.2.2.1. *If R is a commutative unital ring, and M_1 and M_2 are R-modules then the following statement hold.*

1. *If M_1 and M_2 are flat, then $M_1 \otimes_R M_2$ is flat;*
2. *if M_1 and M_2 are projective, then $M_1 \otimes_R M_2$ is projective; and*
3. *if M_1 and M_2 are invertible, then $M_1 \otimes_R M_2$ is invertible.*

Proof. Exercise. □

The above lemma implies that there is a natural binary operation on the set of isomorphism classes of invertible R-modules for a commutative ring R induced by tensor product. This binary operation is naturally associative because of associativity of tensor product, has a unit, given by the free R-module of rank 1, and has an inversion with respect to this choice of unit. Moreover, tensor product is symmetric monoidal, it follows that these structures equip the set of isomorphism classes of invertible R-modules with the structure of a commutative group.

Definition 2.2.2.2. If R is a commutative ring, then $Pic(R)$ is the commutative group of isomorphism classes of invertible R-modules (with the group structure described above).

Lemma 2.2.2.3. *If $f : R \to S$ is any ring homomorphism, then "extension of scalars", i.e., sending $M \to M \otimes_R S$ determines a functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$. Extension of scalars sends*

1. *flat R-modules to flat S-modules,*
2. *(finitely generated) projective R-modules to (finitely generated) projective R-modules, and*
3. *invertible R-modules to invertible R-modules.*

Proof. For the first statement, observe that there is always an isomorphism of functors $(M \otimes_R S) \otimes_S - \cong M \otimes_R (S \otimes_S -)$. Since $S \otimes_S -$ is the identity functor, we conclude that if M is a flat R-module, then $M \otimes_R S$ is a flat S-module.

For the second statement, if P is a (finitely generated) projective R-module, then $P \oplus Q \cong F$ for F a (finitely generated) free R-module. Then, $(P \oplus Q) \otimes_R S \cong P \otimes_R S \oplus Q \otimes_R S \cong F \otimes_R S$. Since $F \otimes_R S$ is a (finitely generated) free S-module, we conclude that $P \otimes_R S$ is (finitely generated) projective.

The third statement amounts to associativity of tensor product and is left as an exercise. □

Corollary 2.2.2.4. *If $\varphi : R \to S$ is a ring homomorphism, then there is an induced homomorphism $\varphi^* : Pic(R) \to Pic(S)$.*

Proof. We already saw that extension of scalars sends invertible modules to invertible modules. Granting that, observe that if L_1 and L_2 are invertible R-modules, then there are isomorphisms of the form

$$(L_1 \otimes_R L_2) \otimes_R S \cong L_1 \otimes_R (L_2 \otimes_R S) \cong (L_1 \otimes_R S) \otimes_S (L_2 \otimes_R S),$$

which yield the statement. □

2.2.3 Projective and locally free modules

Recall that if R is a commutative unital ring, then the Jacobson radical $J(R)$ is equal to the intersection of all maximal ideals of R (the intersection of the annihilators of simple R-modules).

Lemma 2.2.3.1 (Nakayama)**.** *If M is a finitely generated R-module and $M/J(R) \cdot M = 0$, then $M = 0$.*

Example 2.2.3.2. We will essentially always apply Nakayama's lemma in the situation where R is a local ring with maximal ideal $\mathfrak{m}$. In that case, $J(R) = \mathfrak{m}$.

Projective modules over local rings

Using Nakayama's lemma, we can analyze finitely generated projective R-modules over local rings.

Proposition 2.2.3.3. *If R is a local ring, then every finitely generated projective R-module is free.*

Proof. Supppose $\mathfrak{m}$ is the maximal ideal of R and $\kappa := R/\mathfrak{m}$ is the residue field. Assume P is a projective R-module. In that case, $P/\mathfrak{m} = P \otimes_R R/\mathfrak{m}$ is a finitely generated κ-module and thus a finite dimensional κ-vector space.

Take any R-module M and a morphism $\varphi : M \to P$. Set $\bar{\varphi} : M \otimes_R R/\mathfrak{m} \to P/\mathfrak{m}$. Right exactness of tensoring shows that the $\mathrm{coker}(\varphi) \otimes_R R/\mathfrak{m} \cong \mathrm{coker}(\bar{\varphi})$. In particular, if $\bar{\varphi}$ is an epimorphism, then Nakayama's lemma shows that φ is an epimorphism as well.

Now, fix a basis $\bar{e}_1, \ldots, \bar{e}_n$ for $P/\mathfrak{m}$. We may pick elements e_i lifting $\bar{e}_i$. In that case, we get a homomorphism $\psi : R^{\oplus n} \to P$ whose reduction modulo $\mathfrak{m}$ is surjective. By the discussion of the preceding paragraph, ψ is itself an epimorphism. In that case, $\ker(\psi)$ is a direct summand of $R^{\oplus n}$ and therefore also itself finitely generated and projective. Since $\ker(\psi)$ is trivial when reduced mod $\mathfrak{m}$, again by Nakayama's lemma we conclude that $\ker(\psi)$ is trivial. Therefore, we conclude that ψ is an isomorphism and thus P is free. □

Remark 2.2.3.4. Kaplansky showed [**?**] that projective modules over local rings are always free (without finite generation hypotheses). Along the way, Kaplansky established a remarkable structure theorem for projective modules: every projective module is a direct sum of countably generated projective modules.

Local trivializations of projective modules

The next result is a key consequence of the fact that finitely generated projective modules are finitely presented, combined with the results above about freeness of finitely generated projective modules over local rings.

Proposition 2.2.3.5. *Assume R is a commutative unital ring, $\mathfrak{p}$ is a prime ideal in R and P is a finitely generated projective R-module.*

1. *The localization $P_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of some finite rank n.*
2. *There exists an element $s \in R \setminus \mathfrak{p}$ such that the localization of P away from s is free, i.e., $P[\frac{1}{s}]$ is a free $R[\frac{1}{s}]$-module of rank n.*
3. *If $\mathfrak{p}'$ is any prime ideal not containing s, $P_{\mathfrak{p}'}$ is a free $R_{\mathfrak{p}'}$-module of rank n.*

In particular, if L is an invertible R-module, then $L_{\mathfrak{p}}$ is free of rank 1.

Proof. Regarding point (1): since $R_{\mathfrak{p}}$ is a local ring, and $P_{\mathfrak{p}}$ is a finitely generated projective $R_{\mathfrak{p}}$-module it is necessarily free of some finite rank n.

For Point (2): we begin by observing that since P is finitely generated, it is finitely presented, and we can write P as the cokernel of a matrix M with coefficients in R. Now, to say that $P_{\mathfrak{p}}$ is free, is to say that we can find an invertible matrix with coefficients in $R_{\mathfrak{p}}$ such that the product of this invertible matrix (the matrix expressing the change from the standard basis of $P_{\mathfrak{p}}$ that is obatined from writing it as a quotient of a free module to the basis in which it is a direct summand) and the matrix M. By the definition of localization, each element of M can be written in the form f_{ij}/h_{ij} where $f_{ij} \in R$ and $h_{ij} \in R \setminus \mathfrak{p}$. Taking s to be the product of the h_{ij}, we see that $M \in R[\frac{1}{s}]$, but this is what we wanted to show.

Point (3) is a special case of point (2). For the final statement, since invertible R-modules are always finitely presented, and invertible R-modules over local rings are all free of rank 1, the final assertion is a consequence of the previous ones. □

Definition 2.2.3.6. If R is a commutative unital ring, then an R-module M is said to be *locally free* if we can cover $\operatorname{Spec} R$ by basic open sets D_{f_i} such that each localization M_{f_i} is a free R_{f_i}-module. We will say that M is *finite locally free* if M is locally free and we may choose the f_i so that M_{f_i} is a finite rank free R_{f_i}-module.

Remark 2.2.3.7. By Exercise 1.1.1.39 if the f_i generate the unit ideal, then by quasi-compactness of $\operatorname{Spec} R$, we can pick a finite subset $I' \subset I$ that generates the unit ideal.

Proposition 2.2.3.8. *If R is a commutative unital ring and P is a finitely generated projective R-module, then there is a integer r and finitely many elements $f_1, \ldots, f_r \in R$ such that the family $f_1, \ldots, f_r$ generate the unit ideal in R and such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$-module of finite rank for each i. In other words, finitely generated projective R-modules are finite locally free R-modules.*

Proof. This follows by combining the finite presentation of P and the fact that $\operatorname{Spec} R$ is a quasi-compact topological space (see Exercise 1.1.1.39). In more detail, fix a prime ideal $\mathfrak{p}$ in R. By appeal to Proposition 2.2.3.5 we may find an element f_1 such that $P[\frac{1}{f_1}]$ is a free $R[\frac{1}{f_1}]$-module. Now, pick a prime ideal in $R/(f_1)$ and consider the associated prime ideal in R and repeat the procedure. Altogether we obtain a sequence of elements $f_1, f_2, \ldots$ such that $P[\frac{1}{f_i}]$ is a free $R[\frac{1}{f_i}]$-module. By construction, the open sets D_{f_i} cover $\operatorname{Spec} R$ and thus the family f_i generate the unit ideal. Since $\operatorname{Spec} R$ is quasi-compact, it follows that a finite number of these modules already generate the unit ideal, and restricting our attention to these yields the result. □

Remark 2.2.3.9. It is *not* the case that projective modules that are not finitely generated are locally free. In fact, this latter statement fails even for for countably generated projective modules. Indeed, there exists a countably generated ring R and a projective module M that is a direct sum of countably many locally free rank 1 modules such that M is not locally free [**?**, Lemma 88.26.5 Tag 05WG]. For this reason (and due to many other pathologies that appear), we will typically avoid speaking about infinitely generated projective modules.

Finitely presented flat modules

A variation on the above proof can be used to show that finitely presented flat modules are also finite locally free. We give this proof here for the sake of completeness. Before doing this, we record a useful lemma.

Lemma 2.2.3.10. *If R is a commutative ring, and $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of R-modules with M'' flat, then for any R-module N, the sequence*

$$0 \longrightarrow N \otimes_R M' \longrightarrow N \otimes_R M \longrightarrow N \otimes_R M'' \to 0$$

is exact as well.

Proof. We may choose a surjection $R^I \to N$ with kernel K. Tensoring the short exact sequence of the statement with this one, we obtain the following diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & K \otimes_R M' \arrow[r] \arrow[d] & K \otimes_R M \arrow[r] \arrow[d] & K \otimes_R M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'^I \arrow[r] \arrow[d] & M^I \arrow[r] \arrow[d] & M''^I \arrow[r] \arrow[d] & 0 \\
 & N \otimes_R M' \arrow[r] \arrow[d] & N \otimes_R M \arrow[r] \arrow[d] & N \otimes_R M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

where we have used the flatness of R^I and M. The result follows from the snake lemma applied to the terms in the top two rows: indeed, this result implies that the kernel of the map $K \otimes_R M'' \to M''^I$ maps surjectively onto the kernel of the map $N \otimes_R M' \to N \otimes_R M$ and the former is zero. $\square$

Lemma 2.2.3.11. *Finitely presented flat modules are finite locally free.*

Proof. Suppose M is finitely presented and flat. Choose a prime ideal $\mathfrak{p}$. In that case, $M \otimes_R \kappa(\mathfrak{p})$ is a finitely presented flat $\kappa(\mathfrak{p})$-module, i.e., a $\kappa(\mathfrak{p})$-vector space. Pick elements $e_1, \ldots, e_n \in M$ that lift a $\kappa(\mathfrak{p})$-module basis of $M \otimes_R \kappa(\mathfrak{p})$. By Nakayama's lemma, these elements generate the localization $M \otimes_R R_{\mathfrak{p}}$ as well, i.e., there is a surjection $\varphi : R_{\mathfrak{p}}^{\oplus n} \to M_{\mathfrak{p}}$. In fact, since M is finitely presented, there exists an element $f \in R$, $f \notin P$ such that $e_1, \ldots, e_n$ generate M_f. Now, the kernel of $\varphi : R_f^{\oplus n} \to M_f$ is necessarily also finitely generated.

Since M is a flat R-module, it follows that M_f is a flat R_f-module since localizations are flat. The lemma above applied to the exact sequence $0 \to \ker(\varphi) \to R_f^{\oplus n} \to M_f \to 0$ and tensoring with $\kappa(\mathfrak{p})$ implies that $\ker(\varphi) \otimes_R \kappa(\mathfrak{p}) = 0$. Thus, it follows again from Nakayama's lemma that $\ker(\varphi) = 0$. Arguing as above, we get the requisite open cover of $\operatorname{Spec} R$ and the result follows. □

2.3 Locally free modules are projective

Proposition 2.2.3.8 demonstrated that finite projective modules are finite locally free. We would like to know whether the converse holds: is a finite locally free module over a commutative ring R always finite projective? Our analysis of this question amounts to analyzing whether various conditions can be "glued together". Recall that if R is a commutative unital ring, and $f_1, \ldots, f_r$ is a sequence of elements that generated the unit ideal, then in the constructions of the structure sheaf of $\operatorname{Spec} R$, we showed that the sequence

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{r} R_{f_i} \longrightarrow \bigoplus_{i,j} R_{f_i f_j}$$

was exact (recall the first map is the diagonal map arising from the various localization homomorphism $R \to R_{f_i}$ while the second map sent a sequence $a_1, \ldots, a_r$ to the differences $\ldots, a_i - a_j, \ldots$ in the relevant localizations). Moreover, we also shows that if M is any R-module, then the sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{i} M_{f_i} \longrightarrow \bigoplus_{i,j} M_{f_i f_j}$$

is exact. These observations can be used to help *reconstruct* the module M from information about its various localizations.

2.3.1 Zariski descent I: patching modules and homomorphisms

In the above, we considered the ring homomorphism $R \to R_{f_1} \oplus \cdots R_{f_r}$. For notational simplicity, we will write $R' = R_{f_1} \oplus \cdots \oplus R_{f_r}$. The data of the localizations M_{f_i} for each $i = 1, \ldots, r$ amounts to specifying an R'-module. To understand thsi better, let us describe the category of R'-modules more explicitly. There are ring homomorphism $R_{f_i} \to R'$ for each i, and these are furthermore split by ring homomorphism $R' \to R_{f_i}$. Given a sequence $M_1, \ldots, M_r$ of R_{f_i} modules, we may therefore build an R'-module as follows: take the direct sum of the modules obtained by extension of scalars along the ring homomorphism $R_{f_i} \to R'$; we will write $M_1 \boxplus \cdots \boxplus M_r$ for the resulting R'-module. In fact, using extension of scalars along the ring homomorphism $R' \to R_{f_i}$, given an R'-module M we obtain a sequence M_i of R_{f_i}-modules. These constructions are mutually inverse and give an explicit description of the category $\operatorname{Mod}_{R'}$ in terms of the categories $\operatorname{Mod}_{R_{f_i}}$.

Remark 2.3.1.1. We could define the product category $Mod_{R_{f_1}} \times \cdots Mod_{R_{f_r}}$ as the category whose objects consist of sequences M_i of R_{f_i}-modules and where morphisms are sequences as well. The construction we just described gives an equivalence between the category $\operatorname{Mod}_{R'}$ and this category.

Extension of scalars along the ring homomorphism $R \to R'$ corresponds to a functor:

$$\operatorname{Mod}_R \longrightarrow \operatorname{Mod}_{R'}.$$

Notice that $R \to R'$ is a flat ring homomorphism as localizations are flat ring homomorphism and since direct sums of flat modules are flat. It follows that extension of scalars is actually an exact functor $\mathrm{Mod}_R \to \mathrm{Mod}_{R'}$. Of course, not every R'-module is an extension of scalars. Indeed, an arbitrary object of the form $M_1 \boxplus \cdots \boxplus M_r$ does not arise from an R-module unless each M_i is the localization of a *fixed* R-module. We can phrase the compatibility evident in this observation in a different way. If R_{f_i} and R_{f_j} are two different localizations of R, then given an R-module M, we get M_{f_i} and M_{f_j}. There is a canonical identification $(R_{f_i})_{f_j}$ with $(R_{f_j})_{f_i}$: both of these are *equal* to the ring $R_{f_if_j}$. In particular, associativity of tensor product then yields a distinguished isomorphism

$$\theta_{ij} : (M \otimes_R R_{f_i}) \otimes_{R_{f_i}} R_{f_if_j} \cong M \otimes_R R_{f_if_j} \cong (M \otimes_R R_{f_j}) \otimes_{R_{f_j}} R_{f_if_j}.$$

In particular, in order for an R'-module $M_1 \boxplus \cdots \boxplus M_r$ to arise from an R-module, we must specify isomorphisms

$$\theta_{ji} : M_i \otimes_{R_{f_i}} R_{f_if_j} \xrightarrow{\sim} M_j \otimes_{R_{f_j}} R_{f_if_j},$$

for each pair i, j. Geometrically this is just the gluing isomorphism on 2-fold intersections. Note that we have such an isomorphism even when $i = j$: in that case, both the source and target are the same and the map is simply the identity map, so we have the normalization that $\theta_{ii} = id$ for each i.

We can repackage this collection of isomorphisms in the following way. The ring $\bigoplus_{i,j} R_{f_if_j}$ can be identified as $R' \otimes_R R'$ using the distributivity of tensor product over direct sum. From the tensor product description, there are two *different* extensions of scalars $R' \to R' \otimes_R R'$ corresponding to the universal map for the left or right-hand factors. Given an R'-module, we thus get two different $R' \otimes_R R'$-modules by extension of scalars along the left or right-hand factors. The family of isomorphisms $\{\theta_{ji}\}_{i,j}$, then amounts to an isomorphism θ between these two different pullbacks. Given an R'-module M', let us write M'_r for $R' \otimes_R R'$-module obtained by extension of scalars along the right factor and M'_l for the extension of scalars along the left factor. Our isomorphisms θ_{ji}, then amount to specifying an isomorphism $\theta : M'_l \xrightarrow{\sim} M'_r$.

If there are more than two f_i, then there is a natural further condition. Given a third f_k, the isomorphism θ_{ji} yields an isomorphism of $M_i \otimes_R R_{f_if_jf_k}$ with $M_j \otimes_R R_{f_if_jf_k}$. Likewise, θ_{kj} yields an isomorphism of $M_j \otimes_R R_{f_if_jf_k}$ with $M_k \otimes_R R_{f_if_jf_k}$, and θ_{ik} yields an isomorphism of $M_k \otimes_R R_{f_if_jf_k}$ with $M_i \otimes_R R_{f_if_jf_k}$. It thus makes sense to consider the composite $\theta_{ik} \circ \theta_{kj} \circ \theta_{ji}$ as an endomorphism of $M_i \otimes_R R_{f_if_jf_k}$. If each M_i arises as the localization of an R-module, then it is easy to see that this composite is the identity self-map of $M_i M_i \otimes_R R_{f_if_jf_k}$ for every triple of indices i, j, k. Equivalently, $\theta_{kj} \circ \theta_{ji}$ and θ_{ki} have the same source and target and the compatibility condition can also be phrased as saying these two composites are equal for every triple of indices.

The compatibility condition we just wrote can also be repackaged as follows. The ring $\bigoplus_{i,j,k} R_{f_if_jf_k}$ can be identified with $R' \otimes_R R' \otimes_R R'$. There are now *three* different extensions of scalars $R' \otimes_R R' \to R' \otimes_R R' \otimes_R R'$ corresponding to the first and second factors, the first and third factors and the second and third factors. We thus obtain three further extension of scalars maps corresponding to these ring maps. Let us write $p_{12}^*\theta$, $p_{13}^*\theta$ and $p_{23}^*\theta$ for three $R' \otimes_R R' \otimes_R R'$-module isomorphisms we obtain by extending scalars for θ. In that case, our condition amounts to the equality $p_{12}^*\theta \circ p_{23}^*\theta = p_{13}^*\theta$.

Given the ring homomorphism $R \to R'$ above, define a new category $\mathrm{Mod}_R(f_1, \ldots, f_r)$ whose objects consist of pairs (M', θ) where M' is an R'-module, and $\theta : M'_l \xrightarrow{\sim} M'_r$ is an isomorphism such that $p_{12}^*\theta, p_{13}^*\theta = p_{13}^*\theta$ on $R' \otimes_R R' \otimes_R R$". A morphism in $\mathrm{Mod}_R(f_1, \ldots, f_r)$ is an R'-module map $\varphi : M' \to N'$ that is compatible with the isomorphism θ. The category $\mathrm{Mod}_R(f_1, \ldots, f_r)$ admits a functor to $\mathrm{Mod}_{R'}$ (forget the isomorphism θ). The discussion above shows that extension of scalars deifnes a functor

$$\mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R(f_1, \ldots, f_r).$$

Since extension of scalars along a flat homomorphism is exact, it follows that the above functor is actually exact. We claim that this functor is actually an equivalence of categories. To this end, we will construct an explicit quasi-inverse functor.

To construct the quasi-inverse, recall that given an R-module M, there is an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow \bigoplus_{i \in I} M_{f_i} \longrightarrow \bigoplus_{i,j} M_{f_i f_j},$$

where the right hand morphism sends $(m_1, \ldots, m_r)$ to the differences $m_i - m_j$. In particular, M can be recovered either as a kernel or, as the collection of sequences of elements in M_{f_i} such that $(\cdots, m_i, \cdots, m_j, \cdots)$ such that m_i and m_j have the same restrction in $M_{f_i f_j}$. We can phrase this abstractly using the isomorphism θ as follows. If we give ourselves an R'-module M', then we can use the ring homomorphism $R \to R'$ to view M' as an R-module. We get two $R' \otimes_R R'$-modules $M'l$ and M'_r as above, which are identified via an isomorphism θ. An element m' of M' gives rise to an element of M'_r and M'_l by just taking its image under the relevant extension of scalars; we write m'_r and m'_l for the relevant elements. In that case, $\theta(m'_l)$ and m'_r both lie in M'_r and we can ask whether they are equal. Equivalently, we may view θ as giving an R-module homomorphism $M' \to M'_r$, and we can consider the homomorphism $id - \theta : M' \to M'_R$. When M is an R-module, we recover M from M' as the kernel of $id - \theta$.

Given an object of $\mathrm{Mod}_R(f_1, \ldots, f_r)$, the construction just described defines an R-module by sending the pair (M', θ) to the R-submodule of M' defined by $\ker(id - \theta)$. This construction is functorial as well: the restriction of a morphism in $\mathrm{Mod}_R(f_1, \ldots, f_r)$ to $\ker(id - \theta)$ defines an R-module map. In other words, we've constructed a functor

$$\mathrm{Mod}_R(f_1, \ldots, f_r) \longrightarrow \mathrm{Mod}_R.$$

We now summarize some properties of these functors in the following statement.

We already know that the composite of the above functor with extension of scalars $\mathrm{Mod}_R \to \mathrm{Mod}_R(f_1, \ldots, f_r)$ is the identity by the exactness statement above and the fact that the other composite is the identity is a straightforward exercise.

Theorem 2.3.1.2 (Zariski patching I)**.** *Assume R is a commutative ring, and $f_1, \ldots, f_r$ are elements of R that generate the unit ideal. The functors*

$$\mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R(f_1, \ldots, f_r)$$

given by extension of scalars, and

$$\mathrm{Mod}_R(f_1, \ldots, f_r) \longrightarrow \mathrm{Mod}_R$$

induced by sending a pair (M', θ) to $\ker(id - \theta)$ define mutually inverse equivalences of categories.

Proof. First, we show that the composite $\mathrm{Mod}_R \to \mathrm{Mod}_R(f_1,\dots,f_r) \to \mathrm{Mod}_R$ is the identity functor. That this composite is the identity on objects follows immediately from Lemma 1.3.1.3. That it is the identity on morphisms is a diagram chase.

Next, consider the composite $\mathrm{Mod}_R(f_1,\dots,f_r) \to \mathrm{Mod}_R \to \mathrm{Mod}_R(f_1,\dots,f_r)$. Let (M',θ) be an object of $\mathrm{Mod}_R(f_1,\dots,f_r)$ and let $M = \ker(id-\theta)$ viewed as an R-module. Write $M' = (M_1,\dots,M_r)$, where each M_i is an R_{f_i}-module. The map $M \to M_i$ factors through M_{f_i} by the universal property of localization. Therefore, there is an induced R'-module map $M_{f_1} \boxplus \cdots \boxplus M_{f_r} \to M'$. We claim this morphism of R'-modules is an isomorphism. We leave the check that this induced morphism is injective and surjective as an exercise. □

Example 2.3.1.3 (Characteristic polynomials of endomorphisms). The results above show that *elements* of a module are "locally determined". Here is an application of this fact. Suppose P is a finitely generated projective module over a commutative unital ring R of fixed rank n. Given α an endomorphism of P, we describe how to attach a characteristic polynomial to α. Suppose we take a sequence of elements $f_1,\dots,f_r \in R$ such that each P_{f_i} is a free R_{f_i}-module. In that case, choosing a basis of P_{f_i} as an R_{f_i}-module, we can define the characteristic polynomial of α_{f_i} in the usual way as $\det(\alpha_{f_i} - \lambda Id_n) = P(\alpha_{f_i},\lambda) \in R_{f_i}[\lambda]$ (and, as usual, the expression is independent of the choice of basis). Now, taking determinants of matrices commutes with extension of scalars. Since the modules $(P_{f_i})_{f_j}$ and $(P_{f_j})_{f_i}$ are isomorphic, the elements $P(\alpha_{f_i},\lambda)$ and $P(\alpha_{f_j},\lambda)$ necessarily coincide when viewed as elements of $R_{f_if_j}[\lambda]$. Therefore, there we deduce that there is an element $P(\alpha,\lambda) \in R[\lambda]$ that restricts to $P(\alpha_{f_i},\lambda)$. One can establish the existence of characteristic polynomials in general using an inductive argument and the fact that projective modules are locally free. Moreover, one can show by refining covers that the characteristic polynomial so defined is independent of the choice of cover. The characteristic polynomial defined in this fashion has all the usual properties of the characteristic polynomial, e.g., the Cayley–Hamilton theorem holds, i.e., α satisfies $P(\alpha,\lambda)$.

Example 2.3.1.4. If α is an endomorphism of a rank n projective module over a ring R, then we can define $tr(\alpha), \det(\alpha)$ and similar expressions. In particular, if P is a projective module of rank n, then there is a homomorphism $Aut_R(P) \to R^{\times}$ sending α to its determinant.

2.3.2 Zariski descent II: properties of modules

Now that we know how to reconstruct a module from suitable information at localizations, we can ask whether properties of modules can also be patched together. Given a property P of modules that is stable by localization, we can ask the following: if $(M',\theta) \in \mathrm{Mod}_R(f_1,\dots,f_r)$ is such that M' has property P, does the object $M \in \mathrm{Mod}_R$ obtained by the equivalence of Theorem 2.3.1.2, have property P as well?

Definition 2.3.2.1. A property P for R-modules that is stable by localization will be called *local for the Zariski topology on* $\operatorname{Spec} R$ if an R-module M has property P if and only if for any family $\{f_i\}_{i\in I}$ in R that generates the unit ideal, the modules M_{f_i} have property P.

Faithfully flat ring maps detect exactness

If $f_1, \ldots, f_r$ is a family of elements in a ring R, then we know that setting $R' = \bigoplus_{i=1}^r R_{f_i}$, that $R \to R'$ is a flat ring homomorphism. As such, we know that $\otimes_R R'$ *preserves* exactness. We now argue that, in fact, $R \to R'$ *detects* exactness.

Lemma 2.3.2.2. *If $f_1, \ldots, f_r$ is a family of elements in a ring R, then set $R' = \bigoplus_{i=1}^r R_{f_i}$. A sequence $M_1 \to M_2 \to M_3$ is exact if and only if $M_1 \otimes_R R' \to M_2 \otimes_R R' \to M_3 \otimes_R R'$ is exact.*

Proof. The "only if" direction is immediate. Therefore we focus on the "if" direction. Since $\operatorname{Spec} R' \to \operatorname{Spec} R$ is an open cover, it follows that for any maximal ideal $\mathfrak{m}$ of R, $R'/\mathfrak{m}R'$ is non-zero: indeed, since $f_1, \ldots, f_r$ generate the unit ideal, there exists some f_i such that $f_i \notin \mathfrak{m}$.

Take an arbitrary sequence $M_1 \to M_2 \to M_3$ such that $M_1 \otimes_R R' \to M_2 \otimes_R R' \to M_3 \otimes_R R'$ is exact. Consider the R-module $H = \ker(M_2 \to M_3)/\operatorname{im}(M_1 \to M_2)$, which measures the failure of exactness of M. By assumption

$$H \otimes_R R' \cong \ker(M_2 \otimes_R R' \to M_3 \otimes_R R')/\operatorname{im}(M_1 \otimes_R R' \to M_2 \otimes_R R') = 0.$$

Now, take an element $x \in H$. There is an induced R-module map $R \to H$. If $I = \{r \in R | rx = 0\}$ (i.e., the annihilator ideal of x), then this map factors through an injection $R/I \subset H$. Now, $R/I \otimes_R R' \cong R'/IR' \subset H \otimes_R R' = 0$ again by flatness of $R \to R'$. If $I \neq R$, then there is a maximal ideal $\mathfrak{m}$ containing I, which yields a contradiction. $\square$

Definition 2.3.2.3. A ring homomorphism $\varphi : R \to S$ is called *faithfully flat* if a sequence $M_1 \to M_2 \to M_3$ of R-modules is exact if and only if the sequence $S \otimes_R M_1 \to S \otimes_R M_2 \to S \otimes_R M_3$ is exact.

Remark 2.3.2.4. The argument above proves that if $R' = R_{f_1} \oplus \cdots R_{f_r}$ for $f_1, \ldots, f_r$ generating the unit ideal in R, then $R \to R'$ is faithfully flat. In fact, more generally, that argument implies the following.

Proposition 2.3.2.5. *If $R \to S$ is a flat ring homomorphism such that the map* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is surjective on closed points, then $R \to S$ is faithfully flat.*

Proof. As before, take an arbitrary sequence, $M_1 \to M_2 \to M_3$ such that $M_1 \otimes_R R' \to M_2 \otimes_R R' \to M_3 \otimes_R R'$ is exact and let H be the homology module of the first sequence. By assumption $H \otimes_R S = 0$. Take an element $x \in H$ and consider the induced R-module map $R \to H$, and let I be the annihilator ideal of x. As above, the R-module map $R \to H$ factors through $R/I \to H$. In that case, $S/IS \subset H \otimes_R S$. If $\neq R$, then there exists a maximal ideal $\mathfrak{m}$ containing I, i.e., $S/\mathfrak{m}S = 0$. However, the condition that $S/\mathfrak{m}S = 0$ is equivalent to the assertion that $\mathfrak{m}$ does not lie in the image of $\operatorname{Spec} S \to \operatorname{Spec} R$, which contradicts the assumption that $\operatorname{Spec} \varphi$ was surjective on closed points. $\square$

Finite generation and finite presentation

If R is a ring, then the properties of finite generation, finite presentation of an R-module are stable under localization. Indeed, if $R \to R'$ is a localization, then suppose M is a finitely generated

R-module. This means that there is a surjection $R^{\oplus n} \to M$ since extension of scalars along R' is exact, it follows that $R'^{\oplus n} \to M \otimes_R R'$ is again surjective. The same statement holds for finite presentation and also for coherence of R-modules (exercise!).

Proposition 2.3.2.6. *Assume R is a commutative ring and M is an R-module, finite generation, finite presentation and coherence of M are properties local for the Zariski topology.*

Proof. Suppose we have $f_1, \ldots, f_r$ that generate the unit ideal in R and all the localizations M_{f_i} of M are finitely generated. We want to check that M is finitely generated. Choose generators $R_{f_i}^{\oplus m_i} \to M_{f_i}$ for each i. Without loss of generality, we may assume these generators are in the image of the localization map, i.e., we may find elements $x_i(j) \in M$, $j = 1, \ldots m_i$ whose images in M_{f_i} yield the m_i-chosen generators of R_{f_i}. In that case, these elements determine an R-module morphism $R^{\oplus N} \to M$. This morphism is surjective after localization at any maximal ideal $\mathfrak{m}$ since for any given $\mathfrak{m} \subset R$ some $f_i \notin m$. It follows from Corollary B.1.0.6 that the $R^{\oplus N} \to M$ must be surjective.

Next, suppose we know that M_{f_i} is finitely presented for each i. By the first part, we have a surjection $R^{\oplus n} \to M$. Let K be its kernel. Since M_{f_i} is finitely presented, we know that K_{f_i} is finitely generated for each i since localization is an exact functor. It follows from the preceding paragraph that K must again be finitely generated, which is what we wanted to show.

For the final statement, suppose M is an R-module, and suppose M_{f_i} is coherent for each i. Take a finitely generated submodule $M' \subset M$. By assumption $(M')_{f_i}$ is a finitely generated submodule of M_{f_i}. Since M_{f_i} is coherent, it follows that M'_{f_i} is again finitely presented, but then we conclude by appeal to the conclusion of the preceding paragraph. $\square$

The property that an R-module P is finitely generated projective is stable under localization. Indeed, this follows from the characterization of P as a summand of a free module since localization is exact. Thus, it makes sense to ask whether the property of being finitely generated projective is local for the Zariski topology. Thus, assume that P is an R-module such that P_{f_i} is projective. Since P_{f_i} is finitely generated projective, after further localizing, we may even assume that $P_{f_i g}$ is a finite rank free R-module. Thus, it suffices to show that if P_{f_i} is a finite rank free module for each i, then P is a finitely generated projective R-module. To check this latter statement, we will check that $\mathrm{Hom}_R(P, -)$ is actually an exact functor. Before analyzing this question, we will need some preliminary results about the relationship between localization and $\mathrm{Hom}_R(-,-)$.

Given a ring homomorphism $R \to S$, functoriality of extension of scalars yields a natural map $\mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$; this factors through a map

$$\mathrm{Hom}_R(M, N) \otimes_R S \longrightarrow \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S).$$

If S is a flat R-module, then $\otimes_R S$ is an exact functor. First, let us analyze the situation with localizations.

Proposition 2.3.2.7. *Let R be a ring, and $S \subset S$ a multiplicative subset. If M and N are R-modules with M finitely presented, then the canonical maps given by localization induced identifications of the form:*

$$Hom_R(M, N)[S^{-1}] = Hom_{R[S^{-1}]}(M[S^{-1}], N[S^{-1}]) = Hom_R(M[S^{-1}], N[S^{-1}]).$$

Proof. First, observe that the statement is true if M is a finite rank free module. Indeed, in that case,

$$\operatorname{Hom}_R(R^{\oplus i}, N) = N^{\oplus i},$$

since $\operatorname{Hom}_R(R, -)$ is the identity functor. Then,

$$\operatorname{Hom}_R(R^{\oplus i}, N)[S^{-1}] = N^{\oplus i}[S^{-1}] = N[S^{-1}]^{\oplus i} = \operatorname{Hom}_R[S^{-1}](R[S^{-1}]^{\oplus i}, N[S^{-1}]).$$

For the general case, simply choose a presentation $R^{\oplus n} \to R^{\oplus m} \to M$. Applying the functor $\operatorname{Hom}_R(-, N)$ to this sequence yields

$$0 \longrightarrow \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_R(R^{\oplus m}, N) \longrightarrow \operatorname{Hom}_R(R^{\oplus n}, N).$$

Since $\operatorname{Hom}_R(R, -)$ is the identity functor, it follows that $\operatorname{Hom}_R(R^{\oplus i}, N) = N^{\oplus i}$. Using this identification and localizing, we get an exact sequence of the form

$$0 \longrightarrow \operatorname{Hom}_R(M, N)[S^{-1}] \longrightarrow N[S^{-1}]^{\oplus m} \longrightarrow N[S^{-1}]^{\oplus n}.$$

and the result follows from the corresponding statement when M is a finite-rank free module. □

In fact, the above proof actually establishes the following fact.

Proposition 2.3.2.8. *If $\varphi : R \to R'$ is a flat ring homomorphism, and if M and N are R-modules with M finitely presented (resp. finitely generated), then the map*

$$\operatorname{Hom}_R(M, N) \otimes_R S \longrightarrow \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an isomorphism (resp. monomorphism).

Proposition 2.3.2.9. *Assume R is a commutative ring and P is an R-module. The property that P is finitely generated projective is local for the Zariski topology.*

Proof. As before, we may assume that P is an R-module, and that P_{f_i} is actually a finite rank free R_{f_i}-module for each i. In that case, we will check that $\operatorname{Hom}_R(P, -)$ is an exact functor. Since P is finitely presented, it follows from our preceding analysis that

$$\operatorname{Hom}_R(P, N)_{f_i} = \operatorname{Hom}_{R_{f_i}}(P_{f_i}, N_{f_i})$$

If we are given an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of R-modules, then we conclude that the induced sequence of R-modules:

$$0 \longrightarrow \operatorname{Hom}_R(P, N_1) \longrightarrow \operatorname{Hom}_R(P, N_2) \longrightarrow \operatorname{Hom}_R(P, N_3)$$

is exact, and we want to check exactness on the right. Upon localizing at each f_i, this follows from the assertion that P_{f_i} is a finite rank free module by appeal to Proposition 2.3.2.8. It follows that the result holds upon localization at an arbitrary maximal ideal $\mathfrak{m}$ of R, and we conclude. □

Corollary 2.3.2.10. *Assume R is a commutative ring. If M is an R-module, then the following statements are equivalent*

1. *M is finitely generated and projective;*
2. *M is finitely presented and flat;*
3. *M is finite locally free;*

Proof. The first statement clearly implies the second. That the second statement implies the third is the conclusion of Lemma 2.2.3.11. The assertion that the third statement implies the first follows from Proposition 2.3.2.9. □

2.3.3 Rank

If R is a ring and P is a projective R-module, then for any prime ideal $\mathfrak{p} \subset R$, we see that $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus n}$. By Proposition 2.2.3.5, if P is furthermore finitely generated, then there is a Zariski open set contained in $\operatorname{Spec} R \setminus \operatorname{Spec} R/\mathfrak{p}$ on which the rank is constant. Using this observation, one deduces that sending a projective module to the integer n described above yields a continuous function from $rk : \operatorname{Spec} R \to \mathbb{N}$ (the latter viewed as a discrete topological space). Note, in particular, that rk of a projective module is bounded, and locally constant. Thus, if R is a connected ring, then the rank of a projective module is simply an integer.

Exercise 2.3.3.1. *If R is a commutative unital ring, show that $\operatorname{Spec} R$ is connected if and only if R has no non-trivial idempotents.*

Remark 2.3.3.2. Because of the conclusion of the previous exercise, we will call a commutative unital ring R *connected* if it has no non-trivial idempotents. In general, we can attempt to form a decomposition of R using commuting idempotents. However, without some finiteness hypothesis on R, it is possible that $\operatorname{Spec} R$ has infinitely many connected components: e.g., take any (say connected) ring R and form the ring $\bigoplus_{n \in \mathbb{N}} R$. Nevertheless, if we focus attention on finitely generated projective modules, then the rank of any projective module takes only finitely many values. While we focus on connected rings for simplicity, the observation just mentioned will allow us to make statements about general disconnected rings as well.

Example 2.3.3.3. As we observed above, invertible modules always have constant rank 1.

When L is an invertible R-module, $\operatorname{Hom}_R(L, R)$ is again an invertible R-module. There is a canonical evaluation map $M \otimes_R M^{\vee} \longrightarrow R$. Moreover, in this case, the evaluation map $L \otimes \operatorname{Hom}_R(L, R) \to R$ is an isomorphism: the identity map $\operatorname{Hom}_R(\operatorname{Hom}(L, R), \operatorname{Hom}(L, R))$ corresponds under the hom-$\otimes$-adjunction to the evaluation map $\operatorname{Hom}_R(\operatorname{Hom}(L, R) \otimes L, R)$. Alternatively, the evaluation map is evidently locally an isomorphism and therefore must be an isomorphism in general. Thus, L is an invertible module, there is a distinguished choice for a module L' such that $L \otimes L' \cong R$, namely $\operatorname{Hom}_R(L, R)$.

Exercise 2.3.3.4. *If P and Q are projective R-modules of rank m and n, then $rk(P \oplus Q) = m + n$ and $rk(P \otimes Q) = mn$.*

Faithfully flat descent

The proof that we gave for the fact that finite projective modules are local for the Zariski topology can actually be generalized quite a bit. Suppose $\varphi : R \to S$ is a flat ring homomorphism. It follows from the direct sum characterization of projective modules that if P is a projective R-module, then $P \otimes_R S$ is again a projective R-module. We would like to ask about when the converse is true. By what we saw above, that P is a finite projective R-module is equivalent to P being finitely presented and flat. As such, it suffices to inquire about these two conditions.

Proposition 2.3.3.5. *Assume R is a commutative ring and $\varphi : R \to S$ is a faithfully flat ring homomorphism. If M is an R-module, and $M \otimes_R S$ is finitely generated (resp. finitely presented, resp. coherent), then so is M.*

Proof. Assume $M \otimes_R S$ is finitely generated and pick generators $y_1, \ldots, y_r$ of $M \otimes_R S$. Each y_i can be written as a finite sum $m_j \otimes s_j$ for suitable elements m_j of M. These elements define a homomorphism $R^{\oplus n} \to M$ that upon extension of scalars by S has image containing the generators $y_1, \ldots, y_r$. Since $\varphi : R \to S$ is faithfully flat, it follows that the original map $R^{\oplus n} \to M$ must also be surjective, which is what we wanted to show.

The statements about finite presentation and coherence are reduced to the preceding statement. If $M \otimes_R S$ is finitely presented, then we may choose a surjection $R^{\oplus n} \to M$ after the conclusion of the preceding paragraph. Let K be the kernel of this surjection. By flatness, we conclude that $K \otimes_R S$ coincides with the kernel of $S^{\oplus n} \to M \otimes_R S$, which is itself finitely generated. Another appeal to the conclusion of the preceding paragraph guarantees that K is finitely generated as well.

Finally, suppose M is an R-module and $M \otimes_R S$ is coherent. We conclude that M is finitely presented by appeal the conclusion of the preceding paragraph. Take a finitely generated submodule M' of M. In that case, $M' \otimes_R S$ is again a finitely generated submodule of $M \otimes_R S$. However, since the later is coherent, $M' \otimes_R S$ must actually be finitely presented. Again appealing to the conclusion of the preceding paragraph, we conclude that M' is finitely presented as well. $\square$

Proposition 2.3.3.6. *Assume R is a commutative ring and $\varphi : R \to S$ is a faithfully flat ring homomorphism. If M is an R-module such that $M \otimes_R S$ is flat projective, then M is also flat.*

Proof. Suppose $N_1 \to N_2 \to N_3$ is an exact sequence of R-modules. We want to show that this sequence remains exact after tensoring with R. Consider the sequence of R-modules

$$N_1 \otimes_R M \longrightarrow N_2 \otimes_R M \longrightarrow N_3 \otimes_R M.$$

Tensoring this sequence with S we get the sequence

$$N_1 \otimes_R M \otimes_R S \longrightarrow N_2 \otimes_R M \otimes_R S \longrightarrow N_3 \otimes_R M \otimes_R S.$$

Since the module $M \otimes_R S$ is a flat R-module by assumption, the above sequence is flat. However, since $R \to S$ is a faithfully flat ring map, it follows that our initial sequence was exact as well. $\square$

Corollary 2.3.3.7. *If $\varphi : R \to S$ is a faithfully flat ring homomorphism, and P is an R-module, then P is finite projective if and only if $P \otimes_R S$ is finite projective.*

Proof. Since finite projective modules are finitely presented and flat, this follows from the preceeding propositions. $\square$

2.4 Vector bundles

Above, we saw that projective modules are automatically locally free.

2.4.1 Vector bundles on manifolds

Suppose M is a (smooth) closed manifold, and let $C(M)$ (resp. $C^\infty(M)$) be the ring of continuous (resp. smooth) functions on M. Let $\pi : E \to M$ be a (smooth) vector bundle on E. The set of global (smooth) sections $C(\pi)$ (resp. $C^\infty(\pi)$) has a natural $C(M)$ (resp. $C^\infty(M)$)-module structure arising from the fact that E is fiberwise a vector space.

Corollary 2.4.1.1. *If $\pi : E \to M$ is a (smooth) vector bundle, then $C(\pi)$ (resp. $C^\infty(\pi)$) is a projective $C(M)$-module.*

Proof. If $p_2 : \mathbb{R}^n \times M \to M$ is a trivial vector bundle, then the module of sections of p_2 is a free $C(M)$-module (resp. $C^\infty(M)$)-module of finite rank. Now, since M is a (smooth) closed manifold, we can find a finite open cover of M on which the vector bundle trivializes. Indeed, (smooth) closed manifolds have the homotopy type of CW complexes (for smooth manifolds, this follows, e.g., from Morse theory, but that result is also true for metrizable absolute neighborhood retracts by different methods; see Milnor's paper). For a CW complex we leave it as an exercise to build the necessary cover. □

Remark 2.4.1.2. The same proof works for vector bundles on any space having the homotopy type of a finite CW complex. In fact, with a small change in definitions, we can characterize the vector bundles that arise from projective modules. Assume X is a topological space. A finite partition of 1 on X is a sequence $f_1, \ldots, f_r$ of non-negative continuous functions such that $\sum_i f_i = 1$. We will say that a vector bundle E on X has finite type if there is a finite partition of 1 on X such that $E|_{f_i \neq 0}$ is trivial. With this definition, it follows that finite type vector bundles correspond precisely to projective $C(X)$-modules by the results above. In fact, this bijection of sets can be turned into a suitable categorical statement; this result is known as "Vaserstein's Serre-Swan theorem" [reference].

2.4.2 Vector bundles on ringed spaces (e.g., schemes)

Assume $(X, \mathscr{O}_X)$ is a ringed space. We will say that an $\mathscr{O}_X$-module $\mathscr{E}$ is locally free if there exists an open cover U_i of X together with isomorphisms $\mathscr{E}|_{U_i} \cong \mathscr{O}_{U_i}^{I_i}$ forsuitable index sets I_i. An $\mathscr{O}_X$-module is *finite locally free* if the I_i can be chosen to be finite. Based on what we observed above, we will think of finite locally free modules as vector bundles. In particular, this gives a definition of a vector bundle on a scheme. Note that this definition is not particularly geometric, but at least the following result is immediate from the definitions.

Lemma 2.4.2.1. *Any locally free $\mathscr{O}_X$-module is quasi-coherent.*

Before moving on, let's make sure this definition really does generalize projective modules.

Proposition 2.4.2.2. *Assume R is a commutative ring, $\mathscr{F}$ is a rank n locally free sheaf of $\mathscr{O}_{\operatorname{Spec} R}$-modules and set $P := \Gamma(\operatorname{Spec} R, \mathscr{F})$. In that case:*

1. *P is a projective R-module; and*
2. *there is and induced isomorphism $\mathscr{F} \to \widetilde{P}$ of $\mathscr{O}_{\operatorname{Spec} R}$-modules.*

Proof. The statement is evidently true for free R-modules of rank n, so we will reduce to this case. To this end, choose an open cover U_i of $\operatorname{Spec} R$ on which $\mathscr{F}|_{U_i} \cong \mathscr{O}_{U_i}^{\oplus n}$. Refining our open cover if necessary, we may assume that $U_i = D_{f_i}$. In other words, fix isomorphisms $\varphi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \widetilde{R_{f_i}^{\oplus n}}$. The restriction $\varphi_i|_{U_{ij}} \circ \varphi_j^{-1}|_{U_{ij}}$ is an automorphism of $\widetilde{R_{f_if_j}^{\oplus n}}$. Taking global sections, we obtain an isomorphism θ_{ij} of $R_{f_if_j}$. These isomorphisms are necessarily compatible on threefold intersections since they come from a sheaf $\mathscr{F}$ on $\mathscr{O}_{\operatorname{Spec} R}$, so by appeal to Theorem 2.3.1.2, they define an R-module P. Corollary 2.3.2.10 shows that P is projective, which establishes the first point.

Theorem 2.3.1.2 also gives a morphism $\Gamma(\operatorname{Spec} R, \mathscr{F}) \to P$, since in the preceding paragraph we constructed such a morphism locally over D_{f_i}. Since the maps on each D_{f_i} are isomorphisms, it follows that the morphism $\Gamma(\operatorname{Spec} R, \mathscr{F}) \to P$ is again an isomorphism. There is an induced morphism $\mathscr{F} \to \widetilde{P}$. Once again, this morphism is an isomorphism upon restriction to each D_{f_i} by construction, and therefore is an isomorphism of sheaves as well. $\square$

Remark 2.4.2.3. We will use this proposition without mention in the future to identify rank n locally free sheaves of $\mathscr{O}_{\operatorname{Spec} R}$-modules with rank n projective R-modules.

Example 2.4.2.4. Assume k is a field. Let us write $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. By the preceding proposition, rank n locally free sheaves of $\mathscr{O}_{\mathbb{A}^1_k}$-modules correspond to rank n projective $k[x]$-modules. By the structure theorem for finitely generated modules over a PID, such sheaves are free $k[x]$-modules of rank n.

Definition 2.4.2.5. If $(X, \mathscr{O}_X)$ is a ringed space, we will write $\mathscr{V}_n(X)$ for the set of isomorphism classes of rank n locally free $\mathscr{O}_X$-modules. If $X = (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$, then we write $\mathscr{V}_n(R)$ for $\mathscr{V}_n(\operatorname{Spec} R)$; the former is identified with the set of isomorphism classes of rank n projective R-modules.

2.4.3 Vector bundles on $\mathbb{P}^1_k$

In this section, assume k is a field, and take $(X, \mathscr{O}_X) = (\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k})$. We study the classification of rank n locally free $\mathscr{O}_{\mathbb{P}^1_k}$-modules. We use the description of $\mathbb{P}^1_k$ as glued together from a copy of $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ with a copy of $\mathbb{A}^1_k = \operatorname{Spec} k[t]^{-1}$ along the subscheme $\mathbf{G}_{mk} = \operatorname{Spec} k[t, t^{-1}]$. Suppose $\mathscr{F}$ is a rank n locally free sheaf of $\mathscr{O}_{\mathbb{P}^1_k}$-modules. In that case, by the discussion of Example 2.4.2.4, it follows that $\mathscr{F}|_{\operatorname{Spec} k[t]}$ corresponds to a free $k[t]$-module. Fix an isomorphism $\varphi_+ P_+ := \Gamma(\operatorname{Spec} k[t], \mathscr{F}|_{\operatorname{Spec} k[t]}) \xrightarrow{\sim} k[t]^{\oplus n}$, i.e., pick a $k[t]$-module basis of P_+. Likewise, we may fix an isomorphism $\varphi_- P_- := \Gamma(\operatorname{Spec} k[t^{-1}], \mathscr{F}|_{\operatorname{Spec} k[t^{-1}]}) \xrightarrow{\sim} k[t^{-1}]^{\oplus n}$. Now, since $\mathscr{F}$ is a sheaf on $\mathscr{O}_{\mathbb{P}^1_k}$, we know that

$$P_+ \otimes_{k[t]} k[t, t^{-1}] = P_- \otimes_{k[t]} k[t, t^{-1}]$$

since both of these modules coincide with $\Gamma(\operatorname{Spec} k[t, t^{-1}], \mathscr{F}|_{\operatorname{Spec} k[t, t^{-1}]})$. If we use the basis of P_+ to fix an isomorphism $P_+ \otimes_{k[t]} k[t, t^{-1}] \cong k[t, t^{-1}]^{\oplus n}$, then the matrix expressing change of basis to the basis coming from P_- yields an element of $GL_n(k[t, t^{-1}])$. Thus, we have shown that $\mathscr{F}$ gives rise to an element of $GL_n(k[t, t^{-1}])$. However, the relevant matrix depended on the choice of isomorphisms φ_+ and φ_-]. In other words, if we "rigidify" $\mathscr{F}$ by, in addition, fixing φ_+ and φ_-, then we have built a function:

$$(\mathscr{F}, \varphi_+, \varphi_-) \longrightarrow GL_n(k[t, t^{-1}]).$$

We will refer to the element of $GL_n(k[t, t^{-1}])$ obtained in this way as a *clutching function* for $\mathscr{F}$. We can then formulate the following result.

Theorem 2.4.3.1. *The assignment $(\mathscr{F}, \varphi_+, \varphi_-) \to GL_n(k[t,t^{-1}])$ just described factors through a bijection between the set of isomorphism classes of rank n locally free $\mathscr{O}_{\mathbb{P}^1_k}$-modules and elements of the double coset space*

$$GL_n(k[t^{-1}])\backslash GL_n(k[t,t^{-1}])/GL_n(k[t]).$$

Proof. It remains to analyze the dependence of the clutching function construction on the choice of φ_+ and φ_-. If we change the isomorphism φ_+, that amounts to changing the $k[t]$-module basis of P_+, i.e., right multiplying by an element of $GL_n(k[t])$. Likewise, changing the isomorphism φ_- amounts to changing the $k[t^{-1}]$-module basis of P_-, i.e., left multiplying by an element of $GL_n(k[t^{-1}])$. □

Our next goal is to understand whether the double cosets have "good representatives", i.e., whether there are normal forms for matrices in $GL_n(k[t,t^{-1}])$.

Example 2.4.3.2 (Line bundles). Let us first analyze the case $n = 1$. In that case, $GL_1(k[t,t^{-1}]) = k^{\times} \times \mathbb{Z}$: every unit in $k[t,t^{-1}]$ may be written uniquely as αt^n for $\alpha \in k^{\times}$ and some integer n. We also know that $GL_1(k[t]) = k^{\times}$ (e.g., by homotopy invariance of units). Thus, up to left or right multiplying by α^{-1}, every element of $GL_1(k[t^{-1}])\backslash GL_1(k[t,t^{-1}])/GL_1(k[t])$ is represented uniquely by an element of the form t^n. In other words, isomorphism classes of line bundles on $\mathbb{P}^1_k$ are determined uniquely by an integer n. We will write $\mathscr{O}_{\mathbb{P}^1}(n)$ for the line bundle determined by the transition function t^n. We will frequently write $\mathscr{O}(n)$ for this bundle, suppressing the $\mathbb{P}^1_k$ in subscript.

One source of vector bundles of higher rank on $\mathbb{P}^1_k$ is direct sums $\mathscr{O}(a_1) \oplus \cdots \mathscr{O}(a_n)$. Of course, we may permute the a_i at will to obtain isomorphic bundles. Thus, without loss of generality, we may assume that $a_1 \geq a_2 \geq \cdots a_n$, i.e., the sequence of a_i is decreasing. There are two basic questions that arise. First, can we have two different decreasing sequences of n integers that give rise to isomorphic rank n bundles? Second, are there any other rank n vector bundles on $\mathbb{P}^1_k$? We now answer these questions.

Theorem 2.4.3.3. *If k is a field, then every rank n locally free sheaf $\mathscr{F}$ of $\mathscr{O}_{\mathbb{P}^1_k}$-modules is isomorphic to exactly 1 vector bundle of the form $\mathscr{O}(a_1) \oplus \mathscr{O}(a_n)$ for a decreasing sequence $a_1 \geq a_2 \geq \cdots \geq a_n$.*

By means of Theorem 2.4.3.1 we will analyze normal forms for double cosets of $GL_n(k[t,t^{-1}])$. We begin by normalizing clutching functions slightly. Indeed, suppose we fix a clutching function $X(t,t^{-1})$ for a rank n vector bundle. The determinant of $X(t,t^{-1})$ is an element of $GL_1(k[t,t^{-1}]) = k^{\times} \times t^s$ for some integer s. By changing $X(t,t^{-1})$ by an element of $GL_n(k)$ of the form $diag(\alpha^{-1}, 1, \ldots, 1)$, we may thus assume without loss of generality that $X(t,t^{-1})$ has determinant t^s for some integer s. Theorem 2.4.3.3 then follows from the following more precise result.

Proposition 2.4.3.4. *Let $X(t,t^{-1})$ be an element of $GL_n(k[t,t^{-1}])$ whose determinant is t^s for some integer s. There exist matrices $U(t) \in GL_n(k[t])$ and $V(t^{-1}) \in GL_n(k[t^{-1}])$ (with constant non-zero determinant) such that*

$$V(t^{-1})X(t,t^{-1})U(t) = \begin{pmatrix} t^{r_1} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & t^{r_n} \end{pmatrix}$$

with $r_1 \geq \cdots \geq r_n$, $r_i \in \mathbb{Z}$. Moreover, the r_i are uniquely determined by X and if $X \in GL_n(k[t])$, then $r_i \geq 0$, while if $X \in GL_n(k[t^{-1}])$ then $r_i \leq 0$.

Proof. We prove the proposition by induction on n, the case $n = 1$ having been treated in the discussion of line bundles before the theorem statement. Assume inductively that the result holds for $n - 1$. We now make a series of reductions.

Reduction 1. (Positive powers of t) The entries of X are a priori in $k[t, t^{-1}]$, i.e., polynomials in t and t^{-1}. However, by multiplying by a suitable power of t, we may obtain a new element of $GL_n(k[t, t^{-1}])$, call it Y, whose entries all lie in $k[t]$. In other words, we may write $t^m X = Y$ where the entries of Y lie in $k[t]$. We claim it suffices to find U and V as in the theorem statement for Y. Indeed, if there exists a matrix $D = diag(t^{a_1}, \ldots, t^{a_n})$ such that $UYV = D$, then note that $t^m UYV = Ut^m YV = UXV$, while $t^m D$ is again a diagonal matrix. In other words, if we find U and V for Y, then the same matrices U and V will put X in the required diagonal form. Thus, we work with Y in what follows.

Reduction 2. (Clearing the first row) Consider the matrix Y and look at the entries $y_1 1, \ldots, y_{1n}$ in the first row. Let $y'_{11} = gcd(y_{11}, \ldots, y_{1n})$. We claim that we can right multiply Y by an element of $GL_n(k[t])$ to obtain a new matrix Y' whose $(1,1)$-entry is y'_{11} and such that all other in the first row are zero. Before describing the general case, let's treat the 2×2 case.

We may find a Bézout relation: $y'_{11} = z_1 y_{11} + z_2 y_{12}$ where $z_i \in k[t]$. Since $\det Y = y_{11}y_{22} - y_{12}y_{21}$, it follows that $y'_{11} | \det Y = t^N$. In other words, $y'_{11} = t^{k_1}$ for some $k_1 \geq 0$. If we set $w_1 = y_{11}/y'_{11}$ and $w_2 = y_{12}/y'_{11}$ (both in $k[t]$), then the 2×2-matrix given by

$$\begin{pmatrix} z_1 & -w_2 \\ z_2 & w_1 \end{pmatrix}$$

has determinant 1 and entries in $k[t]$ and is invertible (use the explicit formula for the inverse of a 2×2-matrix). In that case, we compute:

$$\begin{pmatrix} y_{11} & y_{12} \\ * & * \end{pmatrix} \begin{pmatrix} z_1 & -w_2 \\ z_2 & w_1 \end{pmatrix} = \begin{pmatrix} y'_{11} & -w_2 y_{11} + w_1 y_{12} \\ 0 & 0 \end{pmatrix}.$$

Observe that since $y'_{11} | y_{11}$ and y_{12}, it also follows that $-w_2 y_{11} + w_1 y_{12} \in k[t]$ is divisible by y'_{11}. Therefore, subtracting a suitable $k[t]$-multiple of the first column from the second (which is achieved by right multiplying by an element of $GL_n(k[t])$ of determinant 1), we may eliminate y_{12}. Finally, since the determinant of the new matrix is unchanged, we conclude that $y'_{11} | \det Y = t^N$, so $y'_{11} = t^{k_1}$. In other words, we have built a matrix $U_0 \in GL_n(k[t])$ such that $Y' := YU_0$ takes the form

$$\begin{pmatrix} t^{k_1} & 0 \\ * & * \end{pmatrix}.$$

The general case reduces to this one. For any $j = 2, \ldots, n$, let $g_j = gcd(y_{11}, y_{1j})$ and pick a Bézout relation between y_{11} and y_{1j}, say $z_1 y_{11} + z_j y_{1j}$. Let $w_1 = y_{11}/g_j$ and $w_j = y_{1j}/g_j$ so that we have $z_1 w_1 + z_j w_j = 1$. In that case, build an $n \times n$-matrix that differs from the $n \times n$-identity matrix in only the first and j-th rows: the first row is $(z_1, \ldots, -w_j, \ldots)$, where w_j appears in the

j-th slot and all other entries are 0, while the j-th row is $(z_j, \ldots, w_1, \ldots)$ where w_1 appears in the j-th place and all other entries are zero. Once again, this $n \times n$-matrix has determinant 1. Right multiplying Y by this matrix changes Y to an $n \times n$-matrix whose first entry is g_j and then a column operation can be used to eliminate the j-th entry of the resulting matrix. Repeating this procedure for the new matrix as j varies through $2, \ldots, n$, we can sequentially eliminate all the entries in the first row, and furthermore make the first entry of the resulting matrix $y'_{11} = gcd(y_{11}, \ldots, y_{1n})$. The formula for the determinant by expansion along the first row shows that $y'_{11} | \det Y$, i.e., $y'_{11} = t^{k_1}$ for some $k_1 \geq 0$. Note: if we wanted, we could continue with this procedure on subsequent rows beyond the first to put our matrix in lower-triangular form with diagonal entries of the form t^{k_i} by multiplication of a suitable element of $GL_n(k[t])$.

