The Jouanolou-Thomason homotopy lemma

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1 Introduction

The goal of this note is to prove what is now known as the Jouanolou-Thomason homotopy lemma or simply "Jouanolou's trick." Our main reason for discussing this here is that i) most statements (that I have seen) assume unncessary quasi-projectivity hypotheses, and ii) most applications of the result that I know (e.g., in homotopy K-theory) appeal to the result as merely a "black box," while the proof indicates that the construction is quite geometric and relatively explicit. For simplicity, throughout the word *scheme* means separated Noetherian scheme.

Theorem 1.1 (Jouanolou-Thomason homotopy lemma)**.** *Given a smooth scheme X over a regular Noetherian base ring k, there exists a pair $(\tilde{X}, \pi)$, where $\tilde{X}$ is an affine scheme, smooth over k, and $\pi : \tilde{X} \to X$ is a Zariski locally trivial smooth morphism with fibers isomorphic to affine spaces.*

Remark 1.2. In terms of an $\mathbb{A}^1$-homotopy category of smooth schemes over k (e.g., $\mathcal{H}(k)$ or $\mathcal{H}_{\text{ét}}(k)$; see [MV99, §3]), the map π is an $\mathbb{A}^1$-weak equivalence (use [MV99, §3 Example 2.4]. Thus, up to $\mathbb{A}^1$-weak equivalence, any smooth k-scheme *is* an affine scheme smooth over k.

2 An explicit algebraic form

Let $\mathbb{A}^n$ denote affine space over $\operatorname{Spec}\mathbb{Z}$. Let $\mathbb{A}^n \setminus 0$ denote the scheme quasi-affine and smooth over $\operatorname{Spec}\mathbb{Z}$ obtained by removing the fiber over 0. Let Q_{2m-1} denote the closed subscheme of $\mathbb{A}^{2m}$ (with coordinates $x_1, \ldots, x_{2m}$) defined by the equation

$$\sum_i x_i x_{m+i} = 1.$$

Consider the following simple situation.

Lemma 2.1. *For any $m \geq 1$, projection onto $x_1, \ldots, x_m$ determines a morphism*

$$\varphi : Q_{2m-1} \to \mathbb{A}^m \setminus 0$$

that is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{m-1}$. In particular, for $m = 1$, π is an isomorphism.

Proof. It is easy to check that π trivializes over the open affine subschemes U_i of $\mathbb{A}^m \setminus 0$ where one coordinate x_i is non-vanishing. □

Proposition 2.2. *Suppose $Z \subset \mathbb{A}^n$ is a closed subscheme defined by the vanishing of functions $f_1, \ldots, f_d$. Consider the morphism $\mathbb{A}^n \to \mathbb{A}^d$ defined by the functions $f_1, \ldots, f_d$. Define a morphism*

$$\mathbf{f} : \mathbb{A}^n \setminus Z \longrightarrow \mathbb{A}^d \setminus 0$$

via $(f_1, \ldots, f_d)$. Then,

i) the fiber product $\widetilde{\mathbb{A}^n \setminus Z} := \mathbb{A}^n \setminus Z \times_{\mathbb{A}^d \setminus 0} Q_{2d-1}$ (via $\mathbf{f}$ and φ) is isomorphic to the closed subscheme of $\mathbb{A}^n \times \mathbb{A}^d$ (say with coordinates $y_1, \ldots, y_n, x_1, \ldots, x_d$) defined by the equation

$$\sum_{i=1}^{d} x_i f_i = 1.$$

and is in particular affine.

ii) The projection morphism $\pi : \widetilde{\mathbb{A}^n \setminus Z} \to \mathbb{A}^n \setminus Z$ is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{d-1}$, and is, in particular, smooth.

Proof. Easy exercise. The key points are that $\mathbf{f} := (f_1, \ldots, f_n)$ defines an affine morphism, and that π can be (explicitly) trivialized along the open affine scheme U_i where $f_i \neq 0$. $\square$

Remark 2.3. Note that this presentation depends only on a choice of generators for the underlying reduced subscheme of Z. E.g., if we replace $f_1, \ldots, f_d$ by $f_1^{a_1}, \ldots, f_d^{a_d}$ for some integers $a_1, \ldots, a_d$, we get a new morphism that is again Zariski locally trivial with affine space fibers. Simple examples show that the resulting schemes *need not* be isomorphic. Even though our construction does not explicitly indicate dependence on the generators, this dependence is implicit and so the construction is *not* functorial. This problem will reappear in all the varations on this result below. Furthermore, it is clear how to extend this to complements of arbitrary closed subschemes of an affine scheme.

