Proof of the Kahn–Saks Conjecture

Max Aires*

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Abstract

Let $\mathbb{P}(x \prec y)$ be the probability that x precedes y in a uniformly random linear extension of an n-element poset P, and define the balancing coefficient to be $\delta(x,y) = \min(\mathbb{P}(x \prec y), \mathbb{P}(y \prec x))$ with $\delta(P) = \max_{x,y} \delta(x,y)$. Building on work in [3], we prove (Theorem 1) that sufficiently large width forces $\delta(P)$ to be arbitrarily close to $1/2$, answering a long-standing conjecture of Kahn and Saks [14]. In fact, we prove the stronger result (Theorem 2) that large width forces one of two configurations in our poset: either a nearly uniform order on k vertices, or an almost fixed order on t vertices with one further vertex inserted uniformly among the $t+1$ slots. We also show that, for fixed k, the first possibility must occur within any antichain X of size $\Omega(n^{2/3})$.

1 Introduction

Let P be a partially ordered set; we shall abuse notation by also using P for its ground set. A **linear extension** of P is a linear order of the elements of P which agrees with the poset relations. See [8] for a good survey on linear extensions.

Much effort over the years has been dedicated to giving lower bounds on $\delta(P)$, and in particular to proving the famed "1/3-2/3 Conjecture" that $\delta(P) \geq 1/3$ whenever P is not a chain, posed independently in [15] and [12]; see [14],[19],[6],[2] for a (non-exhaustive) sampling of partial progress. Our focus here is instead on conditions which force $\delta(P)$ to be close to $1/2$, i.e. that force the existence of a pair of elements which appear in either order with roughly equal probability.

Intuitively, such elements should almost always exist for a "typical" poset, and a variety of general, sufficient conditions have been proposed. Most notably, in 1984, Kahn and Saks [14] conjectured that, if $w(P)$ is the width of P,

$$w(P) \to \infty \qquad \Longrightarrow \qquad \delta(P) \to 1/2.$$

Here and below all limits are along sequences of posets with $n = |P| \to \infty$. Until recently, the best result in this direction was by Komlós, who proved [17] in 1990 that $\delta(P) \to 1/2$ under the stronger hypothesis that $|\min(P)| = \Omega(n)$; in particular, by a result of Kleitman and Rothschild [16] this implies that $\delta(P) \to 1/2$ asymptotically almost surely for a uniformly random poset on $[n]$ (see also [18]). Aires and Kahn [3] obtained $\delta(P) \to 1/2$ when $w(P) = \Omega(n)$ or $|\min(P)| = \omega(\log n)$. For further work, see [10] and [4].

Our main result is a positive answer to this conjecture:

Theorem 1. *If $w(P) \to \infty$ then $\delta(P) \to 1/2$.*

*Department of Mathematics, University of Southern California

We derive Theorem 1 as a corollary of Theorem 2 below.

The difficulty in proving $\delta(P) \to 1/2$ lies in finding where the balanced elements are; even a slight perturbation to a poset may change the location of a balanced pair. Previous results on balancing, especially those on the 1/3-2/3 Conjecture, have focused on pairs with similar height statistics (see Section 2), but this approach appears to work only up to $\delta(P) \geq 1/e - o(1)$. We give a simpler proof that large width implies this weaker lower bound in Section 6.

It turns out the key to finding pairs with balancing coefficient near 1/2 is to actually look for two quite different stronger configurations, which we describe below.

For a set $Z \subseteq P$, let $\mathcal{S}_Z$ be the set of linear orders on Z, and let $\mathcal{L}_{P;Z}$ be the law of a uniform linear extension restricted to Z. For two laws μ, ν on $\mathcal{S}_Z$, put

$$d_{\mathrm{TV}}(\mu, \nu) = \frac{1}{2} \sum_{\sigma \in \mathcal{S}_Z} |\mu(\sigma) - \nu(\sigma)|.$$

Let $\mathcal{U}_k$ be the uniform law on the $k!$ orders of a k-element set. We call a k-element set $Z \subseteq P$ an **ε-pseudo antichain** if $d_{\mathrm{TV}}(\mathcal{L}_{P;Z}, \mathcal{U}_k) \leq \varepsilon$.

Similarly, let $\mathcal{I}_t$ be the law assigning mass $1/(t+1)$ to each of the orders $y_1 \prec \cdots \prec y_j \prec x \prec y_{j+1} \prec \cdots \prec y_t$ for $0 \leq j \leq t$; equivalently $\mathcal{I}_t$ is the uniform extension law of the disjoint union of a t-element chain with an isolated element. We call a $(t+1)$-element set $Z \subseteq P$ an **ε-pseudo insertor** if it has a labelling $Z = \{x, y_1, \ldots, y_t\}$ for which $d_{\mathrm{TV}}(\mathcal{L}_{P;Z}, \mathcal{I}_t) \leq \varepsilon$, identifying the labels of $\mathcal{I}_t$ with those of Z.

The following is our main result, which has a Ramsey-theoretic flavor:

Theorem 2. *For every fixed $k \geq 2$, $t \geq 1$, and $\varepsilon > 0$, there is a $W_{k,t,\varepsilon}$ such that every poset P with $w(P) > W_{k,t,\varepsilon}$ contains at least one of the following:*

1. *an ε-pseudo antichain of size k;*
2. *an ε-pseudo insertor of size $t+1$.*

To deduce Theorem 1, apply Theorem 2 with $k = 2$ and $t = 1$. Since $\mathcal{I}_1 = \mathcal{U}_2$, either alternative gives distinct $x, y \in P$ with $|\mathbb{P}(x \prec y) - 1/2| \leq \varepsilon$, and hence $\delta(P) \geq 1/2 - \varepsilon$. Note that both possibilities in Theorem 2 are necessary: an n-element antichain has no pseudo-insertor with $t \geq 2$, while Theorem 6 shows that a poset may have arbitrarily large width but no pseudo-antichain with $k \geq 3$.

Let $\pi(x)$ be the number of $y \in P$ which are incomparable to x, and let $\pi(P) = \max_{x \in P} \pi(x)$. While we shall not focus on $\pi(P)$, we note that [4, Theorem 1.7] shows that Theorem 1 may be strengthened as follows:

Theorem 3. *If $\pi(P) \to \infty$ then $\delta(P) \to 1/2$.*

Theorem 2 is deduced from Theorem 14 in Section 6; the latter is proved in Section 7.

To state our remaining results, we first must introduce some notation. Let F be uniform in the **order polytope** $\mathcal{O}(P) = \{z \in [0,1]^P : z_x \leq z_y \text{ if } x <_P y\}$, and put

$$Q_x = \max_{y<x} F(y), \qquad R_x = \min_{y>x} F(y), \qquad c_x = \mathbb{E}(F(x) - Q_x) = \mathbb{E}(R_x - F(x)),$$

with $Q_x = 0$ and $R_x = 1$ when x is minimal or maximal, respectively. Following a volume-preserving bijection of Stanley [23], we may equivalently view c as the centroid of the chain polytope $\mathcal{C}(P) = \{z \in [0,1]^P : \sum_{x \in C} z_x \leq 1 \text{ for all chains } C \subseteq P\}$. Write $S_X = \sum_{x \in X} c_x$.

For $X \subseteq P$ with $|X| \geq k$, define

$$\delta_k(X;P) = \max_{\substack{x_1,\ldots,x_k \in X \\ \text{pairwise distinct}}} \min_{\pi \in S_k} \mathbb{P}\left(x_{\pi(1)} \prec \cdots \prec x_{\pi(k)}\right).$$

Write $\delta_k(P) = \delta_k(P;P)$, so that $\delta(P) = \delta_2(P)$. Note that if Z is a k-element set with $\delta_k(Z;P) \geq 1/k! - \beta$, then $d_{\mathrm{TV}}(\mathcal{L}_{P;Z}, \mathcal{U}_k) \leq k!\beta$.

It follows from [3] that

$$S_X = \Omega(n) \quad \Longrightarrow \quad \delta_k(X;P) \geq 1/k! - \varepsilon.$$

This is strengthened in the following:

Theorem 4. *For every fixed $k \geq 2$ and $X \subseteq P$ with $|X| \geq k$,*

$$S_X = \omega(|X|^{1/2}) \quad \Longrightarrow \quad \delta_k(X;P) \to 1/k!.$$

Moreover, for every $\varepsilon > 0$, there is $\eta = \eta(k,\varepsilon) > 0$ such that

$$S_X = \omega(|X|^{1/2-\eta}) \quad \Longrightarrow \quad \delta_k(X;P) \geq 1/k! - \varepsilon. \tag{1}$$

Hence, (6) and Theorem 4 give

Corollary 5. *For every $k \geq 2$ and $\varepsilon > 0$, there is $\eta' = \eta'(k,\varepsilon) > 0$ such that, for every antichain $A \subseteq P$,*

$$|A| = \omega(n^{2/3-\eta'}) \quad \Longrightarrow \quad S_A \geq \frac{|A|^2}{2n} = \omega(|A|^{1/2-\eta}) \quad \Longrightarrow \quad \delta_k(A;P) \geq 1/k! - \varepsilon.$$

In particular, $w(P) = \omega(n^{2/3})$ implies $\delta_k(P) \to 1/k!$.

The next two constructions demonstrate the tightness of Theorem 4.

Theorem 6. *For every fixed $0 < \varepsilon < 1/2$, there exist $\eta_\varepsilon > 0$ and a sequence of posets P, with $n = |P| \to \infty$, such that*

$$S_P = \omega(n^{1/2-\varepsilon}), \qquad \delta_3(P) \leq 1/6 - \eta_\varepsilon.$$

Furthermore, $w(P) = \omega(n^{1/2-\varepsilon})$ and $\delta(P) = 1/2$.

Theorem 7. *For every fixed $0 < \varepsilon < 1/6$, there exist $\eta_\varepsilon > 0$ and a sequence of posets P with maximum antichains A, where $n = |P| \to \infty$, such that*

$$|A| = \omega(n^{2/3-\varepsilon}), \qquad \delta_3(A;P) \leq 1/6 - \eta_\varepsilon.$$

Furthermore, $S_P = \omega(n^{2/3-\varepsilon})$ and $\delta_k(P) = 1/k!$ for fixed k and sufficiently large n.

Theorem 4 is proved in Section 4; Theorems 6 and 7 are proved in Section 5.

The following question remains open:

Question 1 (Width and tuple balance)**.** *Does there exist a fixed $\eta > 0$ such that*

$$w(P) = \omega(n^{2/3-\eta}) \quad \Longrightarrow \quad \delta_3(P) \to 1/6\ ?$$

2 Common Tools

We shall work with the following statistics of F:

$$\mu_x = \mathbb{E}\, F(x), \qquad \sigma_x^2 = \mathrm{Var} F(x).$$

Let $f : P \to [n]$ be a uniform order-preserving bijection, so that we have the coupling $F(x) = U_{(f(x))}$ where $U_{(j)}$ are the order statistics of n independent uniforms, independent of f. It is convenient to work with a scaled multiple of F rather than F itself: let

$$Z_x = (n+1)F(x).$$

Then we have the **height** statistic

$$h_P(x) = \mathbb{E}\, Z_x = \mathbb{E}\, f(x) = (n+1)\mu_x$$

and for distinct x, y,

$$\mathbb{P}(x \prec y) = \mathbb{P}(F(x) < F(y)) = \mathbb{P}(Z_x < Z_y) = \mathbb{P}(f(x) < f(y)).$$

Let

$$\Delta(x,y) := |h(x) - h(y)|, \qquad d(x,y) = (n+1)\sqrt{\mathrm{Var}(F(x) - F(y))} = \sqrt{\mathrm{Var}(Z_x - Z_y)}.$$

We write $h_P, \Delta_P(x,y)$, etc. when the underlying poset is unclear.

