On the Freudenthal suspension theorem in unstable motivic homotopy theory

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Theorem 1. *Suppose k is an algebraically closed field having characteristic 0 and $d \geq 1$ is an integer. If X is a smooth affine k-variety of dimension $d+1$, and E is a rank d vector bundle on X, then E splits off a trivial rank 1 summand if and only if $0 = c_d(E) \in CH^d(X)$.*

Remark 2. Tbe result above answers in the affirmative "Murthy's conjecture" in characteristic 0. The statement is immediate if $d = 1$ in which case it holds without restriction on the characteristic of k. The statement also holds when $d = 2, 3$ if k has characteristic not equal to 2 by previous work of J. Fasel and the first author [AF14, AF15].

Suppose k is a field having characteristic 0. Write Sm_k for the category of smooth k-schemes. We write Spc_k for a suitable category of spaces (e.g., simplicial presheaves on Sm_k) and $\mathrm{H}(k)$ for the Morel–Voevodsky unstable motivic homotopy category of k [MoVo99]. Traditionally, $\mathrm{H}(k)$ is obtained by a two-step Bousfield localization of Spc_k: one first inverts Nisnevich local weak equivalences and then $\mathbb{A}^1$-weak equivalences.

Theorem 1 is established using techniques of motivic homotopy theory. Combining the $\mathbb{A}^1$-homotopy classification of vector bundles and the existence of $\mathbb{A}^1$-fiber sequences of the form

$$\mathbb{A}^n \setminus 0 \longrightarrow BGL_{n-1} \longrightarrow BGL_n,$$

techniques of obstruction theory reduce the verification of Theorem 1 to understanding the $\mathbb{A}^1$-homotopy theory of $\mathbb{A}^n \setminus 0$.

Write S^1 for the space $\mathbb{A}^1/\{0,1\}$ and $\mathbb{G}_m$ for $\mathbb{A}^1 \setminus 0$ pointed by 1. We set $S^p := (S^1)^{\wedge p}$ and $S^{p,q} = S^p \wedge \mathbb{G}_m^{\wedge q}$. For any pointed space (X, x), one defines homotopy (Nisnevich) sheaves $\pi_p^{\mathbb{A}^1}(X, x)$. These sheaves detect $\mathbb{A}^1$-weak equivalences and we may define $\mathbb{A}^1$-n-connectedness by imposing vanishing conditions on homotopy sheaves. One can define, more generally, $\pi_{p,q}^{\mathbb{A}^1}(X, x)$, and these sheaves may be identified with $\pi_p^{\mathbb{A}^1}(\Omega^q_{\mathbb{G}_m} X)$, where $\Omega^q_{\mathbb{G}_m} X$ is the q-fold $\mathbb{G}_m$-loop space of X.

One knows that $\mathbb{A}^n \setminus 0 \sim S^{n-1,n}$ and $\mathbb{P}^1 \cong S^{1,1}$. F. Morel's foundational unstable connectivity theorem asserts that $\mathbb{A}^1$-localization preserves connectivity. One deduces immediately that $\mathbb{A}^n \setminus 0$ is $\mathbb{A}^1$-$(n-2)$-connected. Morel also computed the first non-vanishing homotopy sheaf of $S^{p,q}$ in various situations. For example, if $p \geq 2$ and $q \geq 1$, then $\pi_{p,i}^{\mathbb{A}^1}(S^{p,q}) \cong \mathbf{K}^{MW}_{q-i}$. where $\mathbf{K}^{MW}_r$ is the so-called Milnor–Witt K-theory sheaf (see [Mor12] for all these results).

Granted Morel's computations, the required information to establish Theorem 1 is contained in the next non-vanishing $\mathbb{A}^1$-homotopy sheaf $\pi_n(\mathbb{A}^n \setminus 0)$. Previous work of the first author and J. Fasel (exposed at a previous Oberwolfach meeting) gave a regular form for this result, which could be deduced from a suitable version of the Freudenthal suspension theorem for $\mathbb{P}^1$-suspension.

While Morel established a Freudenthal suspension theorem for S^1-suspension that looks formally identical to the classical case, simple computations show that Freudenthal suspension for $\mathbb{P}^1$-suspension requires further hypotheses. For example, it is easy to see that $\pi_p^{\mathbb{A}^1}(S^p) \cong \mathbb{Z}$ for all $p \geq 1$, and therefore, for $p \geq 2$, the map $S^p \to \Omega_{\mathbb{P}^1} S^{p+1,1}$ is not an isomorphism on homotopy sheaves in degree p.