Reduction 3. (Applying the IH) Now, we appeal to the induction hypothesis: we may find matrices $U_1(t)$ and $V_1(t^{-1})$ such that

$$V_1(t^{-1})Y(t)U_0(t)U_1(t) = \begin{pmatrix} t^{k_1} & 0 & \ldots & 0 \\ c_2 & t^{k_2} & \ldots & 0 \\ & & \ddots & \vdots \\ c_n & \ldots & \ldots & t^{k_n} \end{pmatrix}$$

where $c_2, \ldots, c_n$ lie in $k[t, t^{-1}]$; let us call this product Y''.

Reduction 4. (Bounding the degrees of c_i) Since the first row Y'' has t^{k_1} with $k_1 \geq 0$, by means of row operations between the first and i-th rows we may eliminate all terms in c_i of negative degree. Likewise, since in the i-th row we have c_i in the first spot and t^{k_i} in the i-th spot, by means of column operations, we can eliminate all terms in c_i of degree $\geq k_i$, i.e., we may furthermore assume c_i has degree $< k_i$. Thus, $c_i \in k[t]$ of degree $\leq k_i$.

Maximality. We claim it suffices to show that $k_1 > k_i$ for all i. Indeed, in that case, we could eliminate c_i by a suitable row operations involving the first and i-th rows. To see this, choose among all matrices lying in the same double coset as $Y''(t, t^{-1})$ one of the same form as Y'' with k_1 maximal. First, observe that such a matrix necessarily exists since k_1 is necessarily bounded above by $\deg degY''$ since all the other k_i are positive. Take this representative with maximal k_1 and suppose to the contrary that $k_1 < k_i$ for some i. In that case, by subtracting suitable $k[t^{-1}]$-multiples of the first row, from the i-th row, we may obtain a matrix of the same form with $c_i = s^{k_1+1}c'_i$. Now, exchange the first and i-th row. Repeating the procedure we used to construct Y'' for the matrix just mentioned, would yield a matrix whose first entry is $t^{k'_1}$ with $k'_1 > k_1$, which contradicts maximality of k_1. In other words, $k_1 > k_i$ for all i and we conclude.

Add proof of uniqueness. $\square$

2.4.4 Vector bundles on $\mathbb{P}^1_k$ revisited

Having studied vector bundles on $\mathbb{P}^1_k$ in the case where k was a field. Our analysis relied in a key way on two ingredients: (i) the structure theorem for finitely generated modules over a principal ideal domain in the proof of Theorem 2.4.3.1, and (ii) the fact that $k[t]$ was a Euclidean domain in the proof of Theorem 2.4.3.3. We can ask what happens if k is more general than a field.

Let us analyze what happens with Theorem 2.4.3.1. The basic problem is that if R is not a field, then it is no longer clear whether projective $R[t]$-modules are free. There are two sources of projective $R[t]$-modules.

Example 2.4.4.1. If R is a commutative ring that has non-trivial rank n projective R-modules, then $R[t]$ also has such modules. Indeed, consider the ring homomorphism $R \to R[t]$. If P is a non-free projective R-module, then $P \otimes_R R[t]$ is again a projective R-module. We claim that it is also not free. To see this, simply observe that $R \to R[t]$ is split by (say) the evaluation map $ev_0 : R[t] \to R$. Indeed, if $P \otimes_R R[t]$ were free, then its extension of scalars along ev_0 would again be a free R-module, but the composite map $R \to R[t] \to R$ is the identity, so this only can happen if P was free to begin with.

Furthermore, even if projective R-modules are all free, it is not obvious that projective $R[t]$-modules are free! The argument of the preceding example shows that the map:

$$\mathscr{V}_n(R) \longrightarrow \mathscr{V}_n(R[t])$$

induced by extension of scalars is split injective, with splitting induced by extension of scalars along ev_0. This leads to the following important question.

Question 2.4.4.2. *For which rings R is it the case that projective $R[t]$-modules are free?*

So what can be salvaged from Theorem 2.4.3.1? As usual, $\mathbb{P}^1_{\operatorname{Spec} R}$ may be glued together from $\operatorname{Spec} R[t]$ and $\operatorname{Spec} R[t^{-1}]$ along $\operatorname{Spec} R[t, t^{-1}]$. Now, we may always glue rank n free $R[t]$ modules with rank n free $R[t^{-1}]$-modules by specifying an element of $GL_n(R[t, t^{-1}])$. The proof of Theorem 2.4.3.1 then implies the following more general result.

Theorem 2.4.4.3. *If R is a ring, then there is a function*

$$GL_n(R[t^{-1}])\backslash GL_n(R[t, t^{-1}])/GL_n(R[t]) \longrightarrow \mathscr{V}_n(\mathbb{P}^1_R).$$

If every projective $R[t]$-module is free, then the above function is a bijection.

Example 2.4.4.4. Let us analyze this theorem in arguably the simplest non-trivial case $R = \mathbb{Z}$. Let us write down some interesting transition functions. Consider the element of $GL_n(\mathbb{Z}[t, t^{-1}])$ given by the matrix

$$\begin{pmatrix} t & 2 \\ 0 & t^{-1} \end{pmatrix}.$$

The element 2 is, of course, not a unit in $\mathbb{Z}$. However, if we pass to ring $\mathbb{Z}[\frac{1}{2}]$, then 2 is a unit. In this ring, we may analyze the double coset containing the above transition function: explicit computation shows that the identity

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2}t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 2 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{2}t \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

holds. As the last matrix we wrote down is an element of $GL_2(\mathbb{Z}[\frac{1}{2}])$ it is, in particular, contained in $GL_2(\mathbb{Z}[\frac{1}{2}][t])$, i.e., the resulting double coset is the same as that of the identity matrix. In other words, in the ring $\mathbb{Z}[\frac{1}{2}]$, the coset containing the transition function above is the trivial bundle $\mathscr{O} \oplus \mathscr{O}$

on $\mathbb{P}^1_{\mathbb{Z}}$. On the other hand, if we localize at the prime ideal (2), then we claim the resulting bundle is *non*-trivial. Indeed, consider the local ring $Z_{(2)}$; the fraction field of this local ring is the finite field $\mathbb{F}_2$. Now, if the bundle corresponding to the transition function above were trivial in $\mathbb{P}^1_{\mathbb{Z}_{(2)}}$, then it would give a transition function when we extend scalars along the map $\mathbb{Z}_{(2)} \to \mathbb{F}_2$. However, setting $2 = 0$ in the above transition function, gives the transition function

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

for a bundle in $\mathbb{P}^1_{\mathbb{F}_2}$, i.e., $\mathscr{O}(1) \oplus \mathscr{O}(-1)$. However, by the classification of vector bundles on $\mathbb{P}^1$ over a field, this bundle is non-trivial. In particular, it follows that the classification of vector bundles on $\mathbb{P}^1_k$ for k a field, does not extend to $\mathbb{Z}$!

Note that $\operatorname{Spec} \mathbb{Z}[\frac{1}{2}]$ and $\operatorname{Spec} \mathbb{Z}_{(2)}$ are Zariski open subsets of $\operatorname{Spec} \mathbb{Z}$ as localizations. Moreover, these two open sets form a Zariski open cover of $\operatorname{Spec} \mathbb{Z}$: their intersection is $\operatorname{Spec} \mathbb{Q}$. Thus, we could view the bundle we've just constructed as gluing a trivial bundle on $\mathbb{P}_{\mathbb{Z}[\frac{1}{2}]}$ with a non-trivial bundle over $\mathbb{P}^1_{\mathbb{Z}_{(2)}}$ along an isomorphism on their intersection $\mathbb{P}^1_{\mathbb{Q}}$.

Example 2.4.4.5. The example above can be generalized significantly. If R is any principal ideal domain, then take any non-zero element f and consider the transition function

$$\begin{pmatrix} t & f \\ 0 & t^{-1} \end{pmatrix}.$$

Replace $\mathbb{Z}[\frac{1}{2}]$ with R_f and $\mathbb{Z}_{(2)}$ with $R_{(f)}$ (since f is non-zero, and any non-zero prime ideal is principal since R is a PID, this ring is local). Replacing 2 by f in the matrices above gives a non-trivial vector bundle on $\mathbb{P}^1_R$ that is not of the form $\mathscr{O}(a) \oplus \mathscr{O}(b)$. For example, take $R = k[x]$ and $f = x$. In that case, we see that the assignment $X \mapsto \mathscr{V}_r(X)$ is *not* a naive $\mathbb{A}^1$-homotopy invariant, e.g., for $X = \mathbb{P}^1_k$: there are vector bundles on $\mathbb{P}^1_k \times \mathbb{A}^1_k$ that are not obtained by extension of scalars from vector bundles on $\mathbb{P}^1_k$.

Chapter 3

Picard groups, normality and $\mathbb{A}^1$-invariance

Contents

We introduced the Picard group of a commutative ring earlier. We begin by globalizing this definition for an arbitrary scheme, and analyzing notions related to extension of scalars.

3.1 Functoriality for sheaves of modules

Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are ringed spaces and $(f, f^\sharp)$ is a morphism of ringed spaces. We now want to study functors between the categories of $\mathscr{O}_X$-modules and $\mathscr{O}_Y$-modules. Unwinding the defintions, we may view $f^\sharp$ as a morphism of sheaves $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$.

3.1.1 Pushfoward

If $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules, then $f_*\mathscr{F}$ is a sheaf of $f_*\mathscr{O}_X$-modules, by construction and $f^\sharp$ precomposition of this module structure with $f^\sharp$ gives $f_*\mathscr{F}$ the structure of a sheaf of $\mathscr{O}_Y$-modules. In other words, the functor f_* from sheaves on X to sheaves on Y induces a functor

$$f_*\mathrm{Mod}(\mathscr{O}_X) \longrightarrow \mathrm{Mod}(\mathscr{O}_Y).$$

Simple examples show that this functor does not usually preserve finiteness properties.

Example 3.1.1.1. If $\varphi : R \to S$ is a ring homomorphism, then we get a morphism $f := \operatorname{Spec}\varphi : \operatorname{Spec} S \to \operatorname{Spec} R$. In that case, the morphism $f^\sharp : \mathscr{O}_{\operatorname{Spec} R} \to f_*\mathscr{O}_{\operatorname{Spec} S}$ induces at the level of global sections the morphism $\varphi : R \to S$. If $\mathscr{F}$ is the sheaf $\tilde{M}$ for an S-module M, then by definition, $f_*\tilde{M}(\operatorname{Spec} X) = M$. Thus, $f_*\mathscr{F}$ is simply $\tilde{M}$ on $\operatorname{Spec} R$, where we view M as an R-module via φ.

3.1.2 Pullback

If $f : X \to Y$ is a continuous map of topological spaces, then the functor f^{-1} was constructed as a suitable directed colimit. Suppose $f : X \to Y$ is a continuous map of topological spaces and we have a sheaf of rings $\mathscr{O}_Y$ on Y. In that case, we may consider the sheaf $f^{-1}\mathscr{O}_Y$. By definition, the sections of $f^{-1}\mathscr{O}_Y$ over an open set $U \subset X$ is the colimit of the sections of $\mathscr{O}_Y$ over opens in Y that contain $f(U)$. If $\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules, then it follows from this observation that $f^{-1}\mathscr{F}$ has naturally the structure of an $f^{-1}\mathscr{O}_Y$-module.

Suppose $(f, f^\sharp)$ is a morphism of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. In that case, $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$. This corresponds as well to a morphism of sheaves $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. Indeed, for an open $U \subset X$, consider the open set $f(U) \subset Y$. Take an open neighborhood V of $f(U)$ in Y then $f^\sharp$ gives upon evaluation at V a morphism $\mathscr{O}_Y(V) \to f_*\mathscr{O}_X(V) = \mathscr{O}_X(f^{-1}(V))$. Now, if $U \subset X$,and V is a neighborhood of $f(U)$ in Y, then it follows that $U \subset f^{-1}f(U) \subset f^{-1}(V)$. In particular, there is a restriction map $\mathscr{O}_X(f^{-1}(V)) \to \mathscr{O}_X(U)$ for any such V. Thus, for any open set U and any neighborhood V of $f(U)$, we get a ring homomorphism $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ by composition. Taking colimits yields a morphism of presheaves $f^{-}\mathscr{O}_Y \to \mathscr{O}_X$ and sheafifying yields a morphism of sheaves $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$.

If $\mathscr{F}$ is a sheaf of $\mathscr{O}_Y$-modules, then since $f^{-1}\mathscr{F}$ is a sheaf of $f^{-1}\mathscr{O}_Y$-modules, if we extend scalars along the morphism $f^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ we get a sheaf of $\mathscr{O}_X$-modules; we set:

$$f^*\mathscr{F} := f^{-1}\mathscr{F} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X,$$

and refer to f^* as the pullback of $\mathscr{F}$ along f.

Example 3.1.2.1. Consider the special case where $\varphi : R \to S$ is a ring homomorphism and $f := \operatorname{Spec}\varphi$. In that case, unwinding the constructions above, if M is an R-module, and $\mathscr{F} = \tilde{M}$, then $f^*\tilde{M} = \widetilde{M \otimes_R S}$. Note that, because of this identification, it follows that the functor f^* respects various module theoretic constructions. For example, if M and N are R-modules, then $f^*\widetilde{M \oplus N} \cong \widetilde{M \otimes_R S} \oplus \widetilde{N \otimes_R S}$ since extension of scalars commutes with direct sums as tensor products distribute over direct sums. Likewise, there is an isomorphism $f^*\widetilde{M \otimes_R N} \cong \widetilde{M \otimes_R S} \otimes \widetilde{N \otimes_R S}$ arising from the associativity of tensor products. The next result generalizes this observation.

Proposition 3.1.2.2. *Assume $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. The functor f^* preserves direct sums and tensor products of modules, i.e., the functor f^* respects symmetric monoidal structures.*

Proof. Exercise. □

Proposition 3.1.2.3. *Assume $(f, f^\sharp) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a morphism of ringed spaces. The functor f^* sends locally free $\mathscr{O}_Y$-modules to locally free $\mathscr{O}_X$-modules. Moreover, if $\mathscr{F}$ is finite locally free (resp. locally free of rank n), then so is $f^*\mathscr{F}$.*

Remark 3.1.2.4. Pullbacks of invertible modules are again invertible. Indeed, if $\mathscr{L}$ is an invertible $\mathscr{O}_Y$-module, then there exists $\mathscr{L}'$ such that $\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{L}' \cong \mathscr{O}_Y$. Note that $f^*\mathscr{O}_Y = \mathscr{O}_X$ by definition. Since pullback preserves tensor products, the result follows.

Definition 3.1.2.5. If $(X, \mathscr{O}_X)$ is a ringed space, then we define $Pic(X)$ to be the set of isomorphism classes of invertible $\mathscr{O}_X$-modules.

The next lemma follows from the definition and the preceding remark.

Lemma 3.1.2.6. *If $(X, \mathscr{O}_X)$ is a ringed space, then $(Pic(X), \otimes, \mathscr{O}_X)$ has the structure of an abelian group; this structure is functorial for pullbacks along morphisms of ringed spaces.*

3.2 Line bundles and divisors

Let us now assume that $(X, \mathscr{O}_X)$ is a scheme. We'd like to investigate the Picard group of X from several different points of view: geometric and cohomological.

3.2.1 Line bundles and Cech cohomology

Suppose we give an invertible $\mathscr{O}_X$-module $\mathscr{L}$. The restriction of $\mathscr{L}$ to any affine open $U_i = \operatorname{Spec} R_i$ subscheme gives an invertible R_i-module L_i. We saw that invertible R-modules were precisely the locally free R-modules of rank 1, i.e., line bundles. By patching, we thus see that invertible $\mathscr{O}_X$-modules are precisely the same thing as line bundles. We can therefore describe the Picard group of a scheme X as the group of isomorphism classes of line bundles with respect to the tensor product operation and the unit is the trivial line bundle.

Example 3.2.1.1. If k is a field, then $Pic(\mathbb{A}^1_k) = 0$. Indeed, we know that locally free $\mathscr{O}_X$-modules of rank 1 on $\mathbb{A}^1_k$ are trivial.

Example 3.2.1.2. If k is a field, then $Pic(\mathbb{P}^1_k) = \mathbb{Z}$. By Theorem 2.4.3.3, we see that every line bundle on $\mathbb{P}^1$ is isomorphic to one of the form $\mathscr{O}(n)$. We claim the assignment $\mathscr{O}(n) \mapsto n$, which is bijective, is a group homomorphism and thus an isomorphism. To see this, it suffices to understand how clutching functions change by taking tensor products. In the proof of Theorem 2.4.3.1, the description of line bundles in terms of transition functions comes by fixing a basis of $\mathscr{O}(n)|_{\operatorname{Spec} k[t]}$ and $\mathscr{O}(n)|_{\operatorname{Spec} k[t^{-1}]}$. Choose such a basis for $\mathscr{O}(m)$ as well. A basis for the tensor product of the modules $\mathscr{O}(n)|_{\operatorname{Spec} k[t]} \otimes \mathscr{O}(m)|_{\operatorname{Spec} k[t]}$ is then given by a pure tensor of the basis vectors we chose. It follows immediately that the transition function is simply given by the product of functions.

From line bundles to cohomology classes

Let us abstract this "patching" description of line bundles on $\mathbb{P}^1_k$ to more general spaces. Start with a scheme X and a line bundle $\mathscr{L}$ on X. Now, we may always choose an open cover $\{U_i\}_{i \in I}$ of X along which $\mathscr{L}$ trivializes: for example, if $\{U_i\}_{i \in I}$ form an open cover of X by affine open sets, then by refining each U_i we can find the necessary open cover. Now, fix isomorphisms $\varphi_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i}$, i.e., we rigidify $\mathscr{L}$ with respect to the open cover. On two-fold intersections U_{ij}, the map $\varphi_i|_{U_{ij}} \circ \varphi_j^{-1}|_{U_{ij}}$ defines a morphism $\mathscr{O}_{U_{ij}} \to \mathscr{O}_{U_{ij}}$ of rank 1 free $\mathscr{O}_{U_{ij}}$-modules. Such an element is specified uniquely by a unit α_{ij} in $\mathscr{O}^{\times}_{U_{ij}}$. Note that on U_{ii}, we have $\alpha_{ii} = 1$ by construction. One then checks that $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ on threefold intersections for all triples $i, j, k \in I^3$; in particular, note that $\alpha_{ij}\alpha_{ji} = 1$ as well. In other words, by specifying a trivializing open cover for $\mathscr{L}$ and an explicit trivialization of $\mathscr{L}$ on this open cover, we get a collection of units; these units are the analog of the clutching function we wrote down for line bundles on $\mathbb{P}^1_k$. Since our groups are commutative, we will typically use additive notation and the formula above relating the α_{ij} is

$$\alpha_{ij} + \alpha_{jk} - \alpha_{ik} = 0;$$

the α_{ij} form what we will momentarily call a Cech 1-cocycle valued in the sheaf of units.

Next, we ask: how does the description of $\mathscr{L}$ change if we modify either the trivializing open cover or the chosen trivialization? Let us deal with the latter modification first If we choose different trivializations $\varphi'_i : \mathscr{L}|_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i}$, then $\varphi_i \circ \varphi_i'^{-1}$ is an automorphism of $\mathscr{O}_{U_i}$, i.e., specified by a unit f_i. If α'_{ij} are the units attached to the cover U_{ij} for the trivialization $\{\varphi'_i\}_{i \in I}$, then we may write down a commutative diagram summarizing all the relevant choices:

$$\begin{tikzcd} \mathscr{O}_{U_{ij}} \arrow[r, "\varphi_j^{-1}|_{U_{ij}}"] & \mathscr{L}|_{U_{ij}} \arrow[r, "\varphi_i|_{U_{ij}}"] \arrow[d, "="] & \mathscr{O}_{U_{ij}} \arrow[d] \\ \mathscr{O}_{U_{ij}} \arrow[r, leftrightarrow, "\varphi_j'^{-1}|_{U_{ij}}"] & \mathscr{L}|_{U_{ij}} \arrow[r, "\varphi_i'|_{U_{ij}}"] & \mathscr{O}_{U_{ij}}. \end{tikzcd}$$

The top horizontal composite is given by multiplication by α_{ij} while the bottom horizontal composite is given by multiplication by α'_{ij}. Unwinding the definition of f_i, we see that the rightmost vertical map is given by multiplication by f_i while the leftmost vertical map is given by multiplication by f_j. In other words, $\alpha'_{ij}f_j = f_i\alpha_{ij}$, or equivalently, $\alpha'_{ij} = \alpha_{ij}\frac{f_i}{f}f_j{}^{-1}$. Note that the expressions $\phi_{ij} = f_i/f_j$ automatically satisfy the condition $\phi_{ij} + \phi_{jk} - \phi_{ik} = 0$.

Cech cohomology

Let us formalize all of this as follows. Suppose $\mathscr{F}$ is any sheaf of abelian groups on a topological space X. Fix an open cover $\{U_i\}_{i\in I}$ of X. Set

$$C^n(X, \{U_i\}_{i\in I}, \mathscr{F}) := \prod_{i_0,\ldots,i_n \in I^{n+1}} \Gamma(U_{i_0i_1\cdots i_n}, \mathscr{F}),$$

where $U_{i_0i_1\cdots i_n} = U_{i_0} \cap \cdots U_{i_n}$. In other words, an element $\alpha \in C^n(X, \{U_i\}_{i\in I}, \mathscr{F})$ consists of sections $\alpha_{i_0,\ldots,i_n} \in \mathscr{F}(U_{i_0i_1\cdots i_n})$.

Define a map

$$d_n : C^n(X, \{U_i\}_{i\in I}, \mathscr{F}) \longrightarrow C^{n+1}(X, \{U_i\}_{i\in I}, \mathscr{O}_X^{\times})^{n+1}$$

by the formula:

$$d_n(\alpha)_{i_0,\ldots,i_{n+1}} = \sum_{k=0}^{n+1}(-1)^k \alpha_{i_0,\ldots,\hat{i_k},\ldots,i_{n+1}}|_{U_{i_0,\ldots,i_{n+1}}}.$$

The usual combinatorial check shows that $d_{n+1} \circ d_n = 0$, i.e., this is a complex of abelian groups. We define

$$H^n(\{U_i\}_{i\in I}, \mathscr{F}) := \ker(d_n)/im(d_{n-1}).$$

Note that Cech cohomology is evidently functorial in the sheaf $\mathscr{F}$. An element of $\ker(d_n)$ will be called a Cech n-cocycle, while an element of $im(d_{n-1})$ will be called a Cech n-coboundary.

Example 3.2.1.3. If $\mathscr{F}$ is any sheaf of abelian groups on a topological space X, then by construction $H^0(\mathcal{U}, \mathscr{F})$ coincides with the group $\mathscr{F}(X) = \Gamma(X, \mathscr{F})$ of global sections of $\mathscr{F}$ over X.

The next lemma describes line bundles trivializing over a given open cover in terms of Cech cohomology.

Lemma 3.2.1.4. *Assume $(X, \mathscr{O}_X)$ is a scheme and $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of X. There is a group of isomorphism classes of line bundles on X that trivialize on $\mathcal{U}$ and the group $H^1(\{U_i\}_{i\in I}, \mathscr{O}_X^{\times})$.*

Proof. If $\mathscr{L}$ is a line bundle on a scheme X, and $\mathscr{L}$ trivializes on an open cover $\{U_i\}_{i\in I}$, then the units α_{ij} we described above form a Cech 1-cocyle valued in the sheaf of abelian groups $\mathscr{O}_X^\times$. If we change the trivialization, then elements $\frac{f_i}{f_j}$ we described give rise to a Cech 1-coboundary valued in the sheaf of abelian groups $\mathscr{O}_X^\times$. The fact that the two sets are in bijection is simply the discussion above. To see that it is an isomorphism of groups simply amounts to the observation we made before that tensor products of line bundles correspond to Kronecker products of clutching functions. $\square$

Non-abelian Cech cohomology

If X is a topological space, and $\mathscr{G}$ is a sheaf of non-abelian groups on X, then we may still define Cech cohomology of $\mathscr{G}$ with respect to an open cover U_i of X in small degrees. Define an action morphism

$$\prod_{i\in I}\mathscr{G}(U_i)\times \prod_{i_0,i_1\in I\times I}\mathscr{G}(U_{ij})\longrightarrow \prod_{i_0,i_1\in I\times I}\mathscr{G}(U_{ij})$$

as follows. If $g_i\in\mathscr{G}(U_i)$ is a collection of sections and $\alpha_{ij}\in\mathscr{G}(U_{ij})$ is a collection of sections, then we set

$$a:(\prod_i g_i,\prod_{i,j}\alpha_{ij})\longmapsto \prod_{i,j}(g_i|_{U_{ij}}\alpha_{ij}g_j^{-1}|_{U_{ij}}).$$

This action defines an orbit map $d_0 = a(\prod_i g_i,\prod_{i,j}1)$, which is evidently a pointed map (i.e., takes the identity to the identity). This function fails to be a group homomorphism in general (since inversion is typically not a group homomorphism). By the kernel of d_0, we will simply mean the pre-image of the identity element. Of course, the kernel of d_0 coincides with the global sections $\mathscr{G}(X)$ as above.

Likewise, we may define a pointed function

$$d_1:\prod_{i_0,i_1\in I\times I}\mathscr{G}(U_{ij})\longrightarrow \prod_{i_0,i_1,i_2\in I^3}$$

by $(d_1\alpha)_{i_0i_1i_2} = \alpha_{i_0i_1}|_{U_{i_0i_1i_2}}\alpha_{i_1i_2}|_{U_{i_0i_1i_2}}\alpha^{-1}_{i_0i_2}|_{U_{i_0i_1i_2}}$. The kernel of d_1 will be the set of non-abelian Cech 1-cocycles; we will write $Z^1(\mathcal{U},\mathscr{G})$ for the set of non-abelian Cech 1-cocycles. The set $Z^1(\mathcal{U},\mathscr{G})$ is stable under the action map a by construction; in particular, the condition that two non-abelian Cech 1-cocycles lie in the same orbit for the action map a is an equivalence relation.

We define the non-abelian Cech cohomology by means of the following formulas

$$H^0(\mathcal{U},\mathscr{G}) = \ker(d^0),$$

and

$$H^1(\mathcal{U},\mathscr{G}) = Z^1(\mathcal{U},\mathscr{G})/\sim,$$

where $\sim$ is the equivalence relation defined by requiring that two non-abelian 1-cocycles lie in the same orbit for a. While H^0 is a group, note that H^1 is only a pointed set (pointed by the image of the identity element). These constructions become much more transparent in an example.

If $(X,\mathscr{O}_X)$ is a ringed space, then we may form the sheaf $GL_n(\mathscr{O}_X)$ whose sections over an open set $U\subset X$ consist of the groups $GL_n(\mathscr{O}_X(U))$. One can check that this is again a sheaf on X. The following result generalizes Theorem 2.4.4.3 to describe locally free sheaves of rank n on an arbitrary ringed space.

Lemma 3.2.1.5. *Suppose $(X, \mathscr{O}_X)$ is a ringed space and $\mathcal{U}$ is an open cover of X. There is a bijection between locally free $\mathscr{O}_X$-modules that trivialize on the open cover $\mathscr{U}$ and elements of $H^1(\mathcal{U}, GL_n(\mathscr{O}_X))$.*

Proof. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of rank n that trivalizes on $\mathcal{U}$, and fix isomorphisms $\varphi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{O}_{U_i}^{\oplus n}$. The composites $\varphi_i \circ \varphi_j^{-1}$ are given by elements $\alpha_{ij} \in GL_n(\mathscr{O}_{U_{ij}})$, and one checks that the cocycle condition above is satisfied. If we change the isomorphism φ_i to φ_i', then the composite $\varphi_i \circ \varphi_i'^{-1}$ is given by an element of $GL_n(\mathscr{O}(U_i))$. In that case, $\alpha_{ij}' =$ □

Refining open covers

Finally, we need a way to compare line bundles that trivialize on different open covers. Given two line bundles $\mathscr{L}$ and $\mathscr{L}'$ on a scheme X, if $\mathcal{U}_1$ is a cover on which $\mathscr{L}$ trivializes and $\mathcal{U}_2$ is a cover on which $\mathscr{L}'$ trivializes, then we can always refine $\mathcal{U}_i$ to a cover on which both trivialize. Thus, after Lemma 3.2.1.4 it suffices to analyze what happens to Cech cohomology as we refine open covers. Thus, let us assume $\mathcal{U}$ is an open cover and $\mathcal{V}$ is a refinement of $\mathcal{U}$. More precisely, suppose $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$, each open set V_j is contained in some U_i, so we can choose a function $c : J \to I$ such that $V_j \subset U_{c(j)}$. The function c induces a map:

$$\gamma : C^n(\mathcal{U}, \mathscr{F}) \longrightarrow C^n(\mathcal{V}, \mathscr{F})$$

by sending $\alpha_{i_0,\ldots,i_n} \in C^n(\mathcal{U}, \mathscr{F})$ to $\alpha_{j_0,\ldots,j_n} = \alpha_{c(j_0),\ldots,c(j_n)}|_{V_{j_0,\ldots,j_n}}$. It is straightforward to check that this formula is compatible with the differential and thus defines a morphism of complexes.

Once again, our construction of this map depended on an auxiliary choice: the choice of the function c. However, the choice of c is not unique since a given open in the refinement could be contained in many different opens in the original cover. Suppose we choose a different function $c' : J \to I$ as above. In that case, we get a different function $\gamma' : C^*(\mathcal{U}, \mathscr{F}) \to C^*(\mathcal{V}, \mathscr{F})$. We claim that the difference $\gamma - \gamma'$ is null homotopic, i.e., there exists a chain homotopy $h : C^{*+1}(\mathcal{U}, \mathscr{F}) \to C^*(\mathcal{V}, \mathscr{F})$ such that $\gamma - \gamma' = dh + hd$. Indeed, one may define the map h by means of the formula

$$h(\alpha)_{j_0,\ldots,j_n} = \sum_{a=0}^{n} \alpha_{c(j_0),\ldots,c(j_a),c'(j_a),\ldots,c(j_n)}$$

We leave it as an exercise to check that this formula has the stated property.

Granted this, the map on cohomology induced by c and c' is the same, and we get well-defined maps

$$H^i(\mathcal{U}, \mathscr{F}) \to H^i(\mathcal{V}, \mathscr{F})$$

for any refinement. The collection of all refinements forms a partially ordered set with respect to refinement, and we define

$$\check{H}^i(X, \mathscr{F}) = \operatorname{colim}_{\mathscr{U}} H^i(\mathcal{U}, \mathscr{F})$$

to get a definition independent of the choice of an open cover.

Theorem 3.2.1.6. *If $(X, \mathscr{O}_X)$ is a scheme, then there is a canonical isomorphism:*

$$\check{H}^1(X, \mathscr{O}_X^{\times}) \xrightarrow{\sim} Pic(X).$$

This isomorphism is functorial with respect to pullbacks along morphisms of schemes.

Proof. Since $(X, \mathscr{O}_X)$ is a scheme, every line bundle trivializes on some open cover of X. The result then follows from Lemma 3.2.1.4. $\square$

In fact, the above results admit a non-abelian generalization as well. If $\mathcal{V}$ is a refinement of an open cover $\mathcal{U}$ of a topological space X, then for any sheaf of groups $\mathscr{G}$, a choice of function c as above defines a morphism

$$H^1(\mathcal{U}, \mathscr{G}) \longrightarrow H^1(\mathcal{V}, \mathscr{G}).$$

In fact, one checks directly that a different choice of c yields the same function above, and one defines

$$\check{H}^1(X, \mathscr{G}) = \mathrm{colim}_{\mathcal{U}} H^1(\mathcal{U}, \mathscr{G}).$$

The non-abelian analog of Theorem 3.2.1.6 is the following result.

Theorem 3.2.1.7. *If $(X, \mathscr{O}_X)$ is a scheme, then there is a canonical isomorphism*

$$\check{H}^1(X, GL_n(\mathscr{O}_X)) \xrightarrow{\sim} \mathscr{V}_r(X).$$

3.2.2 The units-Picard sequence

Suppose now that R is an integral domain. In that case, R has a fraction field K. We would like to analyze $Pic(R)$ in a slightly different way now. Suppose L is an invertible R-module. In that case, $L \otimes_R K$ is an invertible K-modules, i.e., a 1-dimensional K-vector space. If we fix a basis for this 1-dimensional K-vector space, that is equivalent to fixing an isomorphism $\varphi : L \otimes_R K \tilde{\to} K$. If L' is another invertible R-module, and we fix an isomorphism $\varphi' : L \otimes_R K \tilde{\to} K$, then we also get an isomorphism $(L \otimes_R L") \otimes_R K \tilde{\to} K$ from $\varphi \otimes \varphi'$ via the canonical isomorphism $(L \otimes_R L") \otimes_R K \cong L \otimes_R K \otimes L'' \otimes_R K$ arising from the associativity and symmetry isomorphisms for tensor product.

Definition 3.2.2.1. If R is an integral domain with fraction field K, then write $Cart(R)$ for group consisting of pairs (L, φ) where L is an invertible R-module and $\varphi : L \otimes_R K \tilde{\to} K$.

Since we may always choose φ, it follows that there is a surjective group homomorphism $Cart(R) \to Pic(R)$ that corresponds to forgetting φ. What is the kernel of this homomorphism? Note that two different lifts of a given L in $Pic(R)$ to $Cart(R)$ give trivializations $\varphi_1 : L \otimes_R K \tilde{\to} K$ and $\varphi_2 : L \otimes_R K \tilde{\to} K$. The composite $\varphi_1 \circ \varphi_2^{-1}$ is thus an isomorphism $K \to K$, i.e., an element of $K^{\times}$. In other words, there is a surjection

$$K^{\times} \longrightarrow \ker(Cart(R) \to Pic(R)).$$

Note that this map is not injective. Indeed, the identity element of $Cart(R)$ corresponds to the inclusion $R \hookrightarrow K$. Multiplying this inclusion by a unit in $R^{\times}$ yields the same inclusion. In other words, the action of $K^{\times}$ on $Cart(R)$ just described has stabilizer isomophic to $R^{\times}$. Putting these observations together, and observing that all of the statements we've made are compatible with extension of scalars along a homomorphism of integral domains, we obtain the following result.

Theorem 3.2.2.2. *If R is any integral domain with fraction field K, then there is an exact sequence of the form*

$$0 \longrightarrow R^{\times} \longrightarrow K^{\times} \longrightarrow Cart(R) \longrightarrow Pic(R) \longrightarrow 0.$$

This exact sequence is functorial with respect to extension of scalars along homomorphisms of integral domains.

Here is another interpretation of this theorem: for any integral domain R, we have built a 2-term complex

$$K^{\times} \longrightarrow Cart(R)$$

whose cohomology computes the cohomology of $\mathscr{O}^{\times}_{\operatorname{Spec} R}$. This complex bears no a priori relationship to the Cech complex we studied earlier, but it still computes the same cohomology groups.

A non-abelian variation

Note that the construction above works more generally for rank n projective R-modules over an integral domain R with fraction field K. If P is a rank n projective R-module, then $P \otimes_R K$ is an n-dimensional K-vector space for which we may fix a basis. Note that $R \to K$ is an injective R-module map, so we also see that $P \hookrightarrow P \otimes_R K$ is injective since tensoring with P is exact. By a *matrix divisor*, we will mean a pair (P, φ) consisting of a rank n projective R-module, and an isomorphism $P \otimes_R K \stackrel{\sim}{\to} K^{\oplus n}$. Let us write $MCart_n(R)$ for the set of such pairs; this is a pointed set, with base-point the free rank n projective R-module $R^{\oplus n}$ together with the induced basis of $K^{\oplus n}$. Note that two different isomorphisms φ and φ' of $P \otimes_R K$ with $K^{\oplus n}$ differ by a unique element of $GL_n(K)$. In other words, there is an action of $GL_n(K)$ on $MCart_n(R)$. Thus, if we consider the action map

$$GL_n(K) \times MCart_n(R) \longrightarrow MCart_n(R)$$

then the set of orbits for this action is $\mathscr{V}_n(R)$. On the other hand, the stabilizer of the identity is $GL_n(R)$. Once, again, the action just described incarnates the degree 0 and 1 cohomology of the sheaf $GL_n(\mathscr{O}_X)$ on $X = \operatorname{Spec} R$.

From Cartier divisor to Cech cohomology classes

A priori, we have no clear link between the Cech cohomology description of line bundles and the picture we just described in terms of Cartier divisors. First, let us restrict attention to the case where $X = \operatorname{Spec} R$ with R an integral domain with fraction field K. Suppose L is an invertible R-module. We know that we can choose finitely many elements $f_1, \ldots, f_r$ that generate the unit ideal in R together with isomorphisms $\varphi_i : L_{f_i} \stackrel{\sim}{\to} R_{f_i}$ as R_{f_i}-modules. Note that since R is an integral domain, so are all of its localizations, and K is again the fraction field of R_{f_i}. Now, suppose we simultaneously fix an isomorphism $L \otimes_R K \to K$. How does this choice interact with the trivialization above?

Localizing, we see that $L \otimes_R K \to K$ also yields isomorphisms $L_{f_i} \otimes_{R_{f_i}} K \to K$. The chosen trivialization $\varphi_i : L_{f_i} \cong R_{f_i}$, then give rise to a sequence of elements $\sigma_i \in K$ (take the image of 1 in R_{f_i} in K under the evident composite of the above isomorphisms.) The resulting elements of K are necessarily non-zero. Now, our choice of trivialization gives rise to a unit α_{ij} on $R_{f_if_j}$; this unit is the element one gets by tracing the isomorphism $R_{f_if_j} \cong L_{f_if_j} \cong R_{f_if_j}$ where the first isomorphism is the inverse of φ_j localized at f_i and the second map is φ_i localized at f_j. These units are related to the elements σ_i as follows: tracing the isomorphisms $\frac{\sigma_i}{\sigma_j} = \alpha_{ij}$. In other words, if we fix a trivialization of L together with a trivialization of $L \otimes_R K$, then we get an open cover $U_i = D_{f_i}$ of X, together with elements $\sigma_i \in K$ such that $\frac{\sigma_i}{\sigma_j}$ differ by an element of $R^{\times}_{f_if_j}$. Since all of the above choices are compatible with taking tensor products, tensor product induces an evident group structure on the collection of such data: i.e., Cartier divisors equipped with a trivialization on some open cover.

Now, let us analyze what happens if we change the trivialization: suppose $\varphi'_i : L_{f_i} \xrightarrow{\sim} R_{f_i}$ of L (with respect to the same open cover). In this case, we get new elements $\sigma'_i \in K^{\times}$ following the procedure above such that $\sigma'_i/\sigma'_j = \alpha'_{ij} \in R^{\times}_{f_if_j}$. Since the composite $R_{f_i} \cong L_{f_i} \cong R_{f_i}$, where the first isomorphism is given by ${\varphi'}_i^{-1}$ and the second morphism is given by φ_i, is determined by a unit $\tau_i \in R^{\times}_{f_i}$ we see that σ'_i differs from σ_i by τ_i.

Let us now describe these statements in sheaf-theoretic terms. Let $\mathscr{K}$ be the constant sheaf of rings on $\operatorname{Spec} R$ associated with the the R-module K, and let $\mathscr{K}^{\times}$ be the sheaf of units in $\mathscr{K}$. In that case, we have a short exact sequence of sheaves of the form

$$0 \longrightarrow \mathscr{O}^{\times}_{\operatorname{Spec} R} \longrightarrow \mathscr{K} \longrightarrow \mathscr{K}^{\times}/\mathscr{O}_{\operatorname{Spec} R} \longrightarrow 0.$$

Taking global sections we get the following exact sequence:

$$0 \longrightarrow R^{\times} \longrightarrow K^{\times} \longrightarrow \Gamma(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R}).$$

Since $\mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R}$ is a cokernel, it is the sheaf associated with the presheaf cokernel. In fact, the failure of surjectivity of the rightmost map in the above sequence is precisely a measure of the extent to which the quotient group $K^{\times}/R^{\times}$ differs from the global sections of the cokernel.

If we unwind the definitions, then we will see that a section of $\Gamma(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R})$ consists precisely of an open cover U_i of $\operatorname{Spec} R$ together with sections σ_i of $\mathscr{K}^{\times}$ over U_i such that $\frac{\sigma_i}{\sigma_j} \in \mathscr{O}^{\times}_{\operatorname{Spec} R}(U_{ij})$. In paticular, assuming $U_i = D_{f_i}$ as above, we see that what we constructed in the previous paragraph was precisely a global section of $\Gamma(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R})$ relative to a specific open cover. We can add any two sections on a common refinement.

Now, any Cartier divisor admits a trivialization on some open cover, changing the trivialization amounts to a new presentation as a Cartier divisor. I leave it as an exercise to check that what we have constructed is an isomorphism of the form:

$$Cart(R) \longrightarrow \Gamma(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R}).$$

Furthermore, given a Cartier divisor, there is an evident function

$$\delta : \Gamma(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}^{\times}_{\operatorname{Spec} R}) \longrightarrow H^1(\operatorname{Spec} R, \mathscr{O}^{\times}_{\operatorname{Spec} R}) \cong Pic(R)$$

obtained by sending (U_i, σ_i) to $\alpha_{ij} = \sigma_i/\sigma_j$. Indeed, the very by definition any such α_{ij} is a Cech 1-cocycle and we simply take the line bundle attached to this cocycle. This function is evidently a homomorphism and we saw above that it is a surjective homomorphism. Summing up, we've established the following result, which yields another intrepretation of the units-Pic sequence.

Theorem 3.2.2.3. *If $X = \operatorname{Spec} R$ for R and integral domain with fraction field K, then the short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{O}_X^{\times} \longrightarrow \mathscr{K}^{\times} \longrightarrow \mathscr{K}^{\times}/\mathscr{O}_X^{\times} \longrightarrow 0$$

induces an exact sequence in Cech cohomology of the form

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(X, \mathscr{O}_X^{\times}) \arrow[r] \arrow[d] & \check{H}^0(X, \mathscr{K}^{\times}) \arrow[r] \arrow[d] & \check{H}^0(\operatorname{Spec} R, \mathscr{K}^{\times}/\mathscr{O}_{\operatorname{Spec} R}^{\times}) \arrow[r] \arrow[d] & \check{H}^1(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R}^{\times}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{\times} \arrow[r] & K^{\times} \arrow[r] & Cart(R) \arrow[r] & Pic(R) \arrow[r] & 0.
\end{tikzcd}$$

We can even say a bit more. Consider the group $\check{H}^1(X, \mathscr{K}^{\times})$ for $X = \operatorname{Spec} R$ as above. Any element of this group is represented by sections $\alpha_{ij} \in U_{ij}$ satisfying the cocycle condition. Without loss of generality, we may assume that $U_i = D_{f_i}$. Each $\alpha_{ij} \in K^{\times}$, and we claim that implies it is actually a Cech 1-coboundary. Indeed, since $\mathscr{K}^{\times}$ is a constant sheaf, the restriction mapping $\mathscr{K}^{\times}(U_i) \to \mathscr{K}^{\times}(U_{ij})$ is always surjective. In that case, we may lift the unit α_{ij} to a section τ_i over U_i. Then $\tau_i|_{U_{ij}} = \tau_j|_{U_{ij}}$ by construction. Thus, we have even shown that $\check{H}^1(X, \mathscr{K}^{\times}) = 0$. There is an evident map $\check{H}^1(X, \mathscr{O}_X^{\times}) \longrightarrow \check{H}^1(X, \mathscr{K}^{\times})$, say defined at the level of cocycles. We can put this in the context above as well.

Suppose, more generally, that X is a topological space, and we are given a short exact sequence of sheaves of abelian groups of the form

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0.$$

Generalizing what we observed above, a global section of $\mathscr{F}''$ can be described as an element $\{U_i, \sigma_i\}$ where σ_i are sections of $\mathscr{F}$ on U_i such that $\sigma_i - \sigma_j \in \mathscr{F}'(U_{ij})$. As above, sending such an element to the differences $\sigma_i - \sigma_j$ defines a morphism $\delta : \check{H}^0(X, \mathscr{F}'') \longrightarrow \check{H}^1(X, \mathscr{F}')$. There is then an exact sequence of abelian groups of the form:

$$\begin{aligned}
0 \longrightarrow & \check{H}^0(X, \mathscr{F}') \longrightarrow \check{H}^0(X, \mathscr{F}) \longrightarrow \check{H}^0(X, \mathscr{F}'') \\
\xrightarrow{\delta} & \check{H}^1(X, \mathscr{F}') \longrightarrow \check{H}^1(X, \mathscr{F}) \longrightarrow \check{H}^1(X, \mathscr{F}'')
\end{aligned}$$

We would like to define cohomology so that a short exact sequence of sheaves gives rise to a corresponding exact sequence of cohomology groups generalizing the above constructions.

Another variation: fractional ideals

Here is a variation on the description of Cartier divisors given above. Suppose R is an integral domain with fraction field K as above, and P is a finitely generated rank n projective R-module. In that case, we may pick a surjection $R^{\oplus r} \to P$ for some integer r. As above, we have an identification $P \otimes_R K \xrightarrow{\sim} K^{\oplus n}$, but the surjection we fixed yields a map $R^{\oplus r} \to K^{\oplus n}$, i.e., a sequence of r elements of $K^{\oplus n}$. In the case where L is an invertible R-module, i.e., $n = 1$, we thus get an identification of L as the R-submodule of K generated by elements $\sigma_1, \ldots, \sigma_r \in K$. Now, each element σ_i can be written as $\frac{r_i}{s_i}$ for elements $r_i \in R$, $s_i \in R \setminus 0$. By clearing denominators, e.g., multiplying through by the least common multiple s of s_i, we see sL is an R-submodule of R, i.e., an ideal of R.

Definition 3.2.2.4. If R is an integral domain, a *fractional ideal* I in R is an R-submodule $I \subset K$ such that there exists a (non-zero) element $r \in R$ with $rI \subset R$.

Thus, any invertible R-module L, together with a choice of isomorphism $L \otimes_R K \xrightarrow{\sim} K$ and a choice of surjection $R^{\oplus r} \to L$ gives rise to a fractional ideal.

Remark 3.2.2.5. We have imposed no finiteness hypotheses on I in the above definition so that ideals are always examples of fractional ideals. If R is Noetherian, then since $rI \subset R$, we conclude that I is necessarily finitely generated.

So far, we have only use the property that L has rank 1, but not that L is actually an invertible R-module. Now, we know $\mathrm{Hom}_R(L, R)$ is an invertible R-module and the evaluation map yields the isomorphism $L \otimes_R \mathrm{Hom}_R(L, R) \to R$. If I' is the invertible ideal attached to $L^\vee$, then $I \otimes_R I' \cong R$.

Definition 3.2.2.6. If R is an integral domain, an *invertible fractional ideal* I in R is a fractional ideal I in R for which there exists an invertible ideal I' with $I \otimes_R I' \cong R$.

Remark 3.2.2.7. Note that invertible fractional ideals are automatically finitely presented ideals, since invertible modules are finitely presented by Lemma 2.2.1.9. If I is an invertible fractional ideal, then forgetting the choice of generators yields an invertible module. In other words, there is a canonical forgetful map from the group of invertible fractional ideals to $Pic(R)$.

Example 3.2.2.8. The theory of fractional ideals is probably most familiar from number theory. A number field K is a finite extension of $\mathbb{Q}$. The ring of integers $\mathscr{O}_K$ in K is the integral closure of $\mathbb{Z}$ in K. In this situation, the Picard group is more commonly known as the ideal class group and measures the failure of unique factorization. Take $K = \mathbb{Q}(\sqrt{-5})$. Note that unique factorization fails in $\mathbb{Q}(\sqrt{-5})$ since $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Let $J = (2, 1 - \sqrt{-5})$. One shows that $J^2 = (2)$, which is principal. In fact, $\mathrm{Pic}(R)$ is cyclic of order 2 generated by J. More generally, it is a fantastic fundamental result in algebraic number theory that $\mathrm{Pic}(\mathscr{O}_K)$ is always a finite abelian group.

Example 3.2.2.9. If R is a principal ideal domain, it follows from the structure theorem that $\mathrm{Pic}(R) = 0$. In particular, $\mathrm{Pic}(k[t]) = 0$. More generally, any localization of a PID is a PID, so we conclude that Picard groups of (non-empty) proper open subsets of $\mathbb{A}^1_k$ also have trivial Picard groups.

3.2.3 The units-Pic sequence for general commutative rings

In the discussion above, we restricted attention to integral domains, but this was only a technical convenience. Rings of continuous functions will not, in general, be integral domains. Moreover, typically they have many zero divisors (e.g., functions with bounded but disjoint supports). We now observe that with slightly more work, the theory developed above holds equally well for rings that are integral domains; we will keep rings of continuous functions in the back of our head.

Definition 3.2.3.1. If R is a commutative ring, the total quotient ring of R, denoted $Frac(R)$, is the localization of R at the multiplicative set of all non-zero divisors.

In this generality, $Frac(R)$ is no longer a field. Nevertheless, since we are inverting precisely the non-zero-divisors in R, the map $R \to Frac(R)$ is injective. Thus, if L is an invertible R-module, the map $L \to L \otimes_R Frac(R)$ remains injective. However, we ca no longer assert anything about $L \otimes_R Frac(R)$; this may be a non-free $Frac(R)$-module! We can still analyze the extension of scalars homomorphism $\mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(Frac(R))$. The kernel of this map consists precisely of those invertible R-modules such that $L \otimes_R Frac(R)$ is a free rank 1 $Frac(R)$-module.

Suppose we are given an invertible R-module such that $L \otimes_R Frac(R)$ is a free rank 1 $Frac(R)$-module. By choosing generators of L, one obtains an R-submodule I of $Frac(R)$ generated by finitely many elements $\sigma_1, \ldots, \sigma_n$. Clearing the denominators, we conclude that $sI \subset R$.

Definition 3.2.3.2. If R is a commutative ring, then an invertible fractional ideal in R is an invertible R-submodule of $Frac(R)$, such that $L \otimes_R Frac(R)$ is a free rank 1 $Frac(R)$-module. Write $\mathbf{I}(R)$ for the set of invertible fractional ideals.

As before the set of invertible fractional ideals is a group under tensor product of R-modules, and there is, by construction an exact sequence of the form

$$\mathbf{I}(R) \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(Frac(R)).$$

The kernel of the map $\mathbf{I}(R) \to \mathrm{Pic}(R)$ once again consists of invertible fractional ideal structures on the trivial R-module. A choice of basis of a free rank 1 R-submodule of $Frac(R)$ is uniquely determined by an element $u \in Frac(R)^{\times}$. Two such choices of basis differ by an an element of $R^{\times}$ and therefore, just as above one obtains an exact sequence of the form

$$1 \longrightarrow R^{\times} \longrightarrow (Frac(R))^{\times} \longrightarrow \mathbf{I}(R) \longrightarrow \mathrm{Pic}(R) \longrightarrow Pic(Frac(R)),$$

which no longer need be exact on the right.

Remark 3.2.3.3. If $\varphi : R \to S$ is any R-module map, then the kernel of $Pic(R) \to Pic(S)$ coincides precisely with the set of invertible R-modules L such that $L \otimes_R S \cong S$; we will call such objects invertible R-submodules of S and we write $Pic(\varphi)$ or $Pic(R, S)$ for the set of isomorphism classes of such objects. This set is a group with respect to tensor product of R-modules. Arguing as above, the kernel of the map $Pic(\varphi) \to Pic(R)$ corresponds to invertible R-submodule structures on the trivial module $R \otimes_R S$, which correspond to elements of $S^{\times}$ module the image of $R^{\times}$ (which need not inject in $S^{\times}$ in general). In other words, one obtains an exact sequence of the form

$$R^{\times} \longrightarrow S^{\times} \longrightarrow \mathrm{Pic}(\varphi) \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(S)$$

If R is a subring of S, then we can even assert that the left hand map is injective. Functoriality of the resulting exact sequence is a consequence of functoriality of extension of scalars.

The identification of the above sequence in terms of Cartier divisors is slightly more complicated, but proceeds as before. Suppose I is an invertible R-submodule of $Frac(R)$ (such a thing is free of rank 1 as a $Frac(R)$-module by "clearing the denominators"). We may choose a local trivialization of I. In other words, we may find elements $f_1, \ldots, f_n$ such that I_{f_i} is a free R_{f_i}-module of rank 1 and such that $\{f_1, \ldots, f_n\}$ generates the unit ideal. The map $R \mapsto R_{f_i}$ induces a homomorphism $Frac(R) \to Frac(R_{f_i})$. Now, there is a commutative square of the form

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_{f_i} \arrow[d] \\ Frac(R) \arrow[r] & Frac(R_{f_i}). \end{tikzcd}$$

From this it follows that there is a canonical isomorphism $I_{f_i} \otimes_R Frac(R_{f_i}) \cong (I \otimes_R Frac(R)) \otimes_{Frac(R)} Frac(R_{f_i})$. Since $I \otimes_R Frac(R)$ is a free $Frac(R)$-module of rank 1, it follows that so is $I_{f_i} \otimes_R Frac(R_{f_i})$. In any case, our choice of trivialization determines an element $\sigma_i \in Frac(R_{f_i})$. As before, the formula $\alpha_{ij}\sigma_j = \sigma_i$-holds in $R_{f_i f_j}$. It follows that $\frac{\sigma_i}{\sigma_j}$ must be an element of $R_{ij}^{\times} \in Frac(R_{f_i f_j})^{\times}$. Therefore, if we define a Cartier divisor on $\operatorname{Spec} R$ to be a collection $D = \{U_i, \sigma_i\}$ where U_i is an open cover by principal open sets and where $\sigma_i \in Frac(R_{f_i})$ such that σ_i/σ_j is a unit on $U_i \cap U_j$, then we see that there is a bijection between Cartier divisors and invertible R-submodules of $Frac(R)$ just as in the case where X is integral. The fundamental difference is that we may no longer be able to identify $Frac(R_{f_i})$ for different values of i.

Example 3.2.3.4. If X is a compact manifold, and we take $R = C(X, \mathbb{R})$ the ring of real-valued continuous functions on X. Given a partition of unity $\{f_i\}_{i=1,\ldots,n}$, the f_i are typically zero divisors: if g is any compactly supported function with support disjoint from f_i, then $f_i g = 0$. For example, $f_i f_j$ might even be zero. If f_i *is* a zero divisor, then the $R \to R_{f_i}$ is not injective. It follows that the map $Frac(R) \to Frac(R_{f_i})$ is not injective in general either.

Picard groups of non-reduced rings

Now, suppose R is a connected commutative unital ring and N is the nilradical of R. The nilradical is always contained in the Jacobson radical $N \subset J(R)$. As a consequence, we may appeal to Nakayama's lemma to compare finitely generated R-modules and finitely generated R/N-modules.

Proposition 3.2.3.5. *If R is a commutative unital ring, and P, P' are projective R-modules, then if $P/N \cong P'/N$ then $P \cong P'$.*

Using this fact, we observe that if we want to study Picard groups of commutative rings, we can always assume that our rings are reduced by passing from R to R/N.

Corollary 3.2.3.6. *If R is a commutative unital ring, then the map $Pic(R) \to Pic(R/N)$ is an isomorphism.*

Example 3.2.3.7. Suppose X is a compact Hausdorff space and $R = C(X, \mathbb{R})$, the ring of real valued continuous functions. If f is a nilpotent element of $C(X)$, then $f^n = 0$ for some integer n. This means $f^n(x) = 0 \in \mathbb{R}$, which means $f(x) = 0$. In other words, in this case the ring of continuous functions on X is reduced. In fact, the maximal ideals in $C(X)$ have been characterized by Gelfand-Kolmogoroff (cf. [**?**, Chapter 7]): they are parameterized by the points $x \in X$: $\mathfrak{m}_x$ is the ideal of functions vanishing at a point. It follows that $\cap_{x \in X} \mathfrak{m}_x = 0$. Thus, in the case of continuous functions, the real differentiating feature is the presence of zero-divisors. The prime ideal structure of such rings is much more complicated (see, e.g., [**?**],[**?**, Chapter 14]).

3.3 More geometry: closed immersions, separated maps and properness

Our next goal is to globalize the above results and to investigate them in various cases.

3.3.1 Integral schemes

Earlier, we observed that a ring R was an integral domain then R was reduced and $\operatorname{Spec} R$ was irreducible as a topological space. In fact, the converse is also true. We now globalize this definition.

Definition 3.3.1.1. A scheme X will be called reduced if $\mathscr{O}_{X,x}$ is reduced for every $x \in X$.

Lemma 3.3.1.2. *A scheme X is reduced if and only if for every open subset $U \subset X$, $\mathscr{O}_X(U)$ is a reduced ring.*

Definition 3.3.1.3. A scheme X is integral if for every open $U \subset X$, $\mathscr{O}_X(U)$ is an integral domain.

Lemma 3.3.1.4. *If X is a scheme, then any irreducible closed subset $Z \subset X$ has a unique generic point.*

Proof. Suppose X is a scheme and $Z \subset X$ is an irreducible closed subset. For any affine open subset $U = \operatorname{Spec} R \subset X$, the subset $Z \cap U = V(I)$ for some radical ideal I of R. Now, $Z \cap U$ is either empty or irreducible, and the latter must happen for at least one $U \subset X$. In that case, I is a prime ideal R, which corresponds to a generic point ξ of $Z \cap U$. It follows that $Z = \overline{\xi}$. If ξ' was another generic point, then $\xi' \in Z \cap U$ and we conclude that $\xi = \xi'$. □

Lemma 3.3.1.5. *A scheme X is integral if and only if it is reduced and irreducible.*

For us, the important statement will be that integral schemes have unique generic points. From that observation, we can deduce the following result about Cech cohomology of constant sheaves.

Exercise 3.3.1.6. *Show that if $\mathscr{C}$ is any constant sheaf of abelian groups on an irreducible scheme X, then $\check{H}^i(X, \mathscr{C})) = 0$ for every $i > 0$.*

3.3.2 Closed immersions and separatedness

Definition 3.3.2.1. A morphism $i : Z \to X$ of schemes is a *closed immersion* if i) i is a homeomorphism of Z onto a closed subset of X, ii) the morphism of sheaves $i^\sharp : \mathscr{O}_X \to i_* \mathscr{O}_Z$ is a surjective morphism.

Remark 3.3.2.2. Frequently, the definition of closed immersion is made for locally ringed spaces, in which case there is a further finiteness hypothesis imposed on the kernel $\mathscr{I}$ of the morphism $i^\sharp$: it should be locally generated by sections. In fact, it turns out for morphisms of schemes, this additional hypothesis is superfluous, but establishing this requires some effort.

Definition 3.3.2.3. A morphism $f : X \to S$ of schemes is *affine* if the pre-image of any affine open subscheme of S under f is affine.

Lemma 3.3.2.4. *Any closed immersion of schemes is quasi-compact; any affine morphism of schemes is quasi-compact.*

The property that a morphism $i : Z \to X$ of schemes is a closed immersion can be checked locally for the Zariski topology: i.e., if we can find an open cover of X by open sets U_i, then i is a closed immersion if and only if the induced maps $i|_{U_i} : Z \cap U_i \to U_i$ are themselves closed immersion. In particular, we may always reduce to the case where U_i is an open affine subscheme of X.

Lemma 3.3.2.5. *A closed immersion is an affine morphism.*

The property that a topological space is Hausdorff can be phrased as saying that the digonal $X \subset X \times X$ is a closed subset. The following is the standard generalization of this definition in algebraic geometry.