Conditional on the other coordinates, $F(x)$ is uniform on $[Q_x, R_x]$. Consequently

$$\sigma_x^2 \geq c_x^2/3, \qquad \mathbb{E}\,|F(x) - F(y)| \geq c_x/2 \quad (x \neq y). \tag{2}$$

For a random variable V whose density g_V is log-concave, we have the following standard bounds (see [20] for the first two bounds and [3, Lemma 4.4] for the third):

$$\|g_V\|_\infty \leq \frac{\sqrt{2}}{\sqrt{\mathrm{Var} V}}, \qquad \mathbb{P}(V \geq \mathbb{E}\, V), \mathbb{P}(V \leq \mathbb{E}\, V) \geq 1/e, \qquad \mathbb{P}(|V| \leq h) \leq \frac{3h}{\mathbb{E}\,|V|}. \tag{3}$$

Since the law of $F(x) - F(y)$ is log-concave, we have $\mathbb{P}(|F(x) - F(y)| \leq h) \leq 6h/c_x$ in particular.

If V has a log-concave density, $\mathbb{E}\, V = 0$, and $\sigma = \|V\|_2 > 0$, then Lovász–Vempala [20, Lemma 5.7] gives

$$\mathbb{P}(|V| > t\sigma) \leq e^{1-t} \qquad (t \geq 0). \tag{4}$$

The case $0 \leq t \leq 1$ is trivial. Integrating gives $\mathbb{E}\, V^4 \leq 24e\,\sigma^4$.

For $x \neq y$, conditional on every coordinate except $F(x)$, the distribution of $Z_x - Z_y$ is uniform on $[(n+1)Q_x - Z_y, (n+1)R_x - Z_y]$. Total variance and Jensen give $d(x,y)^2 \geq (n+1)^2\, \mathbb{E}(R_x - Q_x)^2/12 \geq (n+1)^2 c_x^2/3$. Interchange x, y to get

$$d(x,y) \geq (n+1)\frac{\max(c_x, c_y)}{\sqrt{3}}. \tag{5}$$

For an antichain A, we have the following inequality from [3, Theorem 5.4], which follows from observing that $c_x \geq \frac{e(P)}{2ne(P-x)}$ and applying Cauchy-Schwarz with the inequality $e(P) \geq \sum_{x \in A} e(P - x)$ from [11] or [22] (or an equivalent geometric version from [5]):

$$\sum_{x \in A} c_x \geq \frac{|A|^2}{2n}. \tag{6}$$

For an interval I, let T_I denote clipping to I (so $T_{[a,b]}(x) = \min(b, \max(a, x))$). Then

$$\operatorname{Cov}(T_I(F(x)), T_I(F(y))) \geq 0, \qquad \sum_y \operatorname{Cov}(T_I(F(x)), T_I(F(y))) \leq |I|/4. \tag{7}$$

To see (7), put $H_a(t) = \min(t, a) - at$. Conditioning on all but one uniform gives $\operatorname{Cov}(\mathbf{1}_{\{U_{(j)}>t\}}, \sum_i \mathbf{1}_{\{U_i>a\}}) = p_j(t)H_a(t)$, where p_j is the density of $U_{(j)}$. Integrating in t and averaging over $f(x)$ yields

$$\sum_y \operatorname{Cov}(F(x), \mathbf{1}_{\{F(y)>a\}}) = \mathbb{E}\, H_a(F(x)) \leq 1/4.$$

Now integrate over $a \in I \cap [0, 1]$. FKG applies to the uniform measure on the order polytope; since both $T_I(t)$ and $t - T_I(t)$ are increasing, it gives the claimed bound and nonnegativity of every summand.

The following is a direct consequence of the continuous form of Shepp's XYZ inequality [21] (see also [9] for a version for weak order-preserving maps): for any distinct x, y, z,

$$\operatorname{Cov}(F(y) - F(x), F(z) - F(y)) \leq 0. \tag{8}$$

In particular,

$$d(x, y)^2 \leq d(x, z)^2 + d(z, y)^2. \tag{9}$$

For any event E with $\mathbb{P}(E) > 0$ and any variable V of finite variance, Cauchy–Schwarz gives

$$(\mathbb{E}[V \mid E] - \mathbb{E}\, V)^2 \leq \operatorname{Var}(V)(1/\mathbb{P}(E) - 1), \qquad \operatorname{Var}(V \mid E) \leq \operatorname{Var}(V)/\mathbb{P}(E). \tag{10}$$

Finally, we need a recent result of Haqi [13], solving affirmatively a conjecture of Kahn (see for instance [7]): for a nonempty ideal I of P,

$$\max_{x\in I} h(x) \geq |I| - |\max(I)| + 1 \tag{11}$$

Here $\min(I)$ and $\max(I)$ are the sets of minimal and maximal elements, respectively. Applying (11) to both P and its dual, we derive the following easy corollary: for any nonempty proper ideal I of P,

$$\min_{y\in I^c} h(y) - \max_{x\in I} h(x) \leq |\max(I)| + |\min(I^c)| - 1. \tag{12}$$

3 Komlós selection

The fixed-k Komlós theorem [3, Theorem 8.1], extending [17], yields the following form.

Theorem 8 (Komlós selection)**.** *Fix $k \geq 2$ and $A, B, \varepsilon > 0$. There is an integer $M_{k,A,B,\varepsilon}$ such that the following holds. Let $V_1, \ldots, V_m$ be jointly distributed real random variables, with $m \geq M_{k,A,B,\varepsilon}$, satisfying*

$$\mathbb{E}\, V_i^2 \leq A, \qquad \mathbb{P}(|V_i - V_j| \leq h) \leq Bh \quad (i \neq j,\ h > 0). \tag{13}$$

Then there are distinct $i_1, \cdots, i_k$ such that, for every $\pi \in S_k$,

$$\mathbb{P}(V_{i_{\pi(1)}} < \cdots < V_{i_{\pi(k)}}) \geq 1/k! - \varepsilon.$$

Proof. Fix $k \geq 2$ and $A, B, \varepsilon > 0$. Choose T large and $h > 0$ small enough so that $kA/T^2 + \binom{k}{2}Bh \leq \varepsilon/2$. Partition $[-T, T]$ into intervals of length at most h and add the two tail intervals. Then, for $V_{i_1}, \ldots, V_{i_k}$, the probability that two V_i's are in the same interval is at most

$$\sum_a \mathbb{P}(|V_{i_a}| > T) + \sum_{a<b} \mathbb{P}(|V_{i_a} - V_{i_b}| \leq h) \leq kA/T^2 + \binom{k}{2}Bh \leq \varepsilon/2.$$

Let B_i be the index of the interval containing V_i, with the intervals indexed from left to right. Apply [3, Theorem 8.1] to the finite-valued variables B_i, with error $\varepsilon/4$. For sufficiently large m, it supplies distinct indices $i_1, \ldots, i_k$ for which the probabilities $p_\pi = \mathbb{P}(B_{i_{\pi(1)}} < \cdots < B_{i_{\pi(k)}})$ differ from one another by at most $\varepsilon/2$. The collision estimate gives $\sum_\pi p_\pi \geq 1 - \varepsilon/2$, so $p_\pi \geq \frac{1-\varepsilon/2}{k!} - \varepsilon/2 \geq 1/k! - \varepsilon$. A strict order of interval indices implies the same strict order of the original variables. Hence $\mathbb{P}(V_{i_{\pi(1)}} < \cdots < V_{i_{\pi(k)}}) \geq 1/k! - \varepsilon$ for every π, as required. $\square$

The following consequence of Theorem 8 will be used in Sections 6 and 7. Put

$$\ell(x, y) = \Delta(x, y) + d(x, y).$$

Lemma 9. *If P has no ε-pseudo antichain (in particular if $\delta_k(P) < 1/k! - \varepsilon$), every ℓ-ball of radius r is covered by at most $N_{k,\varepsilon}$ balls of radius $r/2$.*

Proof. Take an $r/2$-separated subset of the ℓ-ball centered at b. Partition the heights into a bounded number of intervals of length $r/8$. Within one interval, distinct points satisfy

$$d(s, t) = \ell(s, t) - \Delta(s, t) > r/2 - r/8 = 3r/8.$$

Apply Theorem 8 with the fixed k to $X_z = (Z_z - Z_b)/r$ for the vertices in this interval. Since they lie in the ball,

$$\mathbb{E}\, X_z^2 = \frac{\Delta(z, b)^2 + d(z, b)^2}{r^2} \leq 1.$$

Moreover, $\mathrm{sd}(X_s - X_t) > 3/8$, so log-concavity and (3) give

$$\mathbb{P}(|X_s - X_t| \leq u) \leq Cu \qquad (u > 0).$$

These are precisely the two hypotheses of Theorem 8. If the interval contained sufficiently many selected vertices, it would give a k-tuple with every order probability at least $1/k! - \varepsilon/(2k!)$, and hence an ε-pseudo antichain, contrary to the hypothesis. Thus the number of selected vertices in each interval, and hence in the ball, is bounded in terms of k, ε. A maximal $r/2$-separated subset supplies centers for the required covering. $\square$

In particular, balls of bounded radius have covering numbers at most Cr^{-D} at scale $0 < r \leq 1$, for constants depending only on k, ε.

4 Proof of Theorem 4

Proof of Theorem 4. Fix $k \geq 2$, and put

$$K_X := \sup_{\substack{0<u\leq 1/2 \\ b\in\mathbb{R}}} u\,|\{x \in X : \mu_x \in [b, b+u),\ u \leq \sigma_x < 2u\}|.$$

We first show that, for every nonempty $X \subseteq P$ and $0 < s \leq 1/2$,

$$|\{x \in X : \sigma_x \geq s\}| \leq C K_X s^{-2}, \tag{14}$$

where C is an absolute constant. At each dyadic scale $u = 2^j s \leq 1/2$, partition $[0,1]$ into $O(1/u)$ intervals of length u. Each contains at most K_X/u elements with $u \leq \sigma_x < 2u$. Summing $O(K_X/u^2)$ proves (14). Since $c_x \leq \sqrt{3}\sigma_x$ and $c_x \leq 1/2$, integration gives

$$S_X = \int_0^{1/2} |\{x \in X : c_x > t\}|\, dt \leq \int_0^{\infty} \min\{|X|, C K_X t^{-2}\}\, dt = O(\sqrt{|X| K_X}).$$

Hence $K_X = \Omega(S_X^2/|X|)$, so $S_X = \omega(|X|^{1/2})$ implies $K_X \to \infty$.

If $K_X \to \infty$, we now show that $\delta_k(X; P) \to 1/k!$. We will find a growing subset of X whose normalized coordinates satisfy the two hypotheses of Theorem 8: bounded second moments and a uniform small-ball bound for every pair.

By the definition of K_X, choose u, b so that

$$A = \{x \in X : \mu_x \in [b, b+u),\ u \leq \sigma_x < 2u\}, \qquad m = |A|, \qquad mu \geq K_X/2 \longrightarrow \infty.$$

For $x \in A$, put $V_x = (F(x) - b)/u$. Then $0 \leq \mathbb{E}\, V_x < 1$ and $1 \leq \mathrm{Var} V_x < 4$, so $\mathbb{E}\, V_x^2 \leq 5$. It remains to find many of these coordinates whose pairwise differences do not concentrate near zero.