3 Jouanolou's geometric lemma

Let V be a free $\mathbb{Z}$-module, and let $V^\vee$ denote the dual $\mathbb{Z}$-module $\hom(V, \mathbb{Z})$. Consider the pairing $V \times V^\vee \to \mathbb{Z}$ given by evaluation. Let $\mathbb{P}(V) := \operatorname{Proj} \operatorname{Sym}^\bullet V$ and $\mathbb{P}(V^\vee) := \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$. Note that $\mathbb{Z}[x_1, \ldots, x_n] \cong \operatorname{Sym}^\bullet V^\vee$ as graded algebras, so in our notation $\mathbb{P}(V)$ corresponds to the projective space whose k-points are *hyperplanes* in $V \otimes_{\mathbb{Z}} k$ (as in Grothendieck's conventions). Consider the closed subscheme H of $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ defined by the incidence hyperplane defined by the pairing above.

Proposition 3.1. *Let $\widetilde{\mathbb{P}(V)}$ denote the complement $\mathbb{P}(V) \times \mathbb{P}(V^\vee) \setminus H$. Then,*

i) The scheme $\widetilde{\mathbb{P}(V)}$ is affine.

ii) The composite morphism

$$\widetilde{\mathbb{P}(V)} \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(V^\vee) \xrightarrow{p_1} \mathbb{P}(V)$$

is a Zariski locally trivial affine morphism with fibers isomorphic to affine space, and is, in particular, smooth.

Proof. The hypersurface H defines a divisor on $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ that is ample. In particular, the associated invertible sheaf is $\mathcal{O}(1) \boxtimes \mathcal{O}(1) := p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. The complement of an ample divisor is always affine (use the closed embedding into projective space and the fact that the complement of a hyperplane in $\mathbb{P}(V)$ is just an affine space).

As a composite of an open immersion and a projection with smooth fibers, it follows that $\pi : \widetilde{\mathbb{P}(V)} \to \mathbb{P}(V)$ is a smooth morphism. Choosing sections $s_1, \ldots, s_n$ on V, we see by explicit computation that π trivalizes on the complement of the loci where each s_i is non-zero. □

Corollary 3.2 (*cf.* [Jou73] Lemma 1.5 or [Wei89] Proposition 4.3)**.** *Suppose X is a* projective *variety over* $\operatorname{Spec} \mathbb{Z}$*, and fix a very ample line bundle $\mathcal{L}$ on X. Then,*

i) If $\tilde{X}$ denotes the fiber product of $X \times_{\mathbb{P}(H^0(X,\mathcal{L}))} \mathbb{P}(\widetilde{H^0(X,\mathcal{L})})$, then $\tilde{X}$ is affine.

ii) The morphism $\pi : \tilde{X} \to X$ induced by projection is Zariski locally trivial with fibers isomorphic to affine spaces.

Proof. Easy exercise. Again, the key point is that the sections $s_1, \ldots, s_n$ of $H^0(X,\mathcal{L})^\vee$ produce a closed embedding, which is in particular an affine morphism. As before, the morphism $\pi : \tilde{X} \to X$ trivializes over the complement of the vanishing locus of the sections s_i. □

Remark 3.3. Note, in particular, that we only care about $\mathbb{Z}$-module structure of these sections, and the particular choice of basis is not important (just as in the quasi-affine case we treated above). Thus, to prove the result for more general schemes, one expects to need only "enough sections."

4 Thomason's extension

As per our previous remark, we need schemes that have "enough open affines." There is a convenient formalization of this concept generalizing the notion of an ample invertible sheaf.

Definition 4.1. A scheme X is said to *divisorial* or to *admit an ample family of invertible sheaves* if there are invertible sheaves $\mathcal{L}_0, \ldots, \mathcal{L}_n$ on X together with global sections $s_i \in H^0(X, \mathcal{L}_i)$, such that the schemes $U_i \subset X$ defined as the complement of the vanishing locus of s_i are affine and the map $U := \coprod_i U_i \to X$ is a Zariski cover.