Definition 3.3.2.6. A morphism $f : X \to S$ of schemes is *separated* if the diagonal morphism $\Delta_{X/S}$ is a closed immersion and *has affine diagonal* if $\Delta_{X/S}$ is an affine morphism.

Remark 3.3.2.7. Note that since closed immersions are affine morphisms, it follows that separated morphisms necessarily have affine diagonal. Moreoever, since affine morphisms are quasi-compact, it follows that morphisms with affine diagonal are quasi-separated as well.

As before, the property that f is separated can be checked Zariski locally on S. In other words, suppose there is an open cover U_i of S and consider the morphisms $f_i : f^{-1}(U_i) := X_i \to U_i$; we claim that f is separated if and only if f_i is for each i. Indeed, the scheme X can be glued together from the X_i, the fiber product $X \times_S X$ is glued from the $X_i \times_{U_i} X_j$. Now, the statement follows from the fact that whether a morphism is a closed immersion can be checked Zariski locally.

Example 3.3.2.8. Any morphism of affine schemes is separated. Indeed, if $\varphi : R \to S$ is a ring homomorphism corresponding to $\operatorname{Spec} S \to \operatorname{Spec} R$, then the diagonal morphism $\Delta_{\operatorname{Spec} S/\operatorname{Spec} R} : \operatorname{Spec} S \to \operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S$ corresponds to the product homomorphism $S \otimes_R S \to S$. Any closed immersion of schemes is a separated morphism. Indeed, this follows from the fact that separatedness can be checked upon passing to an open affine cover of the target and the preceding statement. Any morphism $f : X \to Y$ of schemes that is separated is automatically quasi-separated since if $\Delta_{X/Y}$ is a closed immersion, it is necessarily quasi-compact by the lemma above.

Definition 3.3.2.9. Assume k is a field, and $f : X \to \operatorname{Spec} k$ is a k-scheme. We will say that X is a k-variety if f is separated, has finite type and X is integral.

Stability under base-change

Assume $f : X \to S$ and $\varphi : S' \to S$ are morphisms of schemes. In that case, we may always form the fiber product $X' := X \times_S S'$ and f induces a morphism $f' : X' \to S'$. If P is a property of morphisms of schemes, then we will say that P is stable by base-change if f has property P, then for any φ, the morphism f' again has property P. Many properties of morphisms of schemes are stable under base-change.

Lemma 3.3.2.10. *The following types of morphisms of schemes are stable under (arbitrary) base-change: quasi-compact, finite-type, open immersions, closed immersions, locally closed immersions, or affine.*

Proof. Exercise. □

We will add various classes of morphisms to this list as we move forward. For the time being, observe that there are properties of morphisms that are not stable under arbitrary morphisms. For example, we could call a morphism of schemes open or closed if the underlying map of topological spaces is open or closed.

Example 3.3.2.11. If k is a field, say algebraically closed for simplicity, then the morphism $\mathbb{A}^1_k \to \operatorname{Spec} k$ is closed. Indeed, the closed subsets of $\mathbb{A}^1_k$ are precisely finite sets of points, which are all evidently sent to the closed point of $\operatorname{Spec} k$. Note, however, that closed morphisms are not stable under base-change. Indeed, consider the base-change of the structure morphism $\mathbb{A}^1_k \to \operatorname{Spec} k$ along itself, i.e., consider the morphism $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \to \mathbb{A}^1_k$. In this case, the base-change is identified with the projection morphism onto the (say) second factor. The fiber product is identified with $\mathbb{A}^2_k$ and the subset $xy = 1$ is a closed subset. The image of this subset under the second projection is the locus $y \neq 0$, which is not closed.

Definition 3.3.2.12. A morphism $f : X \to S$ of schemes is *universally closed* or *universally open* if the base-change of f along any morphism $S' \to S$ is closed (resp. open).

Remark 3.3.2.13. Universally closed or universally open morphisms are stable under arbitrary base-change.

In topology, recall that a morphism f is called *proper* if the preimage of any compact set is compact. At least under some Hausdorffness conditions, this notion of properness is equivalent to being universally closed. This observation can be thought of as motivating the following definition.

Definition 3.3.2.14. A morphism $f : X \to S$ of schemes is proper if f is separated, has finite type and is universally closed.

Lemma 3.3.2.15. *Any closed immersion is proper.*

Proof. We already saw that closed immersions are separated. Closed immersions are evidently closed maps, and since closed immersions are stable by base change, it follows that they are also universally closed. It remains to check that closed immersions are automatically finite type. Since closed immersions are quasi-compact, it suffices to check that they are locally of finite type, i.e., for every point $x \in X$, we can find a neighborhood $U = \operatorname{Spec} S$ of x mapping into an affine open $V = \operatorname{Spec} R$ such that $R \to S$ is finite type. However, S is necessarily a quotient of R by an ideal, so automatically finitely type as an R-algebra. $\square$

3.3.3 Separation, properness and valuation rings

Another one of the equivalent characterization of Hausdorfness for topological spaces is in terms of uniqueness of uniqueness of limits (for spaces that are not metrizable, "limits" have to be taken in terms of nets, rather than countable sequences). Likewise, compactness of topological spaces can be phrased in terms of existence of limits for nets (every net has a convergent subnet). We would like to formulate analogous notions of "limits" in algebraic geometry.

Here is a motivating example: the simplest case where we can think about limits is in terms of $\mathbf{G}_m \subset \mathbb{A}^1$. If we have a morphism $\lambda : \mathbf{G}_m \to X$, then we might say that λ "has a limit as $t \to 0$" if λ can be extended to a morphism $\mathbb{A}^1 \to X$, in which case it makes sense to call the image of 0 the "value" of the limit. Now, as the example of the identity morphism $\mathbf{G}_m \to \mathbf{G}_m$ shows, in general we cannot hope that limits like this exist (i.e., the identity morphism does not factor through a morphism $\mathbb{A}^1 \to \mathbf{G}_m$). However, we can always ask, assuming a limit exists, is it unique? In order to phrase this uniqueness question in scheme-theoretic terms, we would like some information about the locus of points on which limits exist. The following result shows that this locus is well-behaved, i.e., it may be described in scheme-theoretic terms.

Proposition 3.3.3.1. *Let X, and Y be schemes over S. Let $a, b : X \to Y$ be morphisms of schemes over S. There exists a largest locally closed subscheme $Z \subset X$ such that $a|_Z = b|_Z$. In fact Z is the equalizer of (a, b). Moreover, if Y is separated over S, then Z is a closed subscheme.*

Proof. The pair (a, b) defines a morphism $(a, b) : X \to Y \times_S Y$. In that case, we may consider the fiber product diagram:

$$\begin{tikzcd} Z := Y \times_{Y \times_S Y} X \arrow[r] \arrow[d, "\Delta_{X/S}"] & X \arrow[d, "(a,b)"] \\ Y \arrow[r, "\Delta_{Y/S}"] & Y \times_S Y. \end{tikzcd}$$

The equalizer of (a, b) is then the scheme Z in this fiber product diagram for categorical reasons. We claim that the $\Delta_{X/S}$ is always a locally closed immersion (exercise). To say that Y is separated over S is to say that $\Delta_{Y/S}$ is a closed immersion. In that case, the morphism $Z \to X$ is necessarily also a closed immersion because closed immersions are stable under base change. $\square$

Valuation rings: definitions

Next, we would like to talk about the kinds of limits we would like to expect are unique. Rather than looking at $\mathbf{G}_m \subset \mathbb{A}^1$, if all we care about is the value of the limit, then we can localize $\mathbb{A}^1 = \operatorname{Spec} k[t]$ at the ideal (t) corresponding to the closed point 0. Then, rather than considering $\mathbf{G}_m \subset \mathbb{A}^1$, we would simply look at the generic point $\eta \in \mathbb{A}^1$. This local ring $\operatorname{Spec} k[t]_(t)$ consists of two points: the generic point η and a closed point corresponding to the maximal ideal $(t)k[t]_{(t)}$. One could, more generally, try to talk about existence of limits using arbitrary maps from local rings, but we will restrict our attention to a special class of local rings contained in the following definition.

Definition 3.3.3.2. Suppose K is a field, and A and B are two local rings contained in K. We will say that B dominates A if $A \subset B$ and $A \cap \mathfrak{m}_B = \mathfrak{m}_A$. If A is a local domain with fraction field K, then we say that A is a valuation ring if A is maximal for the relation of dominance among local rings contained in K. If A is a valuation ring with fraction field K, given a domain $R \subset K$, we will say that A is centered on R if $R \subset A$.

If A is a valuation ring with fraction field K, then since A is a local domain, A has two points: a generic point and a closed point lying in the closure of the generic point. More generally, we will say that a point y is a specialization of a point x if y is contained in the closure of x. This notion of specalization is a partial ordering on the points of X.

Morphisms out of spectra of valuation rings

Now, suppose A is a valuation ring with fraction field K and X is a scheme. We will try to describe the set of morphisms $\operatorname{Spec} A \to X$ in terms of more concrete data. Suppose we have a morphism $\operatorname{Spec} A \to X$. The image of the generic point of $\operatorname{Spec} A$ gives a point x_1 of X and the image of the closed point of $\operatorname{Spec} A$ gives a point x_0 of X; moreover x_0 is necessarily a specalization of x_1. Write Z for the scheme we get by equipping the closure of x_1 with the reduced scheme structure and consider the local ring $R := \mathscr{O}_{Z,x_0}$. By construction $\operatorname{Spec} A \to X$ factors through $\operatorname{Spec} R$, i.e., corresponds to a ring homomorphism $\varphi : R \to A$. Note that R is a domain by assumption.

The generic point of $\operatorname{Spec} R$ corresponds to the zero ideal, while the closed point is precisely $\varphi^{-1}(\mathfrak{m}_A)$. The map $R \to K$ factors through $k(x_1)$. Since the map $R \subset k(x_1)$ is injective, we also conclude that $R \to A$ is injective and thus that the maximal ideal of R is precisely $R \cap \mathfrak{m}_A$. In other words, A dominates R in K in terms of the definition above. These observations lead to the next result.

Proposition 3.3.3.3. *Suppose A is a valuation ring with fraction field K and X is a scheme. There is a bijection between the set of morphisms $\operatorname{Spec} A \to X$ and the set consisting of the following data: a pair (x_1, x_0) of points in X such that x_0 is a specialization of x_1, $\kappa(x_1) \subset K$, and the local ring $\mathscr{O}_{Z,x_0}$ where Z is $\bar{x_1}$ with its reduced scheme structure, is dominated by A.*

Proof. The construction of the function from the set of morphisms $\operatorname{Spec} A \to X$ to the specified data is carried out before the statement of the proposition. Conversely, suppose we have data as in the statement. In that case, the inclusion $\mathscr{O}_{Z,x_0} \subset A$ defines a morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathscr{O}_{Z,x_0}$, which when composed with the inclusion $\operatorname{Spec} \mathscr{O}_{Z,x_0} \subset X$ yields the required morphism. It is straightforward to check that the two functions just described are mutually inverse bijections. $\square$

Valuative criteria I

We can now formulate our statements about existence and uniqueness of limits in terms of lifting properties.

Definition 3.3.3.4 (Valuative criteria)**.** Suppose $f : X \to S$ is a morphism of schemes. If, given a valuation ring A with fraction field K fitting into a diagram of the form

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S, \end{tikzcd}$$

then there exists at most one morphism $\operatorname{Spec} A \to X$ making all triangles commutes, we will say that f satisfies the uniqueness part of the valuative criterion. If, given a valuation ring A with fraction field K and a diagram as above, there exists a morphism $\operatorname{Spec} A \to X$ making all resulting triangles commute, we will say that f satisfies the existence part of the valuative criterion.

Proposition 3.3.3.5. *If $f : X \to S$ is a separated morphism, then f satisfies the uniqueness part of the valuative criterion.*

Proof. Suppose given a diagram of the form:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S, \end{tikzcd}$$

and two morphisms $a, b : \operatorname{Spec} A \to X$ making all resulting triangles commute. In that case, we may form the equalizer scheme of a and b via Proposition 3.3.3.1. Since f is separated, it follows that there exists a closed subscheme of $\operatorname{Spec} A$ on which a and b agree. By assumption, this closed subscheme contains the generic point of A since a and b agree upon restriction of $\operatorname{Spec} K$. Since A is a domain, it follows that the closed subscheme must be all of $\operatorname{Spec} A$, which is exactly what we wanted to show. □

We would like to establish converses to the above statements and *characterize* separatedness and properness in terms of the relevant valuative criteria.

Proposition 3.3.3.6. *If $f : X \to S$ is a universally closed morphism (e.g., a proper morphism), then f satisfies the existence part of the valuative criterion.*

Proof. Suppose given a diagram of the form:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S. \end{tikzcd}$$

The morphism $f : X \to S$ is universally closed by assumption, so consider the base-change of f along the morphism $\operatorname{Spec} A \to S$; write X_A for this base-change, and consider the morphism $f' : X_A \to \operatorname{Spec} A$; this morphism is again universally closed since universally closed morphisms are stable under base-change. On the other hand, by the universal property of fiber products, the morphisms $\operatorname{Spec} K \to X$ and $\operatorname{Spec} K \to \operatorname{Spec} A$ determine a unique morphism $\operatorname{Spec} K \to X_A$ whose composites agree with the morphisms in the diagram above. We will show that f' has a section.

Write ξ_1 for the image of the map $\operatorname{Spec} K \to X_A$, and let Z be $\overline{\xi_1}$ with the reduced-induced subscheme structure. Since f' is universally closed, it follows that the image of Z in $\operatorname{Spec} A$ is a closed subset. Since the composite of f' and the map $\operatorname{Spec} K \to X_A$ coincides with the inclusion of the generic point in $\operatorname{Spec} A$, it follows that the iamge of Z is all of $\operatorname{Spec} A$. Moreover, the residue field of ξ_1 necessarily coincides with K. Let ξ_0 be any point in X_A that maps to the closed point in $\operatorname{Spec} A$. In that case, we get two points ξ_1 and ξ_0 such that ξ_0 is a specalization of ξ_1. The local ring $\mathscr{O}_{Z,\xi_0}$ is contained in K and is necessarily dominated by A since A is maximal among local domains with fraction field K. As such, we obtain the required section $\operatorname{Spec} A \to X_A$ of f'. □

Valuative criteria II

We now aim to prove the valuative criterion of separatedness and properness. Note that the two criteria are closely related since to check that a morphism is separated, we simply have to check that the morphism $\Delta_{X/S}$ is a closed immersion. Before doing this, we further analyze the link between valuation rings and specializations; our goal is to strengthen the analogy between limits and valuation rings.

Lemma 3.3.3.7. *If K is a field and A is a local ring contained in K, then there exists a valuation ring with fraction field K dominating A.*

Proof. The idea for existence is to apply Zorn's lemma to the partially ordered set of local rings in K dominating A. To do this, we need to know that there exists a local ring dominating A that is different from A. Since $A \neq K$, we can always find an element $t \in K$ that lies outside of the fraction field of A. We then analyze various cases. Either t is algebraic over A or it is transcendental over A. If it is transcendental over A, then $A[t] \subset K$, and the localization $A_{(\mathfrak{m},t)}$ does the job. If t is algebraic over A, then for some element $a \in A$, the element at satisfies a monic irreducible polynomial with coefficients in A. The subring $A' := A[at]$ of K is then finite over A. It suffices to know that if $\mathfrak{m}$ is a prime ideal, then there exists a prime ideal $\mathfrak{m}'$ of A that lies over $\mathfrak{m}$. We claim that the localization of A' at $\mathfrak{m}'$ gives the required local ring. Indeed, since the element t lies outside the fraction field of A, it cannot be the case that $A = A'_{\mathfrak{m}'}$. □

Lemma 3.3.3.8. *If S is a scheme, and s' specializes to s, then*

1. *there exists a valuation ring A and a morphism $f : Spec(A) \to S$ such that the generic point η of $\operatorname{Spec}(A)$ maps to s' and the special point maps to s, and*
2. *given a field extension $\kappa(s') \subset K$ we may arrange it so that the extension $\kappa(s') \subset \kappa(\eta)$ induced by f is isomorphic to the given extension.*

Proof. Let s be a specialization of s' in S, and let $\kappa(s') \subset K$ be an extension of fields. Each of these points corresponds to a morphism from the spectrum of a field into K. It follows that there are ring maps $\mathscr{O}_{S,s} \to \kappa(s') \to K$. Let $A \subset K$ be any valuation ring whose field of fractions is K and which dominates the image of $\mathscr{O}_{S,s} \to K$ (such a valuation ring exists by the previous lemma). One checks that the ring map $\mathscr{O}_{S,s} \to A$ induces the morphism $f : Spec(A) \to S$. □

Next, we'd like to further link closedness and specializations.

Lemma 3.3.3.9. *If $f : Y \to X$ is an immersion of schemes, then f is a closed immersion if and only if $f(Y)$ is a closed subset of X.*

Proof. If f is a closed immersion, then $f(Y)$ is homeomorphic to a closed subset of X and hence closed. Conversely, suppose that $f(Y)$ is a closed. Since f is an immersion, by definition, there is an open subscheme $U \subset X$ such that f is the composition of a closed immersion $i : Y \to U$ followed by the open immersion $j : Y \to X$. Let $\mathscr{I} \subset \mathscr{O}_U$ be the sheaf of ideals associated with the closed immersion i (locally finitely generated). In that case, $\mathscr{I}|_{U \setminus i(Y)} = \mathscr{O}_{U \setminus i(Y)} = \mathscr{O}_{X \setminus i(Y)}|_{U \setminus i(Y)}$. Thus, we may glue $\mathscr{I}$ and the trivial sheaf $\mathscr{O}_{X \setminus i(Y)}$ via the identity map on the intersection. The resulting sheaf $\mathscr{J}$ is locally finitely generated by construction. Again by construction, $\mathscr{J}$ is supported on U and equal to $\mathscr{O}_U/\mathscr{I}$. Thus we see that the closed subspaces associated with $\mathscr{I}$ and $\mathscr{J}$ are the same. The result follows. □

Lemma 3.3.3.10. *Suppose $f : X \to Y$ is a quasi-compact morphism of schemes. The subset $f(X) \subset Y$ is closed if and only if it stable under specialization.*

Proof. Assume that $f(X)$ is stable under specialization. Let $U \subset Y$ be an affine open subscheme. It suffices to prove that $U \cap f(X)$ is closed in U. Since $U \cap f(X)$ is stable under specializations in U, this reduces us to the case where Y is affine. Because f is quasi-compact and U is affine, we conclude that X is quasi-compact as well. Thus, we may take a finite open cover of X by open affine subschemes U_i; say $Y = \operatorname{Spec} R$ and $U_i = \operatorname{Spec} A_i$. In that case, $f(X)$ coincides with the image of $\coprod_{i \in I} U_i \to Y$. Since $\coprod_{i\in I} U_i = \operatorname{Spec}(\prod_i A_i)$, we have reduced to proving the result in the case where X is affine as well.

Thus, assume $\varphi : R \to A$ is a ring homomorphism. We want to show that $f(\operatorname{Spec} A)$ is closed if and only if it is stable under specialization. If $f(\operatorname{Spec} A)$ is closed, it is stable under specialization. Thus, let us assume that $f(\operatorname{Spec} A)$ is closed under specialization. Suppose $\mathfrak{p} \subset R$ be a prime ideal such that the corresponding point of $\operatorname{Spec} R$ is in the closure of $f(\operatorname{Spec} A)$. Unwinding the definition of the image of f, this means that for every $r \in R$, $r \notin \mathfrak{p}$, i.e., $D_r \cap f(\operatorname{Spec} A) \neq \emptyset$. Since $D_r \cap f(\operatorname{Spec} A)$ is the image of $\operatorname{Spec} A_r$ in $\operatorname{Spec} R$, we conclude that $A_r \neq 0$. In other words, $1 \neq 0$ in the ring A_r. Since $A_{\mathfrak{p}}$ is the directed colimit of the rings A_r, we conclude that $1 \neq 0$ in $A_{\mathfrak{p}}$. Thus, $A_{\mathfrak{p}} \neq 0$ and considering the image of $\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A \to \operatorname{Spec} R$ we see that there exists $\mathfrak{p}' \in f(\operatorname{Spec} A)$ with $\mathfrak{p}' \subset \mathfrak{p}$. Since $f(\operatorname{Spec} A)$ is closed under specialization, we conclude that $\mathfrak{p}$ is a point of $f(\operatorname{Spec} A)$ as required. □

In the next lemma, we can link closedness and specializations.

Lemma 3.3.3.11. *A quasi-compact morphism $f : X \to Y$ of schemes is universally closed if and only if specializations lift along arbitrary base extensions of f.*

Proof. If $f : X \to Y$ is a closed map of topological spaces, then note that specializations lift along f, i.e., if y specializes to y' and $y = f(x)$, then there exists $x' \in X$ such that x specializes to x' and $f(x') = x$. Indeed, since $y = f(x)$, consider the set $\bar{x} \subset X$ which is closed. Since f is closed, $f(\bar{x})$ is a closed subset of Y that contains y. It must therefore contain $\bar{y}$. Since $y' \in \bar{y}$, we can thus choose the required lift.

Conversely, suppose $f : X \to Y$ is a quasi-compact morphism of schemes, and suppose specalizations lift along f; we claim that means that f is itself closed. Let $Z \subset X$ be a closed subset; give it the reduced induced scheme structure so that $Z \to X$ is a closed immersion. In that case, $Z \to X$ is automatically a quasi-compact morphism so the composite $Z \to Y$ is quasi-compact as well. Since $Z \to X$ is closed, we know that specalizations lift along $Z \to X$. Then it follows that specializations lift along the composite map $Z \to Y$ as well. Thus, we are reduced to proving that $f(X)$ is closed if specializations lift along f. Note that, in particular, this means that $f(X)$ is stable under specialization which implies it is closed by Lemma 3.3.3.10. □

Next, we can link lifting of specializations to the existence part of the valuative criterion.

Lemma 3.3.3.12. *Assume that $f : X \to S$ is a morphism of schemes. We claim that the following statements are equivalent:*

1. *specializations lift along arbitrary base-changes of f;*
2. *the morphism f satisfies the existence part of the valuative criterion.*

Proof. That the first statement implies the second was essentially the argument we gave above about universally closed maps satisfying the valuative criterion. Thus, let us prove that the second statement implies the first. Thus, suppose we have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S. \end{tikzcd}$$

If $\varphi : S' \to S$ is an arbitrary morphism, then consider the base-change of f along φ. We claim that the valuative criterion also holds for the base-change. Indeed, suppose we are given a diagram of the form

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S' \arrow[r] & S. \end{tikzcd}$$

Since the existence part of the valuative criterion holds for f, we can lift the composite map along f to a morphism $\operatorname{Spec} A \to X$ making the resulting triangles commute. In that case, the universal property of fiber products, yields a map $\operatorname{Spec} A \to X'$ making all of the resulting triangles commute. Thus, we are reduced to showing that specializations lift along the original morphism f.

Thus, let s' be a point of S with specialization s and choose a point x' lying over s'. In that case, this specialization corresponds to a valuation ring A with fraction field K and a morphism $\operatorname{Spec} A \to S$. Since the existence part of the valuative criterion holds, it follows that there exists a lift $\operatorname{Spec} A \to X$ making the diagram commute. The image of the closed point of $\operatorname{Spec} A$ in X then yields the required lift of s. $\square$

Finally, we can put everything together.

Theorem 3.3.3.13. *A morphism $f : X \to S$ of schemes is separated if and only if it is quasi-separated for any valuation ring A with fraction field K and any diagram of the form*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S. \end{tikzcd}$$

there exists at most one lift $\operatorname{Spec} A \to X$ making all the relevant triangles commute.

Proof. We have already established the forward implication, so it remains to establish the reverse implication. To show that f is separated, it suffices to show that $\Delta_{X/S}$ is a closed immersion. Since $\Delta_{X/S}$ is an immersion, it suffices by Lemma 3.3.3.9 to check that $\Delta_{X/S}(X)$ is a closed subset of X. Since f is quasi-separated, $\Delta_{X/S}$ is quasi-compact by assumption. Therefore, by Lemma 3.3.3.10 it suffices to check that $\Delta_{X/S}$ is stable by specialization. Then, by Lemma 3.3.3.12 it follows $\Delta_{X/S}$ satisfies the existence part of the valuative criterion, i.e., given a diagram of the form

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "\Delta_{X/S}"] \\ \operatorname{Spec} A \arrow[r] & X \times_S X. \end{tikzcd}$$

there exists a morphism $\operatorname{Spec} A \to X$ making the relevant triangles exist. Now, specifying a morphism $\operatorname{Spec} A \to X \times_S X$ is equivalent to specifying a pair of morphisms $a, b : \operatorname{Spec} A \to X$. Commutativity of the above diagram implies that the two composites $\operatorname{Spec} K \to X$ obtained from a and b agree. The existence of a lift as in the diagram thus says that $a = b$, which is precisely the uniqueness part of the valuative criterion. □

Theorem 3.3.3.14. *A morphism $f : X \to S$ of schemes is proper if and only if it is finite-type, quasi-separated and for any valuation ring A with fraction field K, and any diagram of the form*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S. \end{tikzcd}$$

there exists a unique lift $\operatorname{Spec} A \to X$ making the resulting triangles commute.

Proof. We have already seen the forward implication is true. For the reverse implication, assume f is finite-type and quasi-separated. In that case, since f is quasi-separated by assumption, it is separated by the uniqueness part of the valuative criterion 3.3.3.13. It remains to check that f is universally closed. Since f has finite-type it is quasi-compact by assumption. Therefore, by Lemma 3.3.3.11 to check f is universally closed, it suffices to check that specializations lift along arbitrary base-extensions for f. In that case, specializations lift along arbitrary base extensions if the existence part of the valuative criterion holds by Lemma 3.3.3.12. Since the existence part of the valuative criterion holds by assumption, we conclude. □

Permanence properties

Proposition 3.3.3.15. *Separated, universally closed and proper morphisms are stable under base-change and composition.*

Integral and finite ring extensions

Definition 3.3.3.16. Suppose $\varphi : R \to S$ is a ring homomorphism. An element $s \in S$ is integral over R if s satisfies a monic polynomial with coefficients in R. We say that φ is an integral ring homomorphism if every element of S is integral over R. We will say that φ is a finite ring map, if S is finitely generated as an R-module.

Lemma 3.3.3.17. *Any finite ring map is integral. Conversely, any finite type, integral ring homomorphism is finite.*

Proof. If $\varphi : R \to S$ is a finite ring map, then suppose $x \in S$. In that case, pick a surjection $R^{\oplus n} \to S$, i.e., finitely many elements $x_i \in S$ that generate S as an R-module. In that case, the elements $1, x, \ldots, x^n$ necessarily satisfy some relation (just as in linear algebra) and the result follows. Conversely, if $R \to S$ is a finite-type integral ring homomorphism, then the images of the algebra generators show that S is a finitely generated R-module. Say $x_1, \ldots, xn$ are the algebra generators. In that case, each x_i is integral over R.... □

Lemma 3.3.3.18. *Composites of finite or integral ring maps are again finite or integral.*

Lemma 3.3.3.19. *If $\varphi : R \to S$ is a ring homomorphism, then the subset of elements in S that are integral over R is a subring S' of S that is integral over R; it is the largest integral sub-extension of S.*

Definition 3.3.3.20. If $\varphi : R \to S$ is a ring homomorphism, then the the integral closure of R in S is the subring S' of S consisting of elements integral over R. If $R \to S$ is a ring homomorphism, then we say that R is integrally closed in S if $R = S$.

Lemma 3.3.3.21. *Integral closure commutes with localization.*

Lemma 3.3.3.22. *Integral and finite ring maps are stable under extension of scalars. Composites of integral and finite ring maps are again integral or finite ring maps.*

Proposition 3.3.3.23. *If $\varphi : R \to S$ is an integral ring map, then $\operatorname{Spec} S \to \operatorname{Spec} R$ is surjective. Moreover if $\mathfrak{p} \subset \mathfrak{p}'$ is an inclusion of prime ideals, and $\mathfrak{q}$ is a prime ideal that maps to $\mathfrak{p}$, then there exists a prime ideal $\mathfrak{q}'$ containing $\mathfrak{q}$ and mapping to $\mathfrak{p}'$.*

Proof. Suppose x is a point of $\operatorname{Spec} R$ corresponding to a prime ideal $\mathfrak{p}$. We want to show that $\mathfrak{p}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$ to show that the scheme-theoretic fiber of $\operatorname{Spec}\varphi$ is non-empty. Since integral ring maps are stable under extension of scalars, it suffices to prove that x lies in the image of $\operatorname{Spec}\varphi$ after localization. Thus, considering the map $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$, we can assume that R is local with maximal ideal $\mathfrak{m}$. In that case, it suffices to prove that $\mathfrak{m}S \neq S$. If $\mathfrak{m}S = S$ that means that $\mathfrak{m}$ generates the unit ideal, i.e., we can write $1 = \sum_i f_i s_i$ with $f_i \in \mathfrak{m}$ and $s_i \in S$. In that case, consider the finite R-sub-module S' of S generated by the s_i. By construction $S' = \mathfrak{m}S'$, so Nakayama's lemma implies that $S' = 0$.

For the second statement, the prime ideal $\mathfrak{q}$ exists by appeal to the first point. In that case, consider the map $R \to R/\mathfrak{p}$; the latter is an integral domain and $\mathfrak{p}'$ corresponds to a prime ideal in $R/\mathfrak{p}$. Since the prime ideal $\mathfrak{q}$ maps to $\mathfrak{p}$, it follows that the ring $S/\mathfrak{q}$ coincides with the extension of scalars ring $S \otimes_R R/\mathfrak{p}$, and the ring homomorphism $R/\mathfrak{p} \to S/\mathfrak{q}$ is induced by φ. Since the extension of scalars of an integral ring homomorphism is again integral, it follows that $R/\mathfrak{p} \to S/\mathfrak{q}$ is surjective. Therefore, by appeal to the first part, there exists a prime ideal of $S/\mathfrak{q}$ mapping to $\mathfrak{p}'$. ☐

Remark 3.3.3.24. The second part of this statement is known as *going up* for integral ring homomorphisms. It follows by induction that given an integral ring homomorphism $\varphi : R \to S$ and a chain of prime ideals in R, we can find a chain of prime ideals in S lifting the given chain in R. Of course, this statement admits an interpretation in terms of lifting specializations lifting along the corresponding map $\operatorname{Spec}\varphi$.

We can globalize the notions of integral and finite ring maps.

Definition 3.3.3.25. A morphism $f : X \to S$ of schemes is integral (resp. finite) if it is affine and for every open affine $V \subset S$, the ring homomorphism corresponding to the morphism of affine schemes $f^{-1}(V) \to V$ is an integral (resp. finite) ring homomorphism.

Lemma 3.3.3.26. *Finite and integral morphisms are stable under arbitrary base-change.*

Proposition 3.3.3.27. *If $f : X \to S$ is a finite morphism, then f is proper.*

Proof. Since f is affine by assumption, it is separated and quasi-compact. The definition of finite morphisms also implies that f has finite-type. Thus, it remains to check that f is universally closed. That f is closed follows from going up, i.e., Proposition 3.3.3.23. □

Valuation rings in more detail

We would like to give a characterization of valuation rings that makes the name "valuation" more apparent.

Lemma 3.3.3.28. *If A is a valuation ring with maximal ideal $\mathfrak{m}$ and fraction field K, then if $x \in K$, either $x \in A$ or $x^{-1} \in A$.*

Proof. Assume that $x \notin A$, we want to show that $x^{-1} \in A$. Let A' be the subring of K generated by A and x. Since A is a valuation ring, we claim there is no prime of A' lying over $\mathfrak{m}$. Indeed, if there was a prime $\mathfrak{p} \subset A'$ such that $\mathfrak{p} \cap A = \mathfrak{m}$, then $A_{\mathfrak{p}}$ would be a local ring with fraction field K that dominates A, which contradicts the maximality of A among local rings contained in K. In that case, since $\mathfrak{m}$ is maximal, it follows that $V(\mathfrak{m}A') = \emptyset$, i.e., $\mathfrak{m}A'$ is necessarily the unit ideal. Thus, we can write $1 = \sum_{i=0}^{d} t_i x^i$ for $t_i \in \mathfrak{m}$. Rewriting this equation, we see that $1 - t_0 = t_1 x + \cdots + t_d x^d$. Multiplying both sides by x^{-d}, we see that

$$(1 - t_0)x^{-d} = t_1 x^{1-d} + \cdots t_d,$$

i.e., the element x^{-1} is integral over A. Therefore, the subring A'' of K generated by A and x^{-1} is finite over A. In particular, by Proposition **??** there exists a prime ideal $\mathfrak{m}''$ of A'' lying over $\mathfrak{m}$. Since A is a valuation ring, we conclude that $A''_{\mathfrak{m}''} = A$ and that $x^{-1} \in A$. □

Remark 3.3.3.29. In fact, the above condition characterizes valuation rings: if A is a subring of a field K such that for any $x \in K$ either $x \in A$ or $x^{-1} \in A$, then A is a valuation ring. We won't prove this here.

Suppose A is a valuation ring with fraction field K. Set $\Gamma := K^\times / A^\times$; we write $+$ for the group law on Γ. Write ν for the quotient map $K^\times \to \Gamma$. We define an ordering on Γ by $\gamma \geq \gamma'$ if $\gamma - \gamma'$ lies in the image of $A \setminus 0 \to \Gamma$. Since for any $x \in K$, either x or $x^{-1} \in A$, it follows that $\geq$ is a total order on Γ. Thus, ν is a homomorphism from $K^\times$ to a totally ordered abelian group. By construction, $\nu(a) = 0$ if and only if $a \in A^\times$. Since we have written the group law additively, it also follows that $\nu(ab) = \nu(a) + \nu(b)$. Finally, we claim that $\nu(a + b) \geq min(\nu(a), \nu(b))$. We will say that A is a discrete valuation ring if $\Gamma = \mathbb{Z}$.

Projective space is proper

Proposition 3.3.3.30. *Let $X = \mathbb{P}^n_{\mathbb{Z}} = Proj \mathbb{Z}[x_0, \ldots, x_n]$. We claim $X \to \operatorname{Spec} \mathbb{Z}$ is proper.*

Proof. Projective space has finite type by construction since it has a finite open cover by affine spaces. We will check the existence and uniqueness parts of the valuative criterion. Suppose we have a diagram of the form

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} \mathbb{Z}. \end{tikzcd}$$

Let ξ_1 be the image of $\operatorname{Spec} K$ in X. By induction on n, we can assume that ξ_1 lies outside any of the hyperplanes defined by $x_i = 0$, which are themselves isomorphic to lower-dimensional projective spaces. Thus, we can assume that all the functions x_i are invertible in the local ring $\mathscr{O}_{X,\xi_1}$. Let $f_{ij} \in K$ be the image of $\frac{x_i}{x_j}$. Note that f_{ij} is a non-zero element of K and also that $f_{ij}f_{jk} = f_{ik}$ since the corresponding formulas hold for $\frac{x_i}{x_j}$.

Let ν be the valuation attached to A and set $g_i = \nu(f_{i0})$ for $i = 0, \ldots, n$. Choose k such that g_k is minimal among the set $\{g_0, \ldots, g_n\}$. In that case, $\nu(f_{ik}) = \nu(f_{i0}) - \nu(f_{k0}) = g_i - g_k \geq 0$ by minimality of g_k. In other words, $f_{ik} \in A$ for $i = 0, \ldots, n$. In that case, the map sending $\frac{x_i}{x_j}$ to f_{ij} factors through A and yields the resulting extension. $\square$

Remark 3.3.3.31. Since proper morphisms are stable under base-change we conclude that $\mathbb{P}^n_S$ is proper over S for any scheme S. Likewise, since closed immersions are proper and composites of proper morphisms are proper, we conclude that any closed subscheme of $\mathbb{P}^n_S$ is again proper.

3.3.4 Dimension

Definition 3.3.4.1. Suppose X is a topological space. A chain of irreducible subsets $Z_0 \subset \mathbb{Z}_1 \subset Z_n$ will be said to have length n if each inclusion is proper. The (Krull) dimension of X, denoted $\dim X$, is the supremum of the lengths of chains of irreducible subsets. If X is a scheme, then the (Krull) dimension of X is dimension of the topological space underyling X. If $Y \subset X$ is an irreducible closed subset, then the *codimension* of Y in X, denoted $codim(Y, X)$ is the supremum of the lengths of chains of irreducible subsets containing Y.

Remark 3.3.4.2. By convention the emptyset has dimension $-\infty$. By definition, we see that $\dim Y + codim(Y, X) \leq \dim X$. If $X = \operatorname{Spec} R$, then we know that $V(I)$ is irreducible if and only if $\sqrt{I}$ is prime. Thus the Krull dimension of $\operatorname{Spec} R$ coincides with the maximum length of a chain of prime ideals in R.

Definition 3.3.4.3. If A is a commutative ring, and $\mathfrak{p}$ is a prime ideal, then $ht(\mathfrak{p})$ is the supremum of the lengths of chains of prime ideals contained in $\mathfrak{p}$. For an arbitrary ideal I, $ht(I)$ is the infimum of the heights of prime ideals containing I.

Proposition 3.3.4.4. *If A is a ring and* $\mathfrak{p}$ *is a prime ideal of A, then* $\dim A_{\mathfrak{p}} = ht(\mathfrak{p}) = codim(V(\mathfrak{p}), A)$

Proposition 3.3.4.5. *If* $\varphi : R \to S$ *is an integral ring homomorphism, then* $\dim \operatorname{Spec} S \leq \dim \operatorname{Spec} R$. *If* φ *is injective, then* $\operatorname{Spec} \varphi$ *is surjective, and* $\dim \operatorname{Spec} S = \dim \operatorname{Spec} R$.

3.4 Curves

We now attempt to compute Picard groups of some simple integral varieties. Since we view dimension as a reasonable measure of complexity of a variety, we start with low-dimensional examples. Suppose R is a commutative domain of Krull dimension 0. In that case, we know that (0) is the only prime ideal and furthermore that it is maximal. In other words, R is simply a field. If R is not a domain, then situation is more interesting, but perhaps less geometric, so we leave this for later. Arguably the first geometrically interesting case to consider is that were R has Krull dimension 1.

3.4.1 Normality and curves

Definition 3.4.1.1. If k is a field, then by a curve over k, we will mean a k-variety of dimension 1. We will sometimes use the word curve more generally for a Noetherian, integral, separated scheme of dimension 1.

Example 3.4.1.2. By this definition, $\operatorname{Spec} \mathbb{Z}$ is itself a curve. Likewise $\mathbb{A}^1_k$ is a curve, and $\mathbb{P}^1_k$ is a curve. Moreover, any scheme that is finite over $\mathbb{P}^1_k$ is again a curve over k. If we consider the subscheme of $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ defined by the equation $y^2 = x^3$ or $y^2 = x^3 - x^2$; these are both curves over k by this definition. These latter schemes are rather different than $\mathbb{A}^1_k$ however: if $k = \mathbb{C}$ for example, the relevant spaces do not give rise to manifolds. As a consequence, we want to restrict the kinds of curves we consider.

Definition 3.4.1.3. A domain R with fraction field K will be called *normal* if R is integrally closed in its field of fractions.

Example 3.4.1.4. The ring $k[t]$ is integrally closed in its field of fractions $k(t)$. The ring $k[C] := k[x, y]/(y^2 - x^3)$ is not integrally closed in its field of fractions. Let $C = \operatorname{Spec} k[x, y]/(y^2 - x^3)$. Observe that $k[x, y]/(y^2 - x^3)$ can be viewed as a subring of $k[t]$ by means of the map sending x to t^2 and y to t^3. This corresponds to a morphism of schemes $\mathbb{A}^1_k \to C$. The field of fractions of $k[C]$ coincides with $k(t)$, e.g., by means of the above map. To see that $k[x, y]/(y^2 - x^3)$ fails to be integrally closed in its field of fractions, we need to write down a monic polynomial with coefficients in $k[x, y]/(y^2 - x^3)$ that admits a solution in $k(t)$ but no solution in $k[C]$. Indeed, the element $t = y/x$ is integral over $k[C]$ since $t^2 = y^2/x^2 = x$ but fails to lie in $k[C]$. Likewise, the ring $k[N] := k[x, y]/(y^2 - x^3 + x^2$ fails to be integrally closed in its field of fractions. Once again, the element $t = \frac{y}{x}$ witnesses this failure.

Lemma 3.4.1.5. *Suppose R is a domain. The following conditions are equivalent.*

1. *R is normal;*
2. *then for every prime ideal $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is normal;*
3. *for every maximal ideal $\mathfrak{m} \subset R$, $R_{\mathfrak{m}}$ is normal.*

Proof. That (1) $\Longrightarrow$ (2) follows from the fact that integral closures commute with localizations. That (2) $\Longrightarrow$ (3) is immediate. To show that (3) $\Longrightarrow$ (1), note that since R is a domain, the map $R \to R_{\mathfrak{m}}$ is injective. It follows that $R \to \cap_{\mathfrak{m}} R_{\mathfrak{m}}$ (where the intersection is taken in the fraction field) is again injective. We claim that this map is also surjective. Consider $M := \cap_{\mathfrak{m}} R_{\mathfrak{m}}$ as an

R-module and let C be the cokernel of the injective R-module map $R \to M$. Since $M \subset R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m} \subset R$, it follows that upon localization at $\mathfrak{m}$ we have a sequence of inclusions

$$R_{\mathfrak{m}} \subset M_{\mathfrak{m}} \subset R_{\mathfrak{m}},$$

i.e., $M_{\mathfrak{m}} = R_{\mathfrak{m}}$. It follows that $C_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m}$, which means that $C = 0$ as well. Thus, if R is a domain, then $R = \cap_{\mathfrak{m}} R_{\mathfrak{m}}$. If each $R_{\mathfrak{m}}$ is normal, then it follows from this observation that R is normal as well. □

Definition 3.4.1.6. An integral scheme X is called *normal* if for every $x \in X$, the local ring $\mathscr{O}_{X,x}$ is normal.

Lemma 3.4.1.7. *Assume X is an integral scheme. The following statements are equivalent.*

1. *The scheme X is normal.*
2. *For any open affine cover U_i of X, the rings $\mathscr{O}_X(U_i)$ are normal.*

Definition 3.4.1.8. If k is a field, then a curve X over k is *non-singular* if it is a normal scheme.

3.4.2 Dedekind domains

We now want to being a "local" analysis of curves. To this end, we recall the following classical definition.

Definition 3.4.2.1. A commutative ring R is called a *Dedekind domain* if it a Noetherian normal domain of Krull dimension 1.

Remark 3.4.2.2. To say that R has Krull dimension 1 (see Definition 1.1.1.30) is to say that every chain of prime ideals is of the form $\mathfrak{p}_0 \subset \mathfrak{p}_1$. Since R is an integral domain that has Krull dimension 1, then we know that (0) is a prime ideal, and therefore that any non-zero prime ideal is maximal.

Examples of Dedekind domains

Directly from the definitions, one sees that for any field k, $k[x]$ and $\mathbb{Z}$ are Dedekind domains. We first establish a way to produce new Dedekind domains from old ones.

Proposition 3.4.2.3. *If R is a Dedekind domain with fraction field K and L is a finite separable extension of K, then the integral closure S of R in L is a Dedekind domain as well.*

Proof. First, we prove that S is a Noetherian domain. To this end, we will show that it is a sub-R-module of a finite rank free R-module and therefore Noetherian as well (that it is a domain is left as an exercise). For any extension L/K, we can consider the trace pairing $L \times L \to K$ given by $(x, y) \mapsto Tr_{L/K}(xy)$ (recall that we view L as a K-vector space and take the trace). The separability assumption arises in the following way: the extension L/K is separable if and only if the trace pairing is non-degenerate.

We claim that for any element $x \in L$, if x is integral over R, then $Tr_{L/K}(x) \in R$. This follows from two facts: (i) the minimal polynomial of x has coefficients in R and (ii) if P is the minimal polynomial of x, d is the degree of P, and $[L : K] = ed$ for some integer e, then

$Tr_{L/K}(x) = -ea_1$, where a_1 is the coefficient of x^{d-1} in the minimal polynomial. For (i), if we take any monic polynomial Q with coefficients in R satisfied by x (such a polynomial exists since x is integral over R), then the minimal polynomial P divides Q. In this case, one shows that the coefficients of P are integral over R (exercise!) and since R is integrally closed, must lie in R.

Now, pick $x_1, \ldots, x_n \in L$ that are integral over R and that form a K-basis for L. The integral closure S of R is contained in the module $M := \{y \in L | \langle x_i, y\rangle \in R, i = 1, \ldots, n\}$. There is an induced isomorphism $M \cong R^{\oplus n}$ and since $S \subset R^{\oplus n}$, since R is Noetherian, S is a finitely generated R-module. We conclude that S is also a Noetherian domain.

Since S is integrally closed in its field of fractions it remains to show that S has Krull dimension 1 if R has the same property. To this end, we analyze chains of prime ideals in S. Let $\mathfrak{P}$ be a non-zero prime ideal of S and let $\mathfrak{p} = \mathfrak{P} \cap R$. We claim that $\mathfrak{p}$ is non-zero. Indeed, if we pick a non-zero element x of P, then since x is integral over R, we see that x satisfies a monic polynomial with coefficients in R and we can choose one f of minimal degree. This polynomial necessarily has non-zero constant term (if not, this would contradict minimality). Moreover, the equation shows that the constant term is in the ideal (x).

Now, if $\mathfrak{P} \subset \mathfrak{Q}$ is a proper inclusion of prime ideals in S, then setting $\mathfrak{q} = \mathfrak{Q} \cap R$ we conclude that there is an inclusion $\mathfrak{p} \subset \mathfrak{q}$. One may check that this inclusion is proper as well. Thus, if R has Krull dimension 1, S must have Krull dimension 1 as well. □

Lemma 3.4.2.4. *If L/K is a finite separable extension, then the trace pairing is non-degenerate.*

Proof. Exercise. □

Remark 3.4.2.5. The following result is known as the Krull-Akizuki theorem [**?**, Theorem 11.7]: if R is a Noetherian integral domain with field of fractions K and having Krull dimension 1, L is a finite algebraic extension of K and S is a ring with $R \subset S \subset L$, then B is a Noetherian ring of Krull dimension ≤ 1. From this one deduces [**?**, p. 85], that if R is any Noetherian integral domain of Krull dimension 1, and L is any finite algebraic extension of the fraction field of R, then the integral closure S of R in L is a Dedekind domain. In particular, separability is not necessary in the statement.

Example 3.4.2.6. Since $\mathbb{Z}$ is an integral domain, the integral closure $\mathscr{O}_K$ of $\mathbb{Z}$ in a finite extension K of $\mathbb{Q}$ is a Dedekind domain. Likewise, $k[x]$ is a Dedekind domain for any field k. Given any finite separable extension E of $k(x)$ the integral closure of $k[x]$ in E is a Dedekind domain.

Example 3.4.2.7. Suppose $f \in k[x]$ is a non-zero polynomial, and consider the equation $y^r - f(x)$. Assume r is invertible in k (i.e., the r is coprime to the characteristic exponent of k). If f is a separable polynomial (i.e., f has no repeated roots upon passing to an algebraic closure of k), then you can check that $P := y^r - f(x)$ is irreducible over $k(x)$ and we can consider its splitting field E over $k(x)$. In that case, we can form the integral closure R of $k[x]$ in E.

Note that there is a ring homomorphism $k[x] \to R$ by definition. There is also a ring homomorphism $k[x,y]/(y^r - f(x)) \to R$ by construction. The fraction field of $k[x,y]/(y^r - f(x))$ coincides with E and you can check that $k[x,y]/(y^r - f(x))$ is integrally closed in its field of fractions.

The map $k[x] \to R$ factors as the inclusion $k[x] \to k[x,y] \to k[x,y]/(y^r - f)$. If we set $C = \operatorname{Spec} k[x,y]/(y^r - f)$, then we have the composite map $p : C \to \mathbb{A}^2_k \to \mathbb{A}^1_k$. The composite map is the inclusion follows by the "projection onto x". We now study the fibers of this map. If $\mathfrak{m}$

is a maximal ideal of $k[x]$, then we let $\kappa := k[x]/\mathfrak{m}$. Observe that $R/\mathfrak{m}R$ is a κ-algebra, and we can describe this κ-algebra explicitly. Indeed, the element f has value $\bar{f} \in k[x]/\mathfrak{m}$. In that case, the algebra $R/\mathfrak{m}R$ can be identified as $\kappa[y]/(y^r - \bar{f})$. Note that $\kappa[y]/(y^r - \bar{f})$ is an algebra of dimension precisely r. If $\bar{f} = 0$ (i.e., f vanishes at the closed point corresponding to $\mathfrak{m}$) then this algebra is $\kappa[y]/y^r$, i.e., it has only one closed point with "nilpotent fuzz." On the other hand, if $\bar{f}$ has an r-th root in κ, then $\kappa[y]/(y^r - \bar{f})$ is $\kappa \oplus \cdots \kappa$ as a κ-algebra, and the fiber has r distinct points. In general, the structure of the fiber depends on the roots of $\bar{f}$ in κ.

The picture we are describing here is a slight refinement of the usual ideal from complex analysis that $p : C \to \mathbb{A}^1_k$ is a branched cover of $\mathbb{A}^1_k$ branched along the locus where f vanishes. Indeed, the description above shows the kind of additional information that is kept beyond just keeping track of the number of points in the fiber of p.

3.4.3 Local Dedekind domains: equivalent characterizations

Now, let us analyze local Dedekind domains first, characterize such things, and then attempt to patch the information together.

Local Dedekind domains

We now proceed to characterize local Dedekind domains. If $(R, \mathfrak{m})$ is a local Dedekind domain, then R has a unique non-zero ideal, which is necessarily the maximal ideal $\mathfrak{m}$.

Lemma 3.4.3.1. *If $(R, \mathfrak{m})$ is a local Dedekind domain, then $\mathfrak{m}$ is principal.*

Proof. Let K be the fraction field of R. Suppose we fix an element $\pi \in \mathfrak{m}$. Since R is Noetherian, $\mathfrak{m}$ is finitely generated. By Nakayama's lemma if $\mathfrak{m} = \mathfrak{m}^2$, then $\mathfrak{m} = 0$, so we conclude that $\mathfrak{m} \neq \mathfrak{m}^2$. Analogously, we conclude that $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all $n > 0$. Choose an element $t \in \mathfrak{m} \setminus \mathfrak{m}^2$. In that case, $(t) \subset \mathfrak{m}$ and we claim that equality holds.

In a Noetherian ring, every ideal contains a power of its radical. Since $\mathfrak{m}$ is the unique non-zero ideal of R, it follows that the radical of (t) is $\mathfrak{m}$. Therefore, it follows that $\mathfrak{m}^n \subset (t)$. If $n = 1$, then we are done, so assume that $n > 1$. In that case, we may find $x \in \mathfrak{m}^{n-1}$ that does not lie in (t). Then, $x\mathfrak{m} \subset \mathfrak{m}^n \subset (t)$ and the element $y := \frac{x}{t}$ lies in K. If y was in R, then yt would necessarily lie in $(t) \subset R$, but $yt = x \notin (t)$ by assumption. If y was integral over R, then since R is integrally closed in its field of fractions, then y would be in R, which would contradict the assertion we just made. We thus claim that y is integral over R. Indeed, we know that $x\mathfrak{m} \subset \mathfrak{m}^n \subset (t)$. Thus $y\mathfrak{m} \subset R$ is an ideal. If $y\mathfrak{m} = R$, then we may find $f \in \mathfrak{m}$ such that $yf = 1$. In that case, $xf = ytf = t$; however, $xf \in \mathfrak{m}^2$ which contradicts the assumption that $t \notin \mathfrak{m}^2$. Thus, $y\mathfrak{m}$ is a proper ideal of R and $y\mathfrak{m} \subset \mathfrak{m}$. In that case, choose generators $m_1, \ldots, m_n$ of $\mathfrak{m}$ and we may write $ym_j = \sum_i a_{ij} m_i$ for $a_{ij} \in R$. We can rewrite this equation as

$$\sum_i (\delta_{ij} y - a_{ij}) m_i = 0.$$

Let $d = \det(\delta_{ij} y - a_{ij})$. In that case, Cramer's rule tells us that $dm_i = 0$ for all i. In other words, $d\mathfrak{m} = 0$. Since $\mathfrak{m}$ is non-zero, we conclude that $d = 0$, and this yields the required integral dependence relation for y. $\square$

Lemma 3.4.3.2. *If R is a valuation ring with fraction field K, then R is integrally closed in K*

Proof. Let $\alpha \in K$ be a non-zero element of K that is integral over R. Let f be a monic polynomial that is satisifed by α, i.e.,

$$\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 = 0$$

If α is not already in V, then $\alpha^{-1} \in V$. Multiplying both sides of the equation by α^{1-n}, we see that

$$\alpha = -c_{n-1} + \cdots + c_0\alpha^{1-n}$$

which means that $\alpha \in V$ also. □

Example 3.4.3.3. We begin by analyzing a special case, namely consider the localization of $k[x]$ at the maximal ideal $\mathfrak{m}_0$; this is certainly a local Dedekind domain with maximal ideal $\mathfrak{m}_0 k[x]_{\mathfrak{m}_0}$. We can take the polynomial x as a generator of the ideal $\mathfrak{m}_0$. In that case, any element of $k[x]_{\mathfrak{m}_0}$ can be written uniquely as $x^r u$ and the number r is the order of vanishing of f at 0. This defines a function from $k[x]_{\mathfrak{m}_0} \to \mathbb{N}$ and if we restrict to non-zero elements, it is a surjective monoid homomorphism, i.e., $ord_0(fg) = ord_0(f) + ord_0(g)$. Furthermore, it satisfies $ord(f+g) \geq \min(ord(f), ord(g))$. Now, if $f \in k(x)^{\times}$, then either $f \in k[x]_{\mathfrak{m}_0}$ or $f^{-1} \in k[x]_{\mathfrak{o}}$ and therefore, we can extend ord to a (surjective) group homomorphism $k(x)^{\times} \to \mathbb{Z}$ preserving the additional inequality. It is convenient to define $ord_0(0) = \infty$ so that $ord(x + -x) = \infty \geq min(ord(x), ord(-x))$ (this helps to remember the inequality). (Note: alternatively, we could have spoken about the order of pole of a function; in this case, the inequality would be reversed.) We now abstract these facts.

Lemma 3.4.3.4. *If $(R, \mathfrak{m})$ is a local Dedekind domain with fraction field K, then R is a discrete valuation ring.*

Proof. We first show that R is a valuation ring. Fix a generator π of $\mathfrak{m}$. Since R is a local ring with fraction field K, it follows that R is dominated by some valuation ring V with fraction field K. Suppose $t \in V$ is a non-zero element. Every non-zero element of K can be written as $\frac{u}{\pi^m}$ for some unit $u \in R$, so we may write $t = \frac{u}{\pi^m}$, i.e., $\pi^m t \in R$. If t is not in R, then $m > 0$, and t^{-1} is necessarily in R. Moreover, $t^{-1} \subset \mathfrak{m} = (t)$. In that case, t^{-1} lies in the maximal ideal of V. However, if t^{-1} lies in the maximal ideal of V, then t could not lie in V to begin with. Finally, the value group of R is $K^{\times}/R^{\times}$; every element of $K^{\times}$ can be written uniquely as $u\pi^m$ for some $m \in \mathbb{Z}$. □

Remark 3.4.3.5. Because of the preceding example, we will think of ν as the "order of pole" of a rational function.

We now summarize the conclusions we have drawn.

Theorem 3.4.3.6. *The following are equivalent:*

1. *R is a local Dedekind domain;*
2. *R is a discrete valuation ring;*
3. *R is a local PID.*

Proof. That $(1) \implies (2)$ follows from the discussion above. That $(2) \implies (3)$ is an exercise: one checks that if ν is the valuation attached to R, then ν is a Euclidean norm, and thus R is a principal ideal domain. See [**?**, Theorem 11.2] for the equivalences. □

3.5 Picard groups, Dedekind domains and Weil divisors

Having understood Dedekind domains locally (they are discrete valuation rings), we now attempt to understand how to patch this information together. We begin by studying how the property of being "integrally closed in the fraction field" behaves under localization.

3.5.1 Integral closure and localization

Exercise 3.5.1.1. *Show that any UFD is normal.*

The beginning of the proof of Proposition 3.4.2.3 can be repeated to establish the following result.

Proposition 3.5.1.2. *If R is a normal Noetherian integral domain with fraction field K and L is a finite separable extension of K, then if R' is the integral closure of R in L, the map $R \to R'$ makes R' into a finitely generated R-module.*

3.5.2 Equivalent characterizations of Dedekind domains

Theorem 3.5.2.1. *The following conditions on a commutative integral domain R are equivalent:*

1. *the ring R is a Dedekind domain;*
2. *the ring R is Noetherian, and for each non-zero prime ideal $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is a discrete valuation ring.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from combining the results established above. For the implication (2) $\Longrightarrow$ (1), note that R is a Noetherian domain by assumption, and integrally closed in its field of fractions by Lemma **??**. Therefore, it suffices to show that R has Krull dimension 1. For any (non-zero) prime ideal $\mathfrak{p}$, the ring $R_{\mathfrak{p}}$ has precisely 2 ideals (0) and the ideal $\mathfrak{p}R_{\mathfrak{p}}$. The result follows. □

3.5.3 Picard groups of non-singular curves

We now analyze Picard groups of non-singular curves over a field k. In that case, if X is a non-singular curve over a field k, then for $x \in X$, the ring $\mathscr{O}_{X,x}$ is a discrete valuation ring. Suppose $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module. As before, we know that line bundles on X correspond to elements of $\check{H}^1(X, \mathscr{O}_X^{\times})$. Let $\mathscr{K}_X^{\times}$ be the sheaf of total quotients; we may consider the exact sequence of sheaves

$$0 \longrightarrow \mathscr{O}_X^{\times} \longrightarrow \mathscr{K}_X^{\times} \longrightarrow \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times} \longrightarrow 0.$$

Repeating the arguments we gave earlier, we can identify $H^0(X, \mathscr{K}^{\times}/\mathscr{O}^{\times})$ as Cartier divisors: elements are given by pairs $(U_{i i \in I}, f_i)$ where U_i is an open cover of X (which we can assume affine without loss of generality) and $f_i \in \mathscr{K}_X^{\times}(U_i) = \mathbf{K}^{\times}$ such that $f_i/f_j \in \mathscr{O}_X^{\times}(U_{ij})$. In particular, we have the two-term complex

$$\mathbf{K}^{\times} \longrightarrow H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times})$$

whose cohomology computes the Picard group of X and the units of X.

Fix a Cartier divisor $(U_{i i\in I}, f_i)$. Now, take a point $x \in X$. Since X is a curve, $\mathscr{O}_{X,x}$ is a discrete valuation ring with valuation ν_x. Since $f_i \in K^{\times}$, then we may consider $\nu_x(f_i) \in \mathbb{Z}$. If $x \in U_{ij}$, then $\nu_x(\frac{f_i}{f_j}) = 0$ since $\frac{f_i}{f_j}$ is a unit on U_{ij}. In other words, $\nu_x(f_i) = \nu_x(f_j)$. It follows that there is a well-defined homomorphism

$$H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times}) \longrightarrow \prod_{x\in X; xclosed} \mathbb{Z}\cdot x.$$

We want to analyze this map in more detail.

Let us first analyze the case where X is a local curve, i.e., the spectrum of a discrete valuation ring. Thus, assume R is a discrete valuation ring with fraction field K. In that case, there is a single closed point $x \in \operatorname{Spec} R$, so the group $\bigoplus_{x\in X;xclosed} \mathbb{Z}\cdot x$ is simply the integers. In that case, we have the exact sequence

$$0 \longrightarrow R^{\times} \longrightarrow K^{\times} \longrightarrow K^{\times}/R^{\times} \longrightarrow 0;$$

this follows from the definition of exactness of a sequence of sheaves because local rings are precisely the stalks in the Zariski topology. Alternatively, it follows because $Pic(R) = 0$. In this case, the map

$$K^{\times}/R^{\times} \longrightarrow \mathbb{Z}$$

is the map induced by $\nu : K^{\times} \to \mathbb{Z}$ since $R^{\times}$ lies in the kernel by construction, i.e., the above map is an isomorphism because it is the map that defines the value group. In this case, we can construct an explicit inverse map as follows: simply choose an element $\pi \in K^{\times}$ that generates the maximal ideal of R since such an element necessarily has valuation 1 by the definition of the discrete valuation.