Put $Y_x = T_{[-10,10]}(V_x)$. The density bound in (3) gives $\|g_{V_x}\|_\infty \leq \sqrt{2}$, and Markov's inequality gives $\mathbb{P}(|V_x| > 10) \leq 1/20$. For any interval J of length $1/4$, therefore,

$$\mathbb{P}(Y_x \in J) \leq \mathbb{P}(V_x \in J) + \mathbb{P}(|V_x| > 10) \leq \frac{\sqrt{2}}{4} + \frac{1}{20} < \frac{1}{2}.$$

Taking $J = [\mathbb{E}\, Y_x - 1/8, \mathbb{E}\, Y_x + 1/8]$ shows that $\mathrm{Var} Y_x \geq (1/8)^2/2 =: v = 1/128$.

For $I = [b - 10u, b + 10u]$, we have $Y_x = (T_I(F(x)) - b)/u$. Since every covariance in (7) is nonnegative, restricting its sum to A gives

$$\sum_{y \in A} \mathrm{Cov}(Y_x, Y_y) = u^{-2} \sum_{y \in A} \mathrm{Cov}(T_I(F(x)), T_I(F(y))) \leq u^{-2} \frac{20u}{4} = \frac{5}{u}.$$

Join distinct $x, y \in A$ by an edge when $\mathrm{Cov}(Y_x, Y_y) > v/2$. Each vertex has degree at most $10/(vu)$. Greedily keeping a vertex and deleting its neighbors therefore produces a set $B \subseteq A$ with

$$|B| \geq \frac{m}{1 + 10/(vu)} = \Omega(mu) \longrightarrow \infty, \qquad \mathrm{Cov}(Y_x, Y_y) \leq v/2 \quad (x \neq y \in B).$$

Here the implied constant is absolute, since $u \leq 1/2$. For distinct $x, y \in B$,

$$\mathbb{E}(Y_x - Y_y)^2 \geq \mathrm{Var} Y_x + \mathrm{Var} Y_y - 2\,\mathrm{Cov}(Y_x, Y_y) \geq v.$$

Clipping is 1-Lipschitz and $|Y_x - Y_y| \leq 20$, hence

$$\mathbb{E}\,|V_x - V_y| \geq \mathbb{E}\,|Y_x - Y_y| \geq \frac{\mathbb{E}(Y_x - Y_y)^2}{20} \geq \frac{v}{20}.$$

The difference $V_x - V_y = (F(x) - F(y))/u$ has a log-concave density. Thus (3) gives, for every $h > 0$,

$$\mathbb{P}(|V_x - V_y| \leq h) \leq \frac{3h}{\mathbb{E}\,|V_x - V_y|} \leq \frac{60}{v} h.$$

Together with $\mathbb{E}\, V_x^2 \le 5$ and $|B| \to \infty$, this verifies all hypotheses of Theorem 8. It supplies k elements of $B \subseteq X$ with every ordering probability $1/k! + o(1)$. The common positive rescaling preserves their orders, and the orders under F have the same law as under f. Therefore $\delta_k(X;P) \to 1/k!$. Together with the bound on K_X, this proves the first assertion.

Now fix $\varepsilon > 0$, and suppose $\delta_k(X;P) < 1/k! - \varepsilon$; all constants below may depend on k, ε. The preceding argument gives $K_X = O_{k,\varepsilon}(1)$. Thus (14) implies

$$|\{x \in X : \sigma_x > s\}| \le C s^{-2} \qquad (s > 0). \tag{15}$$

We refine the count by covering in the difference metric d in rank units.

Choose an integer $L \ge 2$ large enough that the finite selection theorem applies to L variables satisfying $\mathbb{E}\, V_i^2 \le 5$ and $\mathbb{P}(|V_i - V_j| \le h) \le 4\sqrt{2}\, h$, with target error $\varepsilon/2$. Put $D = \log_2 L$. We claim that, if $0 < u \le s$ and $Y \subseteq X$ satisfies

$$\mu_x \in [b, b+u), \qquad \sigma_x \le s, \qquad c_x \ge u \quad (x \in Y),$$

then

$$|Y| \le C(s/u)^D. \tag{16}$$

Consider a ball in Y, centered at $a \in Y$, of radius $r \ge (n+1)u/2$. For any subset in this ball with pairwise distances greater than $r/2$, set $V_x = (Z_x - Z_a)/r$. Then

$$\mathbb{E}\, V_x^2 = \frac{d(x,a)^2 + (n+1)^2(\mu_x - \mu_a)^2}{r^2} \le 5, \qquad \mathbb{P}(|V_x - V_y| \le h) \le \frac{2\sqrt{2}\, hr}{d(x,y)} \le 4\sqrt{2}\, h.$$

The second bound follows from the log-concave density bound in (3). Subtracting the same random coordinate preserves all orders. The deficit therefore forces this separated subset to have fewer than L elements. Taking a maximal separated subset covers the ball by at most L balls of radius $r/2$, with centers in Y.

The diameter of Y is at most $2(n+1)s$. Start with a ball of radius $2^J(n+1)u$, where $J = \lceil \log_2(2s/u) \rceil$, and apply the covering step $J+2$ times. The final radii are $(n+1)u/4$. By (5), distinct points of Y have distance at least $(n+1)u/\sqrt{3} > (n+1)u/2$, so each final ball contains at most one point. Hence $|Y| \le L^{J+2} \le C(s/u)^D$, proving (16).

For $0 < u \le 1/2$, separate the elements with $c_x \ge u$ into those with $\sigma_x > s$ and those with $\sigma_x \le s$. Use (15) for the first group and partition the second group's coordinate means into $O(1/u)$ intervals of length u. Equation (16) gives, for any $s \ge u$,

$$|\{x \in X : c_x \ge u\}| \le C \left(s^{-2} + s^D u^{-(D+1)} \right).$$

Choose $s = u^{(D+1)/(D+2)}$. Both terms are u^{-p}, where $p = 2(D+1)/(D+2) \in (1,2)$. Integrating this count, using $c_x \le 1/2$, yields

$$S_X \le \int_0^\infty \min\{|X|, Cu^{-p}\}\, du \le C'|X|^{1-1/p} = C'|X|^{1/2-\gamma}, \qquad \gamma = \frac{1}{2(D+1)} > 0.$$

Taking $\eta = \gamma$, the contrapositive proves (1). $\square$

5 Counterexamples for window and local width thresholds

Let C_t be the chain of size t. We use $P + Q$ to denote the parallel sum (disjoint union) of P and Q, and $P \oplus Q$ for the series sum (i.e. every element of P precedes every element of Q. Both constructions in this section are series-parallel, meaning they can be formed from chains by $+$ and $\oplus$.

Note that both constructions generalize to δ_k, as for every fixed $k \geq 3$, restricting a k-tuple to three elements gives, for any set X of size at least k,

$$\delta_k(X;P) \leq \frac{6}{k!}\delta_3(X;P)$$

5.1 A recursive example without balanced triples

Fix $0 < \theta < 1$ and a sufficiently large integer N. Starting from $P_0 = C_N$, put

$$n_r = |P_r|, \qquad p_r = \lceil \theta n_r \rceil, \qquad B_r = C_{p_r} \oplus P_r \oplus P_r \oplus C_{p_r}, \qquad P_{r+1} = B_r + B_r. \tag{17}$$

Copies in these expressions are disjoint. Clearly $\delta(P_r) = 1/2$ when $r \geq 1$.

Taking an n-element component in a series sum of total size b multiplies each c_x by $(n+1)/(b+1)$, while parallel composition leaves c_x unchanged. Each end-chain element contributes $1/(2n_r + 2p_r + 1)$. Thus

$$n_{r+1} = 4(n_r + p_r), \qquad w(P_r) = 2^r, \qquad S_{P_{r+1}} = \frac{4(n_r+1)}{2n_r + 2p_r + 1} S_{P_r} + \frac{4p_r}{2n_r + 2p_r + 1}.$$

The size multiplier tends to $4(1+\theta)$ and the S_P multiplier to $2/(1+\theta) > 1$, with summable errors $O(1/n_r)$. Consequently, for fixed N,

$$n_r = \Theta\big((4(1+\theta))^r\big), \qquad w(P_r) = \Theta(n_r^{\beta_\theta}), \qquad S_{P_r} = \Theta(n_r^{a_\theta}), \tag{18}$$

where

$$\beta_\theta = \frac{\log 2}{\log(4(1+\theta))}, \qquad a_\theta = 3\beta_\theta - 1 = \frac{\log(2/(1+\theta))}{\log(4(1+\theta))}.$$

Both exponents approach $1/2$ from below as $\theta \to 0$.

Write $U \leq_{\mathrm{st}} V$ if $\mathbb{P}(U > t) \leq \mathbb{P}(V > t)$ for every t. We use the following observation.

Lemma 10 (Nearly uniform triples)**.** *Let X, Y, Z have continuous, stochastically ordered marginal laws, with Z independent of (X, Y). Suppose the conditional law of either of X, Y, given the other, increases stochastically with the conditioning value. If each of the six orders has probability at least $1/6 - \varepsilon$, where $\varepsilon \geq 0$, then, for every $s, t \in \mathbb{R}$,*

$$0 \leq \mathbb{P}(X \leq s, Y \leq t) - \mathbb{P}(X \leq s)\mathbb{P}(Y \leq t) \leq 48\varepsilon. \tag{19}$$

Proof. Write H_X, H_Y, H_Z for the distribution functions, and put $U = H_X(X)$, $V = H_Y(Y)$, $W = H_Z(Z)$. These variables are uniform on $[0,1]$, with W independent of (U, V). Stochastic ordering makes $\{X < Z\}$ and $\{U < W\}$ nested almost surely, so

$$\mathbb{P}(\{X < Z\} \triangle \{U < W\}) = |\mathbb{P}(X < Z) - 1/2| \leq 3\varepsilon.$$

The same holds for Y, Z. Thus $q := \mathbb{P}(\min(U,V) < W < \max(U,V)) \geq 1/3 - 8\varepsilon$. Put $D(u,v) = \mathbb{P}(U \leq u, V \leq v) - uv$. Conditional monotonicity makes D concave in each coordinate, with zero boundary values, so $D \geq 0$. Independence of W gives

$$\int_0^1 D(t,t)\,dt = \frac{1/3 - q}{2} \leq 4\varepsilon.$$

Concavity in the two coordinates also gives $D(t,t) \geq \min\{t, 1-t\}^2 D(u,v)$ for all $u, v, t \in [0,1]$. Integrating yields $D(u,v) \leq 48\varepsilon$; take $u = H_X(s)$ and $v = H_Y(t)$ to obtain (19). $\square$

The order-polytope density is log-supermodular, since its support is closed under coordinatewise minimum and maximum. The Ahlswede–Daykin Four-Functions theorem [1], applied on grids and then by approximation, gives the same property for each two-coordinate marginal density g:

$$g(x,s)g(x',t) \geq g(x,t)g(x',s) \qquad (x < x',\ s < t).$$

Integrating over $s \leq a < t$ and normalizing shows that $\mathbb{P}(Y > a \mid X = x)$ is nondecreasing in x. Exchanging X, Y gives conditional stochastic monotonicity in both directions.

