SET VALUES FOR MEAN FIELD GAMES

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ABSTRACT. In this paper we study mean field games with possibly multiple mean field equilibria. Instead of focusing on the individual equilibria, we propose to study the set of values over all possible equilibria, which we call the set value of the mean field game. When the mean field equilibrium is unique, typically under certain monotonicity conditions, our set value reduces to the singleton of the standard value function which solves the master equation. The set value is by nature unique, and we shall establish two crucial properties: (i) the dynamic programming principle, also called time consistency; and (ii) the convergence of the set values of the corresponding N-player games, which can be viewed as a type of stability result. To our best knowledge, this is the first work in the literature which studies the dynamic value of mean field games without requiring the uniqueness of mean field equilibria. We emphasize that the set value is very sensitive to the type of the admissible controls. In particular, for the convergence one has to restrict to corresponding types of equilibria for the N-player game and for the mean field game. We shall illustrate this point by investigating three cases, two in finite state space models and the other in a continuous time model with controlled diffusions.

1. INTRODUCTION

In this paper we study Mean Field Games (MFG, for short) without monotonicity conditions. There are typically multiple Mean Field Equilibria (MFE, for short) with possibly different values. Instead of focusing on the individual equilibria, we propose to study the set of values over all equilibria, which we call the set value of the MFG. Note that the set value always exists (with empty set as a possible value) and is by definition unique. When the MFE is unique, typically under certain monotonicity conditions, our set value is reduced to the singleton of the standard value function of the game, which solves the so called master equation. So the set value can be viewed as the counterpart of the standard value function for MFGs without monotonicity conditions, and it indeed shares many nice properties. In this paper, we focus particularly on two crucial properties of the set value:

- the Dynamic Programming Principle (DPP, for short), or say the time consistency;
- the convergence of the set values of the corresponding N-player games, which can be viewed as a type of stability result in terms of model perturbation.

For general theory of MFGs, we refer to Caines-Huang-Malhamé[7], Lasry-Lions [34], Lions [36], Cardaliaguet [8], Bensoussan-Frehse-Yam [6], and Camona-Delarue [13, 14].

In standard stochastic control theory, it is well known that the dynamic value function satisfies the DPP. In fact, this is the underlying reason for the PDE approach to work. For MFGs under appropriate monotonicity conditions, the value function (at the unique MFE) also satisfies the DPP, which, together with the Itô formula, leads to the master equation. However, with the presence of multiple equilibria (see, e.g., Bardi-Fischer [2] for some examples), to our best knowledge this is the first work in the literature to study the MFG dynamically and to address the time consistency issue. We show that, when formulated properly, the dynamic set value function satisfies the DPP. This also opens the door to a possible PDE approach for these general games by introducing the so called set valued PDE. We refer to our work [30] for set valued PDEs induced by multivariate stochastic control problems, and Ma-Zhang-Zhang [37] for numerical methods for set valued PDEs, and we leave their extension to mean field games for future research. Our set value approach follows from Feinstein-Rudloff-Zhang [24], which studies non-zero sum games with finitely many players. See also the related works Abreu-Pearce-Stacchetti [1] and Sannikov [42] in economics literature, and Feinstein [23] which studies the set of equilibria instead of values.

We note that the set value of games relies heavily on the types of admissible controls we use. In this paper we shall consider closed loop controls. The open loop equilibria of games are typically time inconsistent, see e.g. Buckdahn's counterexample in Pham-Zhang [40, Appendix E] for a two person zero sum game, and consequently, the set value of games with open loop controls would violate the DPP. For the MFG, noting that the required symmetry decomposes the game problem into a standard control problem and a fixed point problem of measures, and that open loop and closed loop controls yield the same value function for a standard control problem, it is possible that the set value with open loop controls still satisfies the DPP. Nevertheless, bearing in mind the DPP of the set value for more general (non-symmetric) games, as well as the practical consideration in terms of the information available to the players, we shall focus on closed loop controls. There is also a very subtle path dependence issue. While the game parameters are state dependent, we may consider both state dependent and path dependent controls. For general non-zero sum games (not mean field type), [24] shows that DPP holds for the set value for path dependent controls, but in general fails for the set value for state dependent controls. For MFGs with closed loop controls, again due to the required symmetric properties, the set values for both state dependent controls and path dependent controls will satisfy the DPP, but they are in general not equal. For MFGs with closed loop relaxed controls, or say closed loop mixed strategies, however, it turns out that the state dependent controls and the path dependent controls induce the same set value which still satisfies the DPP.