Now, let us return to the general situation described above. Observe that we can give a simple description to the composite map

$$K^{\times} \longrightarrow H^0(X, \mathscr{K}^{\times}.\mathscr{O}^{\times}) \longrightarrow \prod_{x\in X} \mathbb{Z}\cdot x.$$

Indeed, the Cartier divisor defined by an element $f \in K^{\times}$ is simply the global section f. In other words, the composite map sends

$$f \mapsto \sum_{xclosed} \nu_x(f).$$

Let us analyze this sum.

Definition 3.5.3.1. A scheme X will be called *locally Noetherian* if every $x \in X$ has an open affine neighborhood $U = \operatorname{Spec} R$ such that R is a Noetherian ring. We will say that X is Noetherian if X is locally Noetherian and quasi-compact.

If a scheme X is Noetherian, then its underyling topological space is Noetherian, which means every descending chain of closed subspaces stabilizes.

Lemma 3.5.3.2. *Assume X is a locally Noetherian scheme. If $Z \subset X$ is any closed subscheme, then the collection of irreducible components of Z is locally finite.*

Proof. Let $U \subset X$ be a quasi-compact open subscheme (e.g., an affine open subscheme). In that case, U is a Noetherian scheme and has a Noetherian underlying topological space. In particular, that means that any subspace is Noetherian and thus has finitely many irreducible components. □

Lemma 3.5.3.3. *If $f \in K^{\times}$, then $\sum_{xclosed} \nu_x(f)$ is a finite sum, i.e., $\nu_x(f)$ is only non-zero for finitely many points $x \in X$.*

Proof. Suppose X is a non-singular curve over a field k, and $f \in k(X)^{\times}$ is a non-zero rational function. Note that X is a Noetherian scheme by construction. In that case, there is an affine open set $U \subset X$ on which f is actually regular. Then, $Z = X \setminus U$ is a proper closed subset of X and can therefore only consist of finitely many points. □

As a consequence of this lemma, the map

$$H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times}) \longrightarrow \prod_{x \in X; xclosed} \mathbb{Z} \cdot x$$

has image in $\bigoplus_{x \in X; xclosed} \mathbb{Z} \cdot x$.

Definition 3.5.3.4. If X is a locally Noetherian integral scheme, then a prime divisor on X is an integral closed subscheme of codimension 1. A *Weil divisor* D on X is a finite formal linear combination of prime divisors, i.e., an element of the free abelain group on prime divisors; we write $Div(X)$ for the group of Weil divisors on X.

With this definition, we constructed above a map

$$H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times}) \longrightarrow Div(X)$$

for any curve X. We have also constructed a map $K^{\times} \to Div(X)$ as the composite $K^{\times} \to H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times}) \to Div(X)$. Any Weil divisor in the image of the map will be called a *principal* divisor; if f is a non-zero rational function, then we will write (f) for the associated principal divisor.

Definition 3.5.3.5. If X is a non-singular curve, then the (Weil) divisor class group of X, denoted $Cl(X)$ is the quotient $Div(X)/imK^{\times}$.

By construction there is an induced homomorphism

$$Pic(X) \longrightarrow Cl(X)$$

for any non-singular curve. We now analyze this homomorphism.

Proposition 3.5.3.6. *The map $Pic(X) \to Cl(X)$ is an isomorphism.*

Proof. By diagram chasing, the map $Pic(X) \to Cl(X)$ is an isomorphism if and only if the map

$$H^0(X, \mathscr{K}_X^{\times}/\mathscr{O}_X^{\times}) \longrightarrow Div(X)$$

from which it is induced is an isomorphism. Let us construct an explicit inverse map. Since the group on the right is a free abelian group, the inverse can be defined by specifying its value on generators.

Suppose x is a closed point. We attach a Cartier divisor to x as follows. Let f_1 be a choice of uniformizing parameter in $\mathscr{O}_{X,x}$. The function f_1 lies in K and by construction $\nu_x(f_1) = 1$. Since there are only finitely many points $x \in X$ where $\nu_x(f_1) \neq 0$, by shrinking U_1 if necessary, we can assume that x is the only point in U_1 for which $\nu_x(f_1) \neq 0$ and furthermore that U_1 is affine. The complement of U_1 in X consists of finitely many points $y_2, \ldots, y_n$. Let $U_i, i = 2, \ldots, n$ be neighborhoods of each of those points and choose a function $f_i \in K^\times$ on U_i that restricts to a unit in $\mathscr{O}_{X,y_i}^\times$. Since f_i is a unit in $\mathscr{O}_{X,y_i}$, it is a unit in some neighborhood of y_i, thus by shrinking $U_i, i = 2, \ldots, n$ if necessary, we may assume that f_i is a unit on U_i. We may also assume that x does not lie in $U_i, i = 2, \ldots, n$.

To conclude, we have to show that $\{(U_i, f_i)\}$ is a Cartier divisor, i.e., we have to show that $\frac{f_i}{f_j}$ is a unit on U_{ij}. This is immediate if neither i nor j is equal to 1 by the construction we have given. Therefore, it suffices to show that $\frac{f_1}{f_j}$ is a unit on U_{1j}.

Say $U_1 = \operatorname{Spec} R$. Since R is a Dedekind domain, observe that $R = \cap_{xclosed} R_{\mathfrak{m}_x}$ (indeed, this is true for any integral domain). However, since the localizations of a Dedekind domain at closed points are discrete valuation rings, we obtain the following criterion for a non-zero element of K to be a unit: $\nu_y(f) = 0$ for all closed points $y \in \operatorname{Spec} R$. Now, any intersection U_{1j} corresponds to a localization of R, which is again a Dedekind domain. In that case, $\nu_y(\frac{f_1}{f_j}) = \nu_y(f_1) - \nu_y(f_j)$. Since f_j is a unit on U_{1j} it has valuation 0. Thus, it suffices to prove that $\nu_y(f_1) = 0$ for all $y \in U_{1j}$. However, U_{1j} does not contain the point x, and we constructed U_1 so that f had valuation 0 for all points $\neq x$, so we conclude that $\nu_y(f_1) = 0$ on U_{1j} as well. We conclude that $\{(U_i, f_i)\}$ defines a Cartier divisor.

We leave it as an exercise to check that this actually defines the required inverse function. □

Remark 3.5.3.7. Observe that the isomorphism we just constructed gives a *presentation* of $Pic(X)$ for X a non-singular curve. Indeed, $Div(X)$ is a free abelian group by construction, and the subgroup of principal divisors is also a free abelian group (as any subgroup of a free abelian group is free abelian).

3.5.4 In what sense can we "actually" compute Picard groups?

While the above structural results are nice, they perhaps sidestep the question of what the Picard group actually looks like, even for Dedekind domains. Saying that a group is a quotient of a free abelian group of infinite rank by the image of a map that is difficult to understand is perhaps not so helpful. I state a few results that indicate how widely the Picard group can vary. The first result shows that Picard groups of curves over algebraically closed fields can be "very big".

Theorem 3.5.4.1 (Grothendieck(?))**.** *If k is an algebraically closed field, and R is a Dedekind k-algebra then $Pic(R)$ is a divisible abelian group.*

Remark 3.5.4.2. The proof of this result uses techniques that are very different from those we study here. In fact, one shows that $Pic(R)$ is the set of points of an algebraic variety (the Picard variety) that has a natural abelian group structure; this structure is obtained essentially by studying integrals of differential forms. When $k = \mathbb{C}$, one observes that $Pic(R)$ is a compact topological space (in the usual topology), and is isomorphic as a complex manifold to a torus T. If k has positive characteristic, one produces a purely algebraic variant of this complex torus.

Theorem 3.5.4.3 (Mordell-Weil)**.** *If k is a number field, and R is a Dedekind k-algebra, then $Pic(R)$ is a finitely generated abelian group.*

Remark 3.5.4.4. Even saying this raises questions: a finitely generated abelian group is a product of a free part and a torsion part. What does the torsion subgroup look like? What is the rank of the free part? Each of these questions is interesting, and the answers are largely conjectural. E.g., the Birch–Swinnerton-Dyer (BSD) conjecture asserts that the rank of the free part can be calculated purely analytically from an L-function one attaches to the abelian variety in a fashion that generalizes the analytic class number formula.

Finally, we state a result of Claborn, which shows that the Picard groups of Dedekind domains comprise all abelian groups.

Theorem 3.5.4.5 ([**?**, Theorem 7])**.** *Given any abelian group A, there is a Dedekind domain D such that* $\operatorname{Pic}(D) \cong A$.

Remark 3.5.4.6. We refer the reader to [**?**, §14] for a detailed treatment of the above result and a discussion of the strategy of the proof.

3.6 Weil divisors and Picard groups of higher dimensional varieties

The discussion above for curves presages what should happen for higher-dimensional varieties.

3.6.1 Weil divisors

We now analyze the construction made above for integral schemes of higher dimension. We will now assume that X is a Noetherian normal scheme having Krull dimension d. If K is the fraction field of X, then we can study discrete valuations on K. Since discrete valuation rings are always of dimension 1, the kinds of local rings on X we will get will *not* be localizations at arbitrary prime ideals.

Proposition 3.6.1.1. *If X is a Noetherian normal scheme of dimension d, and if $x \in X$ is a point of codimension 1, then $\mathscr{O}_{X,x}$ is a discrete valuation ring.*

Proof. Without loss of generality we can restrict attention to an affine open neighborhood $U = \operatorname{Spec} R$ of x so R is a Noetherian normal domain by the equivalent characterizations of normality, and x corresponds to a prime ideal $\mathfrak{p}$ of R that has height 1. In that case, $R_{\mathfrak{p}}$ is a local Noetherian normal domain, and since $\mathfrak{p}$ has height 1, it follows that $R_{\mathfrak{p}}$ has dimension 1. In that case, as a local Noetherian normal domain of dimension 1, $R_{\mathfrak{p}}$ is a discrete valuation ring by the equivalent characterizations of discrete valuation rings. □

Theorem 3.6.1.2 (Krull)**.** *Suppose R is a Noetherian normal domain.*

1. *The equality $R = \bigcap_{\mathfrak{p} | \operatorname{ht}\mathfrak{p}=1} R_{\mathfrak{p}}$ holds.*

2. *For any $f \in R \setminus 0$, there are only finitely many height 1 prime ideals containing f.*

Proof. See [**?**, Theorem 12.4(i) p.88] □

We now proceed to link the Picard group more closely with the geometry of closed subvarieties of codimension 1, generalizing the situation with curves. Let X be a Noetherian normal scheme of Krull dimension d; for a point x of codimension 1, we will write ν_x for the assoiated discrete valuation. It follows from the discussion above that if K is the fraction field of X (i.e., the residue field at the generic point of X), then for $f \in K^\times$ there at most finitely many codimension 1 points $x \in X$ for which $\nu_x(f) \neq 0$. As before, any element $f \in K^\times$ defines a Weil divisor

$$div(f) := \sum_{xcodim1} \nu_x(f).$$

If (U_i, f_i) is a Cartier divisor, then we can define a Weil divisor D by the formula that $D = \sum_{xcodim1} \nu_x(f_i)$, where f_i is chosen so that x lies in U_i. Thus, we get a function

$$H^0(X, \mathscr{K}^\times/\mathscr{O}^\times) \longrightarrow \bigoplus_{xcodim1} \mathbb{Z} \cdot x,$$

which factors through a morphism

$$Pic(X) \longrightarrow Cl(X).$$

To begin we revisit the notion of Cartier divisor. Suppose $X = \operatorname{Spec} R$ is an integral affine scheme with fraction field K. In that case, a Cartier divisor $D = \{U_i, \sigma_i\}$ with $\sigma_i \in K$ is called *effective* if σ_i is a unit *on* U_i. Note that every Cartier divisor can be written as the difference of two effective divisors: if we write $\sigma_i = \frac{r_i}{s_i}$, with $r_i, s_i \in R$, then $\{U_i, r_i\}$ and $\{U_i, s_i\}$ are both effective Cartier divisors. Indeed, since $\frac{r_i}{s_j}\frac{s_i}{r_j}$ is a unit on $U_i \cap U_j$, we conclude that both $\frac{r_i}{r_j}$ and $\frac{s_i}{s_j}$ are units on $U_i \cap U_j$. Thus, the group of Cartier divisors can be thought of in terms of formal differences in the monoid of all effective Cartier divisors.

Now, if $\{U_i, f_i\}$ is an effective Cartier divisor, then the vanishing of f_i determines a hypersurface in U_i. Let R_i be the ring of functions on U_i (some principal open subset of X). The compatibilities inherent in being a Cartier divisor mean that the local ideals $(f_i) \subset R_i$ patch together to determine an ideal $I(D) \subset R$; this ideal is *locally principal* by construction (i.e., there is a Zariski open cover of X by principal open sets on which this ideal is actually a principal ideal). This construction yields an equivalence between locally principal ideals in R and effective Cartier divisors on R. Thus, effective Cartier divisors on X correspond to certain closed subvarieties of X that are locally cut out by a single equation.

Given an ideal $I(D)$, if we consider the localization of R at a height 1 prime ideal, by Proposition 3.6.1.1 we obtain an ideal in a discrete valuation ring. Such an ideal is necessarily of the form $(\pi_\mathfrak{p})^{r_\mathfrak{p}}$ for some positive integer r and choice of local uniformizing parameter $\pi_\mathfrak{p}$. Therefore, we can attach to each ideal $I(D)$ a formal sum $\sum_{\mathfrak{p}|\mathrm{ht}\mathfrak{p}=1} r_\mathfrak{p} \cdot \mathfrak{p}$. As in the case of Dedekind domains, it follows from Theorem 3.6.1.2(2) that the integer $r_\mathfrak{p}$ is only non-zero for finitely many $\mathfrak{p}$. Since we can write any Cartier divisor as a formal difference, in this way we obtain a function

$$\mathrm{Cart}(R) \longrightarrow \bigoplus_{\mathfrak{p}|\mathrm{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p},$$

just as in the situation for Dedekind domains. By Theorem 3.6.1.2(1), this homomorphism is necessarily injective: indeed, if $I(D)$ is the ideal associated with an effective Cartier divisor, then if $I(D) \cap R_{\mathfrak{p}} = 0$ for all height 1 prime ideals, then $I(D) = 0$ already.

3.6.2 Triviality of the class group

Lemma 3.6.2.1. *If R is a Noetherian domain, then R is a unique factorization domain if and only if R is normal and $Cl(R) = 0$.*

Proof. We use the following fact from ring theory: if R is a Noetherian domain, then R is a UFD if and only if every height 1 prime ideal is principal [**?**, Tag 034O Lemma 10.119.6] or [**?**, Theorem 20.1] (one proof of this result uses Krull's principal ideal theorem: if R is a Noetherian ring, $x \in R$, and $\mathfrak{p}$ is minimal among prime ideals in R containing x, then $\mathfrak{p}$ has height ≤ 1 together with a bit of the theory of primary decomposition).

Suppose every prime ideal of height 1 is principal. In that case, if $\mathfrak{p}$ is a prime ideal of height 1, we can choose a generator f. Then, f lies in the image of the divisor map. It follows that the map div is surjective, and thus that $Cl(R) = 0$.

We leave the other direction as an exercise. □

Corollary 3.6.2.2. *If k is a field, $Cl(k[x_1, \ldots, x_n]) = 0$.*

Definition 3.6.2.3. A ring R is called *locally factorial* if for every prime ideal $\mathfrak{p}$, the localization $R_{\mathfrak{p}}$ is a unique factorization domain.

Proposition 3.6.2.4. *If R is a locally factorial Noetherian normal domain, then the map*

$$\operatorname{Cart}(R) \longrightarrow \bigoplus_{\mathfrak{p} | \operatorname{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p}$$

is an isomorphism. As a consequence, under these hypotheses, the induced map $\operatorname{Pic}(R) \to Cl(R)$ is an isomorphism.

Proof. As before it suffices to construct an inverse map and to do this we proceed exactly as before: beginning with a height 1 prime ideal $\mathfrak{p}$, it suffices to show that $\mathfrak{p}$ is actually an invertible ideal in R. Since R is Noetherian, $\mathfrak{p}$ is automatically finitely presented, and therefore it suffices to check this after localization at every prime. The ideal $\mathfrak{p}$ determines an ideal in $R_{\mathfrak{q}}$ as $\mathfrak{q}$ ranges through the prime ideals of R. By assumption, $Cl(R_{\mathfrak{q}}) = 0$, so locally $\mathfrak{p}$ is a principal fractional ideal. By the finite presentation assumption, we can find a Zariski open neighborhood containing $R_{\mathfrak{q}}$ on which $\mathfrak{p}$ is principal, and we can cover $\operatorname{Spec} R$ by such open sets. □

Example 3.6.2.5. Note that if R is a Dedekind domain, it follows from Lemma **??** that R is normal and locally factorial. We will focus on systematically writing down other examples of such rings soon.

We have shown that, if $X = \operatorname{Spec} R$ is a normal affine variety, then there is a two-term complex

$$K^{\times} \xrightarrow{\operatorname{div}} \bigoplus_{\mathfrak{p} | \operatorname{ht}\mathfrak{p}=1} \mathbb{Z} \cdot \mathfrak{p};$$

the cokernel of the map div is $\operatorname{Pic}(R)$, while the kernel of div is $R^{\times}$.

The Picard groups of a UFD

It is possible to give a direct proof that the Picard group of a UFD is trivial (without passing through the identification afforded by Proposition 3.6.2.4.

Proposition 3.6.2.6. *If R is a UFD, then* $\mathrm{Pic}(R) = 0$.

Proof. See [**?**, Tag 0AFW Lemma 15.84.3]. □

3.6.3 Dominant maps

We are interested in analyzing the extent to which $Pic(-)$ is homotopy invariant, i.e., what can we say about the map

$$Pic(X) \longrightarrow Pic(X \times \mathbb{A}^1)$$

given by pullback along the projection $X \times \mathbb{A}^1 \to X$. Now, there is always a morphism $X \to X \times \mathbb{A}^1$ given by base-change along the morphism $0 \to \mathbb{A}^1$. As a consequence, we conclude that the above map is always split injective. We would like to use what we have learned about Cartier and Weil divisors in order to analyze this map, but we immediately run into some subtleties because the two maps above behave very differently.

Example 3.6.3.1. Assume $X = \operatorname{Spec} R$ is an integral affine scheme with K the fraction field of R. Let us consider the ring maps $R \hookrightarrow R[t]$ corresponding to the projection $X \times \mathbb{A}^1 \to X$ and the $ev_0 : R[t] \to R$ corresponding to the closed immersion $X \to X \times \mathbb{A}^1$. A Cartier divisor on $X \times \mathbb{A}^1$ corresponds to an invertible $R[t]$-module L together with a trivialization $\varphi : L \otimes_{R[t]} K(t) \xrightarrow{\sim} K(t)$. The first homomorphism $R \to R[t]$ induces an inclusion of fields $K \hookrightarrow K(t)$ thus, given a Cartier divisor L' on X, i.e., an invertible R-module L' together with $\varphi' : L' \otimes_R K \xrightarrow{\sim} K$, we get a Cartier divisor on $R[t]$ by extension of scalars: $L := L' \otimes_R R[t]$ is a line bundle, and φ' induces an isomorphism $L \otimes_R [t] K(t) \cong L' \otimes_R K \otimes_K K(t) \cong K(t)$. What is essential here is that the pullback of the zero ideal in $R[t]$ under the ring homomorphism $R \to R[t]$ yields the zero ideal in R. This is evidently false for the homomorphism $R[t] \to R$, since (t) is contained in the pre-image!

Before moving forward we describe some ideal theoretic properties of the image of the morphism of schemes attached to $\varphi : R \to S$. We begin with some equivalent characterizations of the condition that a point lies in the image.

Definition 3.6.3.2. A morphism $\varphi : X \to S$ of schemes will be called dominant if the scheme-theoretic image of φ is dense.

Lemma 3.6.3.3. *A morphism of integral schemes is dominant if and only if the generic point of the target is contained in the image.*

Example 3.6.3.4. Consider the map $SL_2 \to \mathbb{A}^2$ obtained by projection onto the first column. Ring theorretically, if we identify $k[SL_2] = k[x_{11}, x_{12}, x_{21}, x_{22}]/(x_{11}x_{22} - x_{12}x_{21} - 1)$, then map in question is given by the inclusion of $k[x_{11}, x_{21}]$. This ring map is injective, so the morphism in question is dominant, but note that the morphism $SL_2 \to \mathbb{A}^2$ is *not* surjective. Indeed, the first column of an invertible 2×2-matrix over any ring cannot be identically zero. Thus, the scheme-theoretic image of a ring map, and the image of the associated map of spectra need not coincide in general.

Lemma 3.6.3.5. *A morphism $\varphi : R \to S$ of affine schemes is dominant if and only if $\ker(\varphi)$ is nilpotent. In particular, if R is reduced, then φ is dominant if and only if it is injective.*

3.6.4 Dominant maps and pullbacks of divisors

Construction 3.6.4.1. Given a dominant morphism $\varphi : X \to S$ of integral schemes, we can define a pullback of Cartier divisors as follows. First, let us treat the case of affine schemes.

Thus, assume $\varphi : R \to S$ is a homomorphism of integral domains such that the induced map on spectra is dominant. Given an invertible R-module L, we obtain a Cartier divisor by considering $L \hookrightarrow L \otimes_R K \xrightarrow{\sim} K$. Now, if $R \to S$ is dominant with corresponding inclusion $K \hookrightarrow E$, then we see that $S \hookrightarrow E$ is injective as well. In that case, the specified isomorphism $L \otimes_R K \xrightarrow{\sim} K$ induces a isomorphism

$$(L \otimes_R K) \otimes_K E \cong L \otimes_R (K \otimes_K E) \cong L \otimes_R E \xrightarrow{\sim} E$$

and the composite $L \otimes_R S \to L \otimes_R E \xrightarrow{\sim} E$ yields a Cartier divisor. Alternatively, if we think in terms of the $\{U_i, \sigma_i\}$, then we can simply pullback the defining equations.

Proposition 3.6.4.2. *Suppose R and S are integral domains, and $\varphi : R \to S$ is a dominant ring homomorphism with K the fraction field of R, and E the fraction field of S.*

1. *There is a commutative diagram of the form*

$$\begin{tikzcd} K^{\times} \arrow[r, "\mathrm{div}"] \arrow[d] & \mathrm{Cart}(R) \arrow[d] \\ E^{\times} \arrow[r, "\mathrm{div}"] & \mathrm{Cart}(S), \end{tikzcd}$$

   where the left hand map is inclusion $K^{\times} \to E^{\times}$ and the right hand map is the pullback on Cartier divisors from Construction 3.6.4.1.
2. *If R and S are furthermore, locally factorial, Noetherian and normal, then we can replace* $\mathrm{Cart}(-)$ *by* $\mathrm{Div}(-)$.
3. *The induced maps of kernels coincides with the pullback map of unit groups.*
4. *The induced maps of cokernels coincides with the pullback map of Picard groups.*

Proof. For the first statement, the only thing that has to be checked is commutativity of the diagram. Given an element $f \in K^{\times}$, it is sent to the Cartier divisor $Rf \subset K$. Unwinding the definitions, this is sent to $Sf \subset E$, as expected.

For the second statement, we appeal to the identification of Proposition 3.6.2.4 to transport the pullback on Cartier divisors to Weil divisors (this is compatible with the divisor map by construction).

The final two statements follow by unwinding the definitions. □

If $\varphi : X \to Y$ is a dominant morphism of integral schemes, then we can always cover X and Y by open affine schemes such that induced maps of affine schemes are dominant morphisms as above. In that case, given an open affine cover U_i of Y, and a line bundle $\mathscr{L}$, we can realize the Cartier divisor on U_i as a generic trivialization as above. In that case, the pullback is defined on the relevant open cover as above.

3.7 Homotopy invariance, localization and Mayer-Vietoris

Having developed a fair amount of technology for studying Picard groups, we now reap the benefits and deduce various basic properties that show $\mathrm{Pic}(-)$ acts like a cohomology theory: we show it is $\mathbb{A}^1$-invariant, and a suitable form of Mayer-Vietoris holds.

3.7.1 Homotopy invariance

The inclusion map $R \to R[t]$ is a dominant ring homomorphism. Assuming R is a domain, then $R[t]$ is also a domain. Therefore, there is an induced pullback map $\mathrm{Cart}(R) \to \mathrm{Cart}(R[t])$. If we write K for the fraction field of R, then we can identify the fraction field of $R[t]$ with $K(t)$.

Exercise 3.7.1.1. *If R is a locally factorial Noetherian normal domain, then $R[t]$ is as well.*

In that case, there is a morphism of complexes of the following form:

(3.7.1)
$$\begin{tikzcd} K^{\times} \arrow[r] \arrow[d] & \bigoplus_{\{\mathfrak{p}\subset R | \mathrm{ht}(\mathfrak{p})=1\}} \mathbb{Z} \arrow[d] \\ K(t)^{\times} \arrow[r] & \bigoplus_{\{\mathfrak{p}\subset R[t] | \mathrm{ht}(\mathfrak{p})=1\}} \mathbb{Z}. \end{tikzcd}$$

The left vertical map is evidently injective. We now describe the height 1 prime ideals in $R[t]$ more geometrically. If $\mathfrak{p}$ is a prime ideal in $R[t]$ that has height 1, then the pullback under the ring map $R \to R[t]$ is a prime ideal in R, which may not have height 1. For example, the homomorphism $\mathbb{Z} \to R$ induced by the unit, determines a homomorphism $\mathbb{Z}[t] \to R[t]$. Any irreducible polynomial in t with integral coefficients therefore defines an ideal in $R[t]$, which is a height 1 principal ideal. Note that the pullback of this ideal to R under $R \mapsto R[t]$ is R itself.

On the other hand, the evaluation map $R[t] \to R$ defines height 1 prime ideals of the form $\mathfrak{p}[t]$ in $R[t]$; the pullback of such an ideal to R under $R \to R[t]$ is precisely $\mathfrak{p}$. Only these latter prime ideals are in the image of the pullback map. We summarize this observation in the following result.

Lemma 3.7.1.2. *The pullback map*

$$\bigoplus_{\{\mathfrak{p}\subset R | \mathrm{ht}(\mathfrak{p})=1\}} \mathbb{Z} \longrightarrow \bigoplus_{\{\mathfrak{p}\subset R[t] | \mathrm{ht}(\mathfrak{p})=1\}} \mathbb{Z}$$

is injective and its image consists of those height 1 prime ideals of the form $\mathfrak{p}[t]$.

To study the homotopy invariance question, it suffices to show that all height 1 prime ideals in $R[t]$ differ from a sum of those of the form $\mathfrak{p}[t]$ by the divisor of a rational function. To this end, suppose $\mathfrak{q} \subset R[t]$ is a height 1 prime ideal. In that case, we can consider the image of $\mathfrak{q}$ in $K[t] \supset R[t]$. Now, since K is a field, $K[t]$ is a principal ideal domain, so the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$ is necessarily principal. Choose a generator f of the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$. The element f yields an element of $K(t)$.

Now, we analyze the principal divisor attached to f. If we pick generators $f_1, \ldots, f_r$ of $\mathfrak{p}$, then we can write these in the form $f_i(t) = a_{0,i} + \cdots + a_{n_i,i}t^{n_i}$ where each $a_i \in R$. The corresponding element of $K[t]$ is obtained by introducing denominators. Since the ideal $\mathfrak{q} \otimes_{R[t]} K[t]$ is principal, that means after inverting coefficients, $f_i(t) = \alpha_i f(t)$ for $\alpha_i \in R$. From the form of these expressions, we can deduce that $\operatorname{div}(f)$ differs from $\mathfrak{q}$ by prime divisors in the image of the pullback map. Altogether, we have established the following fact:

Theorem 3.7.1.3 (Homotopy invariance)**.** *Assume $X = \operatorname{Spec} R$, with R is a locally factorial Noetherian normal domain.*

1. *the map $\operatorname{Pic}(X) \to \operatorname{Pic}(X \times \mathbb{A}^1)$ is an isomorphism;*
2. *the map of* Equation 3.7.1 *is a quasi-isomorphism of complexes, i.e., induces an isomorphism after taking cohomlogy.*

Proof. Point (1) is established by the discussion just prior to the statement. For Point (2), we simply describe the maps on cohomology. That the map of complexes induces an isomorphism on H^0 (i.e., after taking kernels) follows from Proposition 1.5.1.6 and an isomorphism on H^1 (i.e., after taking cokernels) follows from the conclusion of Point (1). Since the complexes in question have only 2 terms, there are no other possibly non-vanishing cohomology groups. □

Example 3.7.1.4. Suppose R is a principal ideal domain. In that case, R is a UFD, and therefore $Pic(R) = 0$ by the structure theorem for finitely generated modules. Theorem 3.7.1.3 and an induction argument then show that if $X = \operatorname{Spec} R$, $Pic(X \times \mathbb{A}^n) = 0$ as well. For example, we see that $\mathbb{A}^n_{\mathbb{Z}}$ has trivial Picard group.

Example 3.7.1.5. As is the case with units, the Picard group is *not* $\mathbb{A}^1$-invariant on all rings. Moreover, counterexamples to $\mathbb{A}^1$-invariance exist even for reduced rings. For example, one may check that if $R = k[x,y]/(y^2 - x^3)$, then the map $R \mapsto R[t]$ is not an isomorphism on Picard groups.

Remark 3.7.1.6. Theorem 3.7.1.3 is not the best possible $\mathbb{A}^1$-invariance result for Picard groups. Indeed, if R is a ring, then evaluation determines a homomorphism $\operatorname{Pic}(R[t]) \to \operatorname{Pic}(R)$ for any ring R. Therefore, $\mathbb{A}^1$-invariance is equivalent to establishing the kernel of this map is trivial. The kernel corresponds to invertible $R[t]$-modules L such $L/tL \cong R$, and thus one would like to show that if L is a module such that L/tL is trivial, then L is already trivial.

Fix an isomorphism $L/tL \cong R$. On the other hand, if R is a domain, with fraction field K, then we know that $L \otimes_{R[t]} K[t]$ is a trivial rank 1 module. Therefore, we can fix a trivialization $L \otimes_{R[t]} K[t] \cong K[t]$ as well. Because L is finitely presented, we can find $f \in K$ such that the above trivialization extends to an isomorphism $L \otimes_{R[t]} R_f[t] \cong R_f[t]$. In this way, we obtain an isomorphism $R_f \cong L/tL \otimes_R R_f \cong L_f/tL_f \cong R_f$, which corresponds to a unit in R_f. Modifying the trivialization of $L \otimes_R K[t]$ by this unit, we can extend the isomorphism $L/tL \cong R$ over $R_f[t]$. One then wants to show that under suitable hypotheses this isomorphism can be extended over all of $R[t]$. This can be accomplished, e.g., if R is a Noetherian normal domain. However, it holds even more generally for semi-normal rings, which essentially rule out precisely singularities of "cusp" type (cf. Example 3.7.1.5). See [**?**] [**?**] and [**?**] for more details.

3.7.2 The localization sequence

Suppose X is a locally factorial normal integral scheme and $U \subset X$ is an open subscheme. In that case, there is an induced restriction morphism

$$Pic(X) \longrightarrow Pic(U).$$

If U is non-empty, then this homomorphism is dominant so we have pullbacks of divisors as above. Let K be the fraction field of X, which is necessarily also the fraction field of U. In that case, the induced map of complexes takes the form:

$$\begin{tikzcd} K^{\times} \arrow[r] \arrow[d] & \bigoplus_{\{x \in X | codim x = 1\}} \mathbb{Z} \cdot x \arrow[d] \\ K^{\times} \arrow[r] & \bigoplus_{\{x \in U | codim x = 1\}} \mathbb{Z} \cdot x. \end{tikzcd}$$

The left hand vertical map is an isomorphism. Unwinding the definition of pullback of Cartier divisors and the identification with Weil divisors, since all codimension 1 points of U are codimension 1 points of X, it follows that the right hand vertical map is surjective and corresponds simply to the projection onto the factors corresponding to codimension 1 points in U. Now, $X \setminus U$ is a closed subscheme of X and if X is Noetherian, necessarily has finitely many codimension 1 irreducible components. Putting everything together, we obtain the following result.

Theorem 3.7.2.1 (Localization sequence)**.** *Suppose X is a locally factorial Noetherian integral normal scheme and $U \subset X$ is a non-empty open subscheme.*

1. *There is a short exact sequence of complexes of the form:*

$$0 \longrightarrow \begin{pmatrix} 0 \\ \downarrow \\ \bigoplus_{\{x \in X | codim x = 1 x \in X \setminus U\}} \mathbb{Z} \cdot x \end{pmatrix} \longrightarrow \begin{pmatrix} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in X | codim x = 1\}} \mathbb{Z} \end{pmatrix} \longrightarrow \begin{pmatrix} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in U | codim x = 1} \mathbb{Z} \end{pmatrix} \longrightarrow 0.$$

2. *There is an exact sequence of groups of the form:*

$$0 \longrightarrow \Gamma(X, \mathscr{O}_X^{\times}) \longrightarrow \Gamma(U, \mathscr{O}_U^{\times}) \longrightarrow \bigoplus_{\{x \in X | codim x = 1 x \in X \setminus U\}} \mathbb{Z} \cdot x \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U) \longrightarrow 0.$$

Proof. The first point is an immediate consequence of the analysis before the statement. The second point follows immediately from the first by taking the long exact sequence in cohomology associated with a short exact sequence of complexes. □

Corollary 3.7.2.2. *If X is a locally factorial Noetherian normal integral scheme, and $U \subset X$ is a non-empty open subscheme whose complement $X \setminus U$ has codimension ≥ 2 in X, then the pullback maps $Pic(X) \to Pic(U)$ and $\Gamma(X, \mathscr{O}_X^{\times}) \to \Gamma(U, \mathscr{O}_U^{\times})$ are both isomorphisms.*

Example 3.7.2.3. If $X = \mathbb{P}^n_{\mathbb{Z}}$, $n \geq 2$ and $x \in X$ is a $\mathbb{Z}$-point (e.g., the point $[0 : \cdots : 0 : 1]$ in homogeneous coordinates), then $Pic(X) \to Pic(X \setminus x)$ is an isomorphism.

Example 3.7.2.4. If $X = \mathbb{P}^n_{\mathbb{Z}}$, then we can compute $Pic(X)$ using the localization sequence and induction. Indeed, X has an open subscheme $\mathbb{A}^n_{\mathbb{Z}}$ that has trivial Picard group by Example 3.7.1.4. Moreover, by homotopy invariance of units we conclude that $\Gamma(\mathbb{A}^n_{\mathbb{Z}}, \mathscr{O}^{\times}_{\mathbb{A}^n_{\mathbb{Z}}}) = \mathbb{Z}^{\times} \cong \mathbb{Z}/2$. In that case, the localization sequence reads:

$$\mathbb{Z}/2 \longrightarrow \mathbb{Z} \longrightarrow Pic(\mathbb{P}^n_{\mathbb{Z}}) \longrightarrow Pic(\mathbb{A}^n_{\mathbb{Z}}) \longrightarrow 0;$$

where the factor of $\mathbb{Z}$ is the free abelian group of rank 1 corresponding to the codimension 1 point of $\mathbb{P}^n_{\mathbb{Z}}$ defined by the copy of $\mathbb{P}^{n-1}_{\mathbb{Z}}$ complementary to $\mathbb{A}^n_{\mathbb{Z}}$. Since $Pic(\mathbb{A}^n_{\mathbb{Z}}) = 0$ and any homomorphism $Z/2 \to \mathbb{Z}$ is trivial, we conclude that $Pic(\mathbb{P}^n_{\mathbb{Z}}) = 0$.

3.7.3 Zariski patching and Mayer-Vietoris sequences

Suppose X is a locally factorial Noetherian normal integral scheme and $X = U \cup V$ for open subschemes U and V. In that case we have a commutative square of the form

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X. \end{tikzcd}$$

Both the vertical and horizontal maps in this diagram give rise to localization sequences. The relevant restriction maps then give rise to maps

$$Pic(X) \longrightarrow Pic(U) \oplus Pic(V) \qquad \Gamma(X, \mathscr{O}^{\times}_X) \longrightarrow \Gamma(U, \mathscr{O}^{\times}_U) \oplus \Gamma(V, \mathscr{O}^{\times}_V)$$

and we also have "difference" maps

$$Pic(U) \oplus Pic(V) \longrightarrow Pic(U \cap V) \quad \Gamma(U, \mathscr{O}^{\times}_U) \oplus \Gamma(V, \mathscr{O}^{\times}_V) \longrightarrow \Gamma(U \cap V, \mathscr{O}^{\times}_{U \cap V}).$$

These maps are induced by morphisms of complexes, which we now describe.

Assume K is the fraction field of X. In that case, there is a morphism of complexes of the form:

$$\left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in X | codim x = 1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in U | codim x = 1\}} \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in V | codim x = 1\}} \mathbb{Z} \end{array} \right).$$

Likewise, there is a difference map at the level of complexes:

$$\left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in U | codim x = 1\}} \mathbb{Z} \end{array} \right) \oplus \left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in V | codim x = 1\}} \mathbb{Z} \end{array} \right) \longrightarrow \left(\begin{array}{c} K^{\times} \\ \downarrow \\ \bigoplus_{\{x \in U \cap V | codim x = 1\}} \mathbb{Z} \end{array} \right)$$

defined as follows. The map $K^\times \oplus K^\times \to K^\times$ is given by $(f,g) \mapsto fg^{-1}$ and the map

$$\left(\bigoplus_{x \in U | codim x = 1} \mathbb{Z} \cdot x\right) \oplus \left(\bigoplus_{x \in V | codim x = 1} \mathbb{Z} \cdot x\right) \longrightarrow \bigoplus_{x \in U \cap V | codim x = 1} \mathbb{Z}$$

gotten by projection onto the factors corresponding to those codimension 1 points on X lying in $U \cap V$ and on such factors sending send (x,y) to $x - y$. You can check that the map defined componentwise in this fashion is a homomorphism of complexes. The following result is obtained by diagram chasing using localization sequences.

Theorem 3.7.3.1 (Mayer-Vietoris)**.** *Suppose X is a locally factorial Noetherian normal integral scheme, and U and V are open subschemes such that $X = U \cup V$.*

1. *Restriction and difference (as defined above) fit together to give a short exact sequence of complexes of the form:*

$$0 \longrightarrow \begin{pmatrix} K^\times \\ \downarrow \\ \bigoplus_{\{x \in X | codim x = 1\}} \mathbb{Z} \end{pmatrix} \longrightarrow \begin{pmatrix} K^\times \\ \downarrow \\ \bigoplus_{\{x \in U | codim x = 1\}} \mathbb{Z} \end{pmatrix} \oplus \begin{pmatrix} K^\times \\ \downarrow \\ \bigoplus_{\{x \in V | codim x = 1\}} \mathbb{Z} \end{pmatrix} \longrightarrow \begin{pmatrix} K^\times \\ \downarrow \\ \bigoplus_{\{x \in U \cap V | codim x = 1\}} \mathbb{Z} \end{pmatrix} \longrightarrow 0$$

2. *There is an induced* Mayer-Vietoris *exact sequence of the form*

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathscr{O}_X^\times) \longrightarrow \Gamma(U, \mathscr{O}_U^\times) \oplus \Gamma(V, \mathscr{O}_V^\times) \longrightarrow \Gamma(U \cap V, \mathscr{O}_{U \cap V}^\times) \\ \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(U) \oplus \mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(U \cap V) \longrightarrow 0. \end{aligned}$$

The existence of this Mayer–Vietoris sequence has a number of other consequences.

Definition 3.7.3.2. A morphism $f : X \to Y$ of schemes will be called Zariski locally trivial with fibers isomorphic to $\mathbb{A}^n$ if there exists an open cover U_i of Y such that $f^{-1}(U_i)$ form an open cover of X and there are isomorphisms $\varphi_i : f^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^n$.

Corollary 3.7.3.3. *Assume X and Y are locally Noetherian normal integral schemes and $f : X \to Y$ is a Zariski locally trivial morphism with fibers isomorphic to $\mathbb{A}^n$. The pullback map $f : Pic(Y) \to Pic(X)$ is an isomorphism.*

Proof. TBy Mayer–Vietoris this reduces to homotopy invariance for Picard groups in the affine case. □

Corollary 3.7.3.4. *For any integer $n \geq 1$ there is an isomorphism $Pic(\mathbb{P}^n_{\mathbb{Z}}) = \mathbb{Z}$.*

Proof. For $n = 1$, we know that $Pic(\mathbb{P}^1_{\mathbb{Z}}) = \mathbb{Z}$. We claim that for any integer $n \geq 2$, the scheme $\mathbb{P}^n_{\mathbb{Z}}$ pt has the structure of a Zariski locally trivial morphism with $\mathbb{A}^1_k$ fibers over $\mathbb{P}^{n-1}_{\mathbb{Z}}$. Indeed, simply project away from the point (you can write down formulas when $x = [0 : \cdots : 0 : 1]$ and $\mathbb{P}^{n-1}_{\mathbb{Z}}$ is identified with $[x_0 : \cdots : x_{n-1} : 0]$). The result then follows by induction using the fact that $Pic(\mathbb{P}^n_{\mathbb{Z}}) \to Pic(\mathbb{P}^n_{\mathbb{Z}} \setminus pt)$ is an isomorphism for $n \geq 2$. □

Chapter 4

Sheaf cohomology

Contents

In this section, we would like to analyze sheaf cohomology from the point of view of abstract homotopy theory. I could try to do this from the point of view of ∞-categories to give a truly modern treatment, but since this would require introducing a rather long list of preliminaries, I have chosen to follow a middle path: using the theory of model categories to construct derived categories and various related homotopy categories.

4.1 Model categeories

In this section, we want to give the motivation and basic definitions around the theory of model categories. We will keep two examples in mind.

4.1.1 Localizations of categories: motivation

The classical definition of the homotopy category of topological spaces is performed as follows. Write $\mathbf{Top}$ for the category of all topological spaces. A continuous map $f : X \to Y$ of topological spaces is a homotopy equivalence if there exists a homotopy inverse, i.e., a morphism $g : Y \to X$ such that the two composites $g \circ f$ and $f \circ g$ are homotopic as maps to the relevant identity. This notion of homotopy equivalence of maps is an equivalence relation, and one defines the homotopy category Ho as the category as a quotient category: the objects of Ho are the same $\mathbf{Top}$], but for which the morphisms are the quotient by the relevant equivalence relation. By construction, there is a functor

$$\mathbf{Top} \longrightarrow \mathrm{Ho};$$

this functor has the following universal property, which I will phrase rather vaguely: any functor F from $\mathbf{Top}$ to some category $\mathbf{C}$ that takes homotopy equivalences to isomorphisms factors uniquely through Ho. This kind of construction is part of the general theory of localizations of categories: one would like to build a "universal" category in which a prescribed set of morphisms have been forced to be isomorphisms.

If we restrict our attention to CW complexes, then homotopy equivalences can be detected on homotopy groups in the following sense. Recall that one defines a morphism $f : X \to Y$ of topological spaces to be a weak equivalence if the map $\pi_0(X) \to \pi_0(Y)$ is a bijection, and for every point $x \in X$, and every integer $i \geq 1$ the map $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ is an isomorphism of groups. The cellular approximation theorem says that if $f : X \to Y$ is any continuous map, then f is *homotopic* to a cellular map. If f is a cellular map of CW complexes, then f is a weak equivalence if and only if f is a homotopy equivalence, i.e., one may build a homotopy inverse to any cellular map. The CW approximation theorem says that if X is any topological space, then one may build a CW complex Z and a weak equivalence $f : Z \to X$. Thus, up to *weak* equivalence, every space is equivalent to a CW complex. Unfortunately, weak homotopy equivalence is not an equivalence relation: while weak homotopy equivalence is reflexive and and transitive (composites of weak homotopy equivalences are weak homotopy equivalences), it fails to be symmetric as the CW approximation theorem suggests. In that case, we may consider the equivalence relation *generated* by weak homotopy equivalences. Unwinding the notion of equivalence relation generated by a reflexive and transitive relation, we will say that two spaces X and X' have the same weak homotopy type if there is a finite sequence of spaces $X_0 = X, X_1, \ldots, X_n, X_{n+1} = X', Z_1, \ldots, Z_{n+1}$ and maps $f_i : Z_i \to X_i$ and $g_i : Z_i \to X_{i-1}$ that are all weak equivalences.

Note that if $f : X \to Y$ is a weak equivalence of topological spaces, then the induced map on singular homology (or cohomology) is an isomorphism. In fact, there are many functors from topological spaces to abelian groups that transform weak equivalences into isomorphisms. As such, one would like to construct a *weak homotopy category* which is universal in the sense that any functor that transforms weak equivalences into isomorphism factors uniquely through this weak homotopy category. One can build such a category using the ideas sketched above. Around the same time, D. Kan introduced a combinatorialization of homotopy theory using the notion of (abstract) simplicial set. There is a combinatorial notion of weak equivalence of simplicial sets, but it once again fails to be an equivalence relation and one would like to effectively construct the homotopy category.

Suppose R is a commutative ring, and Ch_R is the category of (say, bounded below or bounded above) (co)chain complexes of R-modules, or if X is a topological space, we can write $Ch(\mathscr{O}_X)$ for the category of bounded complexes of sheaves of $\mathscr{O}_X$-modules. In this context, there is the notion of a chain homotopy of R-modules or of complexes of $\mathscr{O}_X$-modules. Chain homotopy is an equivalence relation and one can construct the homotopy category of Ch_R in exactly the same way as one constructed the homotopy category of topological spaces above. Onwhen proving the

Mayer-Vietoris theorem or excision for singular homology, one approach is to replace the the complex of singular chains on a topological space X by a subcomplex of chains adapted to the open covering. The inclusion of this sub-complex is not a chain homotopy equivalence, but it does induce an isomorphism on homology groups, and this induces a finer notion of equivalence that is frequently useful.

Definition 4.1.1.1. If $f : A \to B$ is a morphism of (co)chain complexes of sheaves of $\mathscr{O}_X$-modules, then f is a quasi-isomorphism if f induces an isomorphism on (co)homology sheaves.

As above, we would like to formally invert quasi-isomorphisms; the resulting category, if it exists, is called the derived category of R-modules. The process of constructing the derived category if formally analogous to that of constructing the weak homotopy category, so we would like to introduce a formalism where we can perform all of these constructions together.

4.1.2 Limits and colimits

In order to build the homotopy category in an effective way (i.e., so that we can give an explicit description of morphism sets in the homotopy category) we will need to know that various constructions we would like to perform in our category can actually be performed. For example, we built pushouts of topological spaces via an explicit quotient construction. Likewise, we also built fibered products in topological spaces by an explicit construction. Both of these constructions have universal properties that we discussed (the coproduct for maps out, and the fiber product for maps in). We now abstract these kinds of constructions.

Suppose $\mathbf{I}$ is a category. We will call $\mathbf{I}$ small if it has a set of objects and finite if it has a finite set of objects. Fix a category $\mathbf{C}$. By an $\mathbf{I}$-diagram, we will mean a functor $\mathbf{I} \to \mathbf{C}$. Two simple examples to keep in mind are the categories $\mathbf{I}_0$ and $\mathbf{J}_0$ given pictorially by:

$$* \to * \leftarrow * \qquad * \leftarrow * \to *,$$

where we have drawn the objects and the non-identity arrows in the category. Thus a functor from $\mathbf{I}_0 \to \mathbf{Top}$ consists of a diagram of topological spaces of the form

$$X \longrightarrow Z \longleftarrow Y.$$

The fiber product $X \times_Y Z$ has the universal property that it comes equipped with two projection maps $X \times_Z Y \to X$ and $X \times_Z Y \to Z$given a space W together with maps $W \to X$ and $W \to Y$ such that the two composites $W \to Z$ agree, there is a unique morphism $W \to X \times_Z Y$ making the relevant diagrams commute. We can rephrase this universal property using the diagram category $\mathbf{I}_0$ as follows. The topological space W gives rise to a "constant" $\mathbf{I}_0$-diagram

$$W \xrightarrow{id} W \xleftarrow{id} W.$$

In fact, assigning to a topological space W the corresponding constant diagram determines a functor $c : \mathbf{Top} \to \mathbf{Fun}(\mathbf{I}_0, \mathbf{Top})$. In that case, specifying morphisms $W \to X$ and $W \to Y$ such that the composites to $W \to Z$ agree amounts to specifying a natural transformation from the constant $\mathbf{I}_0$-diagram attached to W. The universal property can then be stated as follows:

$$\mathrm{Hom}_{Fun(\mathbf{I}_0,\mathbf{Top})}(c(W), X \to Z \leftarrow Y) = \mathrm{Hom}_{\mathbf{Top}}(W, X \times_Y Z).$$

The space $X \times_Y Z$ is functorial in the diagram $X \to Z \leftarrow Y$, i.e., it corresponds to a functor

$$D : Fun(\mathbf{I}_0, \mathbf{Top}) \longrightarrow \mathbf{Top}$$

which is characterized by the above property. The functor D is called a *right adjoint* to the functor c. We abstract this discussion as follows.

Definition 4.1.2.1 (Constant diagram)**.** If $\mathbf{I}$ is a small category, and $\mathbf{C}$ is a category, then we write $c_{\mathbf{I}}$ for the constant functor assigning to an object X of $\mathbf{C}$ the diagram $c_{\mathbf{I}}(X)$ that has $c_{\mathbf{I}}(X)(i) = X$ for every $i \in Ob(\mathbf{I})$ and $c_{\mathbf{I}}(X)(i \to i') = id_X$ for every morphism $i \to i'$ in $\mathbf{I}$.

Definition 4.1.2.2 (Limits)**.** If $\mathbf{I}$ is a small category, and $\mathbf{C}$ is a category, then we will say that $\mathbf{I}$-shaped limits exist in $\mathbf{C}$ if the functor c has a right adjoint functor

$$\lim_{\mathbf{I}} : Fun(\mathbf{I}, \mathbf{C}) \to \mathbf{C},$$

i.e., for any $F : \mathbf{I} \to \mathbf{C}$ we have

$$\mathrm{Hom}_{Fun(\mathbf{I},\mathbf{C})}(c_{\mathbf{I}}(X), F) = \mathrm{Hom}_{\mathbf{C}}(X, \lim_{\mathbf{I}} F).$$

We will say that a category $\mathbf{C}$ is *complete* if for any small category $\mathbf{I}$, $\mathbf{I}$-shaped limits exist in $\mathbf{C}$.

Example 4.1.2.3. Here are some other important examples. Consider the category

$$* \rightrightarrows *.$$

Then limits for this diagram are called equalizers. A limit for the empty diagram is what is typically called an *final object*. Inverse limits are limits in the above sense as well: take $\mathbf{I}$ to be the category associated with the ordered set of natural numbers.

We can phrase the pushout construction analogously. The universal property for pushouts of topological spaces is phrased via morphisms from $\mathbf{J}_0 \to \mathbf{Top}$. Indeed, if we have a diagram of spaces of the form $X \leftarrow Z \to Y$, then the universal property of the pushout is that if we have a space W and morphisms $X \to W$ and $Y \to W$ whose restrictions to Z coincide, then the pushout $X \sqcup_Z Y$ comes equipped with two morphisms $i_X : X \to X \sqcup_Z Y$ and $Y \to X \sqcup_Z Y$ and a unique morphism $X \sqcup_Z Y \to W$ making all the relevant diagrams commute. This can be rephrased in terms of the constant diagram functor as well:

$$\mathrm{Hom}_{Fun(\mathrm{op}J_0,\mathbf{Top})}(X \leftarrow Z \to Y, c(W)) = \mathrm{Hom}_{\mathbf{Top}}(X \sqcup_Z Y, W).$$

In fact, this construction is functorial in the input diagram, and the coproduct is a *left adjoint* to the constant diagram functor.

Definition 4.1.2.4. If $\mathbf{I}$ is a small category, and $\mathbf{C}$ is a category, then we will say that $\mathbf{I}$-shaped colimits exist in $\mathbf{C}$ if the constant functor $c_{\mathbf{I}}$ has a left adjoint functor, i.e., for any $F : \mathbf{I} \to \mathbf{C}$ and any object $X \in \mathbf{C}$

$$\mathrm{Hom}_{Fun(\mathbf{I},\mathbf{C})}(F, c_{\mathbf{I}}(X)) = \mathrm{Hom}_{\mathbf{C}}(\mathrm{colim}_{\mathbf{I}}\, F, X).$$

We will say that $\mathbf{C}$ is *cocomplete* if for any small category $\mathbf{I}$, $\mathbf{I}$-shaped colimits exist in $\mathbf{C}$.

Example 4.1.2.5. If one takes the empty category, then a colimit of this shape corresponds to what is usually called an initial object. If one takes the category

$$* \rightrightarrows *,$$

then a colimit for this diagram is called a coequalizer. Direct limits are examples of colimits where one takes $\mathbf{I}$ to be the category attached to the partially ordered set of the natural numbers.

Proposition 4.1.2.6. *A category* $\mathbf{C}$ *is (co)complete if and only if it has all small (co)products and (co)equalizers.*

Proof. One direction here is immediate from our examples. The converse is obtained by factoring an arbitrary category in terms of these constructions. □

The following result will be very useful.

Proposition 4.1.2.7. *If* $F : \mathbf{C} \to \mathbf{D}$ *and* $G : \mathbf{D} \to \mathbf{C}$ *is an adjoint pair of functors, then* F *preserves colimits and* G *preserves limits.*

Proof. MacLane. □

Example 4.1.2.8. The category $\mathbf{S}et$ is a model category: that all small limits and colimits exist can be seen explicitly in the case of (co)products and (co)equlizers. Products are simply cartesian products and equalizers are defined set-theoretically. Coproducts are disjoint unions and co-equalizers are given by the relevant quotient construction.

Example 4.1.2.9. If R is a commutative ring, then the category Mod_R is a model category. Products and equalizers can be computed from the corresponding set-theoretic variants, i.e., are cartesian products or equalizers. The coproduct is the direct sum, while the co-equalizer of two maps is the difference cokernel, i.e., if $f : A \to X$ and $g : A \to X$ are two morphisms, then the coequalizer of f and g is simply the cokernel of the map $f - g$.

Example 4.1.2.10. If R is a commutative ring, then the category of (bounded, bounded above, bounded below, or unbounded) chain complexes of R-modules is a model category. Products, coproducts and equalizers or co-equalizers are inherited from R-modules by working degreewise.

Example 4.1.2.11. The category $\mathbf{T}op$ is a model category. Products and equalizers are defined in the usual way (i.e., Cartesian product with product topology and equalizer with subspaces topology) and likewise for coproducts and coequalizers.

Example 4.1.2.12. If X is a topological space, then the category of sheaves of abelian groups on X is a model category. Coproducts and co-equalizers are defined by sheafifying the relevant presheaf notions. The relevant presheaf notions of products and equalizers coincide with their sheaf variants. Likewise, the category of chain complexes of sheaves of abelian groups on X is a model category (working degreewise).

Underyling a number of these examples is the following general claim.

Proposition 4.1.2.13. *If* $\mathbf{C}$ *is a category that is (co)complete, and if* $\mathbf{D}$ *is any small category, then the category of functors* $Fun(\mathbf{D}, \mathbf{C})$ *is again (co)complete.*

4.1.3 Model categories

We now want to "do" homological algebra. As we discussed in the examples above, we have a class of morphisms that we would like to invert (e.g., weak equivalences of topological spaces, quasi-isomorphisms of chain complexes, etc.). Typically, when we want to compute something (e.g., homology of a space, etc.), we choose a model where we can conveniently make computations. In order to *effectively* invert the relevant class of morphisms, it is helpful to have some supporting morphisms to use.

Lifting properties

E.g., when we study topological spaces, we spend a lot of time discussing Serre fibrations and cofibrations. Serre fibrations can be characterized by a suitable lifting property. Likewise, in homological algebra, projective modules can be characterized by suitable lifting properties.

Definition 4.1.3.1. Suppose we are given a commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y. \end{tikzcd}$$

If there exists a morphism $q : B \to X$ such that all the resulting triangles in the diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, "q"] & Y \end{tikzcd}$$

commute, then we will say that f *has the right lifting property with respect to* i or i *has the left-lifting property with respect to* f.

Example 4.1.3.2 (Projectives and injectives). Suppose R is a commutative ring. The notions of projective and injective R-modules can be phrased in terms of these lifting properties. Indeed, a module P is projective if and only if given any diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ P \arrow[r] & N \end{tikzcd}$$

where f is a surjection there exists a morphism $P \to M$ making the relevant triangles commute. In the terminology above: projective modules are precisely those modules for which the trivial homomorphism $0 \to P$ admits the left-lifting property with respect to any surjective R-module homomorphism. Likewise, an R-module I is an injective R-module if given any diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"] & I \arrow[d] \\ N \arrow[r] & 0 \end{tikzcd}$$

where i is an injection, there exists a morphism $N \to I$ making the resulting diagram commute, i.e., injective R-modules are characterized by the property that the trivial map $I \to 0$ has the right lifting property with respect to an arbitrary injection.

Example 4.1.3.3 (Serre fibrations). A morphism $f : X \to Y$ of topological spaces is called a *Serre fibration* if it has the right lifting property with respect to the morphisms $D^n \to D^n \times I$ where D^n is the n-disc and I is the unit interval.

Retractions

Suppose we are given a morphism $f : X \to Y$. We will say that X is a retract of Y if there exists a morphism $r : Y \to X$ such that the composite $r \circ f = id_X$. Frequently, one will say X is a retract of Y if the identity morphism on X factors through Y.

Example 4.1.3.4. Any retract of a projective R-module is projective. Any retract of an injective R-module is injective.

Since retraction is a statement about factorizing morphisms (namely the identity morphism on X), it will be useful to generalize the notion of retraction of an object to retraction of a morphism.

Definition 4.1.3.5. If $\mathbf{C}$ is a category, we will say that a morphism f is a retract of a morphism g if there exists a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

where the horizontal composites are the identity maps on A and B respectively.

Remark 4.1.3.6. Suppose $\mathbf{C}$ is an arbitrary small category. Let $\mathbf{I}$ be the category

$$\bullet \longrightarrow \bullet.$$

The functor category $Fun(\mathbf{I}, \mathbf{C})$ is the category whose objects are morphisms $f : A \to B$ in $\mathbf{C}$; we will write $\mathrm{Ar}(\mathbf{C})$ for this category. A morphism in $\mathrm{Ar}(\mathbf{C})$ is simply a commutative diagram in $\mathbf{C}$. A retraction of a morphism as described in the preceding definition is simply a retraction of that morphisms considered as an object in $\mathrm{Ar}(\mathbf{C})$.

Factorizations

Any function between sets admits a unique factorization as a surjection followed by an injection. Indeed, any morphism $f : S \to T$ factors as the surjective function $S \to im(f)$ and the inclusion $im(f) \hookrightarrow T$. Likewise, analogous statements can be made for homomorphisms of groups, morphisms of R-modules or morphisms of sheaves on a topological space. In fact, these factorizations are even functorial. The existence of such (functorial) factorizations can be formalized in the notion of a factorization system in a category and is exceedingly useful in analyzing exact sequences of groups (it reduces problems about long exact sequences to corresponding problems about short exact sequences).

The notion of functorial factorization can be formalized as follows. If $\mathbf{C}$ is a small category, then a collection of morphisms that contains the identity and is stable under compositions determines a subcategory of $\mathrm{Ar}(\mathbf{C})$. Suppose we give ourselves two collections of morphisms that contain the identity morphism and are stable under composition; these give rise to two subcategories of $\mathrm{Ar}(\mathbf{C})$. A functorial factorization amounts to specifying two endofunctors L and R from $\mathrm{Ar}(\mathbf{C})$ to itself such that (a) the composite $R \circ L$ is the identity functor and (b) the functors R and L have image in the specified subcategories of $\mathrm{Ar}(\mathbf{C})$. In other words, given a morphism f in $\mathbf{C}$, the morphisms $L(f)$ and $R(f)$ lie in the two distinguished subcategories and their composite in the identity.