Proof of triple bound in Theorem 6. The coordinate laws of P_r are totally stochastically ordered: this holds for a chain, series sums preserve it by their common increasing affine embeddings, and parallel duplication repeats the same list of laws. Put $n = n_r$, $p = p_r$, $b = 2n + 2p$. An old copy preceded by $a \in \{p, p+n\}$ has embedding

$$F_i' = L + TF_i, \qquad (L, T, 1-L-T) \sim \mathrm{Dirichlet}(a, n+1, b-a-n), \tag{20}$$

independently of its old coordinates. Total variance and Dirichlet moments give

$$2b\mathrm{Var}(F_i') \leq \frac{2b(n+1)(n+2)}{n(b+1)(b+2)}\, n\mathrm{Var}(F_i) + 1/2.$$

The coefficient tends to $1/(1+\theta) < 1$, and end-chain coordinates have scaled variance at most $1/2$. For sufficiently large N, iteration therefore gives

$$\sup_{r, x \in P_r} n_r \mathrm{Var}(F_x) \leq C_\theta. \tag{21}$$

Consider an antichain triple with least ordering probability $1/6 - \xi$. Restrict it to its smallest recursive component P_s, preserving its ordering law, and put $n = n_{s-1}$, $p = p_{s-1}$, $b = 2n + 2p$, so $m = |P_s| = 2b$. Two elements x, y share an old copy in one branch, while z lies in the other and is independent of them. In (20), put $R = L + T$. The old covariance is nonnegative by FKG, and $\mathrm{Var}L, \mathrm{Var}R \geq \mathrm{Cov}(L, R)$. Since conditional means are convex combinations of L, R, total covariance gives

$$m\,\mathrm{Cov}(F_x, F_y) \geq 2b\,\mathrm{Cov}(L, R) = \frac{2b\,p(n+p)}{(b+1)^2(b+2)} \geq c_\theta > 0. \tag{22}$$

Put $A = \sqrt{m}(F_x - \mu_x)$ and $B = \sqrt{m}(F_y - \mu_y)$. Then $\mathrm{Var}A, \mathrm{Var}B \leq C_\theta$ and $\mathrm{Cov}(A, B) \geq c_\theta$. The tail bound (4) and Cauchy–Schwarz allow a fixed $K = K_\theta \geq \sqrt{C_\theta}$ such that clipping both variables to $[-K, K]$ leaves covariance at least $c_\theta/2$. Apply Lemma 10 to (F_x, F_y, F_z). Its rectangle bound is preserved by separate increasing affine changes of coordinates. Integrating the indicator covariances over $[-K, K]^2$ therefore gives

$$c_\theta/2 \leq \mathrm{Cov}(T_{[-K,K]}(A), T_{[-K,K]}(B)) \leq 48\xi(2K)^2.$$

Hence $\xi \geq \eta_\theta := c_\theta/(384K_\theta^2) > 0$, uniformly in the depth and in sufficiently large seed sizes. Since $\eta_\theta \leq 1/384$, triples containing a comparable pair satisfy the same bound trivially. Finally, choose θ so that $a_\theta > 1/2 - \varepsilon$, and fix such an N. Equation (18), together with $\delta(P_r) = 1/2$ for $r \geq 1$, proves Theorem 6. □

5.2 A maximum antichain without balanced triples

Proof of Theorem 7. Take $H = P_r$ from the preceding construction, with $M = |H|$ and $w = 2^r = \Theta(M^{\beta_\theta})$. Choose one seed-chain vertex and, at each step, its descendants in the first old copy of each branch. They form a maximum antichain D of size w. Symmetry gives identical marginals on D, and (21) gives

$$\mathbb{E}\,F_x = \mu, \qquad \mathrm{Var}(F_x) \leq C_\theta/M \quad (x \in D).$$

Fix a sufficiently large constant $L = L(\theta)$, and put

$$h = \left\lceil L\sqrt{M} \right\rceil, \qquad T = \lfloor M/(2h) \rfloor.$$

Put

$$Q_j = C_{jh} \oplus H \oplus C_{M-jh} \quad (1 \le j \le T), \qquad P = Q_1 + \cdots + Q_T, \qquad A = \bigcup_{j=1}^{T} D^{(j)}. \tag{23}$$

Since the first vertices of the prefix chains are independent and identically distributed, any k of them have all $k!$ orders equally likely, so $\delta_k(P) = 1/k!$ whenever $T \ge k$.

Every branch has size $2M$ and width w, so A is a maximum antichain. Its groups have coordinate mean spacing $h/(2M+1)$, while all their variances are $O_\theta(1/M)$:

$$\mu_j = \frac{jh + (M+1)\mu}{2M+1}, \qquad \mathrm{Var}(F_x) \le C_1/M \quad (x \in D^{(j)}). \tag{24}$$

Indeed the embedding is $F'_x = U + VF_x$, with $(U, V, 1-U-V) \sim \mathrm{Dirichlet}(jh, M+1, M-jh)$; its added variance is at most $1/[4(2M+2)]$. For $i < j$, independence across branches and Chebyshev give

$$\mathbb{P}(F_y < F_x) \le 18C_1/L^2 \le 1/4 \qquad (x \in D^{(i)},\ y \in D^{(j)}),$$

once L is large enough. Any triple meeting different branches has three orders containing such a reversed pair, so one has probability at most $1/12$. A triple within a branch retains its ordering law in H. Therefore

$$\delta_3(A; P) \le 1/6 - \eta', \qquad \eta' = \min\{\eta_\theta, 1/12\} > 0. \tag{25}$$

Since $T = \Theta(\sqrt{M})$, we have

$$n = 2MT = \Theta(M^{3/2}), \qquad |A| = Tw = \Theta\left(n^{(1+2\beta_\theta)/3}\right).$$

Each branch has M end-chain elements, each contributing $1/(2M+1)$, while the old c_x values are multiplied by $(M+1)/(2M+1)$. Thus

$$S_P = T\frac{(M+1)S_H + M}{2M+1} = \Theta(n^{2\beta_\theta - 1/3}). \tag{26}$$

Choose θ so that $\beta_\theta > 1/2 - \varepsilon/2$. Then both $|A|$ and S_P are $\omega(n^{2/3-\varepsilon})$, as required. □

6 Reduction to small gaps

For every nonempty $B \subseteq P$, define its height span and its diameter in d by

$$H(B) := \max_{x,y\in B} \Delta(x, y), \qquad D(B) := \max_{x,y\in B} d(x, y).$$

We write $H_P(B)$ and $D_P(B)$ when the underlying poset is unclear.

Lemma 11. *Fix $k \ge 2$ and $\varepsilon > 0$. There are constants $C > 0$ and $\eta > 0$ such that, if $\delta_k(P) < 1/k! - \varepsilon$, every nonempty antichain $B \subseteq P$ satisfies*

$$|B| \le C\,(1 + H(B) + D(B))^{1-\eta}. \tag{27}$$

Proof. Put $m = |B|$, $T = H(B) + D(B)$, and

$$A = \{x \in B : (n+1)c_x \geq m/4\}.$$

We have $|A| \geq m/2$: otherwise $q = |B \backslash A| > m/2$, and (6) gives

$$q^2/2 \leq \sum_{x \in B \backslash A} (n+1)c_x < qm/4,$$

a contradiction. By (5), distinct $x, y \in A$ satisfy $\ell(x,y) \geq d(x,y) \geq m/(4\sqrt{3})$, while the ℓ-diameter of A is at most T. The doubling bound in Lemma 9 therefore gives constants $C_0, p > 0$, depending only on k, ε, such that

$$m \leq C_0(1 + T/m)^p.$$

If $T \leq m$, this bounds m by a constant. If $T > m$, it gives $m^{p+1} \leq C_0 2^p T^p$. Thus in both cases $m \leq C(1+T)^{p/(p+1)}$, proving the claim with $\eta = 1/(p+1)$. ☐

If the elements of P are indexed so that $h(x_1) \leq h(x_2) \leq \cdots \leq h(x_n)$, then let the **gap** of P be

$$G(P) = \max(h(x_1), h(x_2) - h(x_1), \ldots, h(x_n) - h(x_{n-1}), n + 1 - h(x_n)).$$

Also, put

$$R(P) = \max(\{1\} \cup \{d(x,y) : \Delta(x,y) \leq d(x,y)\}). \tag{28}$$

We write $R = R(P)$ and $G = G(P)$ when the poset P is clear.

Lemma 12. *Put $M = \max(R, G)$. For all vertices x, y,*

$$d(x,y)^2 \leq 4M(\Delta(x,y) + M), \tag{29}$$

and for every interval I of length M, $|\{z \in P : h(z) \in I\}| \leq 8M + 1$.

Proof. First, the definition of R gives, for any vertices u, v,

$$d(u,v) \leq \max\{R, \Delta(u,v)\}.$$

Indeed, if $d(u,v) \geq \Delta(u,v)$, this pair enters the maximum defining R, so $d(u,v) \leq R$. Otherwise $d(u,v) < \Delta(u,v)$.

The case $x = y$ is immediate. For distinct x, y, interchange them if necessary so that $h(x) \leq h(y)$, and list the vertices between them in nondecreasing height order, breaking ties arbitrarily. Starting at $x_0 = x$, choose the next anchor to be the first vertex whose height is at least M larger than the current anchor's height. If no such vertex occurs before or at y, finish with y as the last anchor. Stop as soon as y has been selected, and write the resulting anchors as $x_0, \ldots, x_q = y$. Every step except possibly the last advances the height by at least M. Since the advances sum to $\Delta(x,y)$, we have

$$(q-1)M \leq \Delta(x,y), \qquad q \leq 1 + \frac{\Delta(x,y)}{M}.$$

A step chosen by the threshold rule advances the height by at most $M + G \leq 2M$: just before the first vertex crossing the threshold, the advance is less than M, and the next gap is at most G. A final step that does not reach the threshold has advance less than M. Thus every anchor step satisfies

$$d(x_{i-1}, x_i) \leq \max\{R, \Delta(x_{i-1}, x_i)\} \leq 2M.$$

Equation (9) now gives

$$d(x,y)^2 \le \sum_{i=1}^{q} d(x_{i-1},x_i)^2 \le q(2M)^2 \le 4M^2\left(1+\frac{\Delta(x,y)}{M}\right) = 4M(\Delta(x,y)+M).$$

This proves (29). For the vertex count, choose an anchor b in a nonempty interval of length M. For each other vertex z there, $d(z,b) \le \max\{R, \Delta(z,b)\} \le M$, hence $\mathbb{E}(Z_z - Z_b)^2 \le 2M^2$ and $\mathbb{P}(|Z_z - Z_b| \le 2M) \ge 1/2$.