We next turn to the convergence issue. Let $\mathbb{V}$ and $\mathbb{V}^N$ denote the set values of the MFG and the corresponding N-player games, respectively, under appropriate closed-loop controls. Our convergence result reads roughly as follows (the precise

form is slightly different):

$$\lim_{N \to \infty} \mathbb{V}^N(0, \vec{x}) = \mathbb{V}(0, \mu), \quad \text{when} \quad \mu_{\vec{x}}^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to \mu. \tag{1.1}$$

In the realm of master equations, again under certain monotonicity conditions and hence with unique MFE, one can show that the values of the N-player games converge to the value of the MFG. See Cardaliaguet-Delarue-Lasry-Lions [10], followed by Bayraktar-Cohen [3], Cardaliaguet [9], Cecchin-Pelino [17], Delarue-Lacker-Ramanan [20, 21], Gangbo-Meszaros [29], and Mou-Zhang [38], to mention a few. So (1.1) can be viewed as their natural extension to MFGs without monotonicities.

We emphasize again that the set value is very sensitive to the types of admissible controls. To ensure the convergence, one simple but crucial observation is that the N-player game and the MFG should use the "same" type of controls (more precisely, corresponding types of controls in appropriate sense). We illustrate this point by considering two cases. Note that in the standard literature each player is required to use the same closed loop control along an MFE. For the first case, we will obtain the desired convergence by restricting the N-player game to homogeneous equilibria, namely each player also uses the same closed loop control. In the second case, we remove such restriction and consider heterogeneous equilibria for the N-player games. Note that a closed loop control means the control depends only on the state. In this heterogeneous case players with the same state may choose different controls, then one can not expect in the limit they will have to use the same control[1]. Indeed, in this case the limit is characterized by the MFG with closed loop relaxed controls, or say closed loop mixed strategies, which exactly means players with the same state may still have a distribution of controls to choose from. However, since our relaxed control for MFG is still homogeneous, namely each player uses the same relaxed control, the controls for N-player game and for MFG appear to be in different forms. Our approach is to introduce a new formulation for the MFG, which embeds the structure of heterogeneous controls and shares the same set value as the relaxed control formulation of the MFG. For the homogeneous case, we will investigate both a discrete time model with finite state space and a continuous time diffusion model with drift controls. But for the heterogeneous case we will investigate the discrete model only. The continuous model in such case involves some technical challenges for the convergence and we shall leave it for future research. We shall point out that, however, the DPP would hold in much more general models without significant difficulties.

To ensure the convergence, another main feature is that we define the set value as the limit of the approximate set values over approximate equilibria, rather than the true equilibria. We call the latter the raw set value, and both the set value and the raw set value satisfy the DPP. However, the raw set value is extremely sensitive to small perturbations of the game parameters, in fact, in general even its measurability is not clear, so one can hardly expect the convergence for the raw set values. In the standard control theory, the value function is defined as the

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[1] When the MFE is unique, under appropriate monotonicity conditions, the set value becomes a singleton and it is not sensitive to the type of admissible controls anymore. Consequently, the convergence becomes possible even if the N-player games and the MFG use different types of controls, see e.g. [10]

infimum of controlled values, which is exactly the limit of values over approximate optimal controls, rather than the value over true optimal controls which may not even exist. So our set value, not the raw set value, is the natural extension of the standard value function in control theory. Moreover, since we are considering infinitely many players, an approximate equilibrium means it is approximately optimal for most players, but possibly with a small portion of exceptions, as introduced in Carmona [11].

We would like to mention that, although it is not the focus of the present paper, the set value is also numerically a lot easier to compute than the raw set value. For example, the duality result for set values in [24, Section 3.4] (for finite player games) is very useful for constructing efficient numerical algorithms, see [37]. However, this is not feasible for the raw set value which lacks regularity and thus is hard to approximate in general.