Model structures

Definition 4.1.3.7. If $\mathbf{C}$ is a category, then a *model structure* on $\mathbf{C}$ consists of specifying three classes of morphisms Cof, Fib and W in $\mathbf{C}$ called the cofibrations, fibrations and weak equivalences that are each (a) stable under composition and (b) contain the identity map on any object satisfying the following axioms:

1. (2 out of 3) Given a pair of composable morphisms f and g, if any two of f, g or gf are weak equivalences, then so is the third.
2. (Retracts) Any retract of a cofibration, fibration or weak equivalences is again a cofibration, fibration or weak equivalence.
3. (Lifting) Say that a morphism is an acyclic cofibration (resp. acyclic fibration) if it is simultaneously a cofibration (resp. fibration) and weak equivalence. A trivial cofibration has the left lifting property with respect to any fibration; a trivial fibration has the right lifting property with respect to any cofibration.
4. (Functorial factorizations) Any morphism can be functorially factored either as a trivial cofibration followed by a fibration and as a cofibration followed by a trivial fibration.

Example 4.1.3.8. Any complete and cocomplete category $\mathbf{C}$ admits a model structure. One can take the isomorphisms as one of the sets of morphisms, and then all morphisms as the other two sets of morphisms.

Definition 4.1.3.9. A category $\mathbf{C}$ is a *model category* if is complete, cocomplete and admits a model structure.

Example 4.1.3.10. If $\mathbf{C}$ and $\mathbf{D}$ are model categories, then $\mathbf{C}\times\mathbf{D}$ has a model structure: the "product" model structure.

Remark 4.1.3.11. The terminology *model category* was introduced by Quillen in his short book "Homotopical algebra"; it is an abbreviation of "category of models for a homotopy theory." It is important to remember that we have just given one particular axiomatization, and the one we have given differs from that given by Quillen. In the literature, there are a number of minor variations on the above axioms. Quillen only required his model categories have finite limits and colimits, and did not require factorizations to be functorial. In most of the examples considered by the theory, the stronger axioms we have chosen are satisfied. Moreover, the axioms above overdetermine the cofibrations and fibrations as the following lemma shows.

Lemma 4.1.3.12. *If $\mathbf{C}$ is a category with a model structure, then a morphism $f : A \to B$ is a cofibration (resp.acyclic cofibration) if and only if it has the left lifting property with respect* *to acyclic fibrations (resp. fibrations). Corresponding statements hold for fibrations and trivial fibrations.*

Proof. One implication is immediate from the definitions. Suppose $f : A \to B$ is a morphism, which we assume has the left lifting property with respect to acyclic fibrations. In that case, we can factor $f = pi$ as a cofibration followed by an acyclic fibration, say $i : A \to C$ and $p : C \to B$. Diagramatically, we are given a diagram of the form:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & C \arrow[d, "p"] \\ B \arrow[r, "="] & B \end{tikzcd}$$

so there is a morphism $j : B \to C$ making the resulting triangles commute. In that case, we can unfold the above diagram to one of the following form:

$$\begin{tikzcd} A \arrow[r, "="] \arrow[d, "f"] & A \arrow[r, "="] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, "j"] & C \arrow[r, "p"] & B, \end{tikzcd}$$

which exhibits f as a retraction of i. As a retract of a cofibration, it follows that f is also a cofibration. The other statements are established in a similar fashion. □

One important consequence of the above lemma is that if we specify the (co)fibrations and weak equivalences in a model category, then the third class of morphisms is uniquely determined.

Corollary 4.1.3.13. *Cofibrations and trivial cofibrations are stable under pushouts (i.e., cobase change), while fibrations and trivial fibrations are stable by pullback (i.e., base change).*

Proof. Suppose we have a pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[d, "i'"] \\ B \arrow[r] & D \end{tikzcd}$$

and a lifting problem of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, "i'"] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

If i is a cofibration, then whenever f is an acyclic fibration our lifting problem has a solution, i.e., there exists a morphism $B \to X$ making the resulting triangles commute. In that case, since D is a push-out, and we are given morphisms $B \to X$ and $C \to X$ whose composites to A agree, there exists a unique morphism $D \to X$ making all resulting diagrams commute. This morphism provides the required solution to the lifting problem. The other cases are established similarly. □

4.1.3.14 (Fibrant and cofibrant objects). If $\mathbf{C}$ is a model category, then $\mathbf{C}$ necessarily has initial and final objects. We will provisionally write $\emptyset$ for the initial object and $*$ for the final object. An object X in $\mathbf{C}$ is called *cofibrant* if the map from the initial object to X is a cofibration and *fibrant* if the map from X to the final object is a fibration.

4.1.3.15 (Fibrant and cofibrant replacements). If $\mathbf{C}$ is a model category, then by the factorization axioms the map $\emptyset \to X$ can be (functorially) factored as a cofibration followed by an acyclic fibration or as an acyclic cofibration followed by fibration. In the former case, our factorization reads:

$$\emptyset \longrightarrow QX \longrightarrow X;$$

here QX is cofibrant while the map $QX \to X$ is an acyclic fibration, in particular a weak equivalence. We will refer to QX as a (functorial) cofibrant replacement for X.

Similarly, if we factor $X \to *$ as an acyclic cofibration followed by a fibration, we get

$$X \longrightarrow RX \longrightarrow *;$$

here RX is fibrant, while $X \to RX$ is an acyclic cofibration, in particular a weak equivalence. We will refer to RX as a (functorial) fibrant replacement for X.

4.1.4 Model structures on chain complexes

Our next goal is to analyze a number of examples. Let us fix some terminology before we move forward. Suppose R is a commutative ring. By a cochain complex of R-modules, we will mean a $\mathbb{Z}$-graded R-module $M^\bullet$ together with maps $d^i : M^i \to M^{i+1}$ such that $d^{i+1} \circ d^i = 0$ (we will say that the differential has degree $+1$). By a chain complex of R-modules, we will mean a $\mathbb{Z}$-graded R-module $M_\bullet$ together with maps $d_i : M_i \to M_{i-1}$ such that $d_i \circ d_{i-1} = 0$ (we say that differential has degree -1). We will say that $M^\bullet$ is bounded above if there exists an integer j such that $M^i = 0$ whenever $j > i$.

We write $Ch_R^{\geq 0}$ for the chain complexes situated in non-negative degrees. Equivalently, these can be thought of cochain complexes situated in non-positive degrees. Likewise, we write $Ch^R_{\geq 0}$ for the cochain complexes situated in non-negative degrees.

Theorem 4.1.4.1 (Projective model structure)**.** *Let R be a commutative ring. The category $Ch_R^{\geq 0}$ admits a model structure where*

1. *weak equivalences are quasi-isomorphisms;*
2. *cofibrations are monomorphisms with degreewise projective cokernel;*
3. *fibrations are surjections in positive (≥ 0) degrees.*

Theorem 4.1.4.2 (Injective model structure)**.** *Let R be a commutative ring. The category $Ch^R_{\geq 0}$ admits a model structure where*

1. *weak equivavalences are quasi-isomorphisms;*
2. *fibrations are epimorphisms with degreewise injective kernel;*
3. *cofibrations are monomorphisms in positive degrees.*

Theorem 4.1.4.3 (Injective model structure; sheaves)**.** *If $(X, \mathscr{O}_X)$ is a ringed space, then the category $Ch^{\mathscr{O}_X}_{\geq 0}$ of non-negatively graded cochain complexes of $\mathscr{O}_X$-modules admits a model structure where*

1. *weak equivalences are quasi-isomorphisms;*
2. *fibrations are epimorphisms with degreewise injective kernel;*
3. *cofibrations are monomorphisms in positive degrees.*

The easy-to-verify axioms

Note that Theorem 4.1.4.2 is a special case of Theorem 4.1.4.3 taking X to be the 1-point space with sheaf of rings R. A number of the constructions involved in establishing Theorems 4.1.4.1 and 4.1.4.3 are so similar that we will establish them simultaneously.

First, observe that $Ch_R^{\geq 0}$ and $Ch_{\geq 0}^{\mathscr{O}_X}$ have all small limits and colimits as these are computed degreewise and these statements are inherited from the underyling categories (either R-modules or $\mathscr{O}_X$-modules).

Likewise, since epimorphisms, monomorphisms and isomorphisms are stable under composition and contain the identities it follows that weak equivalences are stable under composition in either model structure, and that cofibrations in the injective model structure and the fibrations in the projective model structure are stable under composition.

Suppose $f : A \to B$ and $g : B \to C$ are two fibrations in the injective model structure. The composite gf is necessarily also surjective in positive degrees and it remains to check that it has injective kernel. To this end, one checks that there are short exact sequences of the form:

$$0 \longrightarrow \ker(f)_i \longrightarrow \ker(gf)_i \longrightarrow \ker(g)_i \longrightarrow 0.$$

Since $\ker(f)_i$ is injective, this short exact sequence is split. Since $\ker(g)_i$ is injective, it follows that $\ker(gf)_i$ is the direct sum of injective $\mathscr{O}_X$-modules, and is thus itself injective.

Likewise, suppose $f : A \to B$ and $g : B \to C$ are two cofibrations in the projective model structure. The composite gf is necessarily also injective in positive degrees and it remains to check that it has degreewise projective cokernel. In this case, diagram chasing gives a short exact sequence of the form

$$0 \longrightarrow \operatorname{coker}(f)_i \longrightarrow \operatorname{coker}(gf)_i \longrightarrow \operatorname{coker}(g)_i \longrightarrow 0.$$

Once again, this short exact sequence is split and thus $\operatorname{coker}(gf)_i$ is a sum of projective modules and thus projective.

2 **out of** 3. To check that 2 out of 3 property for weak equivalences, simply observe that a morphism $f : A \to B$ of chain complexes is a quasi-isomorphism if the induced maps on homology modules are isomorphisms of R-modules. Now, by functoriality of homology, $H_i(gf) = H_i(g) \circ H_i(f)$, so the 2 out of 3 property for weak equivalences follows from the 2 out 3 property for isomorphisms of R-modules.

Retracts. To see that retracts of cofibrations, weak equivalences or fibrations are again of the same sort, we begin by observing that retracts of projective modules or injective sheaves are again automatically projective or injective and likewise straightforward diagram chasing shows that retracts of epi or monomorphisms are again epi or monomorphisms. Likewise, by functoriality of homology, if f is a retract of g, then $H_i(f)$ is a retract of $H_i(g)$ for each i. Thus, if $H_i(g)$ is an isomorphism for each i, then $H_i(f)$ must again be an isomorphism as retracts of isomorphisms are isomorphisms.

The lifting axiom

Now, we verify the lifting axioms; these are essentially induction arguments using the extension properties of injective or projective modules.

Lemma 4.1.4.4. *Given a commutative diagram of objects in $Ch_{\geq 0}^{R}$ of the form:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is a monomorphism with degreewise projective cokernel and f is simultaneously an epimorphism in positive degrees and a quasi-isomorphism, there exists a morphism $\tilde{f} : B \to X$ solving the lifting problem.

Proof. First assume that i is a monomorphism with degreewise projective cokernel and f is simultaneously an epimorphism in positive degrees and a quasi-isomorphism.

Since i is a monomorphism, there is a short exact sequence of chain complexes of the form

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow \operatorname{coker}(i) \longrightarrow 0,$$

where $\operatorname{coker}(i)$ is a chain complex of projective modules. Since $\operatorname{coker}(i)$ is degreewise projective, these short exact sequences split degreewise. In particular, $B_0 \cong A_0 \oplus \operatorname{coker}(i)_0$.

Now, f is an epimorphism in positive degrees and the map $f_0 : H_0(X) \to H_0(Y)$ is an isomorphism by assumption. We thus have a commutative diagram of the form

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & H_0(X) \arrow[r] \arrow[d] & 0 \\ Y_1 \arrow[r] & Y_0 \arrow[r] & H_0(Y) \arrow[r] & 0. \end{tikzcd}$$

A diagram chase (a.k.a., the four lemma) shows that $X_0 \to Y_0$ must also be surjective in this situation.

In that case, we define $\tilde{f}_0 : B_0 \cong A_0 \oplus \operatorname{coker}(i)_0 \to X_0$ as follows. The diagram gives a morphism $A_0 \to X_0$ and the first component of $B_0 \to X_0$ is this morphism. The diagram gives a morphism $B_0 \to Y_0$, which in conjunction with the splitting of B_0 yields a morphism $\operatorname{coker}(i)_0 \to Y_0$. Since $\operatorname{coker}(i)_0$ is projective and $X_0 \to Y_0$ is surjective, we can choose a morphism $\operatorname{coker}(i)_0 \to X_0$ lifting this morphism.

Now, we work inductively to define the morphism B_l to X_l for $l > 0$. To this end, observe that what we have shown above yields an exact sequence of chain complexes

$$0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow 0,$$

where K is an acyclic complex. As above, we can choose splittings $B_l \cong A_l \oplus \operatorname{coker}(i)_l$ where $\operatorname{coker}(i)_l$ is projective and the morphisms $A_l \to X_l$ are specified by commutativity of the diagram.

The map $X_l \to Y_l$ is surjective, and we have $\mathrm{coker}(i)_l \to Y_l$ determined by the splitting and commutativity of the diagram.

As such, we can always pick a lift $\tilde{f}_l : \mathrm{coker}(i)_l \to X_l$ lifting the given one. Without loss of generality, we assume that $\tilde{f}_j : B_j \to X_j$ commutes with differential on X for $j < l$, and is likewise compatible □

Lemma 4.1.4.5. *Given a commutative diagram in $Ch^R_{\geq 0}$ of the form:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where i is simultaneously a monomorphism in positive degrees and a quasi-isomorphism and f is a degreewise epimorphism with injective kernel, the resulting lifting problem has a solution.

Proof. Since f is a degreewise epimorphism with injective kernel, if $\ker(f)$ is the kernel, then $\ker(f)^i$ is an injective module for each i. Therefore, there are splittings of the form $X^i \cong \ker(f)^i \oplus Y^i$ by the property of injectivity.

Now, we are working with cochain complexes, so the differential has degree $+1$, i.e., $H^0(A)$ is a sub-module of A^0 equal to $\ker(d^0)$. We therefore have a commutative diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & H^0(A) \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d] & A^1 \arrow[d] \\ 0 \arrow[r] & H^0(B) \arrow[r] & B^0 \arrow[r] & B^1. \end{tikzcd}$$

By the 4-lemma since the map $A^1 \to B^1$ is injective and the map $H^0(A) \to H^0(B)$ is an isomorphism, we conclude that $A^0 \to B^0$ must also be injective. By the injectivity of $\ker(f)^0$, it follows that we can extend the given map $A^0 \to \ker(f)^0$ to a morphism $B^0 \to \ker(f)^0$. Our candidate lift $\tilde{f}^0$ is then this chosen morphism on one factor and the given morphism $B^0 \to Y^0$ on the other factor. Working inductively as above gives the required lift. □

In order to establish the other lifting axiom, we introduce some helpful results. If A is a non-negatively graded chain-complex, then assigning to A the term A_n determines a (family of) functor(s) $Ch_R^{\geq 0} \to \mathrm{Mod}_R$. These functors admits left adjoints: indeed, if for an R-module M we define $D_n(M)$ to be the chain complex with $D_n(M)_i = 0$ if $i \neq n, n-1$ and equal to M when $i = n, n-1$; the unique potentially non-trivial differential $D_n(M) \to D_{n-1}(M)$ is the identity map.

Lemma 4.1.4.6. *If R is a commutative ring, and M is an R-module, then the map*

$$\mathrm{Hom}_{Ch_R^{\geq 0}}(D_n(M), N) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M, N_n)$$

is an isomorphism.

Corollary 4.1.4.7. *If R is a commutative ring, and P is a projective R-module, then $D_n(P)$ is a projective object in $Ch_R^{\geq 0}$, i.e., if $M \to N$ is any epimorphism of chain complexes, then given a morphism $D_n(P) \to N$, there exists a lift $D_n(P) \to M$. Likewise, arbitrary direct sums of $D_n(P)$ are again projective chain complexes.*

Proof. If $M \to N$ is an epimorphism, then $M_n \to N_n$ is an epimorphism. A morphism $D_n(P) \to M$ corresponds to a morphism $P \to M_n$ by the preceding lemma. Since P is projective and $M_n \to N_n$ is an epimorphism, there exists a lift $P \to N_n$, which again by the preceding lemma corresponds to a morphism $D_n(P) \to N$. The second statement follows from the first. □

Henceforth, if A is a chain complex, we write $Z(A)$ for the subcomplex of cycles, i.e., the graded abelian group lying in the kernel of the differential, and $B(A)$ for the boundaries, i.e., the image of the differential on A. Now, suppose we have a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is an acyclic cofibration. This means that $i : A \to B$ is degreewise injective with cokernel $\mathrm{coker}(i)$ a complex whose homology vanishes and has degreewise projective terms.

Lemma 4.1.4.8. *If P is an acyclic object of $Ch_R^{\geq 0}$ such that each P_n is projective, then each module $Z_k(P)$ is projective and P is isomorphic to $\oplus_{k>0} D_k(Z_{k-1}(P))$. In particular, P is itself a projective chain complex.*

Proof. If A is a chain complex, then the differential gives a map $d : A_{n-1} \to A_{n-2}$. For any integer $n \geq 1$, define $\tau_{\geq n}A$ to be the complex that agrees with A in degrees $\geq n$ and consists of im d in degree $n-1$. The inclusion $\tau_{\geq n}A \to A$ is a morphism of complexes that induces isomorphisms on homology in degrees $\geq n$, but $\tau_{\geq n}A$ has no homology in degrees $< n$ by construction. In fact, if we define $\tau_{\leq n}A$ to be the cokernel of $\tau_{\geq n}A \to A$, then we have written A as an extension of complexes whose homology is concentrated in degrees $\geq n$ and $< n$.

Now, assume P is an acyclic complex whose elements are projective. In that case, since $H_0(P) = 0$, it follows that $P_1 \to P_0$ is surjective. Because P_0 is projective, we can split this surjection and fix an isomorphism $P_1 \cong \ker(d_1) \oplus P_0$. As a summand of a projective module, $\ker(d_1)$ is again projective. Since $P_1 \to P_0$ is surjective, we also see that $\tau_{\geq 1}P = P$. Putting these two facts together, observe that we get a decomposition

$$P \cong \tau_{\geq 1}P \cong \tau_{\geq 2}P \oplus D_1(P_0) = \tau_{\geq 2}P \oplus D_1(Z_0(P)).$$

Now, we proceed by induction. Since P is acyclic, $\ker(d_1) = im(d_2)$ and allows the induction to proceed. The second statement follows from Corollary 4.1.4.7. □

Putting these facts together, we obtain the relevant lifting statement.

Lemma 4.1.4.9. *Given a diagram of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where f is an epimorphism in positive degrees and i is a monomorphism with projective cokernel and a quasi-isomorphism, there exists a lift $B \to X$ making all triangles commute.

Proof. Since $i : A \to B$ is a monomorphism with projective cokernel and simultaneously a quasi-isomorphism, there is a short exact sequence of chain complexes of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow \operatorname{coker}(i) \longrightarrow 0$$

where $\operatorname{coker}(i)$ is an acyclic chain complex each of whose terms is projective. Since $\operatorname{coker}(i)$ is an acyclic complex of projectives, it follows that $\operatorname{coker}(i)$ is a projective chain complex. Since the map $B \to \operatorname{coker}(i)$ is an epimorphism, we can therefore choose a splitting

$$B \cong A \oplus \operatorname{coker}(i)$$

as complexes. Now, the map $A \to X$ is specified by the diagram, and we get a map $\operatorname{coker}(i) \to Y$ from the morphism $B \to Y$. Even though the map $f : X \to Y$ is not an epi-morphism, it *is* an epimorphism in degrees > 0 and thus an epi-morphism onto its image. It follows that we can lift the map $\operatorname{coker}(i) \to Y$ to a morphism $\operatorname{coker}(i) \to X$, and any such choice yields the required lift. $\square$

Functorial factorizations and the small object argument

To complete the proof of the model category axioms, we need to build our functorial factorizations. While this could be achieved by purely elementary inductive arguments, we will give a more involved argument that works in a number of situations.

Recall that an R-module M is called *compact* if $\mathrm{Hom}_{\mathrm{Mod}_R}(M, -)$ commutes with filtered colimits. We observed earlier that the compact modules are precisely the finitely presented R-modules, but this definition of compactness makes sense in other cocomplete categories as well. This notion was sometimes called "sequential smallness" before the present terminology was adapted, and the argument that follows is called the "small object argument" as a consequence.

Lemma 4.1.4.10. *A chain complex $A \in Ch_R^{\geq 0}$ is compact if and only if each A_i is finitely presented and at most finitely many A_i are non-zero.*

To build the required factorization of a map $f : X \to Y$, we will simply build a new complex out of X by attaching "cells" in a suitable fashion to guarantee that lifts exist. Here is the construction more generally.

Suppose we have a *set* G of maps $\{f_i : A_i \to B_i\}_{i \in I}$ in some category $\mathbf{C}$ that we assume is cocomplete. Now, fix a morphism $f : X \to Y$. For each i, there is a set $S(i)$ indexing pairs (g, h)

fitting into a commutative diagram of the form:

$$\begin{tikzcd} A_i \arrow[r, "g"] \arrow[d, "f_i"] & X \arrow[d, "f"] \\ B_i \arrow[r, "h"] & Y. \end{tikzcd}$$

In that case, since $\mathbf{C}$ is cocomplete, it follows that $\sqcup_{i\in I} \sqcup_{(g,h)\in S(i)} A_i$ and $\sqcup_{i\in I} \sqcup_{(g,h)\in S(i)} B_i$ are both objects of $\mathbf{C}$ and we get a diagram of the form

$$\sqcup_{i\in I} \sqcup_{(g,h)\in S(i)} B_i \longleftarrow \sqcup_{i\in I} \sqcup_{(g,h)\in S(i)} A_i \longrightarrow X,$$

and we define $F_1(G,p)$ as the colimit of the above diagram. In essence, we have built a new object by gluing in all possible lifts. By the universal property of colimits, there is a map $F_1(G,p) \to Y$ as well such that p factors as:

$$X \xrightarrow{i_1} F_1(G,p) \xrightarrow{p_1} Y$$

Note that by functorality of colimits, this construction is evidently functorial in p as well. The new map $F_1(G,p) \to Y$ need not have lifts along the maps in G, so we repeat the construction of $F_1(G,p)$ with X replaced by $F_1(G,p)$ to obtain a new space $F_2(G,p)$. Continuing inductively in this way, we obtain a sequence of spaces $F_i(G,p)$ and there is a factorization

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[d, "p"] & F_1(G,p) \arrow[r, "i_2"] \arrow[d, "p_2"] & F_2(G,p) \arrow[r, "i_3"] \arrow[d, "p_3"] & \cdots \\ Y \arrow[r, "="] & Y \arrow[r, "="] & Y \arrow[r, "="] & \cdots \end{tikzcd}$$

We then let $F_\infty(G,p)$ be the colimit $\operatorname{colim}_n F_n(G,p)$. Once again, the universal property of colimits shows that there is an induced factorization of p

$$X \xrightarrow{i_\infty} F_\infty(G,p) \xrightarrow{p_\infty} Y,$$

and the construction is evidently functorial in p as well. We now observe that because $F_\infty(G,p)$ is constructed as a directed colimit, if we know that the sources of the morphisms f_i are compact, then p_∞ has the right-lifting property with respect to morphisms in G. The above construction, in conjunction with the following result is what is usually called the "small object argument."

Proposition 4.1.4.11. *If for each object $f_i \in G$ the object $\mathbf{C}$ is compact, then the map $F_\infty(G,p)$ has the right lifting property with respect to every map in G.*

Proof. Consider a diagram of the form

$$\begin{tikzcd} A_i \arrow[r, "q"] \arrow[d, "f_i"] & F_\infty(G,p) \arrow[d, "p_\infty"] \\ B_i \arrow[r] & Y \end{tikzcd}$$

Since A_i is compact

$$\mathrm{Hom}(A_i, F_\infty(G,p)) = \mathrm{Hom}(A_i, \mathrm{colim}_n F_n(G,p)) = \mathrm{colim}_n \mathrm{Hom}(A_i, F_n(G,p)).$$

In that case, there exists some integer j such that q arises from a map $A_i \to F_j(G,p)$. In that case, a lift of f_i exists in $F_{j+1}(G,p)$ by construction and composing with the map $F_{j+1}(G,p) \to F_\infty(G,p)$ gives the required lift in the diagram. □

Note that any module is a filtered colimit of its compact sub-objects. Likewise, any chain complex can be written as a filtered colimit of its compact sub-objects. In fact, there are some very simple complexes out of which any complex of R-modules can be built. We already defined the "disc" chain complex $D_n(R)$. Consider now the sphere chain complex

$$S^{n-1}(R)_j := \begin{cases} R & \text{if } j = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

There is an evident inclusion map $S^{n-1}(R) \to D_n(R)$ which is the identity in degree $n-1$ and trivial in degree 0. These chain complexes are sequentially small. By convention, we write $D_0(R)$ for R in degree 0 and $S^{-1}(R)$ is the zero complex, while $j_0 : S^{-1}(R) \to D_0(R)$ is the unique map. By construction, the complexes $S^{n-1}(R)$ and $D_n(R)$ are both compact. Note that the map $S^{n-1}(R) \to D_n(R)$ is evidently a cofibration for every $n \geq 0$ and likewise, the map $0 \to D_n(R)$ is an acyclic cofibration for every $n \geq 0$. It follows that any map of complexes that is a fibration has the right lifting property with respect to $0 \to D_n(R)$ for every $n \geq 0$. Likwise, any map that is an acyclic cofibration has the right lifting property with respect to all the maps $S^{n-1}(R) \to D_n(R)$ for all $n \geq 0$. In fact, these cofibrations and acyclic cofibrations *generate* all cofibrations or acyclic cofibrations in the following sense: every cofibration can be built from these forming small colimits (coproducts and coequalizers) and then taking retracts. We leave this is a straightforward exercise in diagram chasing.

Lemma 4.1.4.12. *A morphism $f : X \to Y$ in $Ch_R^{\geq 0}$ has the right-lifting property with respect to (acyclic) cofibrations if and only if it has the right-lifting property with respect to the set of morphisms $S^{n-1}(R) \to D_n(R), n \geq 0$ (resp. $0 \to D_n(R)$, $n \geq 0$).*

Corollary 4.1.4.13. *If $f : X \to Y$ is a morphism in $Ch_R^{\geq 0}$, then we may factor f either as a trivial cofibration followed by a fibration or as a cofibration followed by a trivial fibration.*

Proof. For the first assertion, we apply the small object argument with respect to the set of morphisms $S^{n-1}(R) \to D_n(R)$, $n \geq 0$. In that case, we get a functorial factorization $X \to X' \to Y$. The preceding lemma guarantees that $X' \to Y$ is a fibration, so it remains to check that $X \to X'$ is an acyclic cofibration. We see this by investigating the construction of X' via the small object argument. Indeed, the space X' is obtained from X by adding direct sums of copies of R in various degrees and X' is a transfinite composition of such things. The other case is similar. □

4.1.5 The injective model structure

We return now to the proof of the injective model structure on $\mathscr{O}_X$-modules. We have to establish the other lifting axiom and the existence of functorial factorizations. The proof of the other lifting axiom is analogous to the projective case.

Lemma 4.1.5.1. *Given a commutative diagram in $Ch_{\geq 0}^{\mathscr{O}_X}$ of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

where p is a quasi-isomorphism and an epimorphism with injective kernels and i is a monomorphism in positive degrees, there exists a lift.

Proof. Since p is an epimorphism with injective kernels, it follows that its kernel is an acyclic complex that is termwise injective. Let us analyze such complexes first; we claim such a chain complex is necessarily injective. Consider the exact sequence

$$0 \longrightarrow K^0 \longrightarrow K^1 \longrightarrow \cdots .$$

Since K^0 is injective and $K^0 \to K^1$ is a monomorphism, it necessarily splits, so we write $K^1 \cong K^0 \oplus K^1/K^0$.

Write $D^n(M)$ for the complex of $\mathscr{O}_X$-modules that is M in degree n and $n-1$ with identity map as differential. Likewise, write $\tau^{\geq i}K$ for the complex that agrees with K in degrees $\geq i$ and is isomorphic to $K^{i-1}/im(d^{i-2})$ in degree 1. There is an evident map $K \to \tau^{\geq i}K$ that induces an isomorphism on cohomology in degrees $\geq i$.

In that case, we have written K as the sum of $D^1(\ker(d^0)) \oplus \tau_{\geq 2}K^{\bullet}$. Note that $D^1(\ker(d^0))$ is itself an injective chain complex, and proceeding inductively one concludes. The remainder of the proof is dual to the case of the projective model structure. $\square$

To finish, we need to establish the existence of functorial factorizations. This is slightly more involved than in the projective case since it is more difficult to write down generating cofibrations and trivial cofibrations. In fact, one way to proceed is simply to establish that such a set exists by means of suitable cardinality counts.

Sketch this

4.2 The homotopy category of a model category

Now, we discuss the homotopy category of a model category; the homotopy category of chain complexes will be what is typically called the derived category. We then talk a bit about derived functors. The presentation below is standard.

4.2.1 The homotopy category via fractions

Suppose $\mathbf{C}$ is a model category and let W be the set of weak equivalences. We write $\mathrm{Ho}(\mathbf{C})$ for the category defined as follows. Form the "free category" $F(\mathbf{C}, W^{-1})$ on the arrows of $\mathbf{C}$ and the reversals of the arrows of W. The objects of the free category are the same as the objects in $\mathbf{C}$. A morphism in $F(\mathbf{C}, W^{-1})$ is a finite string of composable arrows $f_1, \ldots, f_n$ where each f_i is either a morphism in $\mathbf{C}$ or the reversal of a morphism in W. The empty string is the identity. We define $\mathrm{Ho}(\mathbf{C})$ to be the quotient of $F(\mathbf{C}, W^{-1})$ by the relations $id_A = (id_A)$ for all objects of A, $f, g = (g \circ f)$ for all composable arrows in $\mathbf{C}$, and the identifying (w, w^{-1}) with the identity on domain, and (w^{-1}, w) with the identity on the codomain.

A priori, it is not clear that $\mathrm{Ho}(()\mathbf{C})$ is even a category since it is not clear that the hom objects are small. Nevertheless, if it is, then there is a functor $u : \mathbf{C} \to \mathrm{Ho}(()\mathbf{C})$ that is the identity on objects and sends morphisms in W to isomorphisms. In fact, $\mathrm{Ho}(()\mathbf{C})$ is initial amongst categories in which W has been inverted in the following sense.

Lemma 4.2.1.1. *Suppose* $\mathbf{C}$ *is a model category with weak equivalences* W*.*

1. *If* $\mathbf{D}$ *is a category and* $F : \mathbf{C} \to \mathbf{D}$ *is a functor such that* $F(w)$ *is an isomorphism for each* $w \in W$*, then there exists a unique functor* $\mathrm{Ho}(F) : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ *such that* $\mathrm{Ho}(F) \circ \gamma = F$*.*
2. *If* $v : \mathbf{C} \to \mathbf{E}$ *is a functor that enjoys the universal property of Point (1), then there is a unique isomorphism* $F : \mathrm{Ho}(\mathbf{C}) \to \mathbf{E}$ *such that* $F \circ u = v$*.*
3. *The correspondence of Point (1) induces an isomorphism of categories between the categories of functors* $\mathrm{Ho}(()\mathbf{C}) \to \mathbf{D}$ *(and natural transformations) and the category of functors* $\mathbf{C} \to \mathbf{D}$ *that take* $\mathbf{W}$ *into isomorphisms and natural transformations.*

Proof. For the first point, if $\mathrm{Ho}(\mathbf{C})$ is defined as a quotient of the free category as above, then the factorization is defined by F on objects and $F(w^{-1}) = F(w)^{-1}$. That this construction defines a functor is immediate from the definition as a quotient category. The remainder of the statements can be found in [**?**]. □

Lemma 4.2.1.2. *If* $\mathbf{C}$ *is a model category, and* $\mathbf{C}_c$ *is the sub-category of cofibrant objects, then*

$$\mathrm{Ho}(\mathbf{C}_c) \longrightarrow \mathrm{Ho}(\mathbf{C})$$

is an equivalence.

Proof. The inverse functor is constructed by cofibrant replacement; we leave the details as an exercise. □

4.2.2 The homotopy category of a model category

In this section, we follow the standard construction of the homotopy category of a model category. The basic idea is to identify the category $\mathrm{Ho}(\mathbf{C})$ as equivalent to a suitable quotient of the category of cofibrant and fibrant objects by imposing a suitable homotopy equivalence relation on morphisms.

Left homotopy

If A is an object of $\mathbf{C}$, then $A \sqcup A$ exists and comes equipped with a map

$$A \sqcup A \longrightarrow A.$$

We may functorially factor this morphism as a cofibration followed by an acyclic fibration:

$$A \sqcup A \longrightarrow A \wedge I \longrightarrow A;$$

the object $A \wedge I$, will be called a very good cyclinder object for A (note that it is not obtained as a product with anything). We will write $i_0 : A \to A \wedge I$ and $i_1 : A \to A \wedge I$ for the "inclusions of the ends." More generally, by a *cylinder object* for A we will mean *any* factorization of $A \sqcup A \to A$ as a cofibration $A \sqcup A \to A'$ followed by a weak equivalence $A' \to A$.

Definition 4.2.2.1. If $f, g : A \to X$ are two maps in $\mathbf{C}$, then we will say that f and g are *left-homotopic* if there exists a cylinder object A' for A together with a morphism $H : A' \to X$ such that $f = H \circ i_0$ and $g = H \circ i_1$; a choice of such an H will be called a *left-homotopy*.

Note that id_A factors through $A \to A \sqcup A \to A$ in two ways. If A' is any cyclinder object for A, then the composites $A \to A' \to A$ are the identity map and it follows from the 2 out of 3 property of weak equivalences $i_0, i_1 : A \to A'$ are weak equivalences as well. If A is cofibrant, then $A \to A \sqcup A$ is again a cofibration as a cobase change of a cofibration; in that case, the composite $A \to A \sqcup A \to A'$ is also a cofibration. In other words, we have established the following result.

Lemma 4.2.2.2. *If A is cofibrant, and A' is a cyclinder object for A, then the two maps $A \to A'$ are acyclic cofibrations.*

Lemma 4.2.2.3. *If f and g are left homotopic, then f is a weak equivalence if and only if g is a weak equivalence.*

Proof. If f is a weak equivalence, then $f = Hi_0$ by definition, and we observed above that i_0 is a weak equivalence, so by 2 out of 3, it follows that H must be a weak equivalence as well. In that case, $g = Hi_1$ is again a weak equivalence. □

Lemma 4.2.2.4. *If $f, g : A \to X$ are left homotopic maps and $h : X \to Y$ is a map, then hf and hg are left-homotopic.*

Proof. Exercise. □

Lemma 4.2.2.5. *If A is cofibrant, then left homotopy is an equivalence relation on* $\mathrm{Hom}(A, X)$.

Proof. Suppose $f : A \to X$ is a map, that f is left-homotopic to f follows from the definitions. To see that left homotopy is symmetric, consider the swap map $sw : A \sqcup A \to A \sqcup A$ that switches the two factors. Since $f \sqcup g = (g \sqcup f) \circ sw$, the required symmetry follows. Now, let us establish transitivity. Suppose $H : A \wedge I \to X$ is a left homotopy between f and g and $H' : A \wedge I' \to X$ is a left homotopy between g and h; we will assume that $A \wedge I$ and $A \wedge I'$ are very-good cylinder objects for A. In that case, consider the map $i_1 : A \to A \wedge I'$ and the map $i_0 : A \to A \wedge I$. Since A is cofibrant, both of these maps are acyclic cofibrations. The pushout of the diagram

$$A \wedge I \xrightarrow{i_0} A \xrightarrow{i_1} A \wedge I'$$

yields a space $A \wedge I''$ and the universal property of the coproduct gives a map $A \wedge I'' \to A$. Since pushouts of cofibrations and trivial cofibrations are stable by cobase-change, it follows that $A \wedge I' \to A \wedge I''$ is again a trivial cofibration and thus that $A \to A \wedge I''$ is a trivial cofibration, and one checks that $A \wedge I''$ is a cyclider object for A as well.

Now, by construction, the maps H and H' yield a map $H'' : A \wedge I'' \to X$. This doesn't yield a left-homotopy but it can be factored to obtain one. □

Right homotopy

Dual to the above constructions, one can also analyze *right homotopy* when considering maps with fibrant target. If X is an object, then $X \times X$ exists, and the map $X \to X \times X$ that one obtains from the universal property is called the diagonal map. By a very good path object for X we will mean a factorization of $X \to X \times X$ of the form

$$X \longrightarrow X^I \longrightarrow X \times X$$

where the first map is an ayclic cofibration and the second map is a fibration. More generally, by a path object for X we will mean any factorization $X \to X' \to X \times X$ where $X \to X'$ is a weak equivalence and $X' \to X \times X$ is a fibration. The factorization axiom in a model category guarantees that there always exists a very good path object for X.

As above, we can define right homotopy and establish that right homotopy is an equivalence relation on $\mathrm{Hom}(A, X)$ whenever X is fibrant.

Proposition 4.2.2.6. *If $f, g : A \to X$ are maps, then*

1. *if A is cofibrant and f and g are left-homotopic, then f and g are right homotopic as well.*
2. *if X is fibrant and f and g are right-homotopic, then f and g are left homotopic as well.*

Proof. The idea here is to use the lifting axioms to build the required homotopy. We prove the first point, leaving the second as an exercise. Choose a very good cylinder object $A \wedge I \xrightarrow{j} A$ and a left-homotopy $H : A \wedge I \to X$ between f and g. Likewise, choose a path object X^I for X so that $X^I \to X \times X$ is a fibration. In that case, the map $f : A \to X$ yields $A \to X \to X^I$. We then have a diagram of the form:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X^I \arrow[d] \\ A \wedge I \arrow[r, "{fj,H}"] & X \times X, \end{tikzcd}$$

where the left map is by assumption an ayclic cofibration and $X^I \to X$ is a fibration. In that case, there exists a lift $\tilde{H} : A \wedge I \to X^I$ in the diagram. The composite $\tilde{H} i_1 : A \to X^I$ provides the desired right homotopy between f and g. $\square$

Weak equivalence and homotopy equivalence

It follows from the above result that the notions of left and right homotopy agree on $\mathrm{Hom}(A, X)$ whenever A is cofibrant and X is fibrant. We will refer to the resulting equivalence relation as the homotopy equivalence relation. If A and X are both cofibrant and fibrant, then it makes sense to say that a map $f : A \to X$ to be a homotopy equivalence if there exists a homotopy inverse, i.e., a morphism $g : X \to A$ such that the two composites are homotopic to the identity.

Proposition 4.2.2.7. *If A and X are both cofibrant and fibrant, then a map $f : A \to X$ is a weak equivalence if and only if it is a homotopy equivalence in the sense above.*

Proof. Suppose $f : A \to X$ is a weak equivalence. In that case, we can factor f as a cofibration followed by an acyclic fibration. In that case, the first map is an ayclic cofibration as well by 2 out of 3. $\square$

We can then define $\mathrm{Ho}(()\mathbf{C})$ as the quotient of $\mathbf{C}_{cf}$ (the subcategory of cofibrant and fibrant objects) by the homotopy equivalence relation.

Ext and the homotopy category

We now specialize these definitions to the case of the category of chain complexes.

Proposition 4.2.2.8. *Suppose A and B are R-modules; write $B[n]$ for the chain complex which has B in degree n and all differentials the zero map. For any integer $n \geq 0$,*

$$\mathrm{Hom}_{\mathrm{Ho}(Ch_R^{\geq 0})}(A, B[n]) = Ext^n(A, B).$$

Proof. The standard definition of the Ext groups in question is as follows: take a projective resolution $P_\bullet \to A$, and consider the complex $\mathrm{Hom}(P_\bullet, B)$ and take the homology of this complex. In other words, $Ext^n(A, B)$ is defined as follows. Consider the portion of the complex:

$$\cdots \longrightarrow \mathrm{Hom}(P_{n-1}, B) \longrightarrow \mathrm{Hom}(P_n, B) \longrightarrow \mathrm{Hom}(P_{n+1}, B) \longrightarrow \cdots .$$

The homology agree of this complex is the kernel of the map $\mathrm{Hom}(P_n, B) \to \mathrm{Hom}(P_{n+1}, B)$ modulo the image of the $\mathrm{Hom}(P_{n-1}, B) \to \mathrm{Hom}(P_n, B)$.

We can identify some portion of the above in terms of maps of complexes: indeed, a map of complexes $P_\bullet \to B[n]$ is the same thing as a morphism $P_n \to B$ such that the composite $P_{n+1} \to P_n \to B$ is the zero map, i.e., an element in the kernel of $\mathrm{Hom}(P_n, B) \to \mathrm{Hom}(P_{n+1}, B)$. The condition that two maps $\mathrm{Hom}(P_n, B)$ lie in the image of $\mathrm{Hom}(P_{n-1}, B)$ is an equivalence relation.

We now describe the computation in terms of the homotopy category. A map is a cofibration if and only if it is a monomorphism with degreewise projective cokernel. Thus, if we factor the map $0 \to A$ as $0 \to QA \to A$ where QA is cofibrant, then QA is a complex of projectives, and the fact that $QA \to A$ is a weak equivalence says that QA has no homology except in degree 0 in which case it coincides with A, i.e., QA is a projective resolution of A.

Now, every object of $Ch_R^{\geq 0}$ is fibrant by construction. It follows that right homotopy is an equivalence relation on $\mathrm{Hom}(QA, B[n])$ and to understand right homotopy we need a path object for $B[n]$. To this end, we want to factor the diagonal map

$$B[n] \longrightarrow B[n] \oplus B[n]$$

as a weak equivalence followed by a fibration. Since the second complex only has terms in degree n, let us construct a path object X by setting $X_n := B \oplus B$; to guarantee that $B[n] \to X$ is a quasi-isomorphism, we must have $d : X_n \to X_{n-1}$ be the difference map $B \oplus B \to B$. This construction gives an epimorphism $X \to B[n] \times B[n]$ if $n > 0$.

Now, let us unwind the definition of right homotopy with respect to X. Two maps $f, g : QA \to B[n]$ are right homotopic (with respect to X) if there exists a map $H : QA \to X$ lifting the product map $f \times g : QA \to B[n] \times B[n]$. A map $H : QA \to X$ is exactly a pair of maps $f_n, g_n : P_n \to B$ and a map $h : P_{n-1} \to B$ such that if $\partial : P_n \to P_{n-1}$ is the differential, then $f_n - g_n = h \circ \partial$ as maps $P_n \to B$. This is precisely the equivalence relation described above. $\square$

4.3 Derived functors

Previously, we showed that Ext groups were naturally computed as homomorphisms in the homotopy category of projective R-modules. But there is another point of view that will be useful here. If A is an R-module, then we can consider the functor $\mathrm{Hom}_R(A,-)$; this is a covariant endo-functor of the category of R-modules, since the set of R-module homomorphisms $\mathrm{Hom}_R(A,B)$ has naturally the structure of an R-module itself. Given a short exact sequence of R-modules

$$0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0,$$

if we apply the functor $\mathrm{Hom}_{\mathrm{Mod}_R}(A,-)$ we get the sequence

$$0 \longrightarrow \mathrm{Hom}(A,B_1) \longrightarrow \mathrm{Hom}(A,B_2) \longrightarrow \mathrm{Hom}(A,B_3).$$

The definition of projective of module shows that this sequence is exact on the right if and only if A is a projective module. If A is not projective, then it's natural to want to measure the failure of exactness, and classically ext groups were defined precisely to measure the failure of exactness on the right. Likewise, the functor $\mathrm{Hom}_R(-,A)$ is a contravariant functor and takes the short exact sequence above to

$$0 \longrightarrow \mathrm{Hom}(B_3,A) \longrightarrow \mathrm{Hom}(B_2,A) \longrightarrow \mathrm{Hom}(B_1,A),$$

i.e., it once again fails to preserve exactness on the right.

The functors $\mathrm{Hom}_R(A,-)$ and $\mathrm{Hom}_R(-,A)$ are both additive functors, in the sense that they preserve direct sums. In the category Mod_R, we know that finite sums are the same thing as finite products. The assertion that $\mathrm{Hom}_R(A,-)$ preserves injections means that it preserves equalizers. In other words, the functor $\mathrm{Hom}_R(A,-)$ preserves finite limits. Of course, as defined if $A_\bullet$ is a chain complex, it also makes sense to study the functor $\mathrm{Hom}_{Ch_R^{\geq 0}}(A_\bullet,-)$. Classically, one defines the group $Ext^i(A,B)$ in terms of a projective resolution of A. Our identification of

$$Ext^i(A,B) = \mathrm{Hom}_{D(R)}(A,B[i])$$

shows that the Ext groups are naturally objects on the homotopy category of chain complexes, i.e., the ext groups yield a family of functors from $D(R) \to \mathrm{Ab}$, parameterized by the index i.

Analogously, consider the functor $A \otimes_R -$. Again, this is a covariant functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$. Tensoring the above exact sequence we get an exact sequence of the form

$$A \otimes_R B_1 \longrightarrow A \otimes_R B_2 \longrightarrow A \otimes_R B_3 \longrightarrow 0.$$

This sequence remains exact on the left if and only if A is a flat R-module, though exactness on the right is always preserved. Once again, $A \otimes_R -$ preserves direct sums, i.e., it preserves finite coproducts in Mod_R, so the assertion that it preserves exactness on the right can be rephrased as saying that $A \otimes_R -$ preserves finite colimits. Classically, Tor functors are defined to measure the failure of exactness on the right. Extending the functor $A \otimes_R -$ to the category of chain complexes requires a bit more work: if $A_\bullet$ is a chain complex, then $A_\bullet \otimes_R -$ does not *a priori* produce a chain complex. We will address this problem and that of "measuring the failure of exactness" simultaneously in the context of model categories.

4.3.1 Exactness

Suppose we have abelian categories $\mathbf{C}$ and $\mathbf{D}$ that are both finitely complete and cocomplete (think of Mod_R). Given an additive functor $F : \mathbf{C} \to \mathbf{D}$ and an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

then there is an associated sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C).$$

We say that F is *left exact*, if it preserves exactness on the left and *right exact* if it preserves exactness on the right. The axioms of abelian categories imply that such categories are automatically finitely complete and cocomplete (i.e., have all finite limits and colimits). Indeed, in an abelian category, finite products necessarily agree with finite coproducts and existence of kernels and cokernels implies existence of equalizers and coequalizers. Abstracting the notion of exactness, one arrives at the following.

Definition 4.3.1.1. Assume $\mathbf{C}$ and $\mathbf{D}$ are categories that are finitely complete and cocomplete. A functor $F : \mathbf{C} \to \mathbf{D}$ is called *left exact* (resp. *right exact*) if F preserves finite limits (resp. finite colimits).

Remark 4.3.1.2. There are many non-additive functors that naturally arise, even on abelian categories. For example, on Mod_R, we can talk about tensor powers, symmetric powers and exterior powers of modules. E.g., consider the functor that assigns to an R-module A, the R-modules $A \otimes_R \cdots \otimes_R A$. This functor is evidently not additive in general if there are more than 2 factors. Likewise, sending an R-module A to $Aut_R(A)$ is a functor from R-modules to (non-abelian) groups. Note that the category of groups is also finitely complete and cocomplete, so we can ask about exactness here.

The derived functors of a given functor on a model category will be a "best approximation" of that functor to factoring through the homotopy category.

Definition 4.3.1.3. Assume $\mathbf{C}$ is a model category and $\mathbf{D}$ is a category, and $F : \mathbf{C} \to \mathbf{D}$ is a functor. Write $u : \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ for the universal functor. Consider pairs consisting of a functor $G : \mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ and a natural transformation $s : Gu \to F$. A *left derived functor* of F is a pair (LF, t) that is universal "from the left", i.e., if (G, s) is any pair as above, then there is a unique natural transformation $s' : G \to LF$ such that the composite natural transformation

$$Gu \xrightarrow{s' \circ u} LFu \xrightarrow{t} F$$

coincides with s. Likewise, a *right derived functor* of F is a pair (RF, t) that is universal "from the right", i.e., if (G, s) is any pair consisting of a functor $\mathrm{Ho}(\mathbf{C}) \to \mathbf{D}$ and a natural transformation $s : F \to Gu$, then there is a unique natural transformation $s' : RF \to G$ such that the evident composite natural transformation coincides with s.

Remark 4.3.1.4. As usual, the universal properties satisfed by derived functors shows that if a left derived functor exists, then any two are isomorphic up to a unique natural transforomation.

Proposition 4.3.1.5. *Let $\mathbf{C}$ be a model category and suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor that has the property that $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects in $\mathbf{C}$. A total left derived functor (LF, t) exists and for any cofibrant object X of $\mathbf{C}$, the map $t_X : LF(X) \to F(X)$ is an isomorphism. Dually, if $F : \mathbf{C} \to \mathbf{D}$ is a functor that has the property that $F(f)$ is an isomorphism whenever f is a weak equivalence between fibrant objects in $\mathbf{C}$, then a total right derived functor (RF, t) exists and for any fibrant object X of $\mathbf{C}$, the map $t_X : F(X) \to RF(X)$ is an isomorphism.*

Proof of 4.3.1.5. The idea to define the derived functor is straightforward: precompose F with the cofibrant replacement functor Q, i.e., take $LF(X) := F(QX)$. If X is cofibrant, then $QX \to X$ is a weak equivalence on cofibrant objects so the assumption on F guarantees that $F(QX) \to F(X)$ is an isomorphism. Since $QX \to X$ is functorial, we therefore always have a map $t_X : F(QX) \to F(X)$. We just have to check that the pair (LX, t) constructed in this way has the relevant universal property. We leave this as an exercise.

Dually, we can define right derived functors by using fibrant replacements. □

Suppose R is a commutative ring, and A is an R-module. The functor $A \otimes_R -$ extends to a functor on the category of chain complexes $Ch_R^{\geq 0}$. We will build a left derived functor of $A \otimes_R -$ using the result above, but to do this we need to check that $A \otimes_R -$ preserves weak equivalences between cofibrant objects. To check this, we use the following result.

Proposition 4.3.1.6 (Ken Brown's lemma)**.** *Suppose $F : \mathbf{C} \to \mathbf{D}$ is a functor between model categories. If F carries acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects. Dually, if F carries acyclic fibrations between fibrant objects to weak equivalences, then F preserves all weak equivalences between fibrant objects.*

Proof. We prove the first statement, leaving the second as an exercise in dualizing. Let $f : A \to B$ be a weak equivalence in $\mathbf{C}$ between cofibrant objects. In that case, form the pushout diagram.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \sqcup B \end{tikzcd}$$

The map $f : A \to B$ then gives a coproduct map $f \sqcup id_B : A \sqcup B \to B$. Note that $in_0 : A \to A \sqcup B$ and $in_1 : B \to A \sqcup B$ are cofibrations as cobase changes of cofibrations. Since A and B are cofibrant, it follows that $A \sqcup B$ is cofibrant as well.

Now, we can functorially factor $f \sqcup id_B$ as

$$A \sqcup B \xrightarrow{q} C \xrightarrow{p} B$$

where the q is a cofibration and p is an acyclic fibration. Since $A \sqcup B$ is cofibrant, it follows that C is cofibrant as well.

Since q is a cofibration, it follows that qin_0 and qin_1 are both cofibrations as well. Since f and id_B are weak equivalences, it follows that $pqin_i$ are weak equivalences for $i = 0, 1$ as well. Since

p is a weak equivalence by assumption, it therefore follows from 2 out of 3 that qin_i is also a weak equivalence. In other words, qin_i is an acyclic cofibration between cofibrant objects for $i = 0, 1$.

Now F preserves acyclic cofibrations between cofibrant objects, so it follows that $F(qin_i)$ is a weak equivalence for $i = 0, 1$. On the other hand, $F(id_B) = id_{F(B)}$ and is therefore also a weak equivalence. Since $pqin_1 = id_B$, it follows from 2 out of 3 that $F(p)$ is also a weak equivalence. Therefore, $F(f) = F(pqin_0) = F(p)F(qin_0)$ is a weak equivalence as well. □

By the preceding lemma, it suffices to check that $A \otimes_R -$ carries acyclic cofibrations between cofibrant objects to weak equivalences. Thus, suppose $f : B \to B'$ is any acyclic cofibration between cofibrant objects. In that case, B/B' is necessarily an acyclic complex that is degreewise projective, i.e., it is itself a projective chain complex and can be written as a sum of disk modules on cycles; in particular B' splits as a sum $B \oplus B'/B$ and B'/B can be further written as a sum of terms of the form $D_k(P)$. Since $A \otimes_R -$ respects direct sums, it follows that

$$A \otimes_R B \to A \otimes_R B' \cong A \otimes_R B \oplus A \otimes_R B'/B,$$

where the final summand is a sum of terms of the form $A \otimes_R D_k(P)$. Thus, it suffices to check that $A \otimes_R D_k(P)$ has trivial homology, but by inspection, $A \otimes_R D_k(P) = D_k(A \otimes_R P)$ is acyclic and we conclude.

Definition 4.3.1.7. If R is a commutative ring, and A is an R-module, then we define $Tor_i(A, -) := H_i(A \otimes_R Q(-))$, i.e., as homology of the left derived functors of $A \otimes_R -$.

Let us observe that with this definition, Tor does the correct thing on exact sequences...

Change of rings

Suppose $R \to S$ is a ring homomorphism, and A is an R-module. In that case, we have the extension of scalars functor from R-modules to S-modules and therefore from chain complexes of R-modules to chain complexes of S-modules. Note that if QA is a projective resolution of A, then QA is a chain complex of projective R-modules. It follows that $QA \otimes_R S$ is a chain complex of projective S-modules, since projectivity is preserved by extension of scalars, and thus $QA \otimes_R S$ computes $Q(A \otimes_R S)$. We get the change of rings functoriality in this way.

Chapter 5

Grothendieck groups, regularity and $\mathbb{A}^1$-invariance

Contents

In this section, we introduce another invariant of rings coming from projective modules: the Grothendieck of isomorphism classes of finitely generated projectives. We connect this invariant with the Picard group studied in the previous section. We then introduce the notion of regularity of a ring and study some basic properties of this notion as a first step toward understanding "smoothness" in algebraic geometry. We also begin a discussion of the homological theory of projective modules, along the lines initiated by Cartan–Eilenberg [**?**]. In particular, we will discuss projective dimension of rings, and study conditions that guarantee finite projective dimension; these notions are closely connected with regularity by classical results of Auslander-Buchsbaum-Serre.

5.1 Lecture 13: Grothendieck groups

5.1.1 Grothendieck groups

If R is a commutative unital ring, then we can consider the set of isomorphisms classes of projective R-modules. This set has a monoid structure given by direct sum (the unit being the zero R-module), but also a product given by tensor product of R-modules. Unlike the case of invertible R-modules, elements need not have inverses for this group structure (e.g., if R is a field, the dimension of a direct sum of R-modules is the sum of the dimensions of the summands and the dimension is always ≥ 0). Nevertheless, this monoid is still commutative (since $M \oplus M' \cong M' \oplus M$, functorially in the inputs).

Grothendieck observed that there is a universal way to construct an abelian group from a commutative monoid, generalizing the way the integers are built from the natural numbers. More precisely, every integer can be viewed as a "formal difference" of natural numbers. More abstractly, a formal difference can be equated with an element of $\mathbb{N} \times \mathbb{N}$. We define an addition on the set of formal differences componentwise. However, many formal differences correspond to the same integer, thus we need to impose an equivalence relation on the set of pairs to get integers. Say that (a, b) and $(a'b,')$ are equivalent if there exists $k \in \mathbb{N}$ such that $a + b' + k = a' + b + k$. In this form, the procedure works more generally: given a monoid M, consider $M \times M$, define addition componentwise and define an equivalence relation on pairs by saying $(m, n) \sim (m', n')$ if there exists $k \in M$ such that $m + n' + k = m' + n + k$.

Exercise 5.1.1.1. *Suppose A is a commutative monoid.*

1. *The procedure just described defines an abelian group A^+ (the group completion of A); this procedure is functorial with respect to homomorphisms of abelian groups.*
2. *There is a monoid homomorphism $A \to A^+$ (send a to $(a, 0)$) and given any abelian group B and a monoid map $\varphi : A \to B$, there is a unique homomorphism $A^+ \to B$ such that φ factors as $A \to A^+ \to B$.*

Definition 5.1.1.2. If R is a commutative unital ring, then $K_0(R)$ is the Grothendieck group of the monoid of isomorphism classes of projective modules with respect to direct sum.

Remark 5.1.1.3. If X is a topological space, and $C(X)$ is the algebra of real-valued continuous functions, then $K_0(C(X))$ is the Grothendieck group of isomorphism classes of topological vector bundles on X. If X is compact and Hausdorff, this coincides with the notion of topological K-theory as studied by Atiyah [**?**] using the Vaserstein-Serre-Swan theorem.

To really spell things out, consider the following result which explains when the isomorphism classes of projective modules agree in $K_0(R)$.

Lemma 5.1.1.4. *If R is a commutative unital ring and P, P' are finitely generated projective R-modules, the following statements are equivalent:*

1. *$[P] = [P']$ in $K_0(R)$;*
2. *there is a finitely generated projective R-module Q such that $P \oplus Q \cong P' \oplus Q$, i.e., the modules P and P' are* stably isomorphic*;*
3. *there is an integer n such that $P \oplus R^{\oplus n} \cong P' \oplus R^{\oplus n}$.*

Proof. Exercise. □

Lemma 5.1.1.5. *Tensor product of R-modules equips the group $K_0(R)$ with the structure of a commutative unital ring.*

Proof. Exercise. □

Example 5.1.1.6. We can compute $K_0(\mathbb{Z})$ from the definition: via the structure theorem for finitely generated modules, the monoid of isomorphism classes of projective R-modules is isomorphic to $\mathbb{N}$ under addition (via the monoid map sending a projective module to its rank). Thus, $K_0(\mathbb{Z}) = \mathbb{Z}$. More generally, if R is a principal ideal domain, the same argument shows that $K_0(R) \cong \mathbb{Z}$.

Lemma 5.1.1.7. *If $f : R \to S$ is a ring homomorphism, then extension of scalars induces a ring homomorphism $f^* : K_0(R) \to K_0(S)$.*

Example 5.1.1.8. If R is any commutative unital ring, then the map $\mathbb{Z} \to R$ induces a homomorphism $K_0(\mathbb{Z}) \to K_0(R)$; this homomomorphism sends $\mathbb{Z}^{\oplus n} \to R^{\oplus n}$ and is injective. Since any non-zero ring has a maximal ideal $\mathfrak{m}$, there is an induced map $K_0(R) \to K_0(R/\mathfrak{m})$. The composite map $\mathbb{Z} \to R/\mathfrak{m}$ induces an isomorphism $\mathbb{Z} = K_0(\mathbb{Z}) \to K_0(R/\mathfrak{m}) \cong \mathbb{Z}$ and therefore, we conclude that $\mathbb{Z}$ is a summand of $K_0(R)$ for any non-zero commutative unital ring. Thus, $K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$ where $\tilde{K}_0(R)$ is called the *reduced K_0 of R*. Note that $\tilde{K}_0(R) = 0$ if and only if each projective R-module is stably free.

Exercise 5.1.1.9. *Show that if R is a commutative unital ring and $N \subset R$ is the nilradical, then $K_0(R) \to K_0(R/N)$ is an isomorphism.*

5.1.2 Grothendieck groups of schemes

If X is a scheme, then we may define the Grothendieck group of X in a fashion generalizing that above. Doing this requires modifying the definition a bit. Over any affine scheme $X = \operatorname{Spec} R$, a short exact sequence of $\mathscr{O}_X$-modules of the form

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow\longrightarrow \mathscr{F}'' \longrightarrow 0$$

is always associated with a short exact sequence of R-modules (the global sections functor is exact)

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

If $\mathscr{F}''$ is locally free of finite rank, then M'' is finitely generated projective, and the definition of projectivity shows that $M \cong M' \oplus M''$.

Now, suppose X is a scheme and we consider a short exact sequence of locally free sheaves as above. Unlike the situation when X is affine, such a short exact sequence of sheaves need not split as direct sum. We will see non-trivial examples shortly. As a consequence, we will define the Grothendieck group of a scheme differently.

