Rational points up to stable $\mathbb{A}^1$-homotopy

joint with C. Häsemeyer and F. Morel

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August 15, 2010

Outline

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- All algebraic varieties will be assumed smooth ($+$ geometrically integral).

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where

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- ∂ is the "divisor" map.

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that induces the usual degree homomorphism

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- If X has a k-rational point, then X has a 0-cycle of degree 1.
- The converse is false, there exist varieties with points over extensions of coprime degrees, but that have no rational point.

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- There are a number of clasical conjectures related to the above question, e.g., the Cassels-Swinnerton-Dyer conjecture.

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  - we can equip $V \oplus W$ with the symmetric bilinear form $\varphi \oplus \psi$ defined by $\varphi \oplus \psi((v \oplus w), (v' \oplus w')) = \varphi(v, v') + \psi(w, w')$, and

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- Write $\langle a \rangle$ for the 1-dimensional vector space L equipped with symmetric bilinear form $(x, x') = axx'$.

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- We can view L as a symmetric bilinear space over k by means of the *trace form* $(x, y) \to tr_{L/k}(xy)$.
- Or, given a symmetric bilinear form (V, φ) over L, we can obtain a symmetric bilinear form over k by viewing V as a vector space over k and composing φ with $tr_{L/k}$.

Grothendieck-Witt groups

Definition

The *Grothendieck-Witt group of a field* k, denoted $GW(k)$, is the Grothendieck group of the monoid of isomorphism classes of symmetric bilinear forms over k with operation determined by the direct sum.

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- $GW(k)$ is a commutative ring with multiplication induced by tensor product and unit given by the class of the symmetric bilinear form $\langle 1 \rangle$.

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Extension of scalars and “trace” induce corresponding maps of Grothendieck-Witt groups:

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- a finite extension $k \hookrightarrow L$ induces a "transfer" homomorphism $GW(L) \to GW(k)$,
- the transfer homomorphism is a map of $GW(k)$ modules, and
- there is a dimension homomorphism $\dim : GW(k) \to \mathbb{Z}$.

Quadratic obstructions

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- $GW(k)$-linear, and
- factors through a "nice" quotient.

Rational points up to stable $\mathbb{A}^1$-homotopy

Definition

If X/k is an algebraic variety, say X *has a rational point up to stable* $\mathbb{A}^1$*-homotopy* if the map $\widetilde{\deg}$ is surjective; a *rational point up to stable* $\mathbb{A}^1$*-homotopy* is a choice of lift of $\langle 1 \rangle \in GW(k)$.

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- If X has a k-rational point, then X has a rational point up to stable $\mathbb{A}^1$-homotopy.
- If X has a rational point up to stable $\mathbb{A}^1$-homotopy, then X has a 0-cycle of degree 1.

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What are general hypotheses under which the converses to the above statements hold? In other words,

1. *if X has a 0-cycle of degree 1, when does X have a rational point up to stable $\mathbb{A}^1$-homotopy?*
2. *If X has a rational point up to stable $\mathbb{A}^1$-homotopy, when does X have a rational point?*

A partial answer

Theorem

Suppose k is not formally real, i.e., -1 is a sum of squares in k. If X has a 0-cycle of degree 1, then X has a rational point up to stable $\mathbb{A}^1$-homotopy (but X need not have a rational point).

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- The proof of the theorem is not very difficult: it uses the fact (Pfister '66) that if k is not formally real, then the ordinary Witt group $W(k)$ is a local ring.

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- The proof of the theorem is not very difficult: it uses the fact (Pfister '66) that if k is not formally real, then the ordinary Witt group $W(k)$ is a local ring.
- If k is formally real it seems reasonable to expect that there are varieties with a 0-cycle of degree 1, but no rational point up to stable $\mathbb{A}^1$-homotopy.

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Theorem

1. *If X and X' are birationally equivalent smooth proper varieties over k, then X has a rational point up to stable $\mathbb{A}^1$-homotopy if and only if X' does.*

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Theorem

1. *If X and X' are birationally equivalent smooth proper varieties over k, then X has a rational point up to stable $\mathbb{A}^1$-homotopy if and only if X' does.*
2. *Existence of a rational point up to stable $\mathbb{A}^1$-homotopy is a stable $\mathbb{A}^1$-homotopy invariant.*

The $\mathbb{A}^1$-derived category

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- Write $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(k)$ for the category obtained from $D_{-}(k)$ by "formally inverting" the maps $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$.

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The resulting category is an analog of Voevodsky's triangulated category of effective motivic complexes "without transfers."

The $\mathbb{A}^1$-chain complex

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- The $\mathbb{A}^1$-homology (sheaves) of X are the homology sheaves of $C_*^{\mathbb{A}^1}(X)$.
- The category $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(k)$ has many good formal properties, in particular a tensor product.

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- Let $\mathrm{D}_{\mathbb{A}^1}(k)$ be the category where $\mathbb{Z}\langle 1\rangle$ is inverted.
- Formally, morphisms in the new category are colimits of the diagrams

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{eff}(k)}_{\mathbb{A}^1}}(A, B) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{eff}(k)}_{\mathbb{A}^1}}(A \otimes \mathbb{Z}\langle 1\rangle, B \otimes \mathbb{Z}\langle 1\rangle) \to \cdots$$

(just like Voevodsky's triangulated category of motives).

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- initial among (stable) strictly $\mathbb{A}^1$-homotopy invariant sheaves admitting a map from X;
- a birational invariant by combining the two observations above with Morel's stable $\mathbb{A}^1$-connectivity theorem (via the Bloch-Ogus theory as formalized by Colliot-Thélène-Hoobler and Kahn).

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Further properties

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- The hint that stable homology is linked to symmetric bilinear forms stems from a fundamental computation of Morel: for any extension L/k, $\mathbf{H}_0^{s\mathbb{A}^1}(\operatorname{Spec} k)(L) = GW(L)$.
- Our fundamental computation gives a description of the sheaf $\mathbf{H}_0^{s\mathbb{A}^1}(X)$ for smooth proper varieties amalgamating the data from "zero cycles" and "symmetric bilinear forms" above.

The main computation

Theorem

For any smooth proper variety X over a field k (char. 0) and any extension L/k, there is a canonical isomorphism

$$\mathbf{H}_0^{s\mathbb{A}^1}(X)(L) \xrightarrow{\sim} \widetilde{CH}_0(X_L),$$

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where $\widetilde{CH}_0(X_L)$ is the group of "enhanced" 0-cycles on X. (Up to a small twist) the group $\widetilde{CH}_0(X_L)$ is the quotient of $\bigoplus_{x \in X_{(0)}} GW(\kappa_x)$ we discussed earlier, and the pushforward to a point is induced by the map $\widetilde{\deg}'$.

Duality and the proof

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- The "small twist" comes from the fact that essentially none of the varieties we are considering is "orientable" and thus when using duality we have to twist by an "orientation local system."
- The explicit presentation arises from an unwinding of the Gersten resolution.

Thank you!

See `http://www-bcf.usc.edu/~asok` for more information

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- If $h = \eta[-1] + 2$, then $\eta \cdot h = 0$.

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When X has trivial canonical bundle, the cokernel of this map is the group $\widetilde{CH}_0(X)$.