At this point we should mention that, for MFGs without monotonicity conditions, there have been many publications on the convergence of N-player games, in terms of equilibria instead of values. For open loop controls, we refer to Camona-Delarue [12], Feleqi [25], Fischer [26], Fischer-Silva [27], Lacker [31], Lasry-Lions [34], Lauriere-Tangpi [35], and Nutz-San Martin-Tan [39], to mention a few. In particular, [31] provides the full characterization for the convergence: any limit of approximate Nash equilibria of N-player games is a weak MFE, and conversely any weak MFE can be obtained as such a limit. The work [26] is also in this direction. For closed loop controls, which we are mainly interested in, the situation becomes much more subtle. The seminal paper Lacker [32] established the following result:
(1.2)
$$\{\text{Strong MFEs}\} \subset \{\text{Limits of } N\text{-player approx. equilibria}\} \subset \{\text{Weak MFEs}\}.$$

Here an MFE is strong if it depends only on the state processes, and weak if it allows for additional randomness. The left inclusion in (1.2) was known to be strict in general. This work has very interesting further developments recently[2] by Lacker-Flem [33] and Djete [22]. In particular, [22] shows that the right inclusion in (1.2) is actually an equality.

We emphasize again that we are considering the convergence of sets of values, rather than sets of equilibria as in (1.2). For standard control problems, the focus is typically to characterize the (unique) value and to find *one* (approximate) optimal control, and the player is less interested in finding *all* optimal controls since they have the same value. The situation is quite different for games, because different equilibria can lead to different values. Then it is not satisfactory to find just one equilibrium (especially if it is not Pareto optimal). However, for different equilibria which lead to the same value, the players are indifferent on them. So for practical purpose the players would be more interested in finding all possible values[3] and then to find one (approximate) equilibrium for each value. This is one major motivation that we focus on the set value, rather than the set of all equilibria. We also note that in general the set value could be much simpler than the set of equilibria. For example, in the trivial case that both the terminal and the running cost functions are constants, the set value is a singleton, while the set of equilibria consists of all admissible controls.

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[2]These two works [22, 33] were circulated slightly after our present paper.

[3]Another very interesting question is how to choose an optimal (in appropriate sense) value after characterizing the set value. We shall leave this for future research.

We should point out that our admissible controls differ from those in [22,32,33]. Roughly speaking, we put two constraints, due to both practical and technical considerations, on the N-player approximate equilibria so that the left inclusion in (1.2) (in terms of values instead of equilibria) becomes an equality. First, for the N-player games, [22,32,33] use full information controls $\alpha_i(t, X_t^1, \cdots, X_t^N)$, while we consider symmetric controls $\alpha_i(t, X_t^i, \mu_t^N)$, where X_t^i is the state of Player i, and $\mu_t^N := \frac{1}{N} \sum_{j=1}^N \delta_{X_t^j}$ is the empirical measure of all the players' states. Note that, as a principle the controls should depend only on the information the players observe. While both settings are very interesting, since N is large, the full information may not be available in many practical situations.

The second difference is that we assume each control is Lipschitz continuous in μ, while [22,32,33] allow for measurable controls. We shall emphasize though we allow the Lipschitz constant to depend on the control, and thus our set value does not depend on any fixed Lipschitz constant. Roughly speaking, we are considering game values which can be approximated by Lipschitz continuous approximate equilibria. This is typically the case in the standard control theory: even if the optimal control is discontinuous, in most reasonable frameworks, we should be able to find Lipschitz continuous approximate optimal controls. The situation is more subtle for games. There may exist (closed loop) equilibria whose values cannot be approximated by any Lipschitz continuous approximate equilibria. In fact, when considering all measurable equilibria, the convergence of set values in (1.1) fails in general, see Example 7.2 and Remark 7.3. While clearly more general and very interesting mathematically, such measurable equilibria are hard to implement in practice, since inevitably we have all sorts of errors in terms of the information, or say, data. Their numerical computation is another serious challenge. For example, in the popular machine learning algorithm, the key idea is to approximate the controls via composition of linear functions and the activation function, then by definition the optimal controls/equilibria provided by these algorithms are (locally) Lipschitz continuous. That is, the game values falling out of our set value are essentially out of reach of these algorithms, see e.g. [37]. Moreover, as a consequence of our constraints, our proof of (1.1) is technically a lot easier than the compactness arguments for (1.2) used in [22,32,33].