Definition 5.1.2.1. If X is a scheme, then we write $\mathrm{Vect}(X)$ for the category of finite rank locally free $\mathscr{O}_X$-modules. We write $K^0(X)$ for the quotient of the free abelian group on the set of objects of $\mathrm{Vect}(X)$ modulo the ideal generated by relations of the form

$$[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$$

for each short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow\longrightarrow \mathscr{F}'' \longrightarrow 0$$

as above.

If $f : X \to Y$ is a morphism of schemes, then since the pullback of a finite rank locally free $\mathscr{O}_X$-module is again finite rank locally free, it follows that there is a functor $f^* : \mathrm{Vect}(X) \longrightarrow \mathrm{Vect}(Y)$. This pullback functor preserves exact sequences and thus there is an induced homomorphism

$$f^* : K^0(X) \longrightarrow K^0(Y),$$

i.e., the assignment $X \mapsto K^0(X)$ is a contravariant functor on the category of schemes. If X is a connected scheme, then the rank of an $\mathscr{O}_X$-module is additive in exact sequences. As a consequence, there is an induced function

$$rk : K^0(X) \longrightarrow \mathbb{Z}.$$

Note that this homomorphism is always surjective, since it is split by $1 \mapsto [\mathscr{O}_X]$. We write $\widetilde{K}^0(X)$ for the kernel of this map and refer to this abelian group as the reduced K-theory of X.

Example 5.1.2.2. We can compute $K^0(\mathbb{P}^1_k)$ if k is a field. We claim that $K^0(\mathbb{P}^1_k) \cong \mathbb{Z} \oplus \mathbb{Z}$ where the first summand is generated by the rank map. The classification of vector bundles on $\mathbb{P}^1_k$ shows that in a given rank n, the isomorphism classes of bundles are represented by expressions of the form $\mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_n)$ for a weakly decreasing sequence of integers $a_1 \geq \cdots \geq a_n$.

However, there are many non-trivial relations that one may write down. For example, we claim that there is a short exact sequence of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_k} \longrightarrow \mathscr{O}(1) \oplus \mathscr{O}(1) \longrightarrow \mathscr{O}(2) \longrightarrow 0.$$

Indeed, we have global sections of $\mathscr{O}(1)$ given by homogeneous coordinates x_0 and x_1 on $\mathbb{P}^1_k$ which defines the first map, while the second map can be described in terms of clutching functions. Indeed, if we trivialize our vector bundles on the usual open cover of $\mathbb{P}^1_k$, i.e., $\mathrm{Spec}\, k[t]$, $\mathrm{Spec}\, k[t^{-1}]$ with intersection $\mathrm{Spec}\, k[t, t^{-1}]$, then we have maps $k[t, t^{-1}]^{\oplus 2} \to k[t, t^{-1}]$ given by the product map for functions. This product map is evidently a $k[t, t^{-1}]$-module map and is the restriction of the corresponding product maps $k[t]^{\oplus 2} \to k[t]$ and $k[t^{-1}]^{\oplus 2} \to k[t^{-1}]$. At the level of transition functions, the induced map sends the transition function $diag(t,t)$ of $\mathscr{O}(1) \oplus \mathscr{O}(1)$ to t^2. We leave it as an exercise to show that this sequence is short exact, but this follows from the fact that the functions x_0 and x_1 are nowhere vanishing, and thus the above sequence is *locally* split. It is important to note that this exact sequence is not *globally* split. Indeed, if it was globally split, then we would have an isomorphism between $\mathscr{O}(1) \oplus \mathscr{O}(1)$ and $\mathscr{O} \oplus \mathscr{O}(2)$, which would contradict our classification of vector bundles on $\mathbb{P}^1_k$.

In fact, more generally, this kind of argument shows that if a and b are positive integers, then there is an exact sequence of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1_k} \longrightarrow \mathscr{O}(a) \oplus \mathscr{O}(b) \longrightarrow \mathscr{O}(a+b) \longrightarrow 0,$$

where the first map is given by multiplication by any homogeneous degree a function and a homogeneous degree b function and the second map can be described on transition functions as above. Tensoring the exact sequence with $\mathscr{O}(c)$ for any integer c yields another relation. Working inductively, we then conclude that the class of $[\mathscr{O}(a_1) \oplus \mathscr{O}(a_2)] = [\mathscr{O}] + [\mathscr{O}(a_1 + a_2)]$ for any integers a_1 and a_2 (tensor with $\mathscr{O}(c)$ to make the representing integers positive, use the relation and tensor by $\mathscr{O}(-c)$ to get back to the original situation). Proceeding in this fashion, we get the relation

$$[\mathscr{O}(a_1) \oplus \mathscr{O}(a_n)] = [\mathscr{O}] \oplus \cdots \oplus [\mathscr{O}] \oplus [\mathscr{O}(a_1 + \cdots + a_n)].$$

It follows that sending a vector bundle $\mathscr{O}(a_1) \oplus \mathscr{O}(a_n)$ as above to $(n, a_1 + \cdots + a_n)$ extends to a well-defined isomorphism $K^0(\mathbb{P}^1_k) \to \mathbb{Z} \oplus \mathbb{Z}$ as claimed. In particular, observe that $K^0(\mathbb{P}^1_k)$ contains no more information than Picard group in this case!

5.1.3 Determinants of locally free sheaves

If M is any R-module, we can speak of exterior powers of M. Define the tensor algebra $T(M)$ to be the R-module $\bigoplus_{n \geq 0} M^{\otimes n}$ with multiplication given on pure tensor by the formula

$$(x_1 \otimes \cdots x_m) \otimes (y_1 \otimes \cdots \otimes y_n) = (x_1 \otimes \cdots x_n \otimes y_1 \otimes \cdots \otimes y_n)$$

and extended linearly. Define the exterior algebra $\wedge M$ to be the quotient of the graded algebra $T(M)$ by the two-sided (graded) ideal generated by $x \otimes x \in T^2(M)$. The image of the pure tensor $x_1 \otimes \cdots x_n$ in $\wedge M$ is denoted $x_1 \wedge \cdots \wedge x_n$. The k-th graded piece of $\wedge M$ is denoted $\wedge^k M$ and called the k-th exterior power of M. It follows that $\wedge^0 M = R$ (since $M^{\otimes 0} = R$), $\wedge^1 M = M$, and $\wedge^k M$ as the quotient of the k-fold tensor product $M \otimes \cdots \otimes M$ by the submodule generated by terms $m_1 \otimes \cdots \otimes m_k$ with $m_i = m_j$ for some $i \neq j$ (this encodes the "alternating" condition). Exterior powers define endo-functors of the category Mod_R. Moreover, one can establish the following fact using the compatibility of extension of scalars and tensor products.

Exercise 5.1.3.1. *If $\varphi : R \to S$ is a ring homomorphism, then $(\wedge^n M) \otimes_R S \xrightarrow{\sim} \wedge^n (M \otimes_R S)$.*

The exterior power functors have the following properties that we will find useful.

Lemma 5.1.3.2. *If R is a commutative unital ring, then the exterior power functor has the following properties:*

1. *The module $\wedge^n R^{\oplus r}$ is a free module of rank $\frac{r!}{n!(r-n)!}$.*
2. *If $M \oplus N$ is a direct sum decomposition, then there is a natural isomorphism*

$$\wedge^n(M \oplus N) \cong \bigoplus_{i=0}^{n} (\wedge^i M) \otimes (\wedge^{n-i} N).$$

Lemma 5.1.3.3. *Suppose P is a finitely generated projective R-module.*
1. *The module $\wedge^n P$ is a finitely generated projective R-module for any integer n.*
2. *If P has constant rank r, then $\wedge^r P$ is an invertible module that we will call $\det P$.*
3. *If P has constant rank r, then $\wedge^n P = 0$ for $n > r$.*

Proof. For Point (1), note that $\wedge^n P$ is finitely generated by assumption (as a quotient of a finitely generated module). Since exterior powers commute with tensor product, and projective modules are locally free, by localizing we can assume without loss of generality that P is free. Thus, by appeal to Lemma 5.1.3.2(1), we conclude that $\wedge^n P$ is also locally free.

Points (2) and (3) follow from Lemma 5.1.3.2(1) as well since $\wedge^n R^{\oplus r}$ has dimension 1 if $n = r$ and is trivial if $n > r$. □

Lemma 5.1.3.4. *Assume R is a connected commutative unital ring.*
1. *The map sending a finitely generated projective R-module to its determinant extends to a group homomorphism $\det : K_0(R) \to \mathrm{Pic}(R)$.*
2. *The homomorphism of Point (1) is functorial with respect to homomorphism of connected commutative unital rings.*

Proof. If P is a projective module of rank r, Lemma 5.1.3.3(3) tells us that $\wedge^n P = 0$ for $n > r$. If P and Q are projective R-modules of ranks m and n, then

$$\wedge^{m+n}(P \oplus Q) \cong \bigoplus_{i=0}^{m+n} \wedge^i P \otimes \wedge^{m+n-i} Q$$

by Lemma 5.1.3.2(2). Now, since i and $m+n-i$ are both ≥ 0 and $\leq m+n-i$, we conclude that either $\wedge^i P = 0$ or $\wedge^{m+n-i} Q = 0$ unless $i = m$. Thus, we conclude that $\wedge^{m+n}(P \oplus Q) \cong \wedge^m P \otimes \wedge^n Q$.

The functoriality statement is immediate from the fact that forming exterior powers commutes with extension of scalars. □

Remark 5.1.3.5. Using local constancy of rank, one can define the determinant for projective modules with non-constant rank "componenentwise" and drop the assumption that R is connected in the previous statement, but we leave it to the interested reader to work this out.

Theorem 5.1.3.6 (Cancellation for rank 1 modules)**.** *Suppose R is a commutative unital ring.*
1. *If L and L are stably isomorphic invertible R-modules, then $L \cong L'$.*
2. *The map $L \mapsto [L]$ determines an injection $\mathrm{Pic}(R) \to K_0(R)^{\times}$.*

Proof. Suppose $L \oplus R^{\oplus n} \cong L' \oplus R^{\oplus n}$. In that case, we conclude that $\wedge^{n+1}(L \oplus R^{\oplus n}) \cong \wedge^{n+1}(L \oplus R^{\oplus n})$. However, $\wedge^{n+1}(L \oplus R^{\oplus n}) \cong \wedge^1 L \otimes \wedge^n R^{\oplus n} \cong L$, and similarly for L'. Therefore, $L \cong L'$.

For Point (2), observe that the composite function $\mathrm{Pic}(R) \to K_0(R) \stackrel{\det}{\to} \mathrm{Pic}(R)$ is the identity after the conclusion of Point (1) (though note that the second map is a homomorphism with respect to the additive structure on K_0 while the first map uses the multiplicative structure, so the composite is not a group homomorphism). $\square$

Remark 5.1.3.7. A natural generalization of the Point (1) in Theorem 5.1.3.6 is the general "cancellation" problem: if P and Q are stably isomorphic projective R-modules (of the same rank) are P and Q isomorphic? A special case of the cancellation problem is: when are stably-free modules free? These problems motivated some of the early study of the groups $K_0(R)$.

The map $\det : K_0(R) \to \mathrm{Pic}(R)$ is a surjective group homomorphism by the same argument as in Theorem 5.1.3.6(2). Therefore, if $K_0(R) \to K_0(R[t_1, \ldots, t_n])$ is an isomorphism, we see $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t_1, \ldots, t_n])$ is an isomorphism too (it is always split injective and the statement about K_0 guarantees surjectivity). Thus, one cannot expect $K_0(R)$ to be $\mathbb{A}^1$-invariant without a hypothesis on R at least as strong as (semi-)normality (cf. Theorem 3.7.1.3 and Remark 3.7.1.6).

Determinants of vector bundles on schemes

Now, assume X is a scheme. The category of $\mathscr{O}_X$-modules has a tensor product making it symmetric monoidal, and one can define the exterior powers as above. If X is connected, and if $\mathscr{F}$ is a rank n locally free $\mathscr{O}_X$-module, then $\wedge^n \mathscr{F}$ is a rank 1 locally free $\mathscr{O}_X$-module by appeal to the results above. Moreover, the formula for exact sequences shows that if

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow\!\!\!\!\rightarrow \mathscr{F}'' \longrightarrow 0,$$

where the terms have rank r', r and r'', then $r = r' + r''$ and then there is an induced isomorphism

$$\wedge^{r'} \mathscr{F}' \otimes \wedge^{r''} \mathscr{F}'' \cong \wedge^r \mathscr{F}.$$

As above, there is an induced map

$$\det : K^0(X) \longrightarrow \mathrm{Pic}(X).$$

As above, if $f : X \to Y$ is a morphism of schemes, then there is an associated commutative diagram

$$\begin{tikzcd} K^0(X) \arrow[r, "\det"] \arrow[d, "f^*"] & \mathrm{Pic}(X) \arrow[d, "f^*"] \\ K^0(Y) \arrow[r] & \mathrm{Pic}(Y), \end{tikzcd}$$

i.e., the determinant homomorphism is functorial.

Adams and exterior operations

In fact, let us observe that the exterior powers give extra structure to the ring...

5.2 Regular local rings

If M is a smooth manifold of dimension d, one way to define the tangent space at a point $x \in M$ is as follows: consider the ideal $\mathfrak{m}_x \subset C^{\infty}(M)$ consisting of smooth functions vanishing at x. Since locally around x there is a neighborhood of x diffeomorphic to an open subset of Euclidean space $\mathbb{R}^d$, we can pick local coordinates $x_1, \ldots, x_d$ that generate the maximal ideal $\mathfrak{m}_x$. The choice of local coordinates then yields a basis of the real vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. The tangent space is then the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^{\vee}$, which is thus a real vector space of dimension d.

5.2.1 Regular local rings: definitions and examples

We first discuss the notion of regularity locally, essentially by directly reinterpreting the situation in topology. There is one basic problem: in algebraic geometry, if $X = \operatorname{Spec} R$ is an affine k-variety, then given a k-point, there is no reason for one to be able to find an open neighborhood of x in X that can be identified with an open subset of affine space. For the sake of intuition, let us discuss the case $k = \mathbb{C}$, and let us think about the "embedded" point of view and identify $R = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. In that case, non-singularity can be tested using the Jacobian criterion: one writes down the matrix of partial derivatives and smooth points are precisely those where the rank of the matrix $(\frac{\partial f_i}{\partial x_j})$ is maximal, i.e., equal to r. One identifies the tangent space as a subspace of $\mathbb{C}^n$, and thus at points where the rank of the Jacobian drops, the dimension of the tangent space increases.

If R is a ring with a maximal ideal $\mathfrak{m}$, then we can consider the field $k = R/\mathfrak{m}$. There is a natural structure of k-vector space on $\mathfrak{m}/\mathfrak{m}^2$. We first establish a result the provides a purely algebraic version of the intuitive description given above.

Theorem 5.2.1.1. *If R is a Noetherian local ring of Krull dimension d, with maximal ideal $\mathfrak{m}$ and residue field κ, then $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$.*

Proof. This is a consequence of Krull's generalized principal ideal theorem: if R is a Noetherian ring, $I \in R$ is an n-generated ideal and $\mathfrak{p}$ is minimal among prime ideals of R containing I, then $\operatorname{ht}\mathfrak{p} \leq c$ (in the special case where $c = 1$, this says that principal ideals always have height ≤ 1, which is where the name comes from, and the proof in the general case can be reduced to this one).

Now, if R is a Noetherian local ring of Krull dimension d, then the ideal $\mathfrak{m}$ has height d by definition. Krull's generalized principal ideal theorem then and by the previous result cannot be generated by fewer than d elements. Now, to obtain the inequality observe that Nakayama's lemma implies that if R is a local ring, then any basis of $\mathfrak{m}/\mathfrak{m}^2$ as a κ-vector space can be lifted to a minimal generating set of of $\mathfrak{m}$, and every minimal generating set is obtained in this way. Thus, $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \operatorname{ht}\mathfrak{m} \geq \dim R$ as claimed. □

Remark 5.2.1.2. Krull's principal ideal theorem is known to hold in various non-Noetherian settings, e.g., for Krull domains (reference?). However, the generalized principal ideal theorem can fail for Krull domains (reference). While work has been done to understand situations in which it holds, this basic fact is one reason Noetherian assumptions are in place here.

Definition 5.2.1.3. A Noetherian local ring $(R, \mathfrak{m})$ with residue field κ is called *regular* if $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Example 5.2.1.4. Every field is a regular local ring of dimension 0. Local rings like $k[\epsilon]/\epsilon^2$ are not regular: indeed (ϵ) is a non-zero maximal ideal here but $(\epsilon)/(\epsilon)^2$ is a 1-dimensional k-vector space. However, the ring $k[\epsilon]/\epsilon^2$ has dimension 0. More generally, suppose $(R, \mathfrak{m})$ is a regular local ring of Krull dimension 0 and residue field κ. In that case, $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = 0$ as well, i.e., $\mathfrak{m} = \mathfrak{m}^2$. By induction, one concludes that $\mathfrak{m} = \mathfrak{m}^n$ for all $n \geq 0$. However, a Noetherian ring of Krull dimension 0 is automatically Artinian, and therefore $\mathfrak{m}^n = 0$ for n sufficiently large. Therefore, $\mathfrak{m}$ must be the zero ideal, in which case R is a field.

Example 5.2.1.5. Any discrete valuation ring is a regular local ring of dimension 1. Indeed, this is a consequence of one of the equivalent characterizations of discrete valuation rings.

Example 5.2.1.6. Any localization of a polynomial ring $k[x_1, \ldots, x_n]$ at a maximal ideal is a regular local ring of dimension n.

5.2.2 Symmetric algebras and tangent spaces

Given a Noetherian local ring $(R, \mathfrak{m})$ with residue field κ, we considered the κ-vector space $\mathfrak{m}/\mathfrak{m}^2$. We want to enhance this κ-vector space to an actual variety. To this end, we begin by recalling the construction of symmetric powers of a module, and we do this in greater generality than we will need here.

Symmetric powers

Definition 5.2.2.1. If R is a ring and M is an R-module, then the symmetric algebra on M, denoted $\mathrm{Sym} M$ is defined as the quotient

$$\mathrm{Sym} M := T(M)/\langle x \otimes y - y \otimes x | x, y \in M \rangle.$$

As with the exterior algebra, $\mathrm{Sym} M$ is a graded algebra, but it is commutative. By the universal property of the tensor algebra, if M is an R-module, and A is any commutative R-algebra, any R-module homomorphism $M \to A$ extends to an R-algebra homomorphism $T(M) \to A$. The commutativity of A ensures that this homomorphism factors through a homomorphism $\mathrm{Sym} M \to A$. On the other hand, given an R-algebra map $\mathrm{Sym} M \to A$, there is an induced R-module homomorphism $M \to A$. These two constructions are mutually inverse and yield a universal property characterizing the symmetric algebra, which we summarize in the following result.

Lemma 5.2.2.2. *If M is an R-module, and A is an R-algebra, then*

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M, A) = \mathrm{Hom}_{\mathrm{Aff}_R}(\mathrm{Sym} M, A).$$

The symmetric power has other properties analogous to the exterior power.

Lemma 5.2.2.3. *If R is a commutative unital ring, then the symmetric power functor has the following properties:*

1. *The module $\mathrm{Sym}^n R^{\oplus d}$ is a free module of rank* $\binom{d-1+n}{n}$.

2. *If $M \oplus N$ is a direct sum decomposition, then there is a natural isomorphism*

$$\mathrm{Sym}^n(M \oplus N) \cong \bigoplus_{i=0}^{n} (\mathrm{Sym}^i M) \otimes (\mathrm{Sym}^{n-i} N).$$

Corollary 5.2.2.4. *If M is a free R-module of rank n, then a choice of basis $x_1, \ldots, x_n$ of M determines an isomorphism* $\mathrm{Sym} M \xrightarrow{\sim} R[x_1, \ldots, x_n]$.

Tangent cones

If $(R, \mathfrak{m})$ is a Noetherian local ring with residue field κ, then R is *filtered* by the powers of $\mathfrak{m}$: more precisely, if $a \in \mathfrak{m}^r$ and $b \in \mathfrak{m}^s$, then $ab \in \mathfrak{m}^{r+s}$. The associated graded ring of this filtered ring is the κ-algebra $gr_{\mathfrak{m}} R := \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$. The fact that the ring $(R, \mathfrak{m})$ is filtered implies that there is an induced ring homomorphism $R \to gr_{\mathfrak{m}} R$.

Example 5.2.2.5. If we take R to be the localization of a polynomial ring over a field k in d-variables $x_1, \ldots, x_d$ at the maximal ideal $(x_1, \ldots, x_n)$, then it follows that $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ can be identified with the vector space of homogeneous symmetric polynomials of degree r in n variables. In particular, $\mathfrak{m}^r/\mathfrak{m}^{r+1} \cong \mathrm{Sym}^r \mathfrak{m}/\mathfrak{m}^2$, which has dimension $\binom{d-1+r}{r}$. In particular, the identification

$$\binom{d-1+r}{r} = \binom{d-1+r}{d-1}$$

shows that this function grows as a polynomial of degree $d-1$. More precisely, set

$$\binom{t}{n} := \frac{1}{n!} t(t-1) \cdots (t-n+1)$$

and observe that $\binom{d-1+r}{d-1}$ is the value at integers of $\binom{t+(d-1)}{d-1}$. Polynomials that take integer values at integers are called *numerical polynomials*.

We now analyze the situation discussed in the previous example in greater detail.

Definition 5.2.2.6. If $(R, \mathfrak{m})$ is a Noetherian local ring with residue field κ, then the *tangent cone* at $\mathfrak{m}$ is the graded κ-algebra $\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$.

Before discussing the relationship between this notion and regularity, we discuss some facts about graded rings. Each $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite-dimensional κ-vector space, and we can consider its dimension $\dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$; this assignment defines a function $f(n) := \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$. In Example 5.2.2.5, we observed that in one special case this function grew as a polynomial in n. Before studying the general case, we recall some simple facts about numerical polynomials.

Numerical polynomials

We begin with a brief review of integer polynomials. A polynomial $P(t)$ is called a numerical polynomial if it takes integer values at integers. The sum and product of any two numerical polynomials is integer valued, and since 0 and 1 are integer valued, it follows that numerical polynomials form a subring of $Q[t]$. The following binomial polynomials give examples of numerical polynomials of arbitrary degree:

In fact, the above polynomials form a basis of the $\mathbb{Z}$-module of numerical polynomials. Indeed, every numerical polynomial can be written as a rational linear combination of such polynomials (since there is one of each degree). Therefore, if given an integer polynomial f and an equality

$$f = \lambda_0 + \lambda_1 t + \cdots + \lambda_n \binom{t}{n}$$

with $\lambda_i \in \mathbb{Z}$, we conclude first that since $f(0) = \lambda_0$, that $\lambda_0 \in \mathbb{Z}$. Then by induction we may conclude that λ_i are all integers. Therefore, we have established the following fact.

Lemma 5.2.2.7. *The $\mathbb{Z}$-submodule of $\mathbb{Q}[t]$ consisting of numerical polynomials has a basis consisting of the polynomials $\binom{t}{n}$.*

The following result generalizes the observation made in Example 5.2.2.5.

Proposition 5.2.2.8. *If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ, then the assignment $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a numerical polynomial φ. Moreover, the following are equivalent: $\dim R = d$, φ has degree $d-1$ and $\mathfrak{m}$ is generated by d elements.*

Proof. This is established by induction; see [**?**, Tag 00KD Proposition 10.59.8] □

Tangent cones and regularity

The universal property of the symmetric algebra shows that the R-module map $\mathfrak{m}/\mathfrak{m}^2 \to gr_{\mathfrak{m}} R$ induces a homomorphism $\mathrm{Sym}\mathfrak{m}/\mathfrak{m}^2 \to gr_{\mathfrak{m}} R$. We now use this observation to give another characterization of regularity.

Proposition 5.2.2.9. *If R is a regular local ring with maximal ideal $\mathfrak{m}$, the map $\mathfrak{m}/\mathfrak{m}^2 \to gr_{\mathfrak{m}} R := \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ induces an isomorphism $\mathrm{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} gr_{\mathfrak{m}} R$.*

Proof. Suppose R is simply a Noetherian local ring of Krull dimension d and maximal ideal $\mathfrak{m}$. In that case, we get a map

$$\psi : \mathrm{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \longrightarrow gr_{\mathfrak{m}} R;$$

this map is a ring homomorphism by the construction of the product on both sides. The map ψ is surjective essentially by construction: indeed, since R is a Noetherian local ring of dimension d, the ideal $\mathfrak{m}$ is generated by $\geq d$ generators $x_1, \ldots, x_r$ and we can then write down generators for $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ as homogeneous polynomials of degree n in the x_i. To see that ψ is injective it suffices to count dimensions. Indeed, we observed above that if R is regular of dimension d, then $\mathfrak{m}$ is generated by exactly d elements (and cannot be generated by fewer elements). The assignment $n \mapsto \dim_\kappa \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a numerical polynomial and one shows that its degree is precisely $d-1$ by Proposition 5.2.2.8. The kernel of the map ψ is a graded ideal I and we can consider the dimensions of the graded pieces of $\mathrm{Sym}\mathfrak{m}/\mathfrak{m}^2/I$. The dimensions of these graded pieces also form a numerical polynomial [**?**, Tag 00K1 Proposition 10.57.7] whose degree $< d-1$ (see [**?**, Tag 00K3 Lemma 10.57.10]). □

5.3 Lecture 15: Geometry of regular local rings

5.3.1 Structural properties of regular local rings

Suppose k is an algebraically closed field and consider the ring $k[x]$. If $\mathfrak{m} \subset k[x]$ is a maximal ideal, then we can write $\mathfrak{m} = (x-a)$ and $\mathfrak{m}^n = (x-a)^n$. There is an evident sequence of inclusions $\mathfrak{m}^n \subset \mathfrak{m}^{n-1}$. The elements of $\mathfrak{m}^n$ are those functions that have a zero of order $\geq n$ at $x = a$. It is immediate from this observation that $\cap_n \mathfrak{m}^n = 0$. Now, if M is any $k[x]$-module, one can consider the filtration on M by powers of $\mathfrak{m}$. Krull established the following far-reaching generalization of this observation.

Proposition 5.3.1.1. *If R is a Noetherian local ring, and $I \subset R$ is any proper ideal, then for any finitely generated R-module M, $\cap_n I^n M$.*

Proof. Set $N = \cap_n I^n M$; this is a finitely generated R-module. Note that $N = I^n M \cap N$ for any integer n, by definition. We claim that $I^n M \cap N \subset IN$ for n sufficiently large; this is a consequence of the Artin-Rees lemma (which states that if I is an ideal in a Noetherian ring R,M is a finitely generated R-module and $N \subset M$ is a submodule, then there exists an integer $k \geq 1$ such that for $n \geq k$ the equality $I^n M \cap N = I^{n-k}(I^k M \cap N)$ holds). Granting this, the result follows immediately from Nakayama's lemma. □

The fact that regular local rings $(R, \mathfrak{m}, \kappa)$ have $gr_{\mathfrak{m}} R$ isomorphic to a polynomial ring (via Proposition 5.2.2.9) is very useful: we can use the ring map $R \to gr_{\mathfrak{m}} R$ to "lift" statements about polynomial rings to corresponding statements about R itself (this technique works well to study filtered rings whose associated graded rings are "easy to understand", e.g., the universal enveloping algebra of a Lie algebra). Here is an example of this kind of argument.

Proposition 5.3.1.2. *If R is a regular local ring, then R is a normal domain.*

Proof. We first prove that R is a domain. As usual, let $\mathfrak{m}$ be the maximal ideal of R. Take elements $f, g \in R$ such that $fg = 0$. By Proposition 5.3.1.1, since $\cap_n \mathfrak{m}^n = 0$, we can find a and b maximal such that $f \in \mathfrak{m}^a$ and $g \in \mathfrak{m}^b$. The product fg lies in $\mathfrak{m}^{a+b}$, but since it is zero, it lies in $\mathfrak{m}^{a+b+1}$ as well. Thus, we can view $fg \in \mathfrak{m}^{a+b+1}/\mathfrak{m}^{a+b+2}$. Now, $\mathrm{Sym}^{\bullet}\mathfrak{m}/\mathfrak{m}^2 \to gr_{\mathfrak{m}} R$ is an isomorphism by Proposition 5.2.2.9 and $\mathrm{Sym}^{\bullet}\mathfrak{m}/\mathfrak{m}^2$ is isomorphic to a polynomial ring in d variables, and so is a domain. In particular, the condition $0 = fg$ for the images in $\mathrm{Sym}^{\bullet}\mathfrak{m}/\mathfrak{m}^2$ means that $f = 0$ or $g = 0$. If $f = 0$, then that means $f \in \mathfrak{m}^{a+1}$ as well and if $g = 0$, that means $g \in \mathfrak{m}^{b+1}$. In either case, we obtain a contradiction.

Now, we establish that R is integrally closed in its field of fractions. Let $\mathfrak{m}$ be the maximal ideal of R, κ the residue field, and set K to be the fraction field of R. By Proposition 5.2.2.9 we know that $\mathrm{Sym}^{\bullet}\mathfrak{m}/\mathfrak{m}^2$ is isomorphic to a polynomial ring, and is therefore integrally closed in its field of fractions $\kappa(\mathfrak{m}/\mathfrak{m}^2)$. The idea is that one deduces inductively that R is integrally closed in its field of fractions. See [**?**, V.1.4 Proposition 15] for this statement. □

Example 5.3.1.3. Any regular local ring R of Krull dimension 1 is a discrete valuation ring. Indeed, Proposition 5.3.1.2 implies R is a local Noetherian normal domain of Krull dimension 1, so this follows immediately from Theorem 3.4.3.6.

Definition 5.3.1.4. A sequence of elements $(x_1, \ldots, x_n)$ in a ring R is called a *regular sequence* if the ideal $(x_1, \ldots, x_n) \subset R$ is proper, and for each i, x_{i+1} is not a zero-divisor in $R/(x_1, \ldots, x_n)$.

Corollary 5.3.1.5. *If R is a regular local ring with maximal ideal $\mathfrak{m}$, and $x_1, \ldots, x_d$ is a minimal set of generators of R, then $x_1, \ldots, x_d$ is a regular sequence.*

Proof. We proceed by induction on i with the base case being that R is regular. Assume inductively that $R/(x_1, \ldots, x_i)$ is a regular local ring. The images of $(x_{i+1}, \ldots, x_d)$ form a minimal set of generators for $\mathfrak{m}/(x_1, \ldots, x_i)/\mathfrak{m}$, which is the maximal ideal in the Noetherian local ring $R/(x_1, \ldots, x_i)$. Indeed, if one of these elements was zero, then we would have a generating set with fewer than $d - i$ generators, which would contradict the conclusion of Theorem 5.2.1.1. Thus, Proposition 5.3.1.2 guarantees that $R/(x_1, \ldots, x_i)$ is an integral domain, and therefore x_{i+1} is not a zero-divisor. □

We have obtained a number of criteria for regularity of a local ring now, and the following result puts everything together.

Proposition 5.3.1.6. *Suppose R is a Noetherian local ring of Krull dimension d, with maximal ideal $\mathfrak{m}$ and residue field κ. The following conditions are equivalent.*

1. *R is a regular local ring of Krull dimension d;*
2. $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = d$*;*
3. *the ideal $\mathfrak{m}$ admits a system of generators with precisely d elements;*
4. *the map $\mathrm{Sym}^\bullet \mathfrak{m}/\mathfrak{m}^2 \to gr_\mathfrak{m} R$ is an isomorphism.*
5. *the ideal $\mathfrak{m}$ admits a system of generators that is a regular sequence of length d.*

Proof. That $(1) \Leftrightarrow (2)$ was the definition. That $(2) \Leftrightarrow (3)$ was Theorem 5.2.1.1. That (3) ⇒ (4) is Proposition 5.2.2.9. That (4) ⇒ (5) is Corollary 5.3.1.5. It is not hard to show that (5) ⇒ (2). □

5.3.2 Regular rings

Definition 5.3.2.1. Suppose R is a Noetherian ring. Say that $X = \mathrm{Spec}\, R$ is *regular* at a closed point $x \in \mathrm{Spec}\, R$ corresponding to a maximal ideal $\mathfrak{m}$ if $R_\mathfrak{m}$ is a regular local ring and *singular* otherwise. Say that R is *regular* if $\mathrm{Spec}\, R$ is regular at all closed points.

Proposition 5.3.2.2. *If R is a Noetherian regular domain, then R is a normal domain.*

Proof. According to Definition 5.3.2.1, all the localizations of R at maximal ideals are regular local rings. Now, Proposition 5.3.1.2 allows to conclude that the localizations of R at maximal ideas are normal rings. Finally, appealing to Proposition **??**, since R is a Noetherian domain, and every localization of R at a maximal ideal is normal, we can conclude that R is normal as well. □

Example 5.3.2.3. A regular ring of dimension 0 is a product of fields. Indeed, any Noetherian ring of Krull dimension 0 is a product of Artin local rings. A regular domain of dimension 1 is precisely a Dedekind domain. Indeed, if R is a regular ring of Krull dimension 1, then R is a Noetherian normal domain of Krull dimension 1 by Proposition 5.3.2.2.

The following result gives the first geometric consequence of normality: singular points of normal varieties lie in codimension ≥ 2.

Corollary 5.3.2.4. *If R is a Noetherian normal domain, then R is* regular in codimension 1*, i.e., for any height 1 prime ideal $\mathfrak{p} \subset R$, then $R_{\mathfrak{p}}$ is a regular local ring.*

Proof. Since discrete valuation rings are regular local rings, it suffices to observe that if R is a Noetherian normal domain, then $R_{\mathfrak{p}}$ is a discrete valuation ring. □

Proposition 5.3.2.5. *If R is a regular ring of Krull dimension d, then $R[x_1, \ldots, x_n]$ is a regular ring of Krull dimension $d + n$.*

Proof. By induction, it suffices to treat the case where $n = 1$. Suppose $\mathfrak{M}$ is a maximal ideal of $R[x]$ and set $\mathfrak{m} = \mathfrak{M} \cap R$. In that case, $R[x]_{\mathfrak{M}}$ is a localization of $R_{\mathfrak{m}}[x]$ at the maximal ideal $\mathfrak{M}R_{\mathfrak{m}}[x]$, so we can assume without loss of generality that R is a regular local ring.

Assuming now that R is local with maximal ideal $\mathfrak{m}$, let $k = R/\mathfrak{m}$ and consider the homomorphism $R[x] \to k[x]$. The ideal generated by $\mathfrak{M}$ in $k[x]$ is principal, generated by a monic irreducible polynomial $\bar{f}$. Therefore, we can find a monic polynomial in $f \in R[x]$ lifting this element and such that $\mathfrak{M} = (\mathfrak{m}, f)$. Since R is regular, it is an integral domain, and therefore $R[x]$ is an integral domain as well. Since the ideal $\mathfrak{M}$ is maximal, the element f cannot be zero and combining everything is not a zero-divisor. Therefore, $\mathrm{ht}\mathfrak{M} = \mathrm{ht}\mathfrak{m} + 1$. Since $R[x]$ has Krull dimension $d + 1$, it follows that we have constructed a minimal set of generators for $\mathfrak{M}$ and thus $R[x]_{\mathfrak{M}}$ is a regular local ring. □

Remark 5.3.2.6. Regularity is an interesting notion, but from what we said above it is not clear that it captures the intuitive notion of smoothness from differential geometry. For example, we had to work hard to prove that if $X = \operatorname{Spec} R$ is a regular ring then $X \times \mathbb{A}^n$ is regular. One can construct examples to show that if X is regular, then $X \times X$ need not be regular. For example set $k = \mathbb{F}_p(t)$, and take $R = k[x]/(x^p - t)$. In this case, R is a field, namely the purely inseparable extension of $\mathbb{F}_p(t)$ obtained by adjoining a p-th root of t; therefore R is regular. On the other hand $R \otimes_k R$ is a zero-dimensional local ring, which is not a field, and therefore not regular.

5.4 Projective resolutions and Tor

5.4.1 Projective resolutions and K_0

If R is a ring, then we think of elements of $K_0(R)$ as formal differences $([P], [Q])$ of projective R-modules. We now give a more flexible *homological* approach to elements of $K_0(R)$. First, observe that given a short exact sequence of projective modules

$$0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0,$$

since this exact sequence splits, we conclude that $P \cong P' \oplus P''$. We now extend this result slightly, but before doing so we make the following general definition.

Definition 5.4.1.1. If R is a commutative unital ring, and M is an R-module, then a (left) resolution of M is an exact sequence

$$\cdots \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0.$$

Such a resolution is called *finite* if there is an integer $r \geq 0$ such that $E_s = 0$ for all $s > r$, *free* if each E_i is free, and *projective* if each E_i is a projective R-module. We will frequently write $E_\bullet \to M$ is a resolution.

Lemma 5.4.1.2. *If R is a commutative unital ring, and $Q_\bullet$ is a finite resolution of a projective R-module P by finitely generated projective R-modules, i.e.,*

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \longrightarrow P \longrightarrow 0,$$

then $[P] = \chi(Q_\bullet) = \sum_{i=0}^{n}(-1)^i[Q_i]$.

Proof. We proceed by induction on n. If $n = 0$, the result is obvious. Since the map $Q_0 \to P$ is surjective and P is projective, we conclude that $Q_0 \cong P \oplus \ker(Q_0 \to P)$. However, $\ker(Q_0 \to P)$ is projective, and the map $Q_1 \to \ker(Q_0 \to P)$ is surjective, so we obtain a resolution of $\ker(Q_0 \to P)$ of smaller length. Thus, $[Q_0] = [P] + [\ker(Q_0 \to P)]$ and the result follows from the induction hypothesis. $\square$

This easy observation shows that one way to obtain results about the structure of $K_0(R)$ is to show that all modules have projective resolutions by modules of a certain type.

Proposition 5.4.1.3. *If $\varphi : R \to S$ is a ring homomorphism, then say an S-module M is extended from R if $M \cong M' \otimes_R S$ for some R-module M'. If every projective S-module admits a finite projective resolution by modules extended from R, then $K_0(R) \to K_0(S)$ is surjective.*

5.4.2 Properties of Tor

Suppose now that R is a commutative unital ring and M is an R-module, we can consider the functor $M \otimes_R -$ on Mod_R. Since neither this functor, nor $- \otimes_R M$ preserves exactness, we are interested in measuring the failure of exactness. The functor Tor_R is cooked up to measure this failure.

Example 5.4.2.1. Suppose R is a ring and M is an R-module, in that case given an element $r \in R$, multiplication by r determines an R-module map $\cdot r : R \to R$. If r is not a zero-divisor, then this map is injective. In that case, there is the following exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot r} R \longrightarrow R/(r) \longrightarrow 0.$$

Now, if we tensor this exact sequence with M the multiplication by r map induces the multiplication by r map:

$$M \xrightarrow{\cdot r} M.$$

While the initial sequence was exact, this sequence fails to be exact: the cokernel of multiplication by r is $M/rM \cong M \otimes_R R/(r)$, but the kernel of the multiplication by r map consists of those elements $m \in M$ such that $rm = 0$, i.e., it is the r-torsion submodule of M, sometimes written $Tor_1^R(R/(r), M)$.

Example 5.4.2.2. More generally, given a finitely generated ideal $I \subset R$ and generators $(r_1, \ldots, r_n)$ of I, we can study the I-torsion in an R-module M by sequentially comparing torsion with respect to each r_i; as in the previous example, we will need some condition on the r_i to ensure that we actually obtain a resolution. For example, if I is generated by 2 elements r_1 and r_2, then we can consider the map $R^{\oplus 2} \to R$ given by multiplication by $(r_1, r_2)^t$. The cokernel of this map is I. We can analyze this a bit more systematically by tensoring the two complexes $R \stackrel{\cdot r_1}{\to} R$ and $R \stackrel{\cdot r_2}{\to} R$ to obtain a diagram of the form

$$\begin{tikzcd} R \arrow[r, "r_2"] \arrow[d, "r_1"] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"] & R. \end{tikzcd}$$

By changing the signs slightly, this yields the following complex

$$R \stackrel{(r_2,-r_1)}{\longrightarrow} R \stackrel{\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}}{\longrightarrow} R,$$

(the composite is zero precisely because of the sign change). The cokernel of the last map is precisely $R/(r_1, r_2)$, and in good situations, this sequence actually yields a resolution of $R/(r_1, r_2)$. Indeed, if we know that r_1 and r_2 are not zero-divisors, then the first map is injective.

The kernel of the map $R \oplus R \to R$ consists of those pairs (a, b) such that $r_1 a + r_2 b = 0$. Thus, $r_2 b$ lies in the ideal (r_1) and the kernel of the first map can be described as those elements $r \in R$ such that $r(r_2) \subset (r_1)$. This collection of elements is an ideal, called the ideal quotient and often denoted $(r_1 : r_2)$. Since r_1 is assumed to not be a zero-divisor, it follows that a is uniquely determined by b. The image of $R \to R \oplus R$ consists of elements of the form $(r_2 r, -r_1 r)$. Now, such an element is contained in the kernel and the image in $(r_1 : r_2)$ is precisely (r_1). Thus, if r_2 is not a zero-divisor in $R/(r_1)$, we conclude that the sequence is exact in the middle as well. Thus, if (r_1, r_2) is a *regular sequence* in the sense we studied earlier, then we obtain a free resolution of $R/(r_1, r_2)$.

Tensoring this sequence with M we obtain a complex that has non-trivial homology: the zeroth homology still computes $M/(r_1, r_2)M$, but there are higher homology terms. For example, the map $M \oplus M \to M$ one obtains sends $(m_1, m_2) \mapsto r_1 m_1 + r_2 m_2$. Thus the first homology of the complex obtained by tensoring with M is the quotient of the submodule of M annihilated by (r_1, r_2) by certain relations.

Remark 5.4.2.3. The complex described in Example 5.4.2.2 is called the *Koszul complex*, and admits a generalization to regular sequences in an arbitrary commutative ring R. The specific example shows that if $(R, \mathfrak{m}, \kappa)$ is a 2-dimensional regular local ring, then the maximal ideal admits a finite free resolution. The second example points to an ambiguity: there are many possible sequences of generators for an ideal I in a ring R, and the cohomology groups obtained in the example might depend on these choices.

Suppose M is a fixed R-module. If $P_\bullet \to M$ is a projective (flat) resolution of M, then the tensor product $P_\bullet \otimes_R N$ has the structure of a complex of R-modules and thus we can consider the homology of this complex. If $P'_\bullet$ is another projective (flat) resolution of M, then using sign changes in a fashion similar to Example 5.4.2.2, then one can build a complex $Tot(P_\bullet \otimes_R P'_\bullet)$ out of

$P_\bullet \otimes_R P'_\bullet$ (see [**?**, 2.7.1] for details). Since $P_\bullet$ and $P'_\bullet$ are resolutions, one shows that $Tot(P_\bullet \otimes_R P'_\bullet)$ is another resolution of M. Moreover, the maps $P_\bullet \to Tot(P_\bullet \otimes_R P'_\bullet)$ and $P'_\bullet \to Tot(P_\bullet \otimes_R P'_\bullet)$ induce morphisms of complexes after tensoring with some module N. One checks as in [**?**, Theorem 2.7.6] that the maps on homology induced by the morphisms of complexes in the previous sentence are isomorphisms. Therefore, the following deifinition makes sense.

Definition 5.4.2.4. Suppose M is a fixed R-module. If N is an arbitrary R-module, define $\mathrm{Tor}_i^R(M, N)$ as the i-th homology of the complex $P_\bullet \otimes_R M$ for any flat resolution $P_\bullet \to M$.

Lemma 5.4.2.5. *If M is a fixed R-module, then the following statements hold:*

1. *the groups $\mathrm{Tor}_i^R(M, N)$ can be computed using a projective resolution;*
2. *the groups $\mathrm{Tor}_i^R(M, N) = 0$ if $i < 0$;*
3. *the group $\mathrm{Tor}_0^R(M, N) = M \otimes N$;*
4. *there is an isomorphism $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M)$;*
5. *the groups $\mathrm{Tor}_i^R(M, N)$ have a natural R-module structure, functorially in the input modules; moreover the map induced by multiplication by $r \in R$ on M is precisely multiplication by r, i.e., the functor $\mathrm{Tor}_i^R(-, N)$ is an R-linear functor; and*
6. *given a short exact sequence $0 \to M' \to M \to M'' \to 0$, there is a functorially associated short exact sequence of R-modules of the form*

$$\cdots \longrightarrow \mathrm{Tor}_i^R(M', N) \longrightarrow \mathrm{Tor}_i^R(M, N) \longrightarrow \mathrm{Tor}_i^R(M'', N) \longrightarrow \mathrm{Tor}_{i-1}^R(M', N) \longrightarrow \cdots .$$

Remark 5.4.2.6. When we actually use these results, we will assume R is Noetherian and study finitely generated modules. In this case, a resolution by finitely generated flat modules will automatically be a projective resolution, so we will later be sloppy about the distinction. However, without suitable finiteness hypotheses in place, we will need to be careful about the difference between flat and projective resolutions. For example, take $R = \mathbb{Z}$ and consider $M = \mathbb{Q}$. Note that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module and therefore $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Q}, N) = 0$ for $i > 0$ and any $\mathbb{Z}$-module M. However, $\mathbb{Q}$ is not itself a projective $\mathbb{Z}$-module (it is an injective $\mathbb{Z}$-module) and has projective dimension 1.

5.4.3 Change of rings

If $R \to S$ is a ring homomorphism, and M and N are R-modules, then we can extend scalars from R to S to view $M \otimes_R S$ and $N \otimes_R S$ as S-modules. Now if $P_\bullet \to M$ is a projective resolution of M, we can use it to compute $\mathrm{Tor}_i^R(M, N) = H_i(P_\bullet \otimes_R N)$. The tensor product $P_\bullet \otimes_R S$ is *not* in general a projective resolution of $M \otimes_R S$. However, if $R \to S$ is a flat ring homomorphism, then $P_\bullet \otimes_R S$ *is* a flat resolution of $M \otimes_R S$. In that case, we deduce the following result.

Lemma 5.4.3.1. *If $\varphi : R \to S$ is a flat ring homomorphism, and M is an R-module, and N is an S-module then there is a functorial isomorphism*

$$\mathrm{Tor}_i^R(M, N) \otimes_R S \longrightarrow \mathrm{Tor}_i^S(M \otimes_R S, N).$$

In particular, if S is a localization of R, it follows that $\mathrm{Tor}_i^S(M \otimes_R S, N \otimes_R S)$ is a localization of the R-module $\mathrm{Tor}_i^R(M, N)$.

When $R \to S$ is not flat, the Tor-groups are still suitably functorial with respect to change of rings.

Lemma 5.4.3.2. *If $\varphi : R \to S$ is a ring homomorphism, M and N are R-modules, then there is a natural R-module map*

$$\mathrm{Tor}_i^R(M,N) \longrightarrow \mathrm{Tor}_i^R(M \otimes_R S, N \otimes_R S).$$

Definition 5.4.3.3. If M is an R-module, then we say that M has projective (resp. flat) dimension $\leq d$ if M admits a projective (resp. flat) resolution of length $\leq d$. We write $\mathrm{pd}(M)$ (resp. $\mathrm{fd}(M)$) for the minimum of the lengths of finite projective (resp. flat) resolutions of d (or ∞ if no such resolution exists).

Lemma 5.4.3.4. *If R is a Noetherian ring, and M is an R-module, the following conditions are equivalent:*

1. $\mathrm{pd}(M) = d$*;*
2. $\mathrm{Tor}_i^R(M,N) = 0$ *for $i > d$.*

Proof. The second statement implies the first since Tor can be computed using projective resolutions. For the other direction, we leave this as an exercise (for the time being): use the facts (i) that finitely presented flat modules are finitely generated projective and (ii) an arbitrary R-module can be written as a filtered colimit of its finitely presented sub-modules. □

Definition 5.4.3.5. If R is a ring, we say R has *finite global dimension* if $\sup\{\mathrm{pd}(M) | M \in \mathrm{Mod}_R\}$ is finite.

5.5 Homological theory of regular rings

5.5.1 Regular local rings and finite free resolutions

Our goal will be to study projective resolutions over regular local rings. If R is a regular local ring of dimension 0, then R is a field, and therefore every R-module is automatically projective. Therefore, regular local rings of dimension 0 have finite global dimension. If R is a regular local ring of Krull dimension 1, then R is a discrete valuation ring and therefore a principal ideal domain. In that case, every finitely generated module is the direct sum of a finitely generated free module and a finitely generated torsion module. Any finitely generated torsion module admits a free resolution of length 1 and therefore, we conclude that every finitely generated module admits projective resolutions of length ≤ 1. By careful limit arguments, one can show that not necessarily finitely generated modules also admit projective resolutions of length ≤ 1. We now analyze the global dimension of modules over regular local rings in general.

5.5.2 Minimal free resolutions

There are particularly nice free resolutions of finitely generated modules over Noetherian local rings. If M is finitely generated, then we can choose a minimal set of generators of M to obtain a surjection $F_0 \to M$. Continuing inductively, we can choose a minimal set of generators of $\ker(F_i \to F_{i-1})$ to build a free resolution of M. Such a resolution will be called a *minimal* free resolution. If $F_\bullet \to M$ is a minimal free resolution, then $F_i \mapsto F_{i-1}$ is given by an matrix with coefficients in R. Now, the image of $F_i \to F_{i-1}$ surjects onto the kernel of $F_{i-1} \to F_{i-2}$. Now, the kernel of $F_{i-1} \to F_{i-2}$ consists of relations among generators of F_{i-1}. If such a relation is given by a unit in R, then it follows that the two basis vectors are redundant; in other words if a resolution is minimal, then there is no relation with coefficient that is a unit. Thus, the image of $F_i \to F_{i-1}$ is contained in $\mathfrak{m}F_{i-1}$. Using this observation, we deduce the following fact.

Theorem 5.5.2.1. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, \kappa)$. The modules $\mathrm{Tor}_i^R(M, \kappa)$ are finite-dimensional κ-vector spaces and $\dim_\kappa(\mathrm{Tor}_i(M, \kappa))$ is the same rank as the rank of the i-th free module in a minimal free resolution of M. Moreover, the following statements are equivalent:*

- *in a minimal free resolution $F_\bullet$ of M, $F_{n+1} = 0$;*
- *the projective dimension of M is at most n;*
- $\mathrm{Tor}_{n+1}(M, \kappa) = 0$*;*
- $\mathrm{Tor}_i(M, \kappa) = 0$ *for all* $i \geq n + 1$.

It follows that a minimal free resolution is the shortest possible projective resolution of M. In particular, M has finite projective dimension if and only if a minimal free resolution is finite.

Proof. If M is a finitely generated module, we can compute $\mathrm{Tor}_i^R(M, \kappa)$ by taking a free resolution of M. Pick a *minimal* free resolution $F_\bullet \to M$. In that case, $F_\bullet \otimes_R \kappa$ is a complex of finite-rank κ-vector spaces and the finiteness of $\dim_\kappa(\mathrm{Tor}_i(M, \kappa))$ is immediate. Since the image of $F_i \to F_{i-1}$ is contained in $\mathfrak{m}F_{i-1}$, it follows that after tensoring with $\kappa = R/\mathfrak{m}$, the maps $F_i \otimes_R \kappa \to F_{i-1} \otimes_R \kappa$ are trivial. Therefore, $\dim_\kappa(\mathrm{Tor}_i(M, \kappa)) = \dim_\kappa F_i \otimes_R \kappa$.

Now, (1) $\Rightarrow$ (2) since Tor can be computed by a projective resolution. The statement (2) $\Rightarrow$ (3) is immediate from the definition of projective dimension. Note that (3) $\Rightarrow$ (1) as well, since we can compute $\mathrm{Tor}_{n+1}(M, \kappa)$ using a minimal free resolution, and in that case, $0 = \dim_\kappa(\mathrm{Tor}_{n+1}(M, \kappa)) = \dim_\kappa F_{n+1} \otimes_R \kappa$, i.e., $F_{n+1} = 0$. Once one term in a minimal free resolution is zero, we conclude that all higher terms are zero as well.

Note also that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\to$ (1) by a similar argument and using the fact that there cannot exist a resolution of length shorter than a minimal free resolution. $\square$

5.5.3 Tor and regular sequences

If R is a ring and $x \in R$ is not a zero-divisor, then we begin by establishing a connection between Tor_i^R and $\mathrm{Tor}_i^{R/x}$.

Proposition 5.5.3.1. *Let R be a ring and $x \in R$ an element.*

1. *Given an exact sequence $Q_\bullet$ of modules*

$$\cdots \longrightarrow Q_{n+1} \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots$$

such that x is not a zerodivisor on all Q_n, the complex $\bar{Q}_\bullet$ obtained by tensoring with R/xR, i.e.,

$$\cdots \longrightarrow Q_{n+1}/xQ_{n+1} \longrightarrow Q_n/xQ_n \longrightarrow Q_{n-1}/xQ_{n-1} \longrightarrow \cdots$$

is also exact.

2. *If x is not a zerodivisor in R and also not a zerodivisor on the module M, while $xN = 0$, then, for all i $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^{R/xR}(M/xM, N)$*

Proof. For Point (1), observe that since x is not a zero-divisor, the multiplication by x map determines a short exact sequence $Q_n \to Q_n \to Q_n/xQ_n$ for every n. Thus, multiplication by x yields an exact sequence of chain complexes of the form:

$$0 \longrightarrow Q_\bullet \xrightarrow{x} Q_\bullet \longrightarrow \bar{Q}_\bullet \longrightarrow 0.$$

Consider the associated long exact sequence in homology for this chain complex. Since $Q_\bullet$ is an exact sequence, it follows that $H_*(Q_\bullet) = 0$. Therefore, by the five lemma, we conclude that $H_*(\bar{Q}_\bullet) = 0$, i.e., $\bar{Q}_\bullet$ is exact as claimed.

For Point (2), take a free resolution of $F_n \to \cdots \to F_0 \to M \to 0$. By Point (1), this sequence remains exact after applying $\otimes_R R/xR$. Thus, we obtain a free resolution of M/xM over R/xR. Let $F_\bullet$ be the complex obtained by forgetting M in our free resolution. In that case, the homology at the n-th spot of $F_\bullet \otimes_R N$ computes $Tor_n^R(M, N)$. Since x kills N, $(R/xR) \otimes_{R/xR} N \cong N$. Thus, $Tor_n(M, N)$ is the homology at the n-th spot of $(F_\bullet \otimes_R R/(x)) \otimes_{R/(x)} N$. Since $F_\bullet \otimes_R R/(x)$ is a free resolution of M/xM over $R/(x)$ it follows that this gorup coincides with $Tor_n^{R/(x)}(M, N)$. □

Proposition 5.5.3.2. *If R is a regular local ring of Krull dimension $\leq d$, then any finitely generated R-module has projective dimension $\leq d$.*

Proof. Suppose R is a regular local ring; throughout we write $\mathfrak{m}$ for the maximal ideal in R and κ for the residue field. We proceed by induction on the dimension.

If $\dim R = 0$, then R is a field by Example 5.2.1.4. In that case, every finitely generated R-module is already free and the result follows.

Now, suppose $\dim R \geq 1$ and fix a finitely generated R-module M. It suffices to prove that $\mathrm{Tor}_n(M, \kappa) = 0$ for $n > d$ by the equivalent properties of Tor. Now, choose a projective module P and a surjection $P \to M$ and let M_1 be the kernel of this map so we have a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is finitely generated and free. Since $M_1 \subset F$, if we choose a regular parameter $x \in M$, x is not a zerodivisor on M_1 as well.

Therefore by Proposition 5.5.3.1 we conclude that $\mathrm{Tor}_n^R(M_1, \kappa) = \mathrm{Tor}_n^{R/xR}(M_1/xM_1, \kappa)$.

The long exact sequences for Tor associated with the above short exact sequence show that $\mathrm{Tor}_{n+1}^R(M, \kappa) \cong \mathrm{Tor}_n(M_1, \kappa) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, \kappa)$ for $n \geq d$. Since R is a regular local ring with maximal ideal $\mathfrak{m}$ and x is a regular parameter, we conclude that R/xR is again a regular local ring. □

Proposition 5.5.3.3. *If R is a ring, the following conditions are equivalent:*

1. *the ring R has finite global dimension;*
2. *every cyclic module R/I has projective dimension $\leq d$;*
3. *every finitely generated R-module has projective dimension $\leq d$.*

Proof. That (1) $\Rightarrow$ (2) is immediate from the definition.

To see that (2) $\Rightarrow$ (3), first observe that every finitely generated R-module has a finite filtration by cyclic modules. Indeed, we proceed by induction on the number of generators of M. Let $x_1, \ldots, x_r$ be a minimal generating set of M. Set $M' = Rx_1 \subset M$. In that case, M/M' has $r-1$ generators, and $M' \cong R/I_1$ with $I_1 = \{f \in R | fx_1 = 0\}$.

To see that (3) $\Rightarrow$ (2), we use a limit argument and write M as a filtered limit of finitely generated sub-modules. See [**?**, Tag 065T] for more details. □

Theorem 5.5.3.4. *If R is a regular local ring of Krull dimension $\leq d$, then R has finite global dimension.*

Proof. By Proposition 5.5.3.3 it suffices to show that all finitely generated modules have projective dimension $\leq d$, but this follows form Proposition 5.5.3.2. □

5.5.4 Globalizing

In this section, we globalize the results of the previous section and show that for an arbitrary regular ring R, every R-module admits a finite projective resolution. The following result links finiteness of the global dimension for a Noetherian ring to its localizations at maximal ideals.

Proposition 5.5.4.1. *If R is a Noetherian ring, then R has finite global dimension if and only if there exists an integer n such that for every maximal ideal $\mathfrak{m} \subset R$ the ring $R_{\mathfrak{m}}$ has global dimension $\leq n$.*

Localization is exact and preserves projectives. Thus, if R has finite global dimension, then $R_{\mathfrak{m}}$ has finite global dimension for any $\mathfrak{m}$; in fact, more generally, any localization of R has finite global dimension. To establish the converse, we will first state some preparatory lemmas. We begin by recording a useful trick due to Schanuel.

Lemma 5.5.4.2 (Schanuel's lemma)**.** *Suppose R is a ring and M is an R-module. Given two short exact sequences $0 \to K \to P_1 \to M \to 0$ and $0 \to L \to P_2 \to M \to 0$ with P_1 and P_2 projective, $K \oplus P_2 \cong L \oplus P_1$.*

Proof. The maps $P_1 \to M$ and $P_2 \to M$ yield a surjective map $P_1 \oplus P_2 \to M$, and we write N for the kernel of this map. Consider the composite maps

$$N \longrightarrow P_1 \oplus P_2 \longrightarrow P_i.$$

We claim these composites are surjective. Indeed, since the kernel of the map $P_1 \to M$ is K and the kernel of the map $P_2 \to M$ is L, one checks that the composite $N \to P_1$ is surjective with kernel L and $N \to P_2$ is surjective with kernel K. However, since the P_i are projective, the surjections $N \to P_i$ can be split, which yields the required isomorphism. □

Corollary 5.5.4.3. *Suppose R is a ring and M is an R-module of projective dimension d. Given $F_e \to F_{e-1} \to \cdots \to F_0 \to M$ an exact sequence with F_i projective and $e \geq d-1$, the kernel of $F_e \to F_{e-1}$ is projective (or the kernel of $F_0 \to M$ is projective if $e = 0$).*

Proof. We proceed by induction on d. If $d = 0$, then M is projective so given a surjection $F_0 \to M$, we can choose a splitting and identify $F_0 \cong M \oplus \ker(F_0 \to M)$ with both summands projective. Thus, if $e = 0$, we are done. If $e > 0$, then replacing M by $\ker(F_0 \to M)$ we can decrease e so we conclude by induction.

Now assume $d > 0$. Let $0 \to P_d \to P_{d-1} \to \cdots \to P_0 \to M$ be a minimal length finite resolution with P_i projective. By Schanuel's lemma 5.5.4.2 we see that $P_0 \oplus \ker(F_0 \to M) \cong F_0 \oplus \ker(P_0 \to M)$. Thus, the result is true if $d = 1$ and $e = 0$ since the right hand side is $F_0 \oplus P_1$, which is projective. Therefore, we may assume that $e > 0$. In that case, the module $F_0 \oplus \ker(P_0 \to M)$ has a finite projective resolution $0 \to P_d \oplus F_0 \to P_{d-1} \oplus F_0 \to \cdots \to P_1 \oplus F_0 \to \ker(P_0 \to M) \oplus F_0$ of length $d-1$. Thus, by induction on d, we conclude that $\ker(F_e \oplus P_0 \to F_{e-1} \oplus P_0)$ is projective. □

Proof of Proposition 5.5.4.1. Assume that R is a Noetherian ring and M is an R-module. We will prove that if M is finitely generated, and $R_{\mathfrak{m}}$ has finite global dimension for every localization of R at a maximal ideal $\mathfrak{m}$, then M has a finite projective resolution.