Remark 4.2. The notion of a scheme admitting an ample family of line bundles (e.g., [SGA71, Exposé II Définition 2.2.4-5]) was new terminology proposed by Illusie for the old notion of a divisorial scheme (variety) introduced by Kleiman and Borelli (e.g., [Bor63, §3]). See [SGA71, Exposé II Proposition 2.2.3] for a proof of equivalence of various conditions with that given in Borelli's definition.

The basic idea of the proof is to glue together the schemes $U_i \times \mathbb{A}^n$ (note that there are $n+1$-sections) with appropriate transition functions to obtain a morphism $\tilde{X} \to X$ where $\tilde{X}$ is affine.

Proposition 4.3 (Thomason's version of Jouanolou's trick *cf.* [Wei89] Proposition 4.4)**.** *Let k be a commutative unital ring. Suppose X is a k-scheme admitting an ample family of line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_n$. Choose sections $s_i \in H^0(X, \mathcal{L}_i)$. There exists a pair $(\tilde{X}, \pi)$ consisting of an affine scheme $\tilde{X}$ together with a Zariski locally trivial morphism $\pi : \tilde{X} \to X$ that has affine space fibers.*

Proof. View each section s_i as a morphism $s_i : \mathcal{O}_X \to \mathcal{L}_i$. If we set $\mathcal{E} = \oplus_i \mathcal{L}_i$, then the the function $(s_0, \ldots, s_n)$ determines a morphism

$$\mathbf{s} : \mathcal{O}_X \to \mathcal{E}.$$

Over each U_i, the morphism $\mathbf{s}$ is a *split* monomoprhism. Thus, the cokernel of $\mathbf{s}$, which we denote by $\mathcal{F}$, is actually a locally free sheaf. Set $\tilde{X} := \mathbb{P}_X(\mathcal{E}) \setminus \mathbb{P}_X(\mathcal{F})$, and let π be the composite morphism

$$\tilde{X} \hookrightarrow \mathbb{P}_X(\mathcal{E}) \longrightarrow X.$$

As the composite of an open immersion and a smooth projection, π is clearly a smooth morphism. By construction, π trivializes along each U_i and one obtains a product decomposition as advertised before the statement of the Proposition. More precisely, one can check that $\tilde{X}$ can be identified with the closed subscheme of $\mathrm{Spec}_X\, \mathcal{E}$ defined by the equation $s = 1$. In particular, this shows that π is an affine morphism. Let f_i denote the element of $\Gamma(\tilde{X}, \mathcal{O}_X)$ induced by s_i. We see that $\sum_i f_i = 1$, and the complement of $f_i = 0$ is isomorphic to $U_i \times \mathbb{A}^n$, which is in particular an affine scheme. By the local criterion for affineness [Gro61, Proposition 5.2.1], it follows that $\tilde{X}$ is affine. □

Remark 4.4. One can check that the above actually proves more: the morphism π is a so-called *affine vector bundle torsor*. If Lin_n denotes the (linear algebraic) group of affine linear automorphisms of a vector space, then a torsor under a vector bundle of rank n is precisely a torsor, i.e., principal homogeneous space, for Lin_n. A torsor under a vector bundle is called an affine vector bundle torsor if its total space is affine. Thus, the above proof shows that π is a torsor under the vector bundle associated with the locally free sheaf $\mathcal{F}$.

To finish, the proof of Theorem 1.1, we need the following pair of results.

Theorem 4.5. *If X is a regular variety over a field k, or more generally, a separated locally factorial Noetherian scheme, then X admits an ample family of line bundles.*

Proof. According to Borelli, the first statement is due to Zariski [Zar47] (I have not looked for the precise reference). For a proof of a statement slightly more general than this, see [Bor63, Theorem 4.1], but note that he refers to what we would today call a locally factorial variety as simply factorial. The second result is culled from [SGA71, Exposé II Proposition 2.2.7]. The latter proof is significantly more involved than the former, which might be the reason that the result is not usually stated in this generality. □

Corollary 4.6. *If X is smooth over a regular Noetherian base ring k, then X admits an ample family of line bundles.*

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