Condition on the linear extension and write $f(b) = j$. The positive differences from Z_b have distributions $(n+1)\,\mathrm{Beta}(k, n+1-k)$, $1 \le k \le n-j$. Their summed density is bounded by the sum over $1 \le k \le n$, which before scaling is

$$\sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} t^{k-1}(1-t)^{n-k} = n.$$

Thus the summed density is at most $n/(n+1)$ on the positive half-line; the negative half-line is identical. Averaging over the extension gives, for every interval $I \subset \mathbb{R}$,

$$\sum_{z \ne b} \mathbb{P}(Z_z - Z_b \in I) \le \frac{n}{n+1}|I|.$$

Apply this to $I = [-2M, 2M]$. Each vertex in the chosen interval other than b contributes at least $1/2$, so there are at most $8M$ such vertices. Including b gives the asserted bound $8M+1$. $\square$

Lemma 13. *Fix $k \ge 2$ and $\varepsilon, \theta > 0$. For all sufficiently large $M = \max\{R(P), G(P)\}$, if P has an internal height gap of size at least θM, then $\delta_k(P) \ge 1/k! - \varepsilon$.*

Proof. Suppose instead that $\delta_k(P) < 1/k! - \varepsilon$. Let $g \ge \theta M$ be the size of an internal height gap, with endpoints u, v, so $h_P(v) - h_P(u) = g$, and put

$$I = \{z : h_P(z) \le h_P(u)\}, \qquad J = P \setminus I, \qquad L = AM\log^2 M,$$

where A is a sufficiently large absolute constant. Use the original coordinates $Z_z = (n+1)F(z)$ throughout. By Lemma 12 and (4), applied to $Z_y - Z_x - H$, for $H = h_P(y) - h_P(x) \ge CM$,

$$\mathbb{P}_P(Z_y < Z_x) \le \exp\left(1 - \frac{H}{d_P(x,y)}\right) \le C\exp(-c\sqrt{H/M}).$$

Form Q' by adjoining, all at once, the comparisons

$$z < u \quad \text{if } h_P(z) \le h_P(u) - L, \qquad v < z \quad \text{if } h_P(z) \ge h_P(v) + L.$$

These comparisons stay within I and J. Let E be the event that all comparisons are satisfied for F. Partition the values below $h_P(u) - L$ and above $h_P(v) + L$ into intervals of length M. Each interval contains at most $8M+1$ vertices by Lemma 12. The union bound therefore gives

$$\mathbb{P}_P(E^c) \le CM \sum_{j \ge \lfloor L/M \rfloor} e^{-c\sqrt{j}} \le M^{-4},$$

provided A and then M are sufficiently large. Thus E has positive probability, the added comparisons are consistent, and conditioning gives the uniform order-polytope law of Q'. Since the ground set and rank scale are unchanged, $h_{Q'}(z) = \mathbb{E}_P[Z_z \mid E]$. No comparison is added from J to I, so I remains an ideal.

For $s \in I, t \in J$, write $H = h_P(t) - h_P(s) \ge g \ge \theta M$. Equation (29) gives $d_P(s,t)/H \le C_\theta$. Putting $q = \mathbb{P}_P(E^c) \le M^{-4}$ in (10), we obtain

$$\left|\frac{h_{Q'}(t) - h_{Q'}(s)}{h_P(t) - h_P(s)} - 1\right| \le C_\theta\sqrt{\frac{q}{1-q}} \le \frac{1}{2}$$

for sufficiently large M in terms of θ. Hence

$$\min_{t\in J} h_{Q'}(t) - \max_{s\in I} h_{Q'}(s) \geq g/2.$$

Every vertex of I whose original height is at most $h_P(u) - L$ now lies below u, and every vertex of J whose original height is at least $h_P(v) + L$ now lies above v. Thus the original heights from $\max_{Q'} I$ lie in $(h_P(u) - L, h_P(u)]$, and those from $\min_{Q'} J$ lie in $[h_P(v), h_P(v) + L)$. By applying (12) in Q', we have

$$g/2 \leq |\max_{Q'} I| + |\min_{Q'} J| - 1.$$

Choosing the larger of these antichains gives $B \subseteq P$ with

$$|B| \geq g/4 \geq \theta M/4, \qquad H_P(B) \leq L = AM\log^2 M. \tag{30}$$

Equation (29) then gives $D_P(B) = O(M\log M)$. Lemma 11, for a fixed $\eta = \eta(k,\varepsilon) > 0$, instead gives

$$|B| = O_{\theta,k,\varepsilon}((M\log^2 M)^{1-\eta}) = o(M),$$

contradicting (30) for sufficiently large M. □

Theorem 14. *For every fixed $k \geq 2$, $t \geq 1$, and $\varepsilon > 0$, there exists $R_* = R_*(k,t,\varepsilon) > 0$ such that every finite poset P with $G(P) < R(P)$ and $R(P) \geq R_*$ contains either an ε-pseudo antichain of size k or an ε-pseudo insertor of size $t + 1$.*

Theorem 2 follows as a corollary.

Proof of Theorem 2. Fix $k \geq 2$, $t \geq 1$, and $\varepsilon > 0$, and put $\gamma = \min\{\varepsilon/2, 1/(2(t+1))\}$ and $\beta = \gamma/(2k!)$. Suppose P_0 contains no ε-pseudo antichain of size k. Adjoin a unique minimum and maximum to obtain P; its width and the relative-order laws on old vertices are unchanged. Any tuple containing a new vertex has a deterministic pair order, and hence TV distance at least $1/2$ from $\mathcal{U}_k$. Thus P contains no γ-pseudo antichain of size k, and $\delta_k(P) < 1/k! - \beta$.

Let $R_* = R_*(k,t,\gamma)$ be the threshold in Theorem 14, and let M_0 be the threshold in Lemma 13 for $k, \beta, \theta = 1$. Put $M = \max(R,G)$. For every vertex x and a neighbor y in height order, $(n+1)c_x \leq \sqrt{3}\, d(x,y) \leq \sqrt{3}M$. The antichain window bound gives $w(P_0) = w(P) \leq 2\sqrt{3}M$. If

$$w(P_0) > 2\sqrt{3}\max\{M_0, R_*, 1\},$$

then $M > M_0$, $M > R_*$, and $M > 1$. If $G = M$, the largest height gap is internal, since the endpoint gaps are one. Lemma 13 with $\theta = 1$ would then give $\delta_k(P) \geq 1/k! - \beta$, a contradiction. Thus $G < M$, so $M = R$, $G < R$, and $R > R_*$. Theorem 14 now gives a γ-pseudo insertor $Z = \{x, y_1, \ldots, y_t\}$ in P, with its defining labelling.

In the target law $\mathcal{I}_t$, the comparison probabilities are $i/(t+1)$, $1 \leq i \leq t$. Since the TV error is at most $\gamma < \varepsilon$, the selected vertices all belong to P_0, so

$$d_{\mathrm{TV}}(\mathcal{L}_{P_0;Z}, \mathcal{I}_t) \leq \gamma < \varepsilon.$$

□

The same reduction gives a short proof of the $1/e$ bound.

Proof that $w(P) \to \infty$ implies $\delta(P) \geq 1/e - o(1)$. Suppose instead that $w(P) \to \infty$ and $\delta(P) \leq 1/e - \varepsilon$ for some fixed $\varepsilon > 0$. As in the preceding proof, adjoin one universal minimum and maximum. Lemma 13 and the window bounds then give $G = o(R)$ and $R \to \infty$, while $\delta(P)$ is unchanged.

Choose x, y with $d(x, y) = R$ and $\Delta(x, y) \leq R$, and list the vertices from x to y in nondecreasing height order as $u_0 = x, \ldots, u_m = y$, interchanging x, y if necessary. Put $\Delta_i = \Delta(u_{i-1}, u_i)$ and $d_i = d(u_{i-1}, u_i)$. Equation (9) gives

$$R^2 \leq \sum_{i=1}^{m} d_i^2, \qquad \sum_{i=1}^{m} \Delta_i^2 \leq G \sum_{i=1}^{m} \Delta_i = G\Delta(x, y) \leq GR.$$

Consequently some consecutive pair u, v satisfies $\Delta(u, v)/d(u, v) \leq \sqrt{G/R} = o(1)$.

For the log-concave variable $V = Z_u - Z_v$, both sides of its mean have probability at least $1/e$. Indeed, its survival function S is log-concave, so Jensen gives $\log S(\mathbb{E}\, V) \geq \mathbb{E} \log S(V) = -1$; apply the same argument to $-V$. Moving the threshold from $\mathbb{E}\, V$ to zero and using the density bound in (3), we obtain

$$\delta(P) \geq \min\{\mathbb{P}(V < 0), \mathbb{P}(V > 0)\} \geq \frac{1}{e} - \sqrt{2}\,\frac{\Delta(u, v)}{d(u, v)} = \frac{1}{e} - o(1),$$

a contradiction. □

7 Proof of Theorem 14

Proof. It suffices to treat $0 < \varepsilon < \frac{1}{2(t+1)}$. Fix $k \geq 2$ and $t \geq 1$. We prove that there is $R_* = R_*(k, t, \varepsilon)$ such that the second alternative holds for P_0 with $G(P_0) < R(P_0)$, $R(P_0) \geq R_*$, and no ε-pseudo antichain of size k.

Stage 1: Normalize coordinates and count them locally. Writing $m = |P_0|$, form $P = C_{m^2} \oplus P_0 \oplus C_{m^2}$ and put $n = |P| = 2m^2 + m$. Clearly $w(P) = w(P_0)$, $G(P) = G(P_0)$, and P has no ε-pseudo antichain.

For distinct $u, v \in P$, put $D = f(v) - f(u)$. Conditional on the extension, the rank-unit difference $Z_v - Z_u$ has mean D and variance $|D|(n + 1 - |D|)/(n + 2)$. Thus

$$d(u, v)^2 = \mathrm{Var}(D) + \mathbb{E}\,\frac{|D|(n + 1 - |D|)}{n + 2} \leq \mathrm{Var}(D) + \mathbb{E}\,|D|. \tag{31}$$

For $u, v \in P_0$, the law of D and its mean are unchanged by padding, while the conditional variance of $Z_u - Z_v$ at $s = |D|$ is $s - s(s + 1)/(n + 2)$, which increases with n. Thus $d_P(u, v) \geq d_{P_0}(u, v)$ and $\Delta_P(u, v) = \Delta_{P_0}(u, v)$, so $R := R(P) \geq R(P_0)$. Hence it suffices to find an ε-pseudo insertor in P.

For $u, v \notin P_0$, D is a fixed nonzero integer, so $d(u, v)^2 < |D| \leq D^2 = \Delta^2(u, v)$ and the pair cannot enter the maximum defining R. For $u, v \in P_0$, $|D| \leq m$ gives $d(u, v)^2 \leq m^2 + m$. For $u \in P_0, v \notin P_0$, D has constant sign and ranges over an interval of at most m values, so if this pair enters the maximum, then $\mathbb{E}\,|D| = |\mathbb{E}\, D| \leq d(u, v)$, so $d(u, v)^2 \leq m^2 + d(u, v)$ and $d(u, v) \leq m + 1$. Consequently $R \leq m + 1$, and a pair attaining $R > 1$ has a vertex in P_0.

Choose a pair x, y attaining R, with $x \in P_0$; then $\Delta(x, y) \leq d(x, y) = R$. For each vertex $z \in P$, put

$$A_z = \frac{Z_z - Z_x}{R} = b_z + W_z, \qquad b_z = \frac{h(z) - h(x)}{R}, \qquad \mathbb{E}\, W_z = 0. \tag{32}$$

We think of b_z as the mean height of z relative to the anchor x, and W_z as its "centered fluctuation," both in units of R. Write

$$d_0(s, t) = \|W_s - W_t\|_2 = \frac{d(s, t)}{R}, \qquad \ell_0(s, t) = |b_s - b_t| + d_0(s, t).$$

In particular $b_x = W_x = 0$, $|b_y| \le 1$, and $d_0(x,y) = 1$. The joint law of $(W_z)_{z\in P}$ is log-concave, since log-concavity is preserved by the affine maps.

Since $G(P) < R$, Lemma 12 with $M = R$ gives

$$\mathbb{E}\, W_z^2 = \frac{d(z,x)^2}{R^2} \le 4\left(1 + \frac{\Delta(z,x)}{R}\right) = 4(1+|b_z|). \tag{33}$$

Similarly,

$$d_0(s,t) = \frac{d(s,t)}{R} \le \max\left\{1, \frac{\Delta(s,t)}{R}\right\} = \max\{1, |b_s - b_t|\}. \tag{34}$$

The metric ℓ_0 has a uniform doubling constant $N_{k,\varepsilon}$ by Lemma 9.

