Rational connectivity and $\mathbb{A}^1$-connectivity

or geometric applications of the Milnor conjectures
(joint with F. Morel)

Aravind Asok (UCLA)

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The goal

"No doubt topologists will welcome a version which can be read by those not familiar with modern algebraic geometry."

-J.F. Adams
from Math Reviews

Outline

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1. Conventions, definitions and basic examples
2. An elementary example
3. A proposed generalization

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  - take $L = \mathbb{C}(t_1, \ldots, t_n)$ and think of a family of varieties.
- All algebraic varieties will be assumed smooth, connected, and often proper (read: compact).
- Given an algebraic variety X over L, we write $L(X)$ for its field of rational functions.

Basic definitions: rationality

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- Write $\mathbb{P}^n$ for n-dimensional projective space (over L), which is the basic example of a rational variety.
- Think: "most," i.e., a (Zariski) open set, of the solutions to the equations defining X can be rationally parameterized.

Basic question

Question

If $X_d \subset \mathbb{P}^n_{\mathbb{C}}$ is a smooth degree d complex hypersurface, (when) is X_d rational?

Basic example

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- Same argument shows any quadric over a field F having an F-rational point is actually F-rational.

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- Of course, there are many ways to prove that smooth cubic curves are not rational, but let us give another (elementary) argument.
- Begin by defining another invariant.

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- Any discrete valuation ν gives rise to a homomorphism

$$\partial_\nu : L^*/(L^*)^2 \to \mathbb{Z}/2\mathbb{Z}.$$

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- $k_1^{ur}(L/\mathbb{C})$ will be called the group of *unramified square classes*.

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  - Why? Every class in $\mathbb{C}(t)^*/(\mathbb{C}(t)^*)^2$ admits a representative lying in $\mathbb{C}[t]$; use the fundamental theorem of algebra.
- In fact, $k_1^{ur}(\mathbb{C}(t_1, \ldots, t_n)/\mathbb{C}) = 0$.

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whose kernel is generated by f. .

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- The field extension $\mathbb{C}(x) \hookrightarrow L$ gives rise to an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{C}(x)^*/(\mathbb{C}(x)^*)^2 \longrightarrow L^*/(L^*)^2$$

sending $1 \in \mathbb{Z}/2\mathbb{Z}$ to the image of f in $\mathbb{C}(x)^*/(\mathbb{C}(x)^*)^2$.

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  - $\frac{f}{x} = (x+1)(x-1)$ is a square.

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  - Case 3. $\nu(x) < 0$. Exc: Using the equation, show that $2\nu(y) = 3\nu(x)$.

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- constructed an exact sequence, and then
- constructed a non-zero unramified element.
- *Note:* with more work, one can actually determine the group $k_1^{ur}(L/\mathbb{C})$.

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- Case $d = 3$, $n > 4$. No known irrational examples, though some rational examples *are* known (Hassett '99)!

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Remark

If the cubic hypersurface is “more special,” i.e., it posesses a linear subspace of higher dimension, then one can equip it with the structure of a higher dimensional quadric bundle.

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What prevents these varieties from being rational? The quadric bundle need not admit a (rational) section!

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- One possible generalization of the group of square classes goes by way of higher Milnor K-theory.
- The maps induced by discrete valuations can be thought of as "residue" maps in Milnor K-theory.

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- Let $K_n^M(L)$ denote the n-th graded piece of this ring.
- Set $k_n(L) := \operatorname{coker}(K_n^M(L) \xrightarrow{\times 2} K_n^M(L))$; we call this mod 2 Milnor K-theory.

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Goal: apply this invariant to study rationality of quadric bundles.

An exact sequence

- Recall that if $L = \mathbb{C}(x)(\sqrt{f})$, with $y^2 = f(x)$ we had

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow k_1(\mathbb{C}(x)) \longrightarrow k_1(L),$$

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  Question: Can one describe the kernel of this map?

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The pair (f, g) determines an element of $k_2(F)$, which we refer to as the symbol (f, g).

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- Generalize this result.

The Milnor conjecture

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"So you're telling me that two groups, both of which are really hard to understand, are isomorphic?"

- Anonymous

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When $n = 2$, such quadrics reduce to the conics from before.

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- "Geometric" part: Rost's study of small Pfister quadrics.

□

Application to rationality problems I

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Example (Non-rational quadric bundles II)

- Peyre '93: generalized these constructions of unramified elements and non-rational quadrics using k_4^{ur}.

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- One might imagine heirarchies of "higher rational connectivity" to make these notions precise (*cf.* A.J. de Jong-J. Starr).
- Concretely, as n increases, "some kind of mod 2 cohomology" vanishes in higher and higher degrees.

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Corollary

If X/F has a "non-trivial" unramified invariant, then F is not stably rational.

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Homological interpretation

Basic principle: $\pi_0^{\mathbb{A}^1}(X)$ controls all unramified invariants of X.

- Topological fact: if A is a discrete abelian group, and M is a manifold, then continuous maps $M \to A$ are in bijection with group homomorphisms $H_0(M) \to A$.
- Analogous to $\pi_0^{\mathbb{A}^1}$, one can define a notion of $H_0^{\mathbb{A}^1}$, which is a *universal* unramified invariant.
- Let A be an unramified invariant (thought of as a functor on field extensions).
- Concrete incarnation (Morel): Unramified invariants on X correspond bijectively with morphisms of functors $H_0^{\mathbb{A}^1}(X) \to A$.

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- There are many natural generalizations: e.g., so-called norm varieties can be used construct other examples of "bundles" that are rationally connected yet not $\mathbb{A}^1$-connected.

Thank you!

See http://www.math.ucla.edu/~asok for more information