Counting vector bundles

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Vector bundles and projective modules

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- (lifting property) given an R-module map $f : P \to M$, and a surjective R-module map $N \twoheadrightarrow M$, we may always find $\tilde{f} : P \to N$.
- (linear algebraic) if P is also finitely generated, then there exist an integer n, and $\epsilon \in End_R(R^{\oplus n})$ such that $\epsilon^2 = \epsilon$ and $P = \epsilon R^{\oplus n}$.

From now on, all projective modules will be assumed finitely generated (f.g.)

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- f.g. projective modules have a rank

  if $\operatorname{Spec} R$ is connected, then this is just an integer

Serre–Swan correspondence

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Using this dictionary, one transplants intuition from geometry to algebra

Theorem (Serre's splitting theorem '58)

Suppose R is a Noetherian commutative ring of Krull dimension d. If P is a projective R-module of rank $r > d$, then there exists a projective R-module Q of rank d and an isomorphism $P \cong Q \oplus R^{\oplus r-d}$.

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- *Rank 1 projective modules form an abelian group (the Picard group) under tensor product*
- *Minkowski's theorem implies that the Picard group is finite*

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- Real vector bundles on any contractible manifold (e.g., $\mathbb{R}^n$) are trivial

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Theorem (Quillen–Suslin '76)

If R is a PID, then every f.g. projective $R[x_1, \ldots, x_n]$-module is free.

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- We will assume R is regular (an analog of smoothness)

Conjecture (Bass–Quillen '72)

If R is a regular ring of finite Krull dimension, then for any $r \geq 0$

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- Still open in completely generality!

Vector bundles and motivic homotopy theory

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Unfortunately, naive homotopy is not a "good" notion (e.g., it is not an equivalence relation).

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- $\mathscr{H}_{mot}(k)$ for the category obtained by inverting both Nisnevich local and $\mathbb{A}^1$-weak equivalences (this is the Morel–Voevodsky $\mathbb{A}^1$-homotopy category)

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Note: BGL_r and Gr_r are not isomorphic in $\mathscr{H}_{alg}(k)$!

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Nevertheless:

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If k is a field or $\mathbb{Z}$, then for any smooth affine k-scheme $X = \operatorname{Spec} R$,

$$[\operatorname{Spec} R, Gr_r]_{naive} = [\operatorname{Spec} R, Gr_r]_{\mathbb{A}^1} \xrightarrow{\sim} \mathscr{V}_r(\operatorname{Spec} R).$$

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New goal: effectively describe $[\operatorname{Spec} R, Gr_r]_{\mathbb{A}^1}$.

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- We can inductively describe the set of maps $[U,\mathscr{X}]_{\mathbb{A}^1}$ using sheaf cohomology with coefficients in $\mathbb{A}^1$-homotopy sheaves

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There are isomorphisms

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- the map $BSp_\infty \to BGL_\infty$ yields a map $\mathbf{K}_2^{MW} \to \mathbf{K}_2^M$; this map is an epimorphism of sheaves and its kernel may be described via the "fundamental ideal" in the Witt ring (A. Suslin)

Counting vector bundles with motivic homotopy theory

(based on joint work with J. Fasel, M. Hopkins)

Restrict attention to smooth affine varieties over finite fields.

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- Fix the determinant, a.k.a., the first Chern class.

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Are there finitely many vector bundles of a given rank with fixed determinant *on a smooth affine variety over a finite field?*

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If X is a smooth affine surface over a finite field, then there are finitely many isomorphism classes of vector bundles with a given rank and determinant.

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- When working over a finite field, $H^2(X, \mathbf{K}_2^{M})$ is finite by higher-dimensional class field theory (Kato–Saito)

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- *The result actually holds for a "regular affine arithmetic surface" (without using any $\mathbb{A}^1$-homotopy theory), but the $\mathbb{A}^1$-homotopy theoretic argument generalizes.*

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If F is a finite field, characteristic unequal to 2, and X is a smooth affine 3-fold over F, then there are finitely many isomorphism classes of vector bundles with given $c_i \in CH^i(X), i = 1, 2$.

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Conjecture

If X is a smooth affine threefold over a finite field, then there are always finitely many isomorphism classes of vector bundles with a given rank and determinant.

What should we expect in higher dimensions?

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- There are smooth affine 4-folds over a finite field that have infinitely many isomorphism classes of rank 2 vector bundles with fixed rank and determinant (e.g., take the complement of the incidence divisor in $\mathbb{P}^2 \times \mathbb{P}^2$).

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Similar counterexamples suggest that the following is the best-possible statement.

Conjecture

If X is a smooth affine variety of dimension d over a finite field, then there are finitely many isomorphism classes of vector bundles with fixed Chern clases $c_i \in CH^i(X)$, $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

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- Jannsen's version of Beilinson–Tate conjecture, resolution of singularities in positive characteristic and the motivic Bass conjecture on finite generation of motivic cohomology guarantee that $CH^i(X)$ is finite for $i > \lfloor \frac{d}{2} \rfloor$, together with finiteness of a host of other motivic cohomology groups

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- Thus, the conjecture follows if we know that we can always express maps into $[X, BGL_n]_{\mathbb{A}^1}$ purely in terms of motivic cohomology
- The latter follows from Hopkins' "Wilson splitting hypothesis"; loosely the classifying space for algebraic cobordism is "even"; this guarantees that we may write nice "resolutions" of BGL_n

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- What do these numbers mean, what do they measure? We might think of them as some higher rank/higher dimensional version of the class number
- What happens for general regular rings of Krull dimension $d \geq 3$ that are finitely generated as $\mathbb{Z}$-algebras?

Thank you!