Thus, suppose M is finitely generated and $0 \to K_n \to F_{n-1} \to \cdots \to F_0 \to M$ is a resolution with each F_i finitely generated and free (since R is Noetherian we can always build such a resolution: pick generators of M and build a surjection $F_0 \to M$, take the kernel of this map, which is again finitely generated. In that case, since R is Noetherian, K_n is finitely generated. By Corollary 5.5.4.3 we conclude that $K_n \otimes_R R_{\mathfrak{m}}$ is projective for every $\mathfrak{m}$. However, since K_n is finitely generated and locally projective, it must be projective. In other words, we have constructed a finite projective resolution of M. □

Corollary 5.5.4.4. *If R is a regular ring of Krull dimension d, then every finitely generated projective R-module has finite projective dimension. Moreover, R has finite global dimension.*

Proof. If R is a regular ring, then $R_{\mathfrak{m}}$ is a regular local ring of Krull dimension d by definition. Proposition 5.5.3.4 implies that regular local rings of Krull dimension d have projective dimension $\leq d$. Therefore Proposition 5.5.4.1 guarantees that R has finite projective dimension as well. □

5.6 K_0 of regular rings: homotopy invariance and Mayer-Vietoris

Our goal in this lecture is to finally establish homotopy invariance of K_0 over regular rings. If R is a regular ring, then $R[t]$ is a regular ring by Proposition 5.3.2.5, and by induction it suffices to establish that the map $K_0(R) \to K_0(R[t])$ induced by $R \to R[t]$ is an isomorphism. This map is split by the evaluation map $R[t] \to R$, so it is automatically injective and therefore we just need to establish surjectivity. By Proposition 5.4.1.3 it suffices to show that every projective $R[t]$ module has a resolution by modules that are extended from R. The argument we give is due to Swan [**?**] as presented in [**?**, II.5].

5.6.1 $\mathbb{A}^1$-invariance and resolutions of projective modules over $R[t]$

Suppose N is a projective $R[t]$-module. If N admits a finite free resolution, then M is automatically extended from R. However, only stably free $R[t]$-modules admit finite free resolutions (indeed, if a finitely generated projective M admits a finite free resolution, then its class in K_0 is necessarily trivial). Nevertheless, we can start building a free resolution and study the failure of extensibility from R. More precisely, by picking $R[t]$-module generators, we obtain a surjection $R[t]^n \to M$, and the module $R[t]$ is evidently extended from R. The kernel M of $R[t]^n \to N$ is thus a submodule of an $R[t]$-module that is extended from R. While M itself is not evidently extended from R, we will show that we can resolve it by (finitely generated) $R[t]$-modules that *are* so extended. More precisely, following Swan, we establish the following result.

Lemma 5.6.1.1 (Swan)**.** *If R is a Noetherian ring and M is an $R[t]$-submodule of some finitely generated $R[t]$-module N that is extended from R, then there exists a short exact sequence*

$$0 \longrightarrow X \xrightarrow{g} Y \xrightarrow{f} M \longrightarrow 0$$

where X and Y are finitely generated R-modules that are extended from R.

Proof. Write $N = R[t] \otimes_R N_0$ for some $N_0 \in \mathrm{Mod}_R^{fg}$. In that case, set $N_r := \sum_{i=0}^r R{\cdot}t^i \otimes_R N_0$ and set $M_r = M \cap N_r$, $r \geq 0$. Since $N_r \in \mathrm{Mod}_R^{fg}$ and since R is Noetherian, we see that $M_r \in \mathrm{Mod}_R^{fg}$ as well. Since $R[t]$ is also Noetherian and since $N \in \mathrm{Mod}_{R[t]}^{fg}$, we know that $M \in \mathrm{Mod}_{R[t]}^{fg}$.

Pick an integer n large enough so that M_{n+1} contains an $R[t]$-module generating set of M and set $X = R[t] \otimes_R M_n$ and $Y = R[t] \otimes_R M_{n+1}$. Define a map $f : Y \to M$ by $f(t^i \otimes m) = t^i m$ for every $m \in M_{n+1}$ and extend by linearity. Note that M_{n+1} is contained in the image of f by construction and therefore f is automatically a surjective $R[t]$-module map.

We construct an $R[t]$-module homomorphism $g : X \to Y$ as follows. Observe that $tN_n \subset N_{n+1}$ and therefore $tM_n \subset M_{n+1}$ as well. Now, define g by means of the formula

$$g(t^i \otimes m) = t^{i+1} \otimes m - t^i \otimes tm, \qquad m \in M_n,$$

and extend this by linearity to an $R[t]$-module morphism.

It remains to check that the resulting sequence is exact. First, we claim that $fg = 0$. To see this, take $m \in M_n$ and compute:

$$fg(t^i \otimes m) = f(t^{i+1} \otimes m - t^i \otimes tm) = t^{i+1}m - t^{i+1}m = 0.$$

Next, we claim that g is injective. Take $x = t^i \otimes m + t^{i-1} \otimes m' + \cdots$ with $m, m' \in M_n$ and $m \neq 0$. In that case,

$$\begin{aligned} g(x) &= g(t^i \otimes m + t^{i-1} \otimes m' + \cdots) \\ &= (t^{i+1} \otimes m - t^i \otimes tm) + (t^i \otimes m' + t^{i-1} \otimes tm') \\ &= (t^{i+1} \otimes m) + t^i \otimes (m' - tm) + \cdots, \end{aligned}$$

and $t^{i+1} \otimes m \neq 0$ so $g(x) \neq 0$.

To conclude, it remains to show that $\ker(f) = \mathrm{im}(g)$. Suppose $y = \sum_{i=0}^r t^i \otimes m_i \in \ker(f)$, with $m_i \in M_{n+1}$. We will show that $y \in \mathrm{im}(g)$ by induction on r. If $r = 0$, this is clear. Thus, assume $r \geq 0$. Write $m_i = \sum_{j=0}^{n+1} t^j \otimes a_{ij}$ where $a_{ij} \in N_0$. In that case,

$$\begin{aligned} 0 = f(y) = \sum_{i=0}^r t^j m_i &= \sum_{i=0}^r \sum_{j=0}^n t^{i+j} \otimes a_{ij} \\ &= t^{r+n+1} \otimes a_{r,n+1} + t^{r+n} \otimes (\ldots) + \cdots . \end{aligned}$$

Thus, $a_{r,n+1} = 0$, i.e., $m_r \in M_n$. Then,

$$y - g(t^{r-1} \otimes m_r) = y - t^r \otimes m_r + t^{r-1} \otimes tm_r = \sum_{i=0}^{r-1} t^i \otimes m_i' \ldots$$

□

So far, we have not used regularity of R. Under this assumption, using Corollary 5.5.4.4 we may further resolve the X and Y as above by projective modules that are extended from R.

Proposition 5.6.1.2 (Swan)**.** *If R is a regular ring, then every finitely generated $R[t]$-module M admits a finite projective resolution $P_\bullet \to M$ with each P_i extended from R.*

Proof. Suppose M is a finitely generated $R[t]$-module. As above, pick a surjection $R[t]^n \to M$ and let M' be its kernel. In that case, Lemma 5.6.1.1 guarantees the existence of an exact sequence of the form:

$$0 \longrightarrow X \xrightarrow{g} Y \longrightarrow M' \longrightarrow 0,$$

where X and Y are finitely generated $R[t]$-modules that are extended from R.

If $\bar{X}$ and $\bar{Y}$ are finitely generated R-modules such that $X = \bar{X} \otimes_R R[t]$ and $Y = \bar{Y} \otimes_R R[t]$, then since R is regular, by appealing to Corollary 5.5.4.4, we can find a finite projective resolutions $\bar{X}_\bullet \to \bar{X}$ and $\bar{Y}_\bullet \to \bar{Y}$.

Next, note that $R[t]$ is free as an R-module (of countable rank). In particular, it follows that $R[t]$ is flat as an R-module and therefore that $R \to R[t]$ is a flat ring homomorphism. Therefore, it follows that $X_\bullet := \bar{X}_\bullet \otimes_R R[t] \to X$ and $Y_\bullet := \bar{Y}_\bullet \otimes_R R[t] \to Y$ are again projective resolutions of X and Y.

Since $X_\bullet$ and $Y_\bullet$ are projective, we can inductively lift the morphim g to a morphism of complexes (abusing terminology)

$$g : X_\bullet \longrightarrow Y_\bullet.$$

To obtain a resolution of M', we form the *mapping cone* of g. More precisely, we define a new complex $C(g)_\bullet$ whose terms are $C(g)_\bullet := X_{\bullet-1} \oplus Y_\bullet$ and where the differential $X_{i-1} \oplus Y_i \to X_{i-2} \oplus Y_{i-1}$ is given by the matrix

$$\begin{pmatrix} -d_X^{i-1} & 0 \\ g & d_Y^i \end{pmatrix}.$$

Now, one checks that $C(g)$ is actually a chain complex, and that $C(g)$ is exact except in degree 0 where the cohomology is M'. In other words, we have produced a resolution

$$0 \longrightarrow C(g) \longrightarrow R[t]^{\oplus n} \longrightarrow M \longrightarrow 0.$$

By assumption, the terms of $C(g)$ are projective $R[t]$-modules extended from R, and the result follows. □

Putting everything together, we obtain the following result.

Theorem 5.6.1.3 (Grothendieck)**.** *If R is a regular ring, then $K_0(R) \to K_0(R[t])$ is an isomorphism of rings.*

5.6.2 Mayer-Vietoris

We can also deduce a Mayer-Vietoris sequence just as for the Picard group. To begin, recall that if f and g are comaximal elements of a ring R, then there is a fiber product diagram of categories of the form:

$$\begin{tikzcd} \mathrm{Vec}(R_{fg}) \arrow[r] \arrow[d] & \mathrm{Vec}(R_f) \arrow[d] \\ \mathrm{Vec}(R_g) \arrow[r] & \mathrm{Vec}(R). \end{tikzcd}$$

The next result can be obtained from directly from this patching result (and thus could have been established immediately after our definition of K_0).

Proposition 5.6.2.1 (Weak Mayer-Vietoris)**.** *If R is a commutative unital ring and f and g are comaximal elements of R, then there is a short exact sequence of the form*

$$K_0(R) \longrightarrow K_0(R_f) \oplus K_0(R_g) \longrightarrow K_0(R_{fg}).$$

Proof. Suppose P is a projective R-module. Suppose we have projective R_f and R_g modules P_f and P_g whose classes in $K_0(R_{fg})$ agree. In that case, the modules $(P_f)_g$ and $(P_g)_f$ are stably isomorphic. Therefore, we can fix an isomorphism $(P_f)_g \oplus R^n_{fg} \tilde{\to} (P_g)_f \oplus R^n_{fg}$. Since the modules R_{fg} are free, the are obtained via restriction. Therefore, we can glue these modules together to get an R-module. Thus, the image of $K_0(R) \to K_0(R_f) \oplus K_0(R_g)$ surjects onto the kernel of the difference map. $\square$

Remark 5.6.2.2. In general, both the kernel of $K_0(R) \to K_0(R_f) \oplus K_0(R_g)$ and the cokernel of $K_0(R_f) \oplus K_0(R_g) \to K_0(R_{fg})$ are non-trivial. One of the original goals of K-theory was to measure the failure of surjectivity and to turn K-theory into a cohomology theory. For example, we can describe the kernel of $K_0(R) \to K_0(R_f) \oplus K_0(R_g)$ in terms of automorphisms of projective modules on R_{fg}, just by paying attention to patching. Originally, one built ad hoc groups that allowed one to extend the above (very) short exact sequence to the left. Quillen eventually gave a good definition of higher K-theory, but it took longer to obtain Mayer-Vietoris sequences in great generality.

5.7 G-theory and the localization sequence

We introduce here a variant of K-theory of a scheme X. We focus on the case where X is Noetherian and consider the abelian category $Coh(X)$ of coherent sheaves on X.

Definition 5.7.0.1. If X is a scheme, then $G_0(X)$ is the quotient of the free abelian group on isomorphism classes of objects in $Coh(X)$ by the ideal generated by the relations:

$$[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$$

whenever there is a short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0.$$

Example 5.7.0.2. We can study $G_0(\mathbb{Z})$ explicitly using the structure theorem. Any finitely generated $\mathbb{Z}$-module can be written as a sum of a free part and a torsion part. The torsion part is itself a sum of finite cyclic groups. Now, note that there is an exact sequence of the form

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0,$$

which yields a relation of the form $[\mathbb{Z}] \cong [\mathbb{Z}] + [\mathbb{Z}/n\mathbb{Z}]$. Cancelling in the abelian group $G_0(\mathbb{Z})$ we see that $[\mathbb{Z}/n\mathbb{Z}] = 0$. In other words, the class in K-theory of any torsion group is zero. This example generalizes to principal ideal domains.

Example 5.7.0.3. We can study $G_0(\mathbb{Z}/p^n)$ and $K^0(\mathbb{Z}/p^n)$ as well.

Note that there is always a homomorphism $K^0(X) \to G_0(X)$ by considering the subcategory $\mathrm{Vect}(X) \to Coh(X)$; this homomorphism is sometimes called the Cartan homomorphism. The previous example shows that this map is an isomorphism for $\mathbb{Z}$. In fact, that statement is true much more generally.

Theorem 5.7.0.4. *If X is the spectrum of a regular ring, then the canonical map $K^0(X) \longrightarrow G_0(X)$ is an isomorphism.*

Proof. This follows from the fact that any regular ring has a finite projective resolution. □

In fact, the above theorem holds even more generally. If X is a regular Noetherian scheme with affine diagonal (e.g., separated), then the Cartan homomorphism is an isomorphism.

Proposition 5.7.0.5. *If $X = \mathbb{P}^n_{\mathbb{Z}}$, then the map $K^0(\mathbb{P}^n_{\mathbb{Z}}) \to G_0(\mathbb{P}^n_{\mathbb{Z}})$ is an isomorphism.*

The G_0 of a scheme behaves somewhat differently than K^0 from the standpoint of functoriality. First, it's not immediately apparent how to construct a pullback morphism $f^* : G_0(Y) \to G_0(X)$ for an arbitrary morphism $f : X \to Y$ of schemes. Indeed, even though f^* will take coherent sheaves to coherent sheaves, it *fails* to preserve exact sequences in general. For example, the pullback along a closed immersion $i : \operatorname{Spec} R/I \to \operatorname{Spec} R$ corresponds to tensoring R-modules with R/I and failure of exactness is precisely measured by the existence of Tor-functors

Lemma 5.7.0.6. *If $f : X \to Y$ is a flat morphism of schemes, then there is an induced pullback morphism $f^* : G_0(Y) \longrightarrow G_0(X)$.*

Proof. It suffices to observe that if $f : X \to Y$ is a flat morphism of schemes, then $f^* : Coh(Y) \to Coh(X)$ is an exact functor. □

There is one other bit of functoriality that exists for G_0 that would require more effort to construct for K^0: a pushforward. If $f : X \to Y$ is a closed immersion of schemes, then $f_* : X \to Y$ sends coherent sheaves to coherent sheaves. Indeed, suppose $f : \operatorname{Spec} R/I \to \operatorname{Spec} R$ is a closed immersion of affine schemes. In that case, we have the ring homomorphism $R \to R/I$ and the pushforward functor corresponds to viewing an R/I-module as an R-module. In that case, exactness is immediate: if we have an exact sequence of R/I-modules then it remains exact when viewed as a sequence of R-modules. The only thing that remains to be checked is that if we have a coherent R/I-module, then it remains coherent when viewed as an R-module. Let us assume that all schemes in question are Noetherian, so R is a Noetherian ring. In that case, if M is an R/I-module that is finitely presented as an R/I-module.

Proposition 5.7.0.7. *If $i : Z \to X$ is a closed immersion of schemes, cut out by a quasi-coherent sheaf of ideals $\mathscr{I}$, then the functor i_* is an exact and fully-faithful functor $QCoh(\mathscr{O}_Z) \to QCoh(\mathscr{O}_X)$ whose essential image consists of quasi-coherent sheaves $\mathscr{F}$ on $\mathscr{O}_X$ such that $\mathscr{I}\mathscr{F} = 0$. If X is furthermore locally Noetherian, then i_* sends coherent sheaves to coherent sheaves.*

Proof. Add. □

5.7.1 Devissage

Theorem 5.7.1.1. *Let $\mathbf{A}$ be an abelian category and $\mathbf{B}$ be an exact abelian subcategory of $\mathbf{A}$ that is closed under formation of quotients and subobjects. If every object of $\mathbf{A}$ has a finite filtration with subquotients in $\mathbf{B}$, then the map $i_* : K_0(\mathbf{B}) \to K_0(\mathbf{A})$ induced by the inclusion functor $\mathbf{B} \subset \mathbf{A}$ is an isomorphism.*

Proof. We first observe that i_* is surjective. Every object A in $\mathbf{A}$ has a filtration $A = A_0 \supset A_1 \subset \cdots \supset A_n = 0$ such that A_i/A_{i+1} is an object in $\mathbf{B}$. By induction on the length of the filtration we conclude that

$$[A] = [A_0/A_1] + \cdots + [A_{n-1}/A_n]$$

where the latter object lies in the image of $K_0(\mathbf{B})$ by definition.

Next, we claim that i_* is injective. To this end, we will construct an explict inverse function. Since an object $A \in \mathbf{A}$ has a finite filtration with successive subquotients in $\mathbf{B}$, we would like to use the formula above to define the inverse: send $[A]$ to $\sum_{i=0}^{n}[A_i/A_{i+1}]$. What is unclear is that this is well-defined: since we might have another filtration of A which defines an *a priori* different element of $K_0(\mathbf{B})$. However, any two filtrations have an equivalent common refinement and the result follows. □

Definition 5.7.1.2. Suppose X is a Noetherian scheme, and $Z \subset X$ is a closed subscheme, then we write $Coh_Z(X)$ for the category of coherent $\mathscr{O}_X$-modules that are supported on Z, i.e., coherent $\mathscr{O}_X$-modules that are killed by some power of $\mathscr{I}$ where $\mathscr{I}$ is the quasi-coherent sheaf of ideals that cuts out Z in X.

Corollary 5.7.1.3. *Let $Z \subset X$ be a closed subscheme, then the inclusion $G_0(Z) \to K_0(Coh_Z(X))$ is an isomorphism.*

Proof. An immediate consequence of devissage since every $\mathscr{O}_X$-module supported on Z admits a finite filtration with subquotients that are killed by $\mathscr{I}$ and therefore come from Z. □

5.7.2 Localization

Now, suppose X is a Noetherian scheme and let $Z \subset X$ be an closed subscheme with open complement U. In that case, the inclusion $j : U \subset X$ is a flat morphism (as an open immersion) so we have the exact functor

$$j^* : Coh(X) \longrightarrow Coh(U),$$

which induces $j^* : G_0(X) \to G_0(U)$. We claim that this functor is always surjective and we can even describe its kernel. Indeed, a coherent sheaf on X whose restriction to U is trivial is necessarily supported on Z. Thus, we have the sequence of exact functors:

$$Coh_Z(X) \subset Coh(X) \longrightarrow Coh(U).$$

We will see that this sequence gives rise to an exact sequence of Grothendieck groups. The right hand functor is in fact known to be essentially surjective: every coherent sheaf on U extends to a coherent sheaf on X. Thus, the right hand functor is thus akin to a surjective group homomorphism, and by this analogy might be thought of as a quotient of the category $Coh(X)$. The category $Coh_Z(X)$ has the property that it is an abelian full subcategory of $Coh(X)$ but it is furthermore closed under taking sub-objects, quotients, and formation of extensions. We would like to think of such a subcategory as akin to a normal subgroup: in that case we would like to form a suitable quotient category. We now say this more abstractly.

Construction 5.7.2.1 (Quotient category)**.** Suppose $\mathbf{A}$ is an abelian category. A Serre subcategory $\mathbf{B} \subset \mathbf{A}$ is an abelian subcategory that is closed under formations of subobjects, quotients and extensions. We would like to construct a quotient category $\mathbf{A}/\mathbf{B}$; this should come equipped with a functor $\mathbf{A}$ satisfying the universal property that if $\mathbf{C}$ is any abelian category such that the objects and morphisms in $\mathbf{B}$ are sent to 0 in $\mathbf{C}$, then there is a unique functor $\mathbf{A}/\mathbf{B} \to \mathbf{C}$ factoring the given functor.

We define the objects of $\mathbf{A}/\mathbf{B}$ to be the objects of $\mathbf{A}$. A morphism $f : A \to A'$ in $\mathbf{A}$ will be called a $\mathbf{B}$-monomorphism, resp. $\mathbf{B}$-epimorphism if the kernel (resp. cokernel) of f lies in $\mathbf{B}$, and a $\mathbf{B}$-isomorphism if both the kernel and the cokernel of f lie in $\mathbf{B}$. A morphism in $\mathbf{A}/\mathbf{B}$ will an equivalence class of diagrams of the form

$$A_1 \longleftarrow A \longrightarrow A_2$$

where the left and right arrows are $\mathbf{B}$-isomorphisms. The equivalence classes will be given by diamonds. Composition is given by pullbacks of diamonds. There is an induced functor

$$loc : \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{B}$$

that is an isomorphism on objects and sends a morphism to its equivalence class. One has to check that $\mathbf{A}/\mathbf{B}$ is an abelian category and loc is an exact functor with the universal property described above.

Theorem 5.7.2.2 (Gabriel)**.** *If X is a Noetherian scheme, U is an open subscheme and Z is the complementary closed subset with its reduced induced scheme structure, then the restriction functor $j^* : Coh(X) \longrightarrow Coh(U)$ identifies $Coh(U)$ with the quotient category $Coh(X)/Coh_Z(X)$.*

If $\mathbf{A}/\mathbf{B}$ is a quotient, then the localization functor

$$loc : \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{B}$$

preserves exact sequences and thus induces a homomorphism

$$K_0(\mathbf{A}) \longrightarrow K_0(\mathbf{A}/\mathbf{B});$$

this function is evidently surjective. The next theorem identifies its kernel.

Theorem 5.7.2.3 (Localization)**.** *If* $\mathbf{B}$ *is a Serre subcategory of an abelian category* $\mathbf{A}$*, then there is an exact sequence of the form*

$$K_0(\mathbf{B}) \longrightarrow K_0(\mathbf{A}) \longrightarrow K_0(\mathbf{A}/\mathbf{B}) \longrightarrow 0.$$

Proof. Note that any object in $\mathbf{B}$ is sent to the zero object in $\mathbf{A}/\mathbf{B}$ since the map $\mathbf{B} \to 0$ is a $\mathbf{B}$-isomorphism by definition. It follows that the composite map $G_0(\mathbf{B}) \to G_0(\mathbf{A}/\mathbf{B})$ is the zero map and there is an induced surjection

$$K_0(\mathbf{A})/K_0(\mathbf{B}) \to K_0(\mathbf{A}/\mathbf{B}).$$

It suffices to prove that this map is injective, and we do this by constructing an explicit inverse function. To this end, since the objects of $\mathbf{A}/\mathbf{B}$ are the same as the objects of $\mathbf{A}$, it would suffice to show that sending $[A]$ to $[A]$ is additive. It remains to show that if $loc(A_1)$ is isomorphic to $loc(A_2)$ in $\mathbf{A}/\mathbf{B}$, then $[A_1] = [A_2]$ in the quotient $K_0(\mathbf{A})/K_0(\mathbf{B})$. Indeed, in that case we can represent the isomorphism $A_1 \to A_2$ in $\mathbf{A}/\mathbf{B}$ by a diagram $A_1 \leftarrow A \to A_2$ where both the left and right arrows are $\mathbf{B}$-isomorphisms. In other words, there are exact sequences

$$0 \longrightarrow \ker(f) \longrightarrow A \longrightarrow A_1 \longrightarrow \operatorname{coker}(f) \longrightarrow 0,$$

which yield $\ker(f) = [A] + [A/\ker(f)]$ and $[A_1] = [\operatorname{coker}(f)] + [A/\ker(f)]$, i.e.,

$$[A] - [\ker(f)] + [A_1] - [\operatorname{coker}(f)] = 0.$$

Similarly,

$$[A] - [\ker(g)] + [A_2] - [\operatorname{coker}(g)] = 0.$$

Thus, $[A_1] = [A_2]$ in $K_0(\mathbf{A})/K_0(\mathbf{B})$ as claimed.

The additivity can be checked similarly. □

Theorem 5.7.2.4 (Localization sequence)**.** *If* X *is a Noetherian scheme,* U *is an open subscheme and* $Z \subset X$ *is the complement with its reduced induced structure, then there is a localization exact sequence*

$$G_0(Z) \xrightarrow{i_*} G_0(X) \xrightarrow{j^*} G_0(U) \longrightarrow 0.$$

Proof. From the localization theorem combined with Gabriel's theorem, we get an identification

$$K_0(Coh_Z(X)) \longrightarrow G_0(X) \longrightarrow G_0(U) \longrightarrow 0.$$

By devissage, we conclude that $G_0(Z) \to K_0(Coh_Z(X))$ is an isomorphism. Moreover, i_* sends a coherent sheaf on Z to the coherent sheaf on X that is annihilated by $\mathscr{I}$ so we're done. □

Corollary 5.7.2.5. *If* X *is a separated regular Noetherian scheme and* U *is an open subscheme with closed complement* Z *with its reduced-induced structure, then there is an exact sequence of the form*

$$G_0(Z) \longrightarrow K^0(X) \longrightarrow K^0(U) \longrightarrow 0.$$

If Z *is furthermore regular, then the term on the left can be replaced by* $K^0(Z)$ *and the sequence remains exact.*

5.7.3 Computations/Consequences

Diagram chasing, we get a Mayer–Vietoris sequence for K-theory of separated regular Noetherian schemes.

Example 5.7.3.1. The K-theory of $\mathbb{P}^n_{\mathbb{Z}}$ is $\mathbb{Z}^{n+1}$ by induction. We do this by induction on n using the localization sequence and the fact that We know that $K^0(\operatorname{Spec}\mathbb{Z}) = \mathbb{Z}$ and $K^0(\mathbb{A}^n_{\mathbb{Z}}) = \mathbb{Z}$ as well. Now, consider the localization sequence

$$K^0(\mathbb{P}^{n-1}) \longrightarrow K^0(\mathbb{P}^n) \longrightarrow K^0(\mathbb{A}^n) \longrightarrow 0.$$

The right hand morphism is a split surjection (split by pullback along the projection $\mathbb{P}^n \to \operatorname{Spec} k$). We claim the left hand morphism is also injective and the result follows by induction.

5.8 K_1, units and homotopy invariance

At the end of the previous section we observed the existence of a portion of the Mayer-Vietoris sequence for K_0 and we observed that the failure of injectivity of the first map was described, via patching ideas, in terms of automorphisms. We now make this more precise by introducing the functor K_1.

5.8.1 K_1 of a ring: basic definitions

Suppose R is a commutative unital ring. Consider the inclusion maps $GL_n(R) \to GL_{n+1}(R)$ defined by the formula

$$X \longmapsto \left(\begin{array}{c|c} X & 0 \\ \hline 0 & 1 \end{array}\right),$$

and set

$$GL(R) := \operatorname{colim}_n GL_n(R),$$

where the colimit is formed in the category of groups. We will refer to $GL(R)$ as the stable or "infinite" general linear group.

If G is any group, recall that the commutator subgroup $[G, G]$ is subgroup generated by commutators $[g, h] = ghg^{-1}h^{-1}$. The quotient $G/[G, G] = G^{ab}$ is an abelian group. Moreover, if A is any abelian group, then given any homomorphism $\varphi : G \to A$, the composite map $[G, G] \to G \to A$ is trivial so φ factors through a map $G/[G, G] \to A$. In particular, the assignment $G \mapsto G/[G, G]$ is a left adjoint to the forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$.

Definition 5.8.1.1. If R is a commutative unital ring, then

$$K_1(R) := GL(R)/[GL(R), GL(R)];$$

this is an abelian-group valued functor on the category of commutative unital rings.

If R is any ring, then $\det : GL_n(R) \to \mathbf{G}_m(R)$ is a homomorphism. Since $\det$ commutes with the inclusion maps $GL_n(R) \to GL_{n+1}(R)$, it follows that there is an induced map $\det : GL(R) \to \mathbf{G}_m(R)$. Since $\mathbf{G}_m(R)$ is abelian, this map factors uniquely through a homomorphism

$$\det : K_1(R) \longrightarrow R^{\times}$$

The maps $GL_n(R) \to R^{\times}$ are split by the map sending u to the diagonal matrix $\mathrm{diag}(u, 1, \ldots, 1)$. Since these maps are also compatible with stabilization, we conclude that there is an induced splitting $R^{\times} \to GL(R)$ and thus a splitting $R^{\times} \to K_1(R)$ of $\det$. In particular, $\det$ is always surjective. If we write $SK_1(R) = \ker(\det : K_1(R) \to R^{\times})$, then using the splitting we conclude that $K_1(R) \cong R^{\times} \oplus SK_1(R)$.

Proposition 5.8.1.2. *If F is a field, then $\det : K_1(F) \to F^{\times}$ is an isomorphism.*

Next, we develop the link between K_1 and projective modules. Begin by observing that if F is a finite rank free R-module, then the homomorphism $GL_n(R) \to K_1(R)$ shows that any automorphism of F gives rise to an element of $K_1(R)$. Suppose more generally that P is a finitely generated projective R-module. Since P is a summand of a finite rank free R-module, we can write $P \oplus Q = R^n$ for some projective module Q. Suppose $\alpha : P \to P$ is an automorphism of P. The choice of splitting allows us to extend α to the automorphism (α, id_Q) of R^n. We now claim that α has a well-defined class in $K_1(R)$ independent of the splitting $P \oplus Q = R^n$. Indeed, any other splitting differs from this one by an automorphism of R^n. Thus, if X is a matrix representing (α, id_Q), then gXg^{-1} represents the new splitting. This defines an inner automorphism of $GL_n(R)$. Stabilizing, such automorphisms act trivially on the abelianization. Therefore, we conclude that for any f.g. projective R-module P there is a well-defined map $Aut_R(P) \longrightarrow K_1(R)$.

5.8.2 The Bass–Heller-Swan theorem

As before, we can consider the map $K_1(R) \to K_1(R[t])$ for any commutative unital ring R. This map is always split injective. We now observe that in the same situations as for K_0, it is also surjective. Because of the existence of the determinant homomorphism, by appeal to homotopy invariance for units, we see that a necessary condition for surjectivity is that R is a reduced ring. The following result, due to Bass–Heller–Swan [**?**], has a proof very similar to that given for K_0 above.

Theorem 5.8.2.1. *If R is a regular ring, then $K_1(R) \to K_1(R[t_1, \ldots, t_n])$ is an isomorphism.*

Chapter 6

Vector bundles and $\mathbb{A}^1$-invariance

We give Quillen's solution to the Serre problem on freeness of projective modules over polynomial rings over fields (or, more generally, PIDs). The two key tools are "Horrocks' theorem" and Quillen's "local-to-global" principle.

6.1 The Quillen–Suslin theorem

6.1.1 Stably free modules vs. unimodular rows

Proposition 6.1.1.1. *Assume k is a commutative ring. If P is a finitely generated projective k-module of the form $P : \ker(f : k^n \to k^m)$, then P is free if and only if f can be lifted to an isomorphism $\tilde{f} : k^n \xrightarrow{\sim} k^m \oplus k^r$ such that f coincides with $\tilde{f}$ followed by the projection onto the first factor.*

Proof. If f can be lifted to an isomorphism $\hat{f}$ as in the statement, then $\ker(f)$ coincides with the kernel of the projection which is evidently free. Conversely, suppose P is free via an isomorphism $g : P \to k^n$. In that case, $Q = \ker(g)$ is projective and the restriction of f to Q determines an isomorphism $Q \xrightarrow{\sim} k^m$. The map $f_0 \oplus g$ then gives the required lift of f. $\square$

The above proposition can be phrased in terms of matrices as well: if P is a stably free R-module, then since f as is in the statement above is split, it corresponds to the kernel of an $m \times n$-matrix M that is right invertible (in the sense that there exists an $n \times m$-matrix whose product with M yields the $m \times m$-identity matrix). Sometimes, such matrices are called *unimodular* $m \times n$-*matrices*. We already mentioned the special case where $r = 1$: such matrices are called unimodular rows. The isomorphism $k^n \to k^m \oplus k^r$ in the statement means that after change of basis, M can be realized as the first m rows of an invertible $n \times n$-matrix. Matrices of the form M are called completeable unimodular $m \times n$-matrices. Of course, if every stable free module over k is free, then, in particular, every unimodular row is completeable. In fact, the converse holds.

Corollary 6.1.1.2. *If k is a commutative ring, then the following statements are equivalent.*

1. *Any stably free f.g. projective k-module is free.*
2. *Any unimodular row over k is completeable.*

Proof. The second statement implies the first by induction on m. □

For any ring k, we know that $k \to k[x]$ induces an isomorphism $K^0(k) \to K^0(k[x])$. In particular, if $K^0(k) = \mathbb{Z}$, i.e., if all stably free k-modules are free, then the same holds true for $k[x]$. By induction if k is a principal ideal domain, we conclude that all stably free $k[x_1, \ldots, x_n]$-modules are free. The result above then shows that if we want to establish that all f.g., projective $k[x_1, \ldots, x_n]$-modules are free, it suffices to establish this fact for unimodular rows (of arbitrary length).

6.1.2 Vaserstein's proof of the Quillen–Suslin

Suppose R is a commutative ring. Observe that there is an action of $GL_n(R)$ on $Um_n(R)$ given by left multiplication. We will say that two unimodular rows of length n are equivalent if the cosets in $Um_n(R)/GL_n(R)$ coincide. Two elements of $Um_n(R)$ lying in the same orbit for this $GL_n(R)$-action determine isomorphic f.g., projective modules. The properties of this action are summarized in the following proposition, whose proof is left as an exercise.

Proposition 6.1.2.1. *The assignment sending an element of $(f_1, \ldots, f_n) \in Um_n(R)$ to the associated projective module $ker(f_1, \ldots, f_n)$ determines a bijection between the set of orbits $Um_n(R)/GL_n(R)$ and the set of rank $n-1$ projective R-modules such that $P \oplus R$ is free. This bijection sends the orbit of $(1, 0, \ldots, 0)$ to the free module of rank n.*

Now, consider the polynomial ring $R[t]$. Suppose we give ourselves a unimodular row $(f_1, \ldots, f_n)$ over $R[t]$. Each polynomial $f_i = \sum_{j=0}^{n_i} a_{ij} t^j$ with $a_{i,n_i} \neq 0$; we will refer to the term a_{i,n_i} as the leading coefficient of f_i.

Theorem 6.1.2.2 (Vaserstein)**.** *Assume $\mathbf{f} := (f_1, \ldots, f_n)$ is a unimodular row of length n over $R[t]$. If the leading coefficients of the f_i generate the unit ideal in R, then $\mathbf{f}(t)$ lies in the same orbit for $GL_n(R[t])$ as $\mathbf{f}(0)$.*

Let us first show how this result implies that stably free k-modules, k a field, are free.

Theorem 6.1.2.3. *If k is a field, then every f.g. projective $k[x_1, \ldots, x_n]$-module is free.*

Proof. We proceed by induction on d: we already know that every f.g., projective k-module is free so the result is true for $d = 0$. Thus, assume $d > 0$ and suppose $\mathbf{f} = (f_1, \ldots, f_n)$ is a unimodular row of length n over $k[x_1, \ldots, x_d]$. Without loss of generality, we may assume $f_1 \neq 0$. We claim there exists a change of variables

$$t_1 \mapsto t_1, t_i \mapsto t_i + t_1^{r_i}, 2 \leq i \leq d$$

such that $f_1(t_1, t_2 + t_1^{r_2}, \ldots) = ch(t_1, \ldots, t_d)$ with $c \in k \setminus 0$ and $h(t_1, \ldots, t_d)$ a monic polynomial in t_1. Indeed, let

$$f_1 = \sum a_{i_1,\ldots,i_d} t_1^{i_1} \cdots t_d^{i_d}.$$

Then

$$f_1(t_1, t_2 + t_1^{r_2} + \cdots, t_d + t_1^{r_d}) = \sum a_{i_1,\ldots,i_d}(t_1^{i_1 + r_1 i_2 + \cdots + r_d i_d} + \text{terms with lower} t_1\text{-degree})$$

Now, we may choose $r_2, \ldots, r_d$ such that $i_1 + r_1 i_2 + \cdots + r_d i_d$ are distinct for all the intervening d-tuples $(i_1, \ldots, i_d)$. In fact, if m is an integer greater than $max(i_1, \ldots, i_d)$, then we may choose $r_j = m^{j-1}$. In that case, the monomial with the highest non-vanishing coefficient will be have non-zero leading coefficient and the result follows. This argument is part of Nagata's proof of the Noether normalization theorem.

Importantly, note that this change of variables determines a k-algebra automorphism of $k[x_1, \ldots, x_d]$. Thus, after choosing this automorphism, we can assume that f_1 is, up to a scalar multiple, a monic polynomial in $R[t_1]$ where $R = k[t_2, \ldots, t_d]$. Note that the leading coefficient of this polynomial is a unit and therefore certainly generates the unit ideal. By the proposition above, we conclude that our original unimodular row lies in the same $GL_n(R[t])$-orbit as its corresponding constant term, which is a polynomial ring of lower degree. By the induction hypothesis, all stably free modules over $k[x_1, \ldots, x_d]$ are free and therefore this unimodular row is completeable. Thus, our original unimodular row is a free module and we conclude. □

Thus, let us now concentrate on establishing Vaserstein's theorem. First, we know that unimodular rows over local rings are completable, so a test case is a polynomial ring over a local ring.

Proposition 6.1.2.4 (Horrocks)**.** *If R is a local ring and $\mathbf{f} = (f_1, \ldots, f_n)$ a unimodular row of length n in $R[t]$ such that f_1 has leading coefficient a unit in R, then $\mathbf{f}$ is completeable, i.e., equivalent to* $(1, 0, \ldots, 0)$.

Proof. The proof of this fact we give is due to Suslin. We can assume that $n \geq 3$ without loss of generality, since the cases $n = 1, 2$ are immediate. Write

$$f_1 = a_d t^d + \cdots + a_0$$

where a_d is a unit by assumption. By means of elementary row operations, we can assume that $a_d = 1$ without loss of generality. If $d = 0$, then we're done, so assume that $d > 0$ and we proceed by induction on d. By means of further elementary row operations we can eliminate all terms in $f_2, \ldots, f_n$ of degree $\geq d - 1$. Now, since $\mathbf{f}$ is unimodular, we can choose a complement

$$\sum_i f_i g_i = 1.$$

If all the coefficients of all the $f_2, \ldots, f_n$ were in $\mathfrak{m}$ then the above relation could not hold upon reduction modulo $\mathfrak{m}$. Thus, some coefficient of some $f_2, \ldots, f_n$ does not lie in $\mathfrak{m}$. Rearranging the f_i if necessary, we may assume that f_2 has a coefficient not lying in $\mathfrak{m}$, in which case it is necessarily a unit as R is local. Summarizing this discussion, we may write

$$f_2 = b_s t^s + \cdots + b_0,$$

with $s \leq d - 1$ and some b_i a unit.

Now, we claim that if we have two polynomials $f_1, f_2 \in R[t]$, with $\deg f_1 = d$ and f_1 monic and $\deg f_2 \leq d - 1$ with some coefficient a unit, then there is a polynomial $uf_1 + vf_2$ of degree $\leq d-1$ whose leading coefficient is 1. Indeed, consider the ideal I generated by leading coefficients of polynomials of the form $uf_1 + vf_2$; it suffices to show that this ideal is the unit ideal. Observe that $b_s \in I$, and one inductively concludes that b_i lie in I as well. using expressions of the form $x^{d-s}f_2 - b_s f_1(x)$. Since some b_i is a unit, we conclude.

Now, we can use row operations to conclude that $f_i, i \neq 1, 2$ has degree $\leq d-1$ and leading coefficient a unit. In that case, we conclude by appeal to the induction hypothesis. □

Corollary 6.1.2.5. *If R is a local ring, and $\mathbf{f}$ is a unimodular row of length n in $R[t]$ one of whose elements is monic, then $\mathbf{f}$ is equivalent to $\mathbf{f}(0)$ in $GL_n(R[t])$.*

Proof. Since $\mathbf{f}$ is a unimodular row in $R[t]$, we can choose a complement $\mathbf{g}$ such that $\mathbf{f}\mathbf{g}^t = 1$. Since evaluation is a ring homomorphism, we conclude that $ev_0(\mathbf{f}\mathbf{g}^t) = 1$ as well, i.e., $ev_0(\mathbf{f})$ is a unimodular row over R. In that case, since we know that all f.g. projective R-modules are free, since R is local, then $ev_0(\mathbf{f})$ is equivalent to $(1, 0, \ldots, 0)$. Since Horrocks' result above shows that $\mathbf{f}$ is equivalent to $(1, \ldots, 0)$ as well, we conclude. □

The local-to-global principle

Since we have a local solution to the problem implicit in Vaserstein's theorem, we now want to try to patch these local solutions together to obtain a global solution. We begin with some preparatory lemmas.

Lemma 6.1.2.6. *Let R be an integral domain and let S be a multiplicative subset. If $\mathbf{f}(x) \sim f(0)$ over $R[S^{-1}][x]$, then there exists $c \in S$ such that $\mathbf{f}(x + cy) \sim \mathbf{f}(x)$ over $R[x, y]$. Conversely, if there exists $c \in S$ such that $\mathbf{f}(x + cy) \sim f(x)$ over $R[x, y]$, then $\mathbf{f}(x) \sim \mathbf{f}(0)$.*

Proof. Let $M \in GL_n(R[S^{-1}][x])$ such that $\mathbf{f} = M\mathbf{f}(0)$. in that case, $M^{-1}\mathbf{f} = \mathbf{f}(0)$ is constant and thus invariant under translation. Let

$$G(x, y) = M(x)M(x + y)^{-1}.$$

In that case, $G(x, y)f(x, y) = f(x)$. In that case, $G(x, 0) = Id_n$, so $G(x, y) = Id + yH(x, y)$, with $H(x, y) \in R[x, y][S^{-1}]$. Clearing denominators, there exists $c \in S$ such that cH has elements in $R[x, y]$. In that case, $G(x, cy)$ has coefficients in R. Since $\det M$ is a unit in $R[S]^{-1}$ (by homotopy invariance of units), we conclude that $\det M(x+cy)$ is equal to this same constant and thus $G(x, cy)$ has determinant 1.

For the converse, extend scalars to $R[S^{-1}][x, y]$ and specalize the resulting equivalence. □

Theorem 6.1.2.7. *Let R be a commutative ring, and $\mathbf{f}$ a unimodular row of length n in $R[t]$. Let*

$$\begin{aligned} A &= \{a \in R | \mathbf{f}(t) \sim \mathbf{f}(0) \textit{ over } R_a[t]\}, \\ B &= \{b \in R | \mathbf{f}(t + cx) \sim \mathbf{f}(t) \textit{ over } R[t, x]\}. \end{aligned}$$

Then, I and J are ideals in R, with $I = radJ$.

Proof. If $b \in B$ and $c \in R$, then substitution of cx for x gives $f(t + bcx) \sim f(t)$ so we conclude that $bc \in B$ as well. Likewise if $b, b' \in B$, then substituting $t + b'x$ for t gives

$$f(t + b'x + bx) \sim f(t + (b + b')x) \sim f(t).$$

Thus, we conclude that B is an ideal. Lemma 6.1.2.6 guarantees that A is the radical of B. □

Theorem 6.1.2.8. *Suppose R is an integral domain and $\mathbf{f}$ is a unimodular row of length n over $R[t]$. If $f(t) \sim f(0)$ in $R_{\mathfrak{m}}[t]$ for all maximal ideals $\mathfrak{m} \subset R$, then $f(t) \sim f(0)$ in $R[t]$ as well. In particular, Vaserstein's theorem holds.*

Proof. Form the ideal B as in Theorem 6.1.2.7. For any maximal ideal $\mathfrak{m} \subset R$, Lemma 6.1.2.6 implies that for any $R \setminus \mathfrak{m}$ contains an element of B. It follows that $B = R$. Thus $U = R$ as well, and we conclude.

Vaserstein's theorem then follows by our analysis of the local case. □

6.1.3 Cech cohomology of bundles on the projective line

In this section, we compute the Cech cohomology of the bundles $\mathscr{O}(n)$ on $\mathbb{P}^1$.

Proposition 6.1.3.1. *Suppose R is a fixed commutative unital ring. Let V be the 2-dimensional vector space of The following formula hold:*

1. $\check{H}^i({\mathbb{P}^1}_R, \mathscr{O}(n)) = 0$ *if* $i \neq 0, 1$*;*
2. $\check{H}^0({\mathbb{P}^1}_R, \mathscr{O}(n)) = R[x_0, x_1]^{(n)}$ *if* $n \geq 0$ *and vanishes otherwise.*
3. $\check{H}^1({\mathbb{P}^1}_R, \mathscr{O}(n)) = R[x_0, x_1]^{-n-2}$ *if* $n \leq -2$ *and vanishes otherwise.*

Proof. In this case, we may compute Cech cohomology with respect to the open cover of $\mathbb{P}^1$ by two open sets isomorphic to $\mathbb{A}^1$ with intersection $\mathbf{G}_m$. If we choose coordinates $R[x]$ and $R[x^{-1}]$ then, the differential is given by... □

Proposition 6.1.3.2. *If R is a Noetherian ring, and if $\mathscr{F}$ is a locally free sheaf on $\mathbb{P}^1_R$, then $\check{H}^i({\mathbb{P}^1}_R, \mathscr{F})$ is a finitely generated R-module.*

Proof. We proceed by descending induction on i. For $i > 1$ $\check{H}^i(\mathbb{P}^1_R, \mathscr{F}) = 0$ by definition of the Cech complex. Now, we know the result is true for finite direct sums of bundles of the form $\mathscr{O}(i)$ by the previous proposition. Therefore, we deduce the result for any quotient of $\bigoplus_{i=1}^r \mathscr{O}(a_i)$ as follows. Indeed, the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{K} \longrightarrow 0 \bigoplus_{i=1}^{r} \mathscr{O}(a_i) \longrightarrow \mathscr{F} \longrightarrow 0$$

yields a long exact sequence in cohomology of the form

$$H^i(\mathbb{P}^1_R, \bigoplus_{i=1}^{r} \mathscr{O}(a_i)) \longrightarrow H^i(\mathbb{P}^1_R, \mathscr{F}) \longrightarrow H^{i+1}(\mathbb{P}^1_R, \mathscr{K}).$$

The induction hypothesis guarantees that $H^{i+1}(\mathbb{P}^1_R, \mathscr{K})$ and therefore we conclude that $H^i(\mathbb{P}^1_R, \mathscr{F})$ is finitely generated as well.

Therefore, to conclude it suffices to know that $\mathscr{F}$ can be written as a quotient of of a finite direct sum of modules of the form $\mathscr{O}(i)$. Indeed, restrict $\mathscr{F}$ to $\operatorname{Spec} R[t]$ and $\operatorname{Spec} R[t^{-1}]$; we obtain finitely generated free modules M_+ and M_- over each of these open sets. Pick surjections $R^{\oplus n} \to M_+$ and $R^{\oplus m} \to M_-$. By including more generators if necessary, we may assume that $n = m$. □

6.1.4 Horrocks' theorem

As we have seen above, the description of vector bundles on $\mathbb{P}^1$ over a local ring can be complicated even when the local ring is regular of dimension 1. Suppose R a Noetherian local ring and consider $R[x]$. View $\operatorname{Spec} R[x]$ as an open subscheme of $\mathbb{P}^1_R$ as the complement of the section at ∞. If we begin with a finite rank vector bundle on $\operatorname{Spec} R[x]$, equivalently a projective $R[x]$-module, when does this module arise as the restriction of a vector bundle on $\mathbb{P}^1_R$? Of course, any free $R[x]$-module extends (in many ways) to $\mathbb{P}^1_R$ and any free $R[x]$-module is necessarily extended from R. In [**?**], Horrocks analyzed the converse to this statement. What can we say about a finitely generated projective $R[x]$-module that extends to $\mathbb{P}^1_R$?

Theorem 6.1.4.1 ([**?**, Theorem 1])**.** *Suppose R is a Noetherian local ring. If $\mathscr{E}$ is a vector bundle on $\mathbb{A}^1_{\operatorname{Spec} R}$ and $\mathscr{E}$ extends to a bundle on $\mathbb{P}^1_{\operatorname{Spec} R}$, then $\mathscr{E}$ is a trivial bundle.*

Proof. Suppose $\mathfrak{m}$ is the maximal ideal of R and κ is the residue field. Let us suppose that $\mathscr{E}$ extends to a vector bundle $\mathscr{G}$ on $\mathbb{P}^1_R$. The restriction $\mathscr{G}|_{\operatorname{Spec}\kappa}$ is a vector bundle on $\mathbb{P}^1_\kappa$. Therefore, by Corollary **??**, the bundle $\mathscr{G}|_{\operatorname{Spec}\kappa}$ is a direct sum of line bundles over κ. On the other hand, tensoring by a line bundle on $\mathbb{P}^1_{\operatorname{Spec} R}$ will not affect the form of the restriction to $\operatorname{Spec} R[x]$. Therefore, we may assume that $\mathscr{G}|_{\operatorname{Spec}\kappa} \cong \mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \cdots \oplus (a_r)$ where $a_r \geq 0$.

The proof proceeds by induction on the rank of $\mathscr{E}$. Evidently a rank 0 projective module is extended from $\operatorname{Spec} R$, so assume $\mathscr{E}$ has rank > 0. In that case, observe that there is an exact sequence of the form

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{G}|_{\operatorname{Spec}\kappa} \longrightarrow \mathscr{G}|_\kappa/\mathscr{O} \longrightarrow 0.$$

The map $\mathscr{O} \to \mathscr{G}|_{\operatorname{Spec}\kappa}$ is precisely a nowhere vanishing section. If we can extend this section to a nowhere vanishing section of $\mathscr{G}$, then we obtain a short exact sequence of modules of the form

$$0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}/\mathscr{O} \longrightarrow 0,$$

where $\mathscr{G}/\mathscr{O}$ is locally free. The restriction of this exact sequence to $\mathbb{A}^1_{\operatorname{Spec} R}$ then yields a short exact sequence of projective modules, which necessarily splits by the definition of projectivity.

Thus, we will try to lift a non-vanishing section of $\mathscr{G}|_{\operatorname{Spec}\kappa}$ to $\mathscr{G}$. We do this in two steps. First, we can filter R by powers of $\mathfrak{m}$. In doing this, we obtain exact sequences of the form

$$0 \longrightarrow \mathfrak{m}^i/\mathfrak{m}^{i+1} \longrightarrow R/\mathfrak{m}^{i+1} \longrightarrow R/\mathfrak{m}^i \longrightarrow 0.$$

The maps $R \to R/\mathfrak{m}^{i+1}$ induce maps $\operatorname{Spec} R/\mathfrak{m}^{i+1} \to \operatorname{Spec} R$ and we obtain corresponding maps

$$\mathbb{P}^1_{\operatorname{Spec} R/\mathfrak{m}^{i+1}} \longrightarrow \mathbb{P}^1_{\operatorname{Spec} R}.$$

Since $\mathscr{G}$ is locally free, tensoring with the exact sequence above yields an exact sequence

$$0 \longrightarrow \mathscr{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1} \longrightarrow \mathscr{G} \otimes_R R/\mathfrak{m}^{i+1} \longrightarrow \mathscr{G} \otimes_R R/\mathfrak{m}^i \longrightarrow 0.$$

Taking cohomology of this short exact sequence yields a long exact sequence; we examine the portion of the sequence

$$H^0(\mathscr{G} \otimes_R R/\mathfrak{m}^{i+1}) \longrightarrow H^0(\longrightarrow \mathscr{G} \otimes_R R/\mathfrak{m}^i) \longrightarrow H^1(\mathscr{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

However, $\mathfrak{m}^i/\mathfrak{m}^{i+1})$ is a finite-dimensional κ-vector space and thus by simply choosing a basis we obtain isomorphisms $H^1(\mathscr{G} \otimes_R \mathfrak{m}^i/\mathfrak{m}^{i+1}) \cong H^1(\mathscr{G}|_\kappa) \otimes \mathfrak{m}^i/\mathfrak{m}^{i+1}$. In particular, if $H^1(\mathscr{G}|_\kappa)$ vanishes, then we may lift at each stage. By our assumptions $\mathscr{G}|_\kappa$ is a direct sum of bundles of the form $\mathscr{O}(n)$ with $n \geq 0$. In particular, the first Cech cohomology of each such bundle vanishes, and we deduce the required surjectivity.

Now, there are maps $H^0(\mathscr{G}) \to H^0(\mathscr{G} \otimes_R R/\mathfrak{m}^{i+1})$ and taking the inverse limit, we obtain a map

$$H^0(\mathscr{G}) \longrightarrow \lim_n H^0(\mathscr{G} \otimes_R R/\mathfrak{m}^{i+1})$$

On the other hand, $H^0(\mathscr{G})$ is an R-module and thus has a topology induced by powers of $\mathfrak{m}$. If we complete this R-module, we obtain a module $\widehat{H^0(\mathscr{G})}$. The map above factors as

$$H^0(\mathscr{G}) \longrightarrow \widehat{H^0(\mathscr{G})} \longrightarrow \lim_n H^0(\mathscr{G} \otimes_R R/\mathfrak{m}^{i+1}).$$

It is a special case of Grothendieck's theorem on formal functions that the right hand map is an isomorphism (though in this case, we may simply check everything by hand).

Now, $H^0(\mathscr{G})$ is a finite generated R-module. By basic properties of completion, we conclude that

$$H^0(\mathscr{G}) \longrightarrow \widehat{H^0(\mathscr{G})} \longrightarrow H^0(\mathscr{G} \otimes_R \kappa)$$

is surjective. However, since $H^0(\mathscr{G} \otimes_R \kappa)$ has a nowhere vanishing section, we conclude that $\mathscr{G}$ also has a nowhere vanishing section, but this is precisely what we wanted to show. □

Remark 6.1.4.2. While the proof of Horrocks' theorem is rather short and intuitive in this setting, it requires some algebro-geometric machinery. In applications, Quillen used a closely related algebraic version of the result. It is possible to give a purely algebraic proof of this algebraic version of the result: see [**?**, Chapter IV] for more details. We have chosen to give Horrocks' original proof since we found it geometrically appealing.

6.2 Lecture 32: The Quillen–Suslin theorem

It was observed by Murthy that a global version of Horrocks' theorem would imply a solution to the Serre problem about triviality of projective modules over polynomial rings over a field.

6.2.1 Extending vector bundles from $\mathbb{A}^1_R$ to $\mathbb{P}^1_R$

We may use Horrocks' theorem to effectively give a criterion to study when modules are extended. Suppose we begin with a vector bundle on $\mathbb{A}^1_R$ (for what we are about to say, it will not be necessary to assume that R is a Noetherian local ring). If we would like to extend this vector bundle to $\mathbb{P}^1_R$, then we do this by attempting to glue. In order to glue, it suffices to extend $\mathbb{A}^1_R$ to a Zariski open cover of $\mathbb{P}^1_R$. The simplest possible situation would be if we could find an open cover by two sets. The easiest open cover is, of course, the usual open cover by $\operatorname{Spec} R[t]$ and $\operatorname{Spec} R[t^{-1}]$. However, it would suffice to take *any* Zariski open subset of $t^{-1} = 0$ inside $\operatorname{Spec} R[t]$. Now, we give an easy and useful criterion for gluing.

Lemma 6.2.1.1. *Suppose X is a scheme and we have a Zariski open cover of X by two open subschemes U and V. If $\mathscr{E}$ is a rank n vector bundle on U, such that $\mathscr{E}|_{U\cap V}$ is trivial, then $\mathscr{E}$ extends to a vector bundle on X.*

Proof. Take the trivial bundle $\mathscr{O}_V^{\oplus n}$ and observe that by assumption $\mathscr{O}_V^{\oplus n}|_{U\cap V} \cong \mathscr{E}|_{U\cap V}$. Gluing these two vector bundles, we obtain the required extension. $\square$

Rather than attempt to make a choice of a Zariski open subset of $\operatorname{Spec} R[t^{-1}]$ that contains the section at "∞", we will look at all possible refinements of Zariski neighborhoods of ∞. To make this clearer, set $s = t^{-1}$. We want to consider Zariski open subsets of $\operatorname{Spec} R[s]$ that contain $s = 0$. We look at elements of $R[s]$ of the form $1+sR[s]$: such elements have constant term 1 and therefore evidently avoid $s = 0$. We consider the localization:

$$V_\infty := \operatorname{Spec} R[s][(1 + sR[s])^{-1}]$$

Note that the map $R[s] \to R[s](1 + sR[s])^{-1}$ is a localization and thus flat. Essentially V_∞ is the intersection of all open sets that contain ∞. We can therefore cover $\mathbb{P}^1_R$ by the two open sets $\operatorname{Spec} R[t]$ and V_∞. We now give a description of this intersection.

Proposition 6.2.1.2. *The intersection* $\operatorname{Spec} R[t] \cap V_\infty = \operatorname{Spec} R\langle t\rangle$*, where $R\langle t\rangle$ is the localization of $R[t]$ at the multiplicative set of all* monic *polynomials.*

Combining these two results, we deduce the following criterion for extensibility.

Corollary 6.2.1.3. *If P is a projective $R[t]$-module, then if $P \otimes_{R[t]} R\langle t\rangle$ is free, then P extends to $\mathbb{P}^1_R$.*

Proof. If P is a projective $R[t]$-module and $P \otimes_{R[t]} R\langle t\rangle$ is free, then there exits a monic irreducible polynomial f such that P_f is a free $R[t]_f$-module. We can view $R[t]_f$ as the intersection with $\mathbb{A}^1_R$ of an open subset of $\mathbb{P}^1_R$ containing the section at ∞. $\square$

In order to make this result useful, we need to better understand $R\langle t\rangle$-modules. To this end, observe that if $R = k$ is a field, then $R\langle t\rangle = k(t)$, thus $k(t)$ has smaller dimension than $R[t]$. We now observe that this phenomenon is general.

Lemma 6.2.1.4. *Suppose R is a Noetherian ring of Krull dimension d.*

1. *The ring $R\langle t\rangle$ has Krull dimension d.*
2. *If R is PID (resp. a field), then so is $R\langle t\rangle$.*

Proof. We understand prime ideals in $R[t]$ rather well and we know $R[t]$ has Krull dimension $d+1$. To show that $R\langle t\rangle$ has Krull dimension d, we have to show that every prime ideal $\mathfrak{P} \subset R[t]$ of height $d + 1$ localizes to the unit ideal in $R\langle t\rangle$. Equivalently, we have to show that $\mathfrak{P}$ contains a monic polynomial. Following [**?**, IV Proposition 1.2], we give an elementary proof of this fact.

Set $\mathfrak{p} = \mathfrak{P}\cap R$. One knows that if $\mathfrak{P}$ has height $d+1$, then $\mathfrak{P}$ is not pulled back from R, and thus $\mathfrak{p}[t]$ is a proper subset of $\mathfrak{P}$ while $\mathfrak{p}$ has height d. Therefore, $\mathfrak{p}$ is a maximal ideal in R. Now, suppose $f \in \mathfrak{P}$ is some polynomial with coefficients in R that lies outside of $\mathfrak{p}[t]$. Say $f = a_n t^n + \cdots + a_0$. We want to modify f by an element of $\mathfrak{B}$ to be monic.

Without loss of generality, we may assume that a_n does not lie in $\mathfrak{p}$. Now, since $\mathfrak{p}$ is maximal, we can find $c = a_n b - 1 \in \mathfrak{p}$. In that case, $b \cdot f - ct^n$ is a monic polynomial contained in $\mathfrak{B}$, but this is precisely what we wanted to show.