For an interval I (with each endpoint either included or excluded), write $|I|$ for its length and put

$$N(w;I) = \frac{1}{R}\#\{z : b_z + w_z \in I\}.$$

By Lemma 12, every interval I of length 1 contains at most $9R$ of the mean heights b_z. We show that, for every fixed $T, \zeta, q > 0$, for sufficiently large R,

$$\mathbb{P}\left(\sup_{I\subseteq[-T,T]} |N(W;I) - |I|| > \zeta\right) < q. \tag{35}$$

Proof. Proof of (35) Conditional on the linear extension f, let $f(x) = r$. Exchangeability of the $n+1$ uniform spacings gives, for fixed $u > 0$ and sufficiently large R,

$$\#\{z : 0 < A_z \le u\} \overset{\text{law}}{=} \min\{\mathrm{Bin}(n, uR/(n+1)),\, n-r\}.$$

Indeed, consecutive spacings to the right of $U_{(r)}$ have the law of an initial segment of the spacings starting at zero; their partial sums count uniform sample points, stopped after $n-r$ points. Since $x \in P_0$ and $R \le m+1$,

$$\min\{r-1, n-r\} \ge m^2, \qquad m \ge R-1 \ge R/2 \quad (R \ge 2).$$

The probability of truncation is at most $uR/m^2 \le 4u/R$ by Markov's inequality. The binomial mean differs from uR by $uR/(n+1)$, and its variance is at most uR. Once $u/(n+1) \le \zeta/2$, Chebyshev's inequality therefore gives

$$\mathbb{P}\left(\left|\frac{1}{R}\#\{z : 0 < A_z \le u\} - u\right| > \zeta\right) \le \frac{4u}{R} + \frac{4u}{\zeta^2 R}.$$

The left side of the anchor is identical, with truncation at $r-1$; the anchor contributes only $1/R$. Subtraction gives the assertion for each fixed interval. For uniformity, choose a finite grid in $[-T,T]$ of mesh at most $\zeta/16$, including both endpoints. The union bound makes every grid-interval error at most $\zeta/2$ outside an event of probability q. Any interval is sandwiched between grid intervals whose lengths differ from its length by at most four mesh widths; the inner interval may be empty. Monotonicity of counts now gives an error at most $\zeta/2 + \zeta/4 < \zeta$. Open and closed endpoint conventions are covered by the same sandwich, including the anchor. This proves (35). □

Stage 2: Choose the reference vertex. Fix $H > 0$ and $0 < \eta < 1$, and put $s = \eta/2$ and $I_h = (h-s, h+s]$. We choose a vertex p near height zero and finitely many measurements that distinguish vertices far from p. Stage 3 will find a vertex close to p in each I_h, $|h| \le H$.

For $g \in \mathcal{G} := \{0 \le g \le 2,\ \mathbb{E}\, g = 1\}$, put $u_g(z) = \mathbb{E}[gW_z]$. Since $\mathbb{E}\, W_z = 0$ and $\|g-1\|_2 \le 1$, Cauchy–Schwarz and (33) give

$$|u_g(z)| = |\,\mathbb{E}[(g-1)W_z]| \le \|g-1\|_2 \|W_z\|_2 \le 2\sqrt{1+|b_z|}. \tag{36}$$

Likewise,

$$|u_g(v) - u_g(w)| = |\mathbb{E}[(g-1)(W_v - W_w)]| \le \|g-1\|_2 \|W_v - W_w\|_2 \le d_0(v,w). \tag{37}$$

For centered log-concave V, Hölder's inequality and (4) give

$$\mathbb{E}|V|^2 \le (\mathbb{E}|V|)^{2/3}(\mathbb{E}|V|^4)^{1/3} \le (24e)^{1/3}(\mathbb{E}|V|)^{2/3}(\mathbb{E}|V|^2)^{2/3}.$$

Thus $\mathbb{E}|V| \ge c_0\|V\|_2$, where $c_0 = (24e)^{-1/2}$; the zero-variance case is immediate. Put $r_0 = c_0\eta/4$. Choose an ℓ_0-net $v_1, \dots, v_q$ at resolution r_0 for $|b_z| \le H+1$. By (34), this window lies within ℓ_0-distance $2(H+1)$ of x. Doubling therefore gives $q \le Q = Q(k, \varepsilon, H, \eta)$, with Q a fixed positive integer.

Let μ_0 be uniform on $\{z : b_z \in I_0\}$, which contains x. Doubling covers this set by at most $K_0 = K_0(k, \varepsilon, \eta)$ ℓ_0-balls of radius $r_0/2$. Assign each vertex to one containing ball. The resulting clusters have d_0-diameter at most r_0, and a largest cluster $\mathcal{C}_*$ has μ_0-mass at least $\alpha_0 := 1/K_0$. Choose $p \in \mathcal{C}_*$. In particular, $|b_p| \le s < \eta$.

For each representative v_i, set

$$g_i = 1 + \tfrac{1}{2}(\operatorname{sgn}(W_{v_i} - W_p) - \mathbb{E}\operatorname{sgn}(W_{v_i} - W_p)), \qquad t_i(z) = u_{g_i}(z) - u_{g_i}(p),$$

where $\operatorname{sgn}(0) = 0$. Since the sign lies in $[-1,1]$, $0 \le g_i \le 2$ and $\mathbb{E}\, g_i = 1$. Centering $W_z - W_p$ gives

$$t_i(z) = \tfrac{1}{2}\mathbb{E}[\operatorname{sgn}(W_{v_i} - W_p)(W_z - W_p)], \qquad |t_i(z)| \le \tfrac{1}{2}d_0(z,p).$$

At v_i, $t_i(v_i) = \frac{1}{2}\mathbb{E}|W_{v_i} - W_p| \ge \frac{c_0}{2}d_0(v_i,p)$; moving to z changes this value by at most $d_0(z,v_i)/2$. If $|b_z| \le H+1$ and $d_0(z,p) \ge \eta$, choose v_i with $d_0(z,v_i) \le \ell_0(z,v_i) \le r_0$. Then

$$t_i(z) \ge \tfrac{1}{2}\big(c_0 d_0(v_i,p) - d_0(z,v_i)\big) \ge \tfrac{1}{2}\big(c_0(\eta - r_0) - r_0\big) \ge r_0,$$

because $c_0\eta = 4r_0$ and $c_0 \le 1$. Consequently

$$\begin{aligned} z \in \mathcal{C}_* &\implies \max_i |t_i(z)| \le r_0/2, \qquad \mu_0(\mathcal{C}_*) \ge \alpha_0, \\ d_0(z,p) \ge \eta &\implies \max_i t_i(z) \ge r_0 \qquad (|b_z| \le H+1). \end{aligned} \tag{38}$$

Thus a vertex with $b_z \in I_h$ and all $|t_i(z)| < r_0$ is close enough to p. We prove its existence by counting crossings and then coloring vertices.

Stage 3: Weak uniformity of interval counts. We prove that for every fixed $T, \zeta > 0$, for sufficiently large R, each $g \in \mathcal{G}$ and $I \subseteq [-T,T]$ satisfies

$$|N(u_g; I) - |I|| \le \zeta. \tag{39}$$

This includes $N(0; I)$, since the constant weight $g = 1$ has $u_1 = 0$. The first step is to find a configuration close to u_g where the random interval counts hold.

Rescale coordinate z by $1 + |b_z|$ and use the norm

$$\|w\|_*^4 = \frac{1}{R}\sum_z \frac{|w_z|^4}{(1+|b_z|)^4}.$$

The fourth-moment bound following (4), together with (33), gives

$$\mathbb{E}|W_z|^4 \le 24e(\mathbb{E}\,W_z^2)^2 \le 384e(1+|b_z|)^2.$$

For each integer $j \geq 0$, at most $18R$ vertices satisfy $j \leq |b_z| < j+1$, counting multiplicities of equal mean heights. Summing over these bands gives the two bounds needed below:

$$\frac{1}{R}\sum_z (1+|b_z|)^{-2} \leq 18\sum_{j\geq 0}(j+1)^{-2} \leq 18\left(1+\int_1^\infty t^{-2}\,dt\right) = 36,$$
$$\frac{1}{R}\sum_z (1+|b_z|)^{-4} \leq 18\sum_{j\geq 0}(j+1)^{-4} \leq 18\left(1+\int_1^\infty t^{-4}\,dt\right) = 24.$$

Thus linearity of expectation gives

$$\mathbb{E}\,\|W\|_*^4 \leq \frac{384e}{R}\sum_z (1+|b_z|)^{-2} \leq 384e\cdot 36 = 13824e. \tag{40}$$

For $T \geq 1$, the same band count gives

$$\frac{1}{R}\sum_{|b_z|>T}\frac{\mathbb{E}\,|W_z|^4}{(1+|b_z|)^4} \leq 6912e\sum_{j\geq\lfloor T\rfloor}(j+1)^{-2} \leq \frac{6912e}{\lfloor T\rfloor} \leq \frac{13824e}{T}. \tag{41}$$

Here the tail sum is at most $\int_{\lfloor T\rfloor}^\infty t^{-2}\,dt$ and $\lfloor T\rfloor \geq T/2$.

For each $r > 0$ and $0 < \alpha < 1$, we claim that uniformly boundedly many $\|\cdot\|_*$-balls of radius r cover probability at least $1-\alpha$; only their centers depend on the poset.

For $T \geq 1$, (34) places the vertices with $|b_z| \leq T$ within ℓ_0-distance $2T$ of x. Doubling gives an ℓ_0-net at resolution ξ with at most $K_1 = K_1(T,\xi,k,\varepsilon)$ representatives in this height window. Choose $\rho(z)$ with $\ell_0(z,\rho(z)) \leq \xi$. Define $V_z = W_{\rho(z)}$ on this height window and $V_z = 0$ outside it. Since $W_z - W_{\rho(z)}$ is centered and log-concave,

$$\mathbb{E}\,|W_z - W_{\rho(z)}|^4 \leq 24e\,d_0(z,\rho(z))^4 \leq 24e\,\xi^4.$$

The second band sum bounds the contribution inside the window; (41) bounds the contribution outside. Thus

$$\mathbb{E}\,\|W-V\|_*^4 \leq 576e\,\xi^4 + \frac{13824e}{T}.$$

Set

$$\theta = \frac{\alpha r^4}{324}, \qquad T = 1+\frac{27648e}{\theta}, \qquad \xi = \left(\frac{\theta}{1152e}\right)^{1/4}.$$

The last expectation is less than θ, so Markov's inequality gives $\mathbb{P}(\|W-V\|_* \geq r/3) < 81\theta/r^4 = \alpha/4$. Each representative has variance at most $4(1+T)$ by (33). Take $M = \sqrt{8K_1(1+T)/\alpha}$ and $\delta = r/8$. Chebyshev's inequality and the union bound give probability at most $4K_1(1+T)/M^2 = \alpha/2$ that any representative lies outside $[-M,M]$. When all representative values lie in $[-M,M]$, round each to a multiple of δ with error at most δ, and assign its rounded value to the same vertices as before, producing $\widetilde{V}$. Keep $\widetilde{V}_z = 0$ outside the window. Then

$$\left\|V-\widetilde{V}\right\|_*^4 \leq \delta^4\,\frac{1}{R}\sum_z (1+|b_z|)^{-4} \leq 24(r/8)^4 < (r/3)^4.$$

There are at most $(2\lceil M/\delta\rceil+3)^{K_1}$ rounded vectors. Except on an event of probability less than $\alpha/4+\alpha/2 < \alpha$, one is within $2r/3$ of W, proving the cover claim.