Intuitively speaking, S^p is not sufficiently "$\mathbb{G}_m$-connected". To make this precise, one proceeds as follows, roughly mimicking one construction of the classical Postnikov tower using Bousfield localization. Consider the left Bousfield localization of $\mathrm{H}(k)$ generated by the maps $\mathbb{G}_m^{\wedge n+1} \times X \to X, X \in \mathrm{Sm}_k$; write L_n for the resulting localization functor.

Definition 3. *We will say that a space X is $\mathbb{G}_m$-n-connected if $X \sim \mathrm{L}_n X$ and $\mathbb{G}_m$-n-truncated if $\mathrm{L}_n X \cong *$.*

By construction, any pointed space that is of the form $\mathbb{G}_m^{\wedge n+1} \wedge X$ is $\mathbb{G}_m$-n-connected. One says that a space X is (p,q)-connected if it is p-connected and $\mathbb{G}_m$-q-connected. In particular, $\mathbb{A}^n \setminus 0$ is $(n-2, n-1)$-connected. From the definitions, it is not hard to show that a pointed, connected space X is $\mathbb{G}_m$-n-truncated if and only if the $\mathbb{G}_m$-loop space $\Omega^{n+1}_{\mathbb{G}_m} X$ is contractible. In particular, by a result of Morel it follows that such an X is $\mathbb{G}_m$-n-truncated if and only if $\pi_i^{\mathbb{A}^1}(X,x)_{-n-1} = 0$ for all i. It follows that the motivic Eilenberg–Mac Lane space $K(\mathbb{Z}(n), 2n)$ is $\mathbb{G}_m$-n-truncated for any $n \geq 0$.

This notion of $\mathbb{G}_m$-connectivity is well-behaved in that $\mathbb{G}_m$-connectedness and truncatedness is preserved by taking suitable fibers and cofibers. To see this, one appeals to a comparison between unstable and S^1-stable homotopy theory. In the S^1-stable context one uses results about existence of $\mathbb{G}_m$-deloopings based on the work of the second author and M. Yakerson [BY20, Bac21]. One nice consequence of these results is that one can establish a Whitehead theorem using motives, at least over fields k having finite étale 2-cohomological dimension: a map f of $(1,1)$-connected spaces such that $H\mathbb{Z} \wedge f$ is an isomorphism is an $\mathbb{A}^1$-weak equivalence.

Theorem 4 ($\mathbb{P}^1$-Freudenthal suspension theorem)**.** *Assume $1 \leq p \leq q$ are integers. If (X,x) is a pointed (p,q)-connected space, then the unit map*

$$X \longrightarrow \Omega_{\mathbb{P}^1} \Sigma_{\mathbb{P}^1} X$$

has $\mathbb{A}^1$-homotopy fiber that is at least $(2p, 2q+1)$-connected.

Theorem 4 can be reduced to the case of motivic Eilenberg–Mac Lane spaces. The assembly maps

$$a_n : \mathbb{P}^1 \wedge K(\mathbb{Z}(n), 2n) \longrightarrow K(\mathbb{Z}(n+1), 2n+2)$$

defining the motivic Eilenberg-Mac Lane spectrum can be used to factor the identity map on $K(\mathbb{Z}(n), 2n)$ through the unit of the loop suspension adjunction:

$$K(\mathbb{Z}(n), 2n) \xrightarrow{u} \Omega_{\mathbb{P}^1} \Sigma_{\mathbb{P}^1} K(\mathbb{Z}(n), 2n) \xrightarrow{\Omega_{\mathbb{P}^1} a_n} \Omega_{\mathbb{P}^1} K(\mathbb{Z}(n+1), 2n+2) \cong K(\mathbb{Z}(n), 2n).$$

In that case, there is a fiber sequence $\mathrm{fib}(u) \longrightarrow * \longrightarrow \mathrm{fib}(\Omega_{\mathbb{P}^1} a_n)$, i.e., $\mathrm{fib}(u) \cong \Omega\,\mathrm{fib}(\Omega_{\mathbb{P}^1} a_n)$. To establish the result, it then suffices to establish a suitable connectivity bound on $\mathrm{fib}(a_n)$. By Blakers–Massey style results, one then reduces to establishing the following bound on the connectivity of $\mathrm{cof}(a_n)$.