For the second point, it suffices to observe that if R is a UFD, then $R[t]$ is also a UFD and then $R\langle t\rangle$ is also a UFD. Since a $\dim R[t] = \dim R$, if R has dimension 1, it suffices to observe that UFDs are normal. □

Combining this result with Horrocks' result, we may establish a preliminary result about projective modules over rings that are not principal ideal domains.

Corollary 6.2.1.5. *If R is a discrete valuation ring, then every finitely generated projective $R[t]$-module is free.*

Proof. By assumption R is a local PID. Thus, by the lemma above $R\langle t\rangle$ is also a PID. In particular, every f.g. projective module over $R\langle t\rangle$ is free. Now, suppose P is a f.g. projective $R[t]$-module. By what we just said $P \otimes_{R[t]} R\langle t\rangle$ is a free $R\langle t\rangle$-module. Therefore, by the proposition above, we may extend P to a vector bundle over $\mathbb{P}^1_R$. In that case, it follows from Horrocks' theorem, that P is extended from an R-module P_0. But since R is a local PID, it follows that P_0 is free itself. Therefore, P is also free. □

Remark 6.2.1.6. This discussion makes it clear that if one has a "global" version of Horrocks' theorem, then one would inductively be able to understand vector bundles on polynomial rings over a PID. That this is true, was more-or-less observed by Murthy shortly after Horrocks' theorem was published. Quillen's solution to the Serre problem proceeds precisely in this fashion by allowing one to prove a global version of Horrocks' theorem.

6.2.2 Quillen's patching theorem

Following Quillen, we now search for a global version of Horrocks' theorem. Recall that descent theory tells us that if R is a commutative unital ring and f and g are a pair of comaximal elements of R, then one way to build a projective R-module is by specifying projective R_f and R_g-modules together with suitable gluing data. Suppose we would like to tell if a given R-module is trivial. Know that the associated R_f and R_g-modules obtained by localization are trivial is certainly not sufficient to guarantee triviality. However, we could ask if, perhaps, one can modify the isomorphism over R_{fg} to guarantee that the glued module is trivial. Quillen's local-to-global principle precisely addresses this problem.

Theorem 6.2.2.1. *If M is a finitely presented $R[T]$-module, and $M_{\mathfrak{m}}$ is an extended $R_{\mathfrak{m}}[t]$-module for each maximal ideal $\mathfrak{m} \subset R$, then M is extended.*

Proof. Our argument follows the presentation of [**?**, Theorem V.1.6]. Let $Q(M)$ be the set of $f \in R$ such that M_f is an extended $A_f[t]$-module. We claim that $Q(M)$ is an ideal in A. We must show that if $f_0, f_1 \in Q(M)$, then $f = f_0 + f_1$ is also in $Q(M)$. After replacing R by R_f, we may assume that f_0 and f_1 are comaximal in R. If we set $N = M/tM$, then we will try to show that $M \cong N[t]$.

We can assume that M_{f_i} is extended from $N_{f_i}[t]$ and thus we may fix automorphisms u_i : $M_{f_i} \to N_{f_i}[t]$, $i = 0, 1$. After composing with a suitable automorphism of $N_{f_i}[t]$ if necessary, we may assume that u_i reduces modulo t to the identity map of N_{f_i}. Pictorially, we have the following situation:

$$\begin{tikzcd}
M_{f_0} \arrow[rr] \arrow[d] & & M_{f_0 f_1} \arrow[dl, "(u_0)_{f_1}"'] \arrow[dr, "(u_1)_{f_0}"] & & M_{f_1} \arrow[ll] \arrow[d] \\
N_{f_0}[t] \arrow[r] & N_{f_0 f_1}[t] \arrow[rr, dashed, "\theta"] & & N_{f_0 f_1}[t] & N_{f_1}[t]. \arrow[l]
\end{tikzcd}$$

If $(u_0)_{f_1} = (u_1)_{f_0}$, then by Zariski descent, these two isomoprhism patch together to give a module isomorphism $M \cong N[t]$ and we are done.

Quillen's idea is to modify the choices of u_0 and u_1 by suitable automorphisms to guarantee that we may patch. Note that the element

$$\theta = (u_1)_{f_0} \circ ((u_0)_{f_1})^{-1} \in End_{R_{f_0 f_1}[t]}(N)_{f_0 f_1}[t] \cong End_R(N)_{f_0 f_1}[t].$$

Set $E = End_R(N)$. By assumption θ reduces to the identity modulo t, i.e., $\theta \in (1 + tE_{f_0 f_1}[t])^{\times}$. Therefore, it suffices to show that θ may be rewritten as $(v_1)_{f_0}^{-1} \circ (v_0)_{f_1}$ for suitable $v_i \in E_{f_i}[t]$. Granting this for the moment (it will be established in Lemma 6.2.2.2) then $(v_0 u_0)_{f_1} = (v_1 u_1)_{f_0}$ and so after replacing u_i by $v_i u_i$, we may patch together our local extensions as observed above.

To establish the result, it suffices then to show that $Q(M)$ is the unit ideal. Set $M' = R[t] \otimes_R M/tM$; this is a finitely presented $R[t]$-module that is extended from M. For any maximal ideal $\mathfrak{m} \subset R$, there exists an isomorphism $\varphi : M_{\mathfrak{m}} \cong M'_{\mathfrak{m}}$. Since φ is a map of finitely presented modules, by clearing the denominators we conclude that there is an element $g \in R \setminus \mathfrak{m}$ such that φ is the localization of an isomorphism of $R_g[t]$-modules $M_g \to M'_g$. In that case, $g \in Q(M) \setminus \mathfrak{m}$ and therefore $Q(M)$ is an ideal that is not contained in $\mathfrak{m}$ which means that $Q(M) = R$. □

Lemma 6.2.2.2. *Let R be a commutative unital ring, and suppose E is an R-algebra (not necessarily commutative!). If $f \in R$ and $\theta \in (1 + TE_f[T]^{\times})$, then there exists an integer $k \geq 0$ such that for any $g_1, g_2 \in R$ with $g_1 - g_2 \in f^k R$, there exists $\psi \in (1 + TR[T])^{\times}$ such that $\psi_f(T) = \theta(g_1 T)\theta(g_2 T)^{-1}$.*

Proof. To be added. For the moment, see [**?**, Corollary V.1.2-3] □

6.2.3 Globalizing Horrocks' theorem and the Quillen–Suslin theorem

Combining the results so far, we may give the "global" version of Horrocks' theorem.

Corollary 6.2.3.1. *If M is a finitely generated projective $R[t]$-module that is the restriction of a vector bundle on $\mathbb{P}^1_{\operatorname{Spec} R}$, then M is extended.*

Proof. To check whether M is extended, it suffices to check whether M is exteded after localizing at every maximal ideal $\mathfrak{m} \subset R$. However, if M is a vector bundle on $R_{\mathfrak{m}}[t]$ that extends to $\mathbb{P}^1_{R_{\mathfrak{m}}}$, then M is extended by Horrocks' theorem. Therefore, M is extended. □

Finally, we may establish the Quillen–Suslin theorem.

Theorem 6.2.3.2 (Quillen–Suslin)**.** *If R is a principal ideal domain, then $\mathscr{V}_r(R) \to \mathscr{V}_r(R[x_1, \ldots, x_n])$ is an isomorphism.*

Proof. We proceed by induction on n. The result is true if $n = 0$ by the structure theorem. Take $A = k[t_1, \ldots, t_{n-1}]$ and set $t = t_n$. Then, $B = A \otimes_{R[t]} R\langle t\rangle$ is a polynomial ring in $n-1$-variables over $R\langle t\rangle$. However, since R is a principal ideal domain, so is $R\langle t\rangle$. Therefore, $M \otimes_R [t]R\langle r\rangle$ is free over B by the induction hypothesis. Thus, $M \otimes_{A[t]} A\langle t\rangle$ is free over $A\langle t\rangle$ and therefore M is free by the results above. □

6.3 Lecture 33: Lindel's theorem on the Bass–Quillen conjecture

In this section, we turn our attention to $\mathbb{A}^1$-invariance of the functor $\mathscr{V}_r(X)$. We already know that if $X = \mathbb{P}^1$, then $\mathbb{A}^1$-invariance fails. However, buoyed by the Quillen–Suslin theorem, we consider the problem of $\mathbb{A}^1$-invariance for $X = \operatorname{Spec} R$ with R a regular ring. That this should be true was conjectured by Bass and became known as the Bass–Quillen conjecture. We begin by giving a mild strengthening Quillen's patching theorem 6.2.2.1. Using this version of patching, to establish the Bass-Quillen conjecture in general, it suffices to establish it for a regular local ring. Beyond Quillen's patching theorem, Lindel's key idea was to reduce the result to the case of polynomial rings by using his étale neighborhood theorem and a refined étale descent result for vector bundles (though in the form we will state the result it will not be a special case of 'etale descent).

6.3.1 Quillen's patching revisited and Roitman's "converse"

Once again, our treatment follows [**?**, Theorem 1.6]. We generalize Quillen's theorem to treat two special cases of the Bass–Quillen conjecture.

Theorem 6.3.1.1. *If R is a commutative unital ring, and M is a finitely presented $R[t_1, \ldots, t_n]$-module, then following statements hold.*

(A_n) The set $Q(M)$ consisting of elements $g \in R$ such that M_g is extended from an R_g-module is an ideal in R (sometimes called the Quillen ideal*).*

(B_n) If $M_{\mathfrak{m}}$ is extended from an $R_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \subset R$, then M is extended.

Proof. To be added. □

Corollary 6.3.1.2. *If R is a Dedekind domain, then every $R[t_1, \ldots, t_n]$-module is extended from R.*

Proof. By Quillen's patching theorem, it suffices to prove this when R is a local Dedekind domain, i.e., when R is a local PID, but this follows immmediately from the Quillen–Suslin theorem. □

The above result admits a rather strong generalization, due to Roitman (without assuming the Quillen–Suslin theorem).

Theorem 6.3.1.3 (Roitman)**.** *Suppose R is a commutative unital ring and $S \subset R$ is a multiplicative set. Fix an integer $n \geq 1$. If every finitely generated projective $R[t_1, \ldots, t_n]$-module is extended from R, then every finitely generated $R[S^{-1}][t_1, \ldots, t_n]$-module is extended from $R[S^{-1}]$.*

Proof. This is [**?**, Proposition 2] (see also [**?**, Theorem V.1.11]). By induction on n, it suffices to treat the case where $n = 1$. Therefore, assume every finitely generated projective $R[t]$-module is extended from R and suppose P is a finitely generated projective $R[S^{-1}][t]$-module. By Quillen's patching theorem, we may replace $R[S^{-1}]$ by $(R[S^{-1}])_{\mathfrak{m}}$ for $\mathfrak{m}$ a maximal ideal of $R[S^{-1}]$. Equivalently, we may find a prime $\mathfrak{p} \subset R$ such that $(R[S^{-1}])_{\mathfrak{m}} = R_{\mathfrak{p}}$ and therefore, we may assume without loss of generality that $R[S^{-1}] = R_{\mathfrak{p}}$.

Thus, suppose P is a finitely generated projective $R_{\mathfrak{p}}[t]$-module. We want to show that P is free. Since P is a direct summand of a finitely generated free module $P = R_{\mathfrak{p}}[x]^{\oplus n}$, it is determined by a projection operator on $R_{\mathfrak{p}}[t]^{\oplus n}$. We want to show that this projection operator is conjugate in $Aut_{R_{\mathfrak{p}}[t]}(R_{\mathfrak{p}}[x]^{\oplus n})$ to the matrix $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$ where the number of 1s that appear is given by the rank of P.

Write $e(t)$ for the projection operator associated with P. Since $R_{\mathfrak{p}}$ is local, the module P/tP is free, and therefore $e(0)$ is conjugate in $Aut_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\oplus n})$ to $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$. Thus, by conjugation by an element of $Aut_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\oplus n})$ we can assume without loss of generality that $e(0)$ is equal to the standard projection operator.

By clearing the denominators, we may find an element $r \in R \setminus \mathfrak{p}$ such that $e(rt)$ lies in the image of

$$\mathrm{M}_n(R[t]) \to \mathrm{M}_n(R_{\mathfrak{p}}[t])$$

(the constant terms $e(0)$ are 0 or 1, which already lie in R). Thus, we may fix $e_0(t) \in \mathrm{M}_n(R[t])$ that localizes to $e(rt)$ such that $e_0(0)$ is the standard operator above. Since $e(rt)$ is a projection operator, $e(rt)^2 - e(rt) = 0$. Therefore, since $e_0(t)$ localizes to $e(rt)$, we conclude that $e_0(t)^2 - e_0(t)$ localizes to zero and therefore is killed by some element $s \in R \setminus \mathfrak{p}$.

Since $e_0(0)^2 = e_0(0)$, we conclude that $e_0(t)^2 - e_0(t)$ has the form $t\epsilon(t)$ for some matrix $\epsilon(t)$ in $R[t]$. Now, t is not a zero-divisor in $R[t]$. Therefore, $st\epsilon(t) = 0$ implies $s\epsilon(t) = 0$. Therefore, $s\epsilon(st) = 0$ as well. Thus,

$$e_0(st)^2 - e_0(st) = st\epsilon(st) = 0 \in \mathrm{M}_n(R[t]),$$

and therefore, $e_0(st)$ determines a finitely generated projective $R[t]$-module as well. Because every $R[t]$-module is extended from R, it follows that this module is extended from R as well. Therefore, we may find $\sigma(t) \in Aut_{R[t]}(R[t]^{\oplus n})$ such that

$$\sigma(t)^{-1} e_0(st) \sigma(x) = e_0(0).$$

Localizing to $\mathrm{M}_n(\mathfrak{R}_{\mathfrak{p}}[t])$, this becomes

$$\sigma(t)^{-1} e(rst) \sigma(t) = e(0),$$

and dividing by rs yields the formula we want:

$$\sigma(\frac{t}{rs})^{-1} e(t) \sigma(\frac{t}{rs}) = e(0).$$

□

Combining Roitman's theorem and the Quillen–Suslin theorem, we may deduce another special case of the Bass–Quillen conjecture: the conjecture holds for R the localization of a polynomial ring over a field or PID.

Corollary 6.3.1.4. *If R is a principal ideal domain (or a field), and A is a localization of a polynomial ring over R, then any finitely generated projective $A[t_1, \ldots, t_n]$-module is extended.*

Proof. By the Quillen–Suslin theorem, if B is a polynomial ring over R, then every finitely generated projective B-module is free. Thus, every finitely generated $B[t_1, \ldots, t_n]$-module is extended from B, since every such module is free, again by the Quillen–Suslin theorem. Now, write A as a localization of suitable B and apply Roitman's theorem. □

6.3.2 Lindel's patching theorem

Essentially, Lindel's approach to the Bass–Quillen conjecture was to try to reduce it to the two results established above. The key step in this reduction was a patching result that rests on Lindel's Nisnevich neighborhood theorem: this is the place where, unlike the proof of the Quillen–Suslin theorem, one is forced to assume that one is considering regular rings containing a field. Indeed, Lindel's theorem shows that any regular local ring containing a field (such that the residue field is separable over the base) is a Nisnevich neighborhood the localization of a polynomial ring at a maximal ideal. The idea is then to use induction on the dimension combined with validity of the conjecture over localizations of polynomial rings to conclude. There is one further technical issue that arises: we cannot, without some restrictions, guarantee that residue field extensions at maximal ideals are always separable: one way to guarantee this is to assume one is working with regular varieties over a perfect field. It is possible to remove this assumption, but we treat this afterwards so as not to complicate the essential geometric idea of the proof.

Suppose k is a perfect field, R is a localization of a finite-type regular k-algebra of dimension d at a maximal ideal $\mathfrak{m}$. If κ is the residue field of R at $\mathfrak{m}$, we may find a polynomial ring $\kappa[x_1, \ldots, x_d] \subset R$ such that, setting $\mathfrak{n} = \kappa[x_1, \ldots, x_d] \cap \mathfrak{m}$ and $S = k[x_1, \ldots, x_d]_{\mathfrak{n}}$, the map $S \to R$ is an étale neighborhood. In fact, without too much work we may refine this neighborhood to a covering.

Lemma 6.3.2.1. *Let R be an étale neighborhood of a local ring S. There exists an element $f \in \mathfrak{n}$ such that*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S_f \arrow[d] \\ R \arrow[r] & R_f \end{tikzcd}$$

is an affine étale cover.

Proof. This is a consequence of local structure of étale morphisms. Essentially we may factor $\operatorname{Spec} R \to \operatorname{Spec} S$ as the composite $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} S[t] \longrightarrow \operatorname{Spec} S$ where the first map is a closed immersion defined by a polynomial $h(t) \in S[t]$ such that $h(0)$ lies in the maximal ideal of S and $h'(0)$ is a unit. In that case, we may take $f = h(0)$ and it suffices to check the remaining properties are satisfied. □

If we take any non-zero element f of $\mathfrak{n}S$ and we invert it, then the resulting rings S_f and R_f have dimension *smaller* than d. Note also that R_f is actually regular as the localization of a regular

k-algebra is again regular (unfortunately, we have not proven this statement in this generality). Thus, we have the following picture:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S_f \arrow[d] \\ R \arrow[r] & R_f \end{tikzcd}$$

is an affine étale cover of S by S_f and R. Similarly, for any integer $n \geq 0$, we obtain an affine étale cover $S[t_1, \ldots, t_n]$ of the form

$$\begin{tikzcd} S[t_1, \ldots, t_n] \arrow[r] \arrow[d] & S_f[t_1, \ldots, t_n] \arrow[d] \\ R[t_1, \ldots, t_n] \arrow[r] & R_f[t_1, \ldots, t_n]. \end{tikzcd}$$

Therefore, by étale descent, we may build projective $S[t_1, \ldots, t_n]$-modules by patching together projective $R[t_1, \ldots, t_n]$-modules and projective $S_f[t_1, \ldots, t_n]$-modules that agree upon extension of scalars to $R_f[t_1, \ldots, t_n]$.

A finitely generated projective $R[t_1, \ldots, t_n]$-module P determines an $R_f[t_1, \ldots, t_n]$-module P'. If we work inductively with respect to the dimension of R, we may assume that P' is extended from an R_f-module P_0'. Note that $P_0' \cong P'/(t_1, \ldots, t_n)P'$. We claim that P' is actually free. To see this, observe that $P_0' \cong P'/(t_1, \ldots, t_n)P' \cong (P/(t_1, \ldots, t_n)P)_f$. Since R is local, $(P/(t_1, \ldots, t_n)P)$ is already a free R-module. On the other hand, finitely generated projective $S_f[t_1, \ldots, t_n]$-modules are always extended from S_f by the corollary to Roitman's theorem established above. In fact, such modules are free. Therefore, étale descent tells us that we may glue P and a free $S_f[t_1, \ldots, t_n]$-module to obtain an $S[t_1, \ldots, t_n]$-module $\tilde{P}$. However, S is the localization of a polynomial ring and therefore, again by appeal to Roitman's theorem, we conclude that $\tilde{P}$ is again extended from an S-module $\tilde{P}_0$. Since $P \cong \tilde{P} \otimes_{S[t_1,\ldots,t_n]} R[t_1, \ldots, t_n]$, we conclude by associativity of tensor product that $P/(t_1, \ldots, t_n)P \cong \tilde{P}_0 \otimes_S R$, i.e., that P is extended as well. Thus, putting everything together with Quillen's patching theorem, we have established the following fact.

Theorem 6.3.2.2 (Lindel)**.** *If k is a perfect field, and R is a finite-type regular k-algebra, then every finitely generated $R[t_1, \ldots, t_n]$-module is extended from R.*

6.3.3 The Bass-Quillen conjecture: the geometric case and beyond

Finally, we eliminate the hypothesis on perfection of the base field.

Theorem 6.3.3.1. *Suppose R is a regular k-algebra, essentially of finite type over k. For any integer $r \geq 0$ and any integer $n \geq 0$, the map*

$$\mathscr{V}_r(R) \longrightarrow \mathscr{V}_r(R[t_1, \ldots, t_n])$$

is a bijection.

Proof. It suffices to reduce to the case where k is perfect; this reduction was sketched by Mohan Kumar. Let k_0 be the prime field of k. We may write R as a quotient of a polynomial algebra over k: $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Since P is a finitely generated projective R-module, it is the image of an idempotent endomorphism of a free R-module of finite rank. Let k' be the subfield of k generated by the coefficients of $f_1, \ldots, f_r$ and of the entries of α. Set $R' = k'[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. By construction, there is a projective module P' such that P is obtained by extending scalars from P. Note also that $R' \to R' \otimes_{k'} k = R$ is a faithfully flat ring map. Since R is regular, it follows that R' is regular as well (use faithfully flat descent to show that it has finite global dimension). Since k'/k_0 is a finite extension, it follows that R' has essentially finite type over k_0 as well. Thus, replacing R by R', we may assume that the base field is perfect, in which case the result follows from the version of Lindel's theorem established above. □

6.3.4 Popescu's extension of Lindel's theorem

Popescu explained how to use approximation theorems to establish the Bass-Quillen conjecture in certain mixed-characteristic situations. In particular, if R is a Dedekind domain with perfect residue fields, he generalized Lindel's étale neighborhood theorem in a fashion that it could be applied to certain regular R-algebras A. We now state and prove the Lindel–Popescu's étale neighborhood theorem.

Theorem 6.3.4.1 ([**?**, Proposition 2.1])**.** *Let R be a discrete valuation ring, p a local parameter in R, and $(A, \mathfrak{m})$ a regular local R-algebra, essentially of finite type. Set $\kappa = A/\mathfrak{m}$, and $k = \mathrm{Frac}(R/(\mathfrak{m} \cap R))$. If*

1. *$k \subset \kappa$ is separable;*
2. *$p \notin \mathfrak{m}^2$; and*
3. *$\dim A \geq 2$,*

then A is an étale neighborhood of a localization of a polynomial R-algebra.

6.4 Lecture 34: Grassmannians and naive $\mathbb{A}^1$-homotopies

Our goal in this section is to show that Lindel's theorem may be translated into a statement about naive $\mathbb{A}^1$-homotopy classes of maps to a suitable Grassmannian variety. To begin, we recall the construction of Grassmannian varieties in algebraic geometry. One key point here is that we describe maps from an arbitrary affine scheme to a Grassmann variety.

6.4.1 Finite-dimensional Grassmannians

Classically, the Grassmannian is an object of linear algebra. Fix a field k, and let V be an n-dimensional vector space over a field k. As a set $Gr_{n,N}$ parameterizes n-dimensional quotients (or sub-spaces) of an N-dimensional k-vector space. We begin by giving a construction of $Gr_{n,N}$ as a scheme (over $\mathrm{Spec}\,\mathbb{Z}$, since this adds no additional complication). The idea of the construction can be thought of as a generalization of homogeneous coordinates, analogous to the construction of projective space. We will show that $Gr_{n,N}$ can be obtained by gluing together copies of affine space. The construction follows one of the standard constructions in differential geometry and simply observes that all the defining maps are given by polynomials.

We would like to show that $Gr_{n,N}$ is naturally the set of k-rational points of a smooth projective k-scheme. To this end, we follow the usual description of coordinate charts. Fix a basis $e_1, \ldots, e_N$ of V. If $W \subset V$ is an n-dimensional subspace, then by picking a basis $w_1, \ldots, w_n$ of W and writing v_i in terms of the basis $e_1, \ldots, e_n$, we may associate with W an $n \times N$-matrix of rank precisely n. Now, the space of $n \times N$-matrices of rank precisely n is an algebraic variety: it is an open subscheme of $\mathbb{A}^{nN}$ whose closed complement is defined by the vanishing of the $n \times n$-minors.

We define $V_{n,N}$ to be the open subscheme of $\mathbb{A}^{nN}$ complementary to the closed subscheme whose ideal is given by the vanishing of $n \times n$-minors of V. Observe that many different $n \times N$-matrices of rank n give rise to the same subspace: indeed, the redundancy is precisely the choice of basis of W. At the level of k-points, the change of basis of W corresponds to left multiplying by an element of $GL_n(k)$. However, by means of such multiplications, we can always reduce an $n \times N$-matrix to one where a fixed $n \times n$-minor is the identity. Thus, we look at the closed subscheme of $\mathbb{A}^{nN}$ with coordinates X_{ij} where a fixed $n \times n$-minor is the identity matrix. The resulting subscheme is isomorphic to $\mathbb{A}^{n(N-n)}$ (with coordinates given by the non-constant entries. Set-theoretically, these subsets form a cover of $Gr_{n,N}$. We may explicitly write down the transition maps on overlaps using matrix inverses and by Cramer's rule, these maps are algebraic. Even better, they are polynomial and all coefficients are $0, \pm 1$. Gluing these copies of $\mathbb{A}^{n(N-n)}$ together gives $Gr_{n,N}$ the structure of a scheme.

Note that $Gr_{n,N}$ carries several "tautological" vector bundles γ_n of rank n. Geometrically: if V is an N-dimensional vector space, we may consider the quotient bundle of $Gr_{n,N} \times V$ whose fiber over $x \in Gr_{n,N}$ is the quotient R_x of V corresponding to x. This definition does not suffice to build a vector bundle over the scheme $Gr_{n,N}$, but it is easy to soup it up to define such a bundle. We build a geometric vector bundle by gluing copies of the trivial bundle of rank n over each open set in the previous section.

Now, if X is any scheme, a map $X \to Gr_{n,N}$ defines a rank n vector bundle on X together with an epimorphism from a trivial bundle of rank N. If $X = \operatorname{Spec} R$ is affine, this corresponds to a rank n projective R-module, together with a set of N R-module generators. In fact, we claim this map is a bijection: given a rank n vector bundle on X together with a surjection from a rank N trivial bundle, we can reconstruct the map $X \to Gr_{n,N}$.

Suppose we are given a surjection $R^{\oplus N} \to P$. Pick a Zariski cover of $\operatorname{Spec} R$ over which P and Q trivialize. In that case, if we fix a subset of $1, \ldots, N$ of size n, then there is an induced inclusion map $R^{\oplus n} \to R^{\oplus N}$, and we can ask that the composite map $R^{\oplus n} \to P$ is an isomorphism. This determines a collection of regular functions on R_f and thus a map $R_f \to \mathbb{A}^{n(N-n)}$. Varying through the subsets of size n, we obtain maps that may be glued to obtain a map $R_f \to Gr_{n,N}$. Varying through the open cover, we may patch to determine our map $X \to Gr_{n,N}$.

The construction we have just outlined goes by many different names and is very robust. In differential geometry, the construction above is sometimes called the "Gauss map attached to a vector bundle" and it is one step in a standard argument relating isomorphism classes of vector bundles to homotopy classes of maps to Grassmannians. In our context, we have just described the "functor of points" of the Grassmannian $Gr_{n,N}$.

6.4.2 Infinite Grassmannians

If V is an N-dimensional vector space, and V' is an $N+1$-dimensional vector space, then any injective map $V \to V'$ defines a map $Gr_{n,N} \to Gr_{n,N+1}$. These maps may be defined scheme-theoretically. Indeed, we simply want to specify a rank n-vector bundle on $Gr_{n,N}$ together with a surjection from a trivial bundle of rank $N+1$. However, we may simply take the universal bundle γ_n equipped with its standard surjection and add an additional trivial summand that maps trivially to γ_n. Now, we would like to define an analog of the infinite Grassmannian that appears in topology.

We set

$$Gr_n := \operatorname{colim}_N Gr_{n,N},$$

but we need to work a bit to make sense of the object on the right hand side. For our purposes here, we may view $Gr_{n,N}$ as a presheaf on the category of schemes. In that case, we may take the colimit in the category of presheaves.

6.4.3 Naive homotopy classification

More precisely, suppose we define Gr_n to be the ∞-Grassmannian. There is a rank n vector bundle on Gr_n. Given any smooth affine scheme X, by the definition of the colimit, a morphism $X \to Gr_n$ corresponds to a morphism $X \to Gr_{n,N}$ for N sufficiently large. Since $X = \operatorname{Spec} R$ is affine, such a morphism corresponds to a rank n projective module P over R together with a surjection $R^{\oplus N} \to P$, i.e., N generators of P. Thus, there is an evident surjective map

$$\operatorname{Hom}(X, Gr_n) \longrightarrow \mathscr{V}_n(X).$$

Here, the left hand side corresponds to natural transformations of functors. Now, the right hand side is $\mathbb{A}^1$-invariant. The left hand side is evidently not $\mathbb{A}^1$-invariant: if we take two different sets of N generators of a given projective module of rank n over R yield different maps to the Grassmannian. Therefore, we would like to form the quotient of the left hand side by the relation generated by naive $\mathbb{A}^1$-homotopy.

Theorem 6.4.3.1. *If k is a field and X is a smooth affine k-scheme, then the map*

$$\operatorname{Hom}(X, Gr_n)/\sim_{\mathbb{A}^1} \longrightarrow \mathscr{V}_n(X)$$

that sends a map $X \to Gr_n$ to its naive $\mathbb{A}^1$-homotopy class is a bijection.

Proof. It suffices to demonstrate injectivity. Therefore, consider two maps $\varphi : X \to Gr_n$ and $\varphi' : X \to Gr_n$ that yield the same vector bundle. The map φ corresponds to a pair $(P, e_1, \ldots, e_r)$ where P is a rank n projective module and $e_1, \ldots, e_r$ are r-generators of P, while the map ψ corresponds to $(P, f_1, \ldots, f_s)$. We want to show that the two resulting maps are naively $\mathbb{A}^1$-homotopic. By adding copies of 0, we may view φ and ψ as N-generated projective modules where $N = r+s$. Thus, we want to construct a homotopy between the generators $(e_1, \ldots, e_r, 0, \ldots, 0)$ and $(0, \ldots, 0, f_1, \ldots, f_s)$.

To this end, consider the $R[t]$-modules $P[t]$ obtained by extending scalars to $R[t]$. The set of elements $e_1, \ldots, e_r$ defines a set of generators for the $R[t]$-module Pt and so does $f_1, \ldots, f_s$. However,

$$(e_1, \ldots, e_r, tf_1, \ldots, tf_s) \text{ and } ((1-t)e_1, \ldots, (1-t)e_r, f_1, \ldots, f_s)$$

also define generators of $P[t]$. These two maps define a naive $\mathbb{A}^1$-homotopy connecting the two different sets of generators, which is precisely what we wanted to prove. $\square$

This result may be improved in several different ways. Given a commutative unital ring R, it is not clear there is a uniform bound on the number of generators of a projective R-module of a fixed rank. If R is not finitely generated, such a bound need not exist for a given module. However, if R is finitely generated, then rank + dimension of $\operatorname{Spec} R$ is a bound by a result of Forster-Swan. Passing to a larger Grassmannian was essential in the argument about to build a homotopy. As a consequence, the naive $\mathbb{A}^1$-homotopy clases of maps to a fixed finite dimensional Grassmannian, do not obviously coincide with maps to the infinite Grassmannian.

Naive homotopy classes of maps to spheres: unimodular rows and complete intersection ideals...

Appendix A

Sets, Categories and Functors

Contents

A.1 Categories

For the most part, we use naive set theory, though we will differentiate between sets and classes. We will assume the axiom of choice. We will use category theoretic language to attempt to keep track of "structure" present in the objects under consideration. Neverthless, it is undoubtedly the case that there can come a point where "structure" becomes so refined as to be unwieldy.

A.1.1 Sets

We will not pay too much attention to set theory, but for the most part it will suffice to think "intuitively" about such things (though perhaps even saying this is unintuitive). As most people have probably heard, we should not talk about the set of all sets, since one runs into paradoxical constructions like "the set of all sets that do not contain themselves" (Russell's paradox). We require that the following constructions can be performed with sets; I hope you agree that all these constructions are reasonable.

1. For each set X and each "property" P, we can form the set $\{x \in X | P(x)\}$ of all members of X that have the property P;
2. For each set X, the collection $\{x | x \in X\}$, is a set; this set is sometimes denoted 2^X or $\mathscr{P}(X)$ and referred to as the *power set* of X.
3. Given any pair of sets X and Y, we can form the following sets:
   a) the set $\{X, Y\}$ whose members are exactly X and Y;
   b) the (ordered) pair (X, Y) with first coefficient X and second coefficient Y; more generally for any natural number n and sets $X_1, \ldots, X_n$ we may form the ordered n-tuple $(X_1, X_2, \ldots, X_n)$;
   c) the union $X \cup Y := \{x | x \in X \text{ or } x \in Y\}$;0
   d) the intersection $X \cap Y := \{x | x \in X \text{ and } x \in Y\}$;
   e) the Cartesian product $X \times Y := \{(x, y) | x \in X \text{ and } x \in Y\}$;
   f) the relative complement $X \setminus Y := \{x | x \in X \text{ and } x \notin Y\}$;
   g) a *function* $f : X \to Y$ is a triple (X, Y, f) consisting of a subset of $f \subset X \times Y$ with the property that for each $x \in X$, there is a unique y such that $(x, y) \in f$; the set Y^X of all functions $X \to Y$ is a set.
4. For any set I and any family of sets X_i indexed by I (write $\{X_i\}_{i \in I}$, we can form the following sets:
   a) the image $\{X_i | i \in I\}$ of the indexing function;
   b) the union $\cup_i X_i := \{x | x \in X_i \text{ for some } i \in I\}$;
   c) the intersection $\cap_{i \in I} X_i := \{x | x \in X_i \text{ for all } i \in I\}$, provided $I \neq \emptyset$;
   d) the Cartesian product $\prod_{i \in I} X_i := \{f : I \to \cup_{i \in I} X_i | f(i) \in X_i \text{ for each } i \in I\}$;
   e) the disjoint union $\coprod_{i \in I} X_i := \cup_{i \in I}(X_i \times \{i\}))$.
5. We can form the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of all natural numbers, integers, rational numbers, real numbers, complex numbers.

Remark A.1.1.1. With the above requirements, each topological space is a set, i.e., it is a pair (X, τ) consisting of a set X and a topology τ on X: the topology τ, which is given by the set of open sets in X, is a subset of $\mathscr{P}(\mathscr{P}(X))$. Likewise, each group is a set, each ring is a set, etc.. While spelling everything out in terms of sets is possible in principle, in practice, it would be extremely cumbersome.

While I hope you agree that whatever notion of set one takes one should be able to perform the above constructions, requiring that one can perform such operations is closely related with the notion of a Grothendieck universe, whose definition we now recall.

Definition A.1.1.2. A *Grothendieck universe* is a set $\mathcal{U}$ with the following properties:
1. If $X \in \mathcal{U}$ and if $y \in X$, then $y \in \mathcal{U}$;
2. If $X, Y \in \mathcal{U}$, then $\{X, Y\} \in \mathcal{U}$;
3. If $X \in \mathcal{U}$, then $\mathscr{P}(X) \in \mathcal{U}$;
4. If $\{X_i\}_{i \in I}$ is a family of elements of $\mathcal{U}$ and if $I \in \mathcal{U}$, then $\cap_{i \in I} X_i \in \mathcal{U}$.

Example A.1.1.3. Grothendieck universes are difficult to construct in general: the empty set gives an example. There is another example of a countable universe (that of hereditarily finite sets). If we want, as we do, to work in a universe that contains an uncountable set, then this amounts to a "largeness hypothesis" on our universe. In any case, it turns out that positing the existence of such a universe requires adjoining an axiom to the usual axioms of set theory, and for this reason some people prefer to avoid using universes.

If we fix a Grothendieck universe $\mathcal{U}$, an element $X \in \mathcal{U}$ is called a $\mathcal{U}$-small set, or simply a set. We will also need to consider "larger" constructions, and for this one introduces the notion of a *class*. We require that (1) the members of each class are sets, and (2) for any property "P", one can form the class of all sets with property P, (3) every set is a class. Classes that are not sets are called *proper classes*. Thus, one speaks of the class of all sets, or the class of all topological spaces.

A.1.2 Categories and Functors

Loosely speaking, categories are structures we introduce to keep track of mathematical structures (objects) and the relations between them (morphisms). One can compose morphisms, and there is an identity morphism from any object to itself. More formally, one makes the following definition.

Definition A.1.2.1. A category $\mathscr{C}$ is a quadruple $(\mathrm{Ob}, \mathrm{Hom}, id, \circ)$ consisting of

1. A class $\mathrm{Ob}_{\mathscr{C}}$ of objects;
2. For each pair $X, Y \in \mathrm{Ob}_{\mathscr{C}}$, a set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$;
3. For each object X, a morphism $id_X \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$;
4. For each triple of objects X, Y, Z, a function

$$\circ : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(Y, Z);$$

these data are subject to the following axioms:

1. composition is associative, i.e., given four objets W, X, Y, Z, and morphism $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$, $h \circ (g \circ f) = (h \circ g) \circ f$;
2. id_X is an identity, i.e., for $f \in \mathrm{Hom}_{\mathscr{C}}(W, X)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, $id_X \circ f = f$ and $g \circ id_X = g$;
3. the sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are pairwise disjoint.

If $\mathcal{U}$ is a universe, and if $\mathrm{Ob}_{\mathscr{C}}$ is a $\mathcal{U}$-small set, then $\mathscr{C}$ will be called a $\mathcal{U}$-small category.

Remark A.1.2.2. What we are calling *categories* are often called *locally small* categories in the literature.

Definition A.1.2.3. If $\mathscr{C}$ is any category, then we can define the opposite category $\mathscr{C}^{\circ}$ to be the category where objects are those of $\mathscr{C}$ and the direction of morphisms is reversed.

Definition A.1.2.4. If $\mathscr{C}$ and $\mathscr{D}$ are categories, then a functor $F : \mathscr{C} \to \mathscr{D}$ consists of a function that assigns to each object X in $\mathscr{C}$ an object $F(X)$ in $\mathscr{D}$, and to each pair of objects X, Y, assigns a function $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(F(X), F(Y))$ (also denoted F) such that

1. F preserves composition, i.e., given $f : X \to Y$ and $g : Y \to Z$, $F(g \circ f) = F(g) \circ F(f)$;
2. F preserves identities, i.e., $F(id_X) = id_{F(X)}$.

Example A.1.2.5. If $\mathscr{C}$ is any category, we write $id_{\mathscr{C}}$ for the functor $\mathscr{C} \to \mathscr{C}$ that is the identity on objects and morphisms. The composite of two functors is a functor.

Definition A.1.2.6. If $F, G : \mathscr{C} \to \mathscr{D}$ are functors, then a *natural transformation* $\theta : F \to G$ (or a morphism of functors) is a rule that assigns to each object X of $\mathscr{C}$ a morphism $\theta_X : F(X) \to G(X)$ such that, if $f : X \to Y$ is any morphism in $\mathscr{C}$, then the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\theta_Y"] & G(Y) \end{tikzcd}$$

If for every object X of $\mathscr{C}$ the morphism θ_X is an isomorphism in $\mathscr{D}$, then θ is called a *natural equivalence* (or *natural isomorphism* or *isomorphism of functors*).

Example A.1.2.7. Given any functor $F : \mathscr{C} \to \mathscr{D}$, there is an identity natural transformation $id : F \to F$, which is simply the identity map $id_X : F(X) \to F(X)$ for every object X in $\mathscr{C}$. If F, G are two functors, and θ and θ' are natural transformations, it makes sense to compose $\theta \circ \theta'$, with composition given by objectwise composition.

Example A.1.2.8. If $\mathscr{C}$ and $\mathscr{D}$ are categories, then we can form a new category $F(\mathscr{C}, \mathscr{D})$ where objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and morphisms are natural transformations of functors; the identity is given by the identity functor, and composition is composition of natural transformations as described in Example A.1.2.7. Even if $\mathscr{C}$ and $\mathscr{D}$ are small categories, the functor category $F(\mathscr{C}, \mathscr{D})$ is typically not small.

Definition A.1.2.9. Suppose $\mathscr{C}$ and $\mathscr{D}$ are categories and $F : \mathscr{C} \to \mathscr{D}$ is a functor. We say that F is

- *faithful* if for any pair of objects $X, Y \in \mathscr{C}$, the function $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is injective;
- *reflects isomorphisms* (or *conservative*) if for any arrow $f \in \mathscr{C}$, $F(f)$ is an isomorphism implies f is an isomorphism.
- *an embedding* if it is faithful and injective on objects;
- *full* if for any pair of objects $X, Y \in \mathscr{C}$, the function $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is surjective;
- *fully faithful* if it is both full and faithful;
- *essentially surjective* (or *isomorphism dense*) if for any object $D \in \mathscr{D}$, there exists an object $C \in \mathscr{C}$ and an isomorphism $F(C) \xrightarrow{\sim} D$;
- *an equivalence of categories* if there exists a functor $G : \mathscr{D} \to \mathscr{C}$ and natural equivalences $FG \xrightarrow{\sim} id_{\mathscr{C}}$ and $GF \xrightarrow{\sim} id_{\mathscr{C}}$

Proposition A.1.2.10. *If $F : \mathscr{C} \to \mathscr{D}$ is a functor that is fully faithful and essentially surjective, then there exists a functor $G : \mathscr{D} \to \mathscr{C}$ and isomorphisms of functors $FG \xrightarrow{\sim} id_{\mathscr{C}}$ and $GF \xrightarrow{\sim} id_{\mathscr{D}}$. In other words, F is an equivalence of categories.*

A.1.3 Indexing categories

When we speak about limits and colimits, we will use "indexing categories". Sometimes indexing categories are drawn as diagrams. For example, if we want to speak about pullbacks, we can think of the category pictured as follows:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet;$$

this category has three objects, and we have drawn the non-identity morphisms. Similarly, any directed graph defines a category: one has an object for each vertex and one non-identity morphism for each arrow pictured.

Example A.1.3.1. If $(P, \leq)$ is a category, we can define a directed graph by creating one vertex for each element of P and where there is a unique non-identity morphism $a \to b$ if $a \leq b$.

We now describe further diagram categories.

Definition A.1.3.2. A category $\mathbf{I}$ is *filtered* if:

1. $\mathbf{I}$ is non-empty;
2. for every pair of objects $i, i' \in I$, there exists an object j and two maps $i \to j$ and $i' \to j$; pictorially:

$$\begin{tikzcd} i \arrow[dr] & & i'; \arrow[dl] \\ & j & \end{tikzcd}$$

3. for every pair of morphisms $\alpha, \beta : i \to j$, there exists an object k and an arrow $\gamma : j \to k$, pictorially:

$$\begin{tikzcd} i \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & j \arrow[r, dashed, "\exists\gamma"] & k, \end{tikzcd}$$

such that $\gamma\alpha = \gamma\beta$.

Analogously, a category $\mathbf{I}$ is *cofiltered* if $\mathbf{I}^{op}$ is filtered.

Example A.1.3.3. If $(D, \leq)$ is a partially ordered set, then we may view $(D, \leq)$ as a category whose set of objects is D and where there is a unique morphism $a \to b$ if $a \leq b$. This category if a filtered category in the sense above.

Notation A.1.3.4. Typically, indexing categories take the form described above, but formally, any category can be viewed as an indexing category.

Definition A.1.3.5. If $\mathbf{C}$ is a category, and $\mathbf{I}$ is a category, then an $\mathbf{I}$-*diagram* is a functor $\mathbf{I} \to \mathbf{C}$. The category $\mathrm{Fun}(\mathbf{I}, \mathbf{C})$ is called the category of $\mathbf{I}$-diagrams in $\mathbf{C}$ (i.e., morphisms are natural transformations of functors).

Example A.1.3.6. If $\mathbf{C}$ is a category, $A \in \mathbf{C}$ is an object, and $\mathbf{I}$ is a category, then the *constant* $\mathbf{I}$-*diagram* (with value A) is the functor that assigns to each object $i \in \mathbf{I}$ the object A and to each morphism $i \to i' \in \mathbf{I}$ the identity morphism. Sending an object A to the constant $\mathbf{I}$-diagram defines a functor

$$\Delta : \mathbf{C} \longrightarrow \mathrm{Fun}(\mathbf{I}, \mathbf{C});$$

this functor is typically called the diagonal.

Remark A.1.3.7. One point of view on limits and colimits is that an $\mathbf{I}$-indexed limit is simply a right adjoint to the diagonal functor while an $\mathbf{I}$-indexed colimit is a left adjoint to the diagonal functor.

Filtered colimits versus directed colimits; every filtered category admits a cofinal functor from a directed category. A category has filtered colimits if and only if it has directed colimits.

A.2 Monoidal categories

A.2.1 Monoidal categories

Definition A.2.1.1. A *monoidal category* $(\mathscr{C}, \otimes, \mathbb{1}, a, l, r)$ consists of

- a category $\mathscr{C}$,
- a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- a distinguished *unit* object $\mathbb{1} \in \mathscr{C}$,
- natural isomorphisms $l_X : \mathbb{1} \otimes X \longrightarrow X$, $r_X : X \otimes \mathbb{1} \longrightarrow X$, and
- natural associativity isomorphisms $a_{X,Y,Z} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$;

these data are supposed to satisfy two *coherence* axioms:

1. given a pair of objects $X, Y \in \mathscr{C}$, the diagram

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "a_{XIY}"] \arrow[dr, "r_X \otimes id"'] & & X \otimes (I \otimes Y) \arrow[dl, "id \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

   commutes;

2. given four objects W, X, Y, Z, the diagram

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "a_{W \otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y} \otimes 1_Z"] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "a_{W,X,Y \otimes Z}"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "a_{W,X \otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "1_W \otimes a_{X,Y,Z}"']
\end{tikzcd}$$

   commutes.

Remark A.2.1.2. A priori, there are infintely many more diagrams whose commutativity we could request (e.g., the associativity relations for 5 or greater arrows). There is a "coherence theorem" that shows that requesting commutativity of the above diagrams guarantees commutativity of various more complicated diagrams...

Example A.2.1.3. The category Set of sets with Cartesian product is monoidal. The category Grp of groups with the usual product of groups is monoidal. Likewise, the category Ab is a monoidal subcategory of Grp. The category Cat of categories with product of categories is monoidal. the category Top of topological spaces with the Cartesian product (equipped with the product topology) is monoidal.

A.2.2 Enriched categories

Given a category $\mathscr{C}$ and three objects X, Y, Z, a priori one has a set of homomorphisms $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and composition determines a function $\mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X, Z)$ via the formula $(f, g) \mapsto f \circ g$. In many cases of interest, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ has additional structure, e.g., it is an abelian group or a vector space over a field, and the composition operation respects this additional structure. We now introduce some the standard terminology one uses to keep track of all of the compatibilities inherent in such a structure.

Definition A.2.2.1. Suppose $(\mathscr{E}, \otimes, \mathbb{1}, a, l, r)$ is a symmetric monoidal category. We will say that a (locally small) category $\mathscr{C}$ is $\mathscr{E}$-enriched (or simply an $\mathscr{E}$-category) if

- for every pair of objects X, Y, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is an object of $\mathscr{E}$;
- for each object $X \in \mathscr{C}$, there is an identity element $i_X : \mathbb{1} \to \mathrm{Hom}_{\mathscr{C}}(X, X)$;
- for every triple of objects X, Y, Z, there is a composition law $M_{X,Y,Z} : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \otimes \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$;

the following axioms hold:

1. composition is associative, i.e., for any four objects W, X, Y, Z the diagram

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathscr{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathscr{C}}(X,Y)) \otimes \mathrm{Hom}_{\mathscr{C}}(W,X) \arrow[rr, "a"] \arrow[d, "M\otimes id"] & & \mathrm{Hom}_{\mathscr{C}}(Y,Z)(\otimes \mathrm{Hom}_{\mathscr{C}}(X,Y) \otimes \mathrm{H} \arrow[d, "id\otimes M"] \\
\mathrm{Hom}_{\mathscr{C}}(X,Z) \otimes \mathrm{Hom}_{\mathscr{C}}(W,X) \arrow[r, "M"] & \mathrm{Hom}_{\mathscr{C}}(W,Z) & \mathrm{Hom}_{\mathscr{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathscr{C}}(W \arrow[l, "M"']
\end{tikzcd}$$

   commutes;

2. composition is compatible with units, i.e., for any pair of objects X, Y in $\mathscr{C}$ the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(Y,Y) \otimes \mathrm{Hom}_{\mathscr{C}}(X,Y) \arrow[r, "M"] & \mathrm{Hom}_{\mathscr{C}}(X,Y) & \mathrm{Hom}_{\mathscr{C}}(X,Y) \otimes \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[l, "M"'] \\
\mathbb{1} \otimes \mathrm{Hom}_{\mathscr{C}}(X,Y) \arrow[u, "i\otimes id"] \arrow[ur, "l"] & & \mathrm{Hom}_{\mathscr{C}}(X,Y) \otimes \mathbb{1} \arrow[u, "id\otimes i"'] \arrow[ul, "r"']
\end{tikzcd}$$

   commutes.

Example A.2.2.2. Every (locally small) category is a Set-enriched category.

Definition A.2.2.3. A category $\mathscr{C}$ is called

- *pre-additive* if it is an Ab-enriched category;
- *pre-R-linear* if it a Mod_R-enriched category, with R a commutative unital ring;
- *topological* if it is a Top-enriched category; and
- *simplicial* if it is an sSet-enriched category.

Given an enriched category, it will be important to consider functors that preserve the additional structure present on morphism sets. This notion is summarized in the next definition.

Definition A.2.2.4. Given a monoidal category $(\mathscr{E}, \otimes, \mathbb{1}, a, l, r)$ and two $\mathscr{E}$-enriched categories $\mathscr{C}$ and $\mathscr{D}$, a functor $F : \mathscr{C} \to \mathscr{D}$ will be called an $\mathscr{E}$-enriched functor, or simply an $\mathscr{E}$-functor if for any pair of objects $X, Y \in \mathscr{C}$, the map $F_{AB} : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ is a morphism in $\mathscr{E}$, and the following conditions are satisfied:

1. the functor is compatible with the monoidal structure, i.e., given three objects X, Y, Z the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(Y,Z) \otimes \mathrm{Hom}_{\mathscr{C}}(X,Y) \arrow[r, "M"] \arrow[d, "F\otimes F"] & \mathrm{Hom}_{\mathscr{C}}(X,Z) \arrow[d, "F"] \\
\mathrm{Hom}_{\mathscr{D}}(F(Y),F(Z)) \otimes \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)) \arrow[r, "M"] & \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))
\end{tikzcd}$$

   commutes;

2. the functor is compatible with units, i.e., given any object $X \in \mathscr{C}$, the diagram

$$\begin{tikzcd} & \mathbb{1} \arrow[dl] \arrow[dr] & \\ \mathrm{Hom}_{\mathscr{C}}(X,X) \arrow[rr, "F"] & & \mathrm{Hom}_{\mathscr{D}}(F(X),F(X)) \end{tikzcd}$$

commutes.

Example A.2.2.5. An $\mathscr{E}$-functor $F : \mathscr{C} \to \mathscr{D}$ of $\mathscr{E}$-categories is called

- *pre-additive* if $\mathscr{E} = \mathrm{Ab}$;
- *pre-R-linear* if $\mathscr{E} = \mathrm{Mod}_R$ for R a commutative unital ring.

A.2.3 Symmetric monoidal categories

Definition A.2.3.1. If $(\mathscr{C}, \otimes, \mathbb{1}, a, l, r)$ is a monoidal category, a *symmetric* structure on $\mathscr{C}$ is the data of a natural isomorphism $c_{XY} : X \otimes Y \to Y \otimes X$ (the commutativity isomorphism) satisfying the following coherence axioms:

1. $c^2 = id_{\mathscr{C}}$, i.e., for every pair of objects $X, Y \in \mathscr{C}$, the diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "c"] \arrow[rr, bend left, "id"] & Y \otimes X \arrow[r, "c"] & X \otimes Y \end{tikzcd}$$

commutes;

2. compatibility with the unit, i.e., for every object $X \in \mathscr{C}$ the diagram

$$\begin{tikzcd} \mathbb{1} \otimes X \arrow[rr, "c"] \arrow[dr, "l_X"] & & X \otimes \mathbb{1} \arrow[dl, "r_X"] \\ & X & \end{tikzcd}$$

commutes;

3. compatibility between commutativity and associativity, i.e., for every triple X, Y, Z of objects in $\mathscr{C}$ the diagram

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "c\otimes id"'] & X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "a_{Y,Z,X}"] \\ (Y \otimes X) \otimes X \arrow[r, "a_{Y,X,Z}"'] & Y \otimes (X \otimes Z) \arrow[r, "id\otimes c"'] & Y \otimes (Z \otimes X) \end{tikzcd}$$

commutes.

A monoidal category equipped with a symmetric structure will be called a *symmetric monoidal category*.

Example A.2.3.2. The category of abelian groups equipped with the isomorphism $c_{A,B} : A \times B \to B \times A$ given by switching the two factors is a symmetric monoidal category. The same holds for the category of R-modules over a commutative unital ring R.

A.3 Other types of categories

A.3.1 Abelian categories

Definition A.3.1.1. An triple $(\mathscr{C}, 0)$ consisting of an Ab-enriched category $\mathscr{C}$ and a distinguished object 0 is called *abelian* if

1. the object 0 is a zero object, i.e., it is both initial and final;
2. for any two objects X, Y, a biproduct $X \times Y$ exists in $\mathscr{C}$;
3. every morphism in $\mathscr{C}$ has a kernel and a cokernel;
4. every monomorphism is a kernel of its cokernel, and every epimorphism is a cokernel of its kernel.

A.3.2 Exact categories

Exact categories were initially defined by Quillen [**?**, §2]. With time, simplifications of Quillen's axioms were observed (cf. [**?**, §9.1]). The following definition is due to Keller [**?**, Appendix A]. See [**?**] for a more detailed treatment.

Definition A.3.2.1. Given an additive category $\mathscr{A}$, a pair of composable morphisms

$$X \xrightarrow{i} Y \xrightarrow{p} Z$$

is called *exact* if i is a kernel of d and d is a cokernel of i. We will refer to a diagram as a pair (i, p) as above as an *exact pair*; the morphism i will be called an *admissible monomorphism* and the morphism p will be called an admissible epimorphism.

Definition A.3.2.2. Given an additive category $\mathscr{A}$, an *exact structure* on $\mathscr{A}$ consists of a a class $\mathscr{E} \subset \mathscr{A}$ of exact pairs closed under isomorphisms and satisfying the following axioms:

E0 the identity morphism on the zero object is an admissible epimorphism;
E1 admissible epimorphisms are stable by composition;
E2 admissible epimorphisms are stable by pullback;
E2° admissible monomorphisms are stable by pushout;

A pair $(\mathscr{A}, \mathscr{E})$ consisting of an additive category and an exact structure will be called an *exact category*.

Lemma A.3.2.3. *If $(\mathscr{A}, \mathscr{E})$ is an exact category, then for any pair of objects X, Y of $\mathscr{C}$, the pair*

$$X \xrightarrow{(0\ 1)} X \oplus Y \xrightarrow{\binom{1}{0}} Y$$

is an exact pair.

Lemma A.3.2.4. *If $(\mathscr{A}, \mathscr{E})$ is an exact category, then admissible monomorphisms are stable under composition.*

Appendix B

Some algebraic facts

Here, I want to include some algebra facts that we will use repeatedly, collected for convenient reference.

B.1 Localization

In this section, we review some basic properties of localization of a ring that will be used repeatedly in the main body of the text. Localization is a way of inverting elements in a ring or a module over a ring.

Definition B.1.0.1. Suppose R is a commutative unital ring. A multiplicative subset $S \subset R$ is a subset such that $1 \in S$, and if $s, s' \in S$, then $ss' \in S$. If $S \subset R$ is a multiplicative subset, then the localization $R[S^{-1}]$ (or sometimes $S^{-1}R$) is quotient of $R \times S$ by the following equivalence relation: $(r,s) \sim (r',s')$ if and only if there exists $u \in S$ such that $(rs' - sr')u = 0$.

The set $R[S^{-1}]$ is a ring with multiplicative unit $(1,1)$ addition defined by the usual formula for adding fractions $(r,s) + (r',s') = (rs' + sr', ss')$ and multiplication defined componentwise, i.e., $(r,s)(r',s') = (rr', ss')$. For this reason, we will frequently write $\frac{r}{s}$ for the element (r,s) in $R[S^{-1}]$. Of course, this notation should be taken with a grain of salt since S might have zero-divisors. Since $1 \in S$ there is an evident ring homomorphism $R \to R[S^{-1}]$ sending r to $(r,1)$, which we will refer to as the *localization map*. Since S might have zero divisors, the ring homomorphism $R \to R[S^{-1}]$ can fail to be injective. More precisely, $(r,1) = 0$ in $R[S^{-1}]$ if and only if there exists $u \in S$ such that $ru = 0$, i.e., the localization map is injective if and only if S has no zero divisors.

Example B.1.0.2. If $f \in R$ is any element, then the multiplicative subset generated by f is the subset $\{1, f, f^2, \ldots, \}$, we write R_f for the corresponding localization. If $\mathfrak{p}$ is a prime ideal, then $R \setminus \mathfrak{p}$ is a multiplicative set by definition of a prime ideal and we write $R_{\mathfrak{p}}$ for the associated localization.

A key property of localization that we will use repeatedly is the universal property: the localization of a ring R at a multiplicative set is the smallest ring in which the elements of S are invertible.

Proposition B.1.0.3. *Assume R is a ring, and $S \subset R$ is a multiplicative set. If $\varphi : R \to A$ is a ring homomorphism that sends every element $s \in S$ to a unit in A, then φ factors uniquely through the* *localization map, i.e., there is a unique ring homomorphism $R[S^{-1}] \to A$ such that the composite $R \to R[S^{-1}] \to A$ coincides with φ.*

Proof. For the existence statement, we send (r, s) to $\varphi(r)\varphi(s)^{-1}$, which is defined since $\varphi(s)$ is a unit in A by assumpion. It is straightforward to check that this is a ring homomorphism. Since $\varphi(1) = 1$, it follows immediately that this homomorphism factors φ as claimed. □

Proposition B.1.0.4. *If R is a commutative ring, and M is an R-module. For an element $x \in R$, the following statements are equivalent*

1. $x = 0$;
2. *x maps to zero in $M_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \subset R$;*
3. *x maps to zero in $M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$.*

In particular, the map $M \to \prod_{\mathfrak{m} \subset R} M_{\mathfrak{m}}$ is injective.

Proof. It is immediate that (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3). We will establish that (3) $\Longrightarrow$ (1). To this end, take $x \in M$ and consider the ideal $I = \{f \in R | fx = 0\}$ (i.e., the annihilator ideal of x). Now, the assumption that x maps to zero in each localization $M_{\mathfrak{m}}$ means that for every maximal ideal $\mathfrak{m}$, there exists an element $f \in R \backslash \mathfrak{m}$ such that $fx = 0$. In other words, $V(I)$ contains no closed points. If that is the case, it follows from Lemma 1.1.1.3(2) that I must be the unit ideal and so x must be zero. □

Corollary B.1.0.5. *If R is a commutative ring and M is an R-module, the following statements are equivalent:*

1. *M is zero;*
2. *$M_{\mathfrak{p}}$ is zero for all prime ideals $\mathfrak{p} \subset R$;*
3. *$M_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m} \subset R$.*

Proof. This follows immediately from the preceding proposition applied to every element $x \in M$. □

Localization preserves exact sequences of R-modules by appeal to Theorem 2.1.3.11. As a consequence, we deduce the following result by observing that the localization of the homology of a sequence $M_1 \to M_2 \to M_3$ of R-modules is the homology of the localization.

Corollary B.1.0.6. *If R is a commutative ring, then the following statements are equivalent:*

1. *a sequence $M_1 \to M_2 \to M_3$ of R-modules is exact;*
2. *the sequence $(M_1)_{\mathfrak{p}} \to (M_2)_{\mathfrak{p}} \to (M_3)_{\mathfrak{p}}$ is exact for all prime ideals $\mathfrak{p} \subset R$;*
3. *the sequence $(M_1)_{\mathfrak{m}} \to (M_2)_{\mathfrak{m}} \to (M_3)_{\mathfrak{m}}$ is exact for all maximal ideals $\mathfrak{m} \subset R$.*

Lemma B.1.0.7. *Let R be a ring. Let $S \subset R$ be a multiplicative subset, and let M, and N be R-modules. Assume all the elements of S act as automorphisms on N. Then the canonical map*

$$\mathrm{Hom}_R(M[S^{-1}], N) \longrightarrow \mathrm{Hom}_R(M, N)$$

induced by the localization map is an isomorphism.

Bibliography

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