We next show that

$$\text{for each } r>0 \text{ there is } \tau>0 \text{ such that } \mathbb{P}(\|W-u_g\|_* < r) \geq \tau \text{ for all } g\in\mathcal{G}. \tag{42}$$

Proof of (42). Put $B_* = (13824e)^{1/4}$ and $\alpha = \min\{1/16, r^2/(6400B_*^2)\}$. The cover gives at most $K = K(r, \alpha, k, \varepsilon)$ closed balls of radius $r/4$ covering probability at least $1-\alpha$. Discard balls of probability below α/K. Their union has probability at most α, so the retained balls cover probability at least $1-2\alpha$.

Define

$$B = \{w : \|w\|_* \le r/2\}, \qquad \Phi(c) = \mathbb{P}(W \in c + B), \qquad A = \{c : \Phi(c) \ge \alpha/K\}.$$

If w belongs to a retained ball, that entire ball lies in $w + B$. Thus $w \in A$ and $\mathbb{P}(W \in A) \ge 1 - 2\alpha$.

Let f_W be the density of W on its affine support S. Then

$$\Phi(c) = \int_S f_W(w) \mathbf{1}_B(w - c)\, dw.$$

The integrand is log-concave jointly in (w, c), because f_W is log-concave and $\{(w, c) : w - c \in B\}$ is convex. Integrating out w preserves log-concavity by [20, Theorem 5.1]. Thus $\log \Phi$ is concave wherever $\Phi > 0$.

Fix $g \in \mathcal{G}$. Put $D = \{W \notin A\}$, $\delta = \mathbb{E}[g\mathbf{1}_D] \le 2\mathbb{P}(D) \le 4\alpha$, and $m = 1 - \delta \ge 1/2$, and define

$$v = \frac{\mathbb{E}[gW\mathbf{1}_{D^c}]}{m}.$$

Jensen's inequality for $\log \Phi$, under the probability measure with density $g\mathbf{1}_{D^c}/m$, gives

$$\log \Phi(v) \ge \frac{1}{m}\,\mathbb{E}[g\mathbf{1}_{D^c} \log \Phi(W)] \ge \log(\alpha/K).$$

Thus $\Phi(v) \ge \alpha/K$.

The fourth-moment bound (40) gives $\mathbb{E}\,\|W\|_*^2 \le B_*^2$ and $\mathbb{E}\,\|W\|_* \le B_*$. Consequently $\|v\|_* \le 2B_*/m \le 4B_*$. Since $u_g = mv + \mathbb{E}[gW\mathbf{1}_D]$, Cauchy–Schwarz gives

$$\begin{aligned}
\|u_g - v\|_* &\le \delta\,\|v\|_* + \mathbb{E}[g\,\|W\|_*\,\mathbf{1}_D] \\
&\le 16B_*\alpha + 2(\mathbb{E}\,\|W\|_*^2)^{1/2}\mathbb{P}(D)^{1/2} \\
&\le 16B_*\alpha + 2B_*\sqrt{2\alpha} \le 20B_*\sqrt{\alpha} \le r/4 < r/2.
\end{aligned}$$

Here we used $\alpha < 1$ and $\alpha \le r^2/(6400B_*^2)$. Hence $\|W - v\|_* \le r/2$ implies $\|W - u_g\|_* < r$, and

$$\mathbb{P}(\|W - u_g\|_* < r) \ge \Phi(v) \ge \alpha/K.$$

This proves (42) with $\tau = \alpha/K$. □

Proof of (39). Fix $T, \zeta > 0$ and put $B = 64 + 4(T+1)$. For $u > B$, $2\sqrt{1+u} \le u/3$ and $T + 1 \le u/4$, so

$$u - 2\sqrt{1+u} - (T+1) \ge 5u/12 \ge (1+u)/4.$$

By (36), a vertex with $|b_z| > B$ cannot have $b_z + u_g(z) \in [-T-1, T+1]$. If $b_z + w_z$ lies there, $e_z := w_z - u_g(z)$ must satisfy $|e_z| \ge (1 + |b_z|)/4$. For $0 < \delta < 1$, the normalized number of these far vertices, together with those having $|b_z| \le B$ and $|e_z| > \delta$, is at most

$$\left(256 + \frac{(1+B)^4}{\delta^4}\right) \|e\|_*^4.$$

Indeed, each far vertex contributes at least $1/(256R)$ to the fourth power of the norm, and each of the other exceptional vertices contributes more than $\delta^4/[R(1+B)^4]$.

Set

$$\delta = \min\{1/2, \zeta/16\}, \qquad r = \left(\frac{\zeta}{4(256 + (1+B)^4/\delta^4)}\right)^{1/4}.$$

The exceptional count is then at most $\zeta/4$ whenever $\|e\|_* < r$. Obtain $\tau > 0$ from (42). By (35), the event that every interval in $[-T-1, T+1]$ has counting error at most $\zeta/4$ has probability greater than $1 - \tau/2$ for large R. For each g, it therefore meets $\{\|W - u_g\|_* < r\}$, whose probability is at least τ. Choose w in the intersection.

For $I \subseteq [-T, T]$, enlarge each endpoint by δ to obtain I^+, and shrink by δ to obtain I^-; use closed outer and open inner intervals, with $I^- = \varnothing$ if necessary. Every nonexceptional vertex contributing to these counts has $|b_z| \le B$ and displacement at most δ. Hence

$$N(w; I^-) - \zeta/4 \le N(u_g; I) \le N(w; I^+) + \zeta/4.$$

The interval-length error is at most 2δ on either side (if I^- is empty, $|I| \le 2\delta$). Hence $|N(u_g; I) - |I|| \le 2\delta + \zeta/2 \le 5\zeta/8 < \zeta$, proving (39) □

Stage 4: Uniform boundedness of the weighted means. Assign each $z \in P$ a (deterministic) "starting position" x_z and "velocity" v_z, so that its position at time t is $x_z + tv_z$. Assume $|v_z| \le M$ with $M > 0$, fix weights $0 \le w_z \le 1$, and let $\tau > 0$. Suppose, for some $c \ge 0$, that the normalized weighted counts satisfy

$$\left|\frac{1}{R}\sum_z w_z \mathbf{1}_{\{x_z + tv_z \in I\}} - c|I|\right| \le \varepsilon \qquad (t \in \{0, \tau\}) \tag{43}$$

for every interval I in some fixed window. Put $E = M(4c\tau M + 2\varepsilon)$. We show that, for intervals I, J whose τM-neighborhoods lie within that window,

$$\left|\frac{1}{R}\sum_{x_z \in I} w_z v_z - \frac{|I|}{|J|}\frac{1}{R}\sum_{x_z \in J} w_z v_z\right| \le E\left(1 + \frac{|I|}{|J|}\right) + \frac{2\varepsilon|I|}{\tau}. \tag{44}$$

Proof of (44). For each point, count movements during $[0, \tau]$ from at or below it to above it with weight w_z/R, and movements in the reverse direction with weight $-w_z/R$. For $a < b$ in the window, the signed crossing count at b minus that at a is

$$\frac{1}{R}\sum_z w_z\big(\mathbf{1}_{\{x_z \in (a,b]\}} - \mathbf{1}_{\{x_z + \tau v_z \in (a,b]\}}\big).$$

Each of these two interval counts differs from $c(b-a)$ by at most ε. Thus signed crossing counts at any two points differ by at most 2ε.

For a positive-length interval I, partition it at all trajectory endpoints $x_z, x_z + \tau v_z$ lying inside it. The signed crossing count is constant on each open piece. Average these counts with weights equal to the piece lengths divided by $|I|$. Exchanging the finite sums over pieces and vertices shows that this average equals the normalized weighted sum of signed trajectory lengths inside I, divided by $|I|$. To that sum, a trajectory wholly inside contributes exactly $\tau w_z v_z/R$. Discrepancies from counting the full trajectory of each vertex initially in I can arise only within τM of the endpoints of I. The two endpoint strips have total normalized weight at most $4c\tau M + 2\varepsilon$, and each trajectory has length at most τM.

The average crossing count of I therefore differs from $\frac{\tau}{R|I|}\sum_{x_z \in I} w_z v_z$ by at most $\tau E/|I|$. Vertices at endpoints are included in these strips, so either endpoint convention is allowed. The averages for any two intervals differ by at most 2ε, since all the signed crossing counts do. For positive-length intervals I, J whose τM-neighborhoods lie in that window, the triangle inequality gives a total error $\tau E/|I| + 2\varepsilon + \tau E/|J|$

between the corresponding velocity sums scaled by $\tau/(R|I|)$ and $\tau/(R|J|)$. Multiplying by $|I|/\tau$ gives (44). $\square$

Now for every fixed $T > 0$, we claim that, for sufficiently large R,

$$g \in \mathcal{G},\ |b_z| \le T \qquad \Longrightarrow \qquad |u_g(z)| \le 3. \tag{45}$$

Proof of (45). Put $T_1 = T + 3$. We shall apply (44) to $\{z \in P : |b_z| \le 68 + 4T_1\}$. By (36), discarded vertices cannot enter $[-T_1, T_1]$ under $b_z + \lambda u_g(z)$, $0 \le \lambda \le 1$. On the retained set take

$$x_z = b_z, \qquad v_z = u_g(z), \qquad w_z = 1, \qquad M = 2\sqrt{69 + 4T_1}.$$

The weight $1 + \lambda(g - 1)$ belongs to $\mathcal{G}$, so (39) supplies (43) with $c = 1$ and any prescribed $\varepsilon > 0$.

Write $J_h = (h - 1/4, h + 1/4]$. Set $\tau = 1/(128M^2)$ and $\varepsilon = \tau/32$. Since $M \ge 1$, we have $\tau \le 1$, $\tau M \le 1$, $4M \le 1/\tau$, and $\varepsilon \le 1/8$. For $|h| \le T$, the τM-neighborhoods of J_h and J_0 lie in $[-T_1, T_1]$. Apply (44) with $I = J_h$ and $J = J_0$, both of length $1/2$, to obtain

$$\left| \frac{1}{R} \sum_{b_z \in J_h} u_g(z) - \frac{1}{R} \sum_{b_z \in J_0} u_g(z) \right| \le 8M^2\tau + (4M + 1/\tau)\varepsilon \le \frac{1}{16} + \frac{2\varepsilon}{\tau} = \frac{1}{8}.$$

Since $u_g(x) = 0$, (34) and (37) give $|u_g(z)| \le 1$ on J_0. Thus the normalized sum over J_0 has absolute value at most $N(0; J_0) \le 1/2 + \varepsilon$, and the sum over J_h has absolute value at most $1/2 + \varepsilon + 1/8$. For a vertex z with $|b_z| \le T$, every vertex in J_{b_z} has measurement within 1 of $u_g(z)$, again by these two inequalities. Since $N(0; J_{b_z}) \ge 1/2 - \varepsilon > 0$, it follows that

$$|u_g(z)| \le 1 + \frac{\left|\frac{1}{R}\sum_{b_y \in J_{b_z}} u_g(y)\right|}{N(0; J_{b_z})} \le 1 + \frac{1/2 + \varepsilon + 1/8}{1/2 - \varepsilon} \le 1 + \frac{3/4}{3/8} = 3.$$

This proves (45). $\square$

Stage 5: Use finite coin flips to detect copies.

Fix $B > 0$ (to be defined below, and put, for $z \in P$ with $|b_z| \le B$ and $1 \le i \le q$,

$$p_i(z) := \frac{u_{g_i}(z) + 3}{6}$$

For fixed weights $0 \le w_z \le 1$ and $a \in \mathbb{R}^q$, put

$$x_z(a) = b_z + \sum_{i=1}^{q} a_i p_i(z), \qquad N_w(a; I) = \frac{1}{R} \sum_z w_z \mathbf{1}_{\{x_z(a) \in I\}}.$$

where the second sum runs only over z for which $x_z(a)$ is defined.