Lemma 5. *The space* $\mathrm{cof}(a_n)$ *is* $(2n+1, 2n)$*-connected;*

Sketch of proof. The proof of Lemma 5 proceeds by analyzing the geometry of symmetric powers. Following Voevodsky, write Quot_{Σ_r} for the unique colimit preserving extension of the functor on quasi-projective Σ_r-schemes to motivic spaces with Σ_r-action given by taking the quotient. One defines $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n}) \cong \mathrm{Quot}_{\Sigma_r}(\mathbb{P}^{1\wedge n \times r})$. To analyze the connectivity of a_n, one uses Voevodsky's motivic Dold-Thom theorem: $K(\mathbb{Z}(n), 2n) \cong \mathrm{Sym}^\infty(\mathbb{P}^{1\wedge n})$ and the fact that the latter space is a colimit of spaces of the form $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n})$.

Let us write ρ for the standard r-dimensional representation of Σ_r, which decomposes as $\bar{\rho} \oplus \mathbf{1}$, where $\mathbf{1}$ is the trivial representation. To make explicit the Σ_r-action we identify $\mathrm{Sym}^r(\mathbb{P}^{1\wedge n})$ with $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\rho^{\oplus n})))$, viewing $\mathbb{A}(\rho^{\oplus n})$ as a trivial vector bundle over a point. The identification $\rho^{\oplus n} \cong \mathbf{1}^{\oplus n} \oplus \bar{\rho}^{\oplus n}$ and the usual properties of Thom spaces imply $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\rho^{\oplus n}))) \cong \Sigma^n_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(\bar{\rho}^{\oplus n})))$.

The assembly map a_n arises from the sequence of inclusions $\mathbf{1} \oplus n\rho \subset \rho \oplus n\rho \cong (n+1)\rho$ by applying $\mathrm{Quot}_{\Sigma_r}(Th(\mathbb{A}(-)))$, i.e., it is a map

$$\Sigma^{r+1}_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r} Th(\mathbb{A}(\bar{\rho}^{\oplus n})) \longrightarrow \Sigma^{r+1}_{\mathbb{P}^1}\mathrm{Quot}_{\Sigma_r} Th(\mathbb{A}(\bar{\rho})^{\oplus n+1}).$$

We may then identify $\mathrm{cof}(a_n)$ with a colimit of spaces of the form

$$\Sigma\mathrm{Quot}_{\Sigma_r} Th(\mathbb{A}_{\mathbb{A}(\bar{\rho})\setminus 0}(\bar{\rho}^{\oplus n} \oplus 1)),$$

viewing $\mathbb{A}(\bar{\rho}^{\oplus n} \oplus 1)$ as a trivial vector bundle over $\mathbb{A}(\bar{\rho}) \setminus 0$.

Following Nakaoka and Voevodsky (see [AD01] and [Voe04, §4]), we analyze the Thom space above by stratifying $\mathbb{A}(\bar{\rho}) \setminus 0$ by stabilizer type. The connectivity of the Thom space in question can be bounded below by the Thom spaces of normal bundles to each of the stabilizers. One analyzes these normal bundles using a bit of representation theory of the symmetric group.

Since $\bar{\rho}$ is an irreducible representation of Σ_r, the only fixed point in $\mathbb{A}(\bar{\rho})$ is the origin, i.e., every point of $\mathbb{A}(\bar{\rho}) \setminus 0$ has a non-trivial stabilizer. Each stabilizer in Σ_r is a partition subgroup and the the stabilizers appearing in $\mathbb{A}(\bar{\rho}) \setminus 0$ are proper partition subgroups. A proper partition subgroup of Σ_r is of the form $\Sigma_{r_1} \times \cdots \times \Sigma_{r_s}$ where $\sum_i r_i = r$; in particular, a non-trivial such subgroup necessarily has more than 1 factor. If H is a partition subgroup of Σ_r, then $\mathrm{Res}^{\Sigma_r}_H(\rho)$ decomposes as a direct sum of the standard r_i-dimensional representation of Σ_{r_i}, each of which splits off a trivial summand. By induction, this observation yields the required connectvity estimate. □

Remark 6. The restriction on the characteristic of the base field arises because of our need to analyze symmetric powers, which may be singular. More generally, a version of the suspension theorem also holds after inverting the exponential characteristic of the base field.

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