Suppose, for $0 < A \le 1$, $0 \le c \le 2$, $0 < \varepsilon \le 1$, and $L \ge H_1 \ge 1$, that

$$|N_w(a; I) - c|I|| \le \varepsilon \quad (\|a\|_1 \le A,\ I \subseteq [-L - 3, L + 3]).$$

Assume also that $N_1(0; I) \le |I| + 1$ in this window; this will follow from the initial unweighted counting estimate. Take $0 < \tau < A$, and for fixed i replace w_z by either $w'_z = w_z p_i(z)$ or $w'_z = w_z(1 - p_i(z))$. By (45), for sufficiently large R, $p_i(z) \in [0, 1]$.

For $\|a\|_1 \le A-\tau$, if e_i is the indicator vector for the ith coordinate, moving with velocity p_i changes a to $a+\tau e_i$; moving with velocity $1-p_i$ changes it to $a-\tau e_i$ and translates all positions by τ. Both movements satisfy the assumed count bound. The extra margin of 3 covers these translations and the endpoint strips.

Apply (44) with $M=1$, $J=(-L,L]$, and $I\subseteq[-H_1,H_1]$. Here $E\le 8\tau+2\varepsilon$. Set $c'=N_{w'}(0;J)/(2L)$. Because $w'\le 1$ and $L\ge 1$, $0\le c'\le(2L+1)/(2L)\le 2$. Also $|x_z(a)-b_z|\le\|a\|_1\le 1$; changing a to zero can alter membership in J only in its two endpoint strips of length 2. Since $w'\le w$, their total normalized weight is at most $4c+2\varepsilon$. Hence

$$|N_{w'}(a;I)-c'|I||\le(8\tau+2\varepsilon)\left(1+\frac{H_1}{L}\right)+\frac{4H_1\varepsilon}{\tau}+\frac{H_1}{L}(4c+2\varepsilon)\le 16\tau+4\varepsilon+\frac{4H_1\varepsilon}{\tau}+\frac{10H_1}{L}.$$

The last inequality uses $H_1/L\le 1$, $c\le 2$, and $\varepsilon\le 1$. Thus the new counts are uniform, with a density c' independent of the movement parameter a.

For any fixed number $D\ge 1$ of rounds, target window $[-H_D,H_D]$ with $H_D\ge 1$, and error $0<\varepsilon_D\le 1$, the preceding estimate can be iterated uniformly over every sequence of color choices. Working backwards for $j=D,D-1,\dots,1$, define

$$\tau_j=\frac{\varepsilon_j}{64D},\qquad L_j=\frac{40H_j}{\varepsilon_j},\qquad H_{j-1}=L_j+3,\qquad \varepsilon_{j-1}=\frac{\varepsilon_j\tau_j}{16H_j}.$$

These definitions preserve $H_j\ge 1$ and $0<\varepsilon_j\le 1$, so $\tau_j\le 1/(64D)\le H_j$. The one-coloring error is at most ε_j, because

$$16\tau_j=\frac{\varepsilon_j}{4D}\le\frac{\varepsilon_j}{4},\qquad \frac{10H_j}{L_j}=\frac{\varepsilon_j}{4},$$
$$\left(4+\frac{4H_j}{\tau_j}\right)\varepsilon_{j-1}=\frac{\varepsilon_j}{4}\left(1+\frac{\tau_j}{H_j}\right)\le\frac{\varepsilon_j}{2}.$$

Start with $w=1$, density $c=1$, and counting error ε_0 on $[-H_0,H_0]$ for $\|a\|_1\le 1$. After round j, the allowed parameter radius is $1-\sum_{r=1}^{j}\tau_r\ge 1-D/(64D)=63/64$; in particular, every step has $\tau_j<A$. Each new density lies in $[0,2]$, as just proved; the required unweighted upper bound holds throughout because all reference windows lie in the initial window and $\varepsilon_0\le 1$. Induction therefore gives, for every resulting weight w and some $c_w\in[0,2]$,

$$|N_w(0;I)-c_w|I||\le\varepsilon_D\qquad(I\subseteq[-H_D,H_D]).\tag{46}$$

Let $\alpha=s\alpha_0$, set

$$\gamma=\frac{\alpha}{8(2s+1)},\qquad n_*=\left\lceil\frac{144Q}{\gamma r_0^2}\right\rceil,$$

and take $D=n_*q$, $H_D=H+1$, and $\varepsilon_D=\alpha/(8\cdot 2^{n_*Q})$ in the finite construction above.

Let H_0 and ε_0 be the resulting initial parameters, and let $B=64+4(H_0+2)$.

For $\|a\|_1\le 1$, the weight $g_a=1+\frac{1}{6}\sum_i a_i(g_i-1)$ lies in $\mathcal{G}$, because $|g_a-1|\le 1/6$ and $\mathbb{E}\,g_a=1$. Moreover,

$$b_z+\sum_i a_ip_i(z)=b_z+u_{g_a}(z)+\tfrac{1}{2}\sum_i a_i.$$

The last translation has magnitude at most $1/2$. The cutoff calculation in the proof of (39), with $T=H_0+1$, excludes discarded vertices even after this translation. Thus (39), applied on $[-H_0-1,H_0+1]$ with error ε_0, supplies all the initial counting hypotheses for (46).

For each vertex, flip n_* coins of success probability $p_i(z)$ for each i, all n_*q coins independent. Let $\widehat{p}_i(z)$ be the success frequency and put $\widehat{u}_i(z)=6\widehat{p}_i(z)-3$. The expectation of $\widehat{u}_i(z)$ is $u_{g_i}(z)$, and

$$\mathrm{Var}(\widehat{u}_i(z))=36p_i(z)(1-p_i(z))/n_*\le 9/n_*.$$

Chebyshev's inequality and the union bound give

$$\mathbb{P}\left(\max_i |\widehat{u}_i(z) - u_{g_i}(z)| \geq r_0/4\right) \leq \frac{144q}{n_* r_0^2} \leq \gamma.$$

Accept the outcomes if $\max_i |\widehat{u}_i(z) - u_{g_i}(p)| < 3r_0/4$. By (38), a vertex in $\mathcal{C}_*$ is accepted with probability at least $1-\gamma$. A vertex in $|b_z| \leq H+1$ with $d_0(z,p) \geq \eta$ is accepted with probability at most γ: on the complementary event in the last display, some measured difference is greater than $r_0 - r_0/4 = 3r_0/4$.

Write $F(h)$ for the expected number of accepted vertices with $b_z \in I_h$, divided by R. The acceptance rule specifies a fixed subset of the $2^{n_* q}$ outcome sequences, the same for every vertex. For each sequence, (46) compares its expected counts in I_h and I_0 with error at most $2\varepsilon_D$, since both intervals have length $2s$. Summing gives

$$|F(h) - F(0)| \leq 2^{n_* q} \cdot 2\varepsilon_D \leq \alpha/4 \qquad (|h| \leq H).$$

Increase R so that (39) also has error at most $s < 1$ on these intervals. Then $N(0; I_0) \geq s$, so $|\mathcal{C}_*|/R \geq \alpha_0 s = \alpha$, and $N(0; I_h) \leq 2s + 1$. Since $\gamma \leq 1/8 < 1/4$,

$$F(h) \geq F(0) - \alpha/4 \geq (1-\gamma)\alpha - \alpha/4 \geq \alpha/2.$$

If every vertex in I_h had $d_0(z,p) \geq \eta$, the acceptance bound would instead give $F(h) \leq \gamma N(0; I_h) \leq \gamma(2s+1) = \alpha/8$, a contradiction. We have proved

$$\text{for every } h \in [-H, H] \text{ there is } z \in P \text{ with } |b_z - h| < \eta, \quad \|W_z - W_p\|_2 < \eta. \tag{47}$$

Indeed $b_z \in I_h$ gives $|b_z - h| \leq s < \eta$.

Stage 6: Construct the pseudo insertor. Put $H = 1 + 2\sqrt{t+1}$. Choose $\eta > 0$ sufficiently small (in terms of t, ε) as specified below. Apply stages 2–5 with these parameters to obtain a vertex p and the copies guaranteed by (47). Since $d_0(x,y) = 1$, one $v \in \{x, y\}$ satisfies $d_0(v,p) \geq 1/2$. Put $V = A_v - W_p$. Then (34) gives

$$\mathbb{E}\, V = b_v, \qquad |b_v| \leq 1, \qquad 1/2 \leq \mathrm{sd}(V) = d_0(v,p) \leq \max\{1, |b_v - b_p|\} \leq 1 + \eta.$$

Its nondegenerate log-concave law has a continuous distribution function. By (3), its density is bounded by $\sqrt{2}/\,\mathrm{sd}(V) \leq 2\sqrt{2}$; put $K = 2\sqrt{2}$.

Now for $1 \leq i \leq t$, choose the quantile q_i with $\mathbb{P}(V < q_i) = i/(t+1)$. Chebyshev's inequality and (3) give

$$|q_i| \leq 1 + (1+\eta)\sqrt{t+1} < H, \qquad q_{i+1} - q_i \geq \frac{1}{K(t+1)} \quad (1 \leq i < t).$$

Choose z_i from (47) at height q_i. Thus

$$A_v = W_p + V, \qquad A_{z_i} = W_p + q_i + e_i, \qquad e_i = b_{z_i} - q_i + W_{z_i} - W_p.$$

$$\mathbb{E}\, e_i^2 = (b_{z_i} - q_i)^2 + \|W_{z_i} - W_p\|_2^2 \leq 2\eta^2$$

The z_i's are distinct if $2\eta < \frac{1}{K(t+1)}$, since $|b_{z_i} - q_i| < \eta$. Also $z_i \neq v$, since $d_0(z_i, p) < \eta < 1/2 \leq d_0(v,p)$.

Put $Z = \{v, z_1, \ldots, z_t\}$. Couple its actual order to the ideal order obtained by inserting V among $q_1 < \cdots < q_t$; the latter has law $\mathcal{I}_t$. If $h = \sqrt{\eta} < \frac{1}{2K(t+1)}$, the two orders agree whenever

$$\max_i |e_i| \leq h \qquad \text{and} \qquad \min_i |V - q_i| > h.$$

Indeed, the leaves remain in threshold order and each comparison with v keeps its sign. The union bound, Chebyshev's inequality, and (3) therefore give

$$d_{\mathrm{TV}}(\mathcal{L}_{P;Z},\mathcal{I}_t) \le \sum_{i=1}^{t}\mathbb{P}(|e_i|>h)+\sum_{i=1}^{t}\mathbb{P}(|V-q_i|\le h)\le \frac{2t\eta^2}{h^2}+2Kth=2t\eta+2Kt\sqrt{\eta}. \tag{48}$$

Note that this does not require independence between V and the errors. Hence, for sufficiently small $\eta > 0$, we have

$$d_{\mathrm{TV}}\left(\mathcal{L}_{P_0;Z},\mathcal{I}_t\right) < \varepsilon.$$

This proves the second alternative and completes the proof. $\square$

Remark. By calculating the dependence on $w(P)$ in the proof of Theorem 1, it can be shown that

$$\delta(P) \ge 1/2 - O((\log\log\log\log\log w(P))^{-1/4}) \tag{49}$$

No attempt has been made at optimizing this.

AI Usage

ChatGPT 6 Astra was used primarily to aid in the discovery of the results here, and also lightly in the preparation of this manuscript. As usual, the author takes full responsibility for the contents of the paper.

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