Finally we would like to mention some other approaches for MFGs with multiple equilibria. One is to add sufficient (possibly infinite dimensional) noise so that the new game will become non-degenerate and hence have unique MFE, see e.g. Bayraktar-Cecchin-Cohen-Delarue [4,5], Delarue [18], Delarue-Foguen Tchuendom [19], Foguen Tchuendom [28]. Another approach is to study a special type of MFEs, see e.g. Cecchin-Dai Pra-Fisher-Pelino [15], Cecchin-Delarue [16], and [19]. Another interesting work is Possamai-Tangpi [41] which introduces an additional parameter function Λ such that the MFE corresponding to any fixed Λ is unique and then the desired convergence is obtained.

The rest of the paper is organized as follows. In Section 2 we introduce the set value for an MFG in a discrete time model on finite state space and establish the DPP, and in Section 3 we prove the convergence for the corresponding N-player games with homogeneous equilibria. Sections 4 and 5 are devoted to MFGs with relaxed controls and the corresponding N-player games with heterogeneous equilibria. In Section 6 we study a continuous time model with controlled diffusions.

Finally in Appendix we provide some examples, discuss the subtle path dependence issue, and complete some technical proofs.

2. MEAN FIELD GAMES ON FINITE SPACE WITH CLOSED LOOP CONTROLS

In this section we consider an MFG on finite space (both time and state are finite) with closed loop controls, and for simplicity we restrict to state dependent setting. Since the game typically has multiple MFEs which may induce different values, see Example 7.1 for an example, we shall introduce the set value of the game over all MFEs. Our goal is to establish the DPP for the MFG set value, and we shall show in the next section that the set values of the corresponding N-player games converge to the MFG set value.

2.1. **The basic setting.** Let $\mathbb{T} := \{0, \cdots, T\}$ be the set of discrete times; $\mathbb{T}_t := \{t, \cdots, T\}$ for $t \in \mathbb{T}$; $\mathbb{S}$ the finite state space[4] with size $|\mathbb{S}| = d$; $\mathcal{P}(\mathbb{S})$ the set of probability measures on $\mathbb{S}$, equipped with the 1-Wasserstein distance W_1. Since $\mathbb{S}$ is finite, W_1 is equivalent to the total variation distance[5] which is convenient for our purpose: by abusing the notation W_1,

$$(2.1) \qquad W_1(\mu, \nu) := \sum_{x \in \mathbb{S}} |\mu(x) - \nu(x)|, \quad \mu, \nu \in \mathcal{P}(\mathbb{S}).$$

Let $\mathcal{P}_0(\mathbb{S})$ denote the subset of $\mu \in \mathcal{P}(\mathbb{S})$ which has full support, namely $\mu(x) > 0$ for all $x \in \mathbb{S}$. Moreover, let $\mathbb{A} \subset \mathbb{R}^{d_0}$ be a measurable set from which the controls take values; and $q : \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \times \mathbb{A} \times \mathbb{S} \to (0, 1)$ be a transition probability function:

$$\sum_{\tilde{x} \in \mathbb{S}} q(t, x, \mu, a; \tilde{x}) = 1, \quad \forall (t, x, \mu, a) \in \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \times \mathbb{A}.$$

We shall use the weak formulation which is more convenient for closed loop controls. That is, we fix the canonical space and consider controlled probability measures on it. To be precise, let $\Omega := \mathbb{X} := \mathbb{S}^{T+1}$ be the canonical space; $X : \mathbb{T} \times \Omega \to \mathbb{S}$ the canonical process: $X_t(\omega) = \omega_t$; $\mathbb{F} := \{\mathcal{F}_t\}_{t \in \mathbb{T}} := \mathbb{F}^X$ the filtration generated by X; and $\mathcal{A}_{state}$ the set of state dependent admissible controls $\alpha : \mathbb{T} \times \mathbb{S} \to \mathbb{A}$. Introduce the concatenation for controls:

$$(2.2) \qquad (\alpha \oplus_{T_0} \tilde{\alpha})(s, x) := \alpha(s, x)\mathbf{1}_{\{s < T_0\}} + \tilde{\alpha}(s, x)\mathbf{1}_{\{s \geq T_0\}}, \quad \alpha, \tilde{\alpha} \in \mathcal{A}_{state}.$$

It is clear that $\alpha \oplus_{T_0} \tilde{\alpha} \in \mathcal{A}_{state}$. Given $(t, \mu, \alpha) \in \mathbb{T} \times \mathcal{P}(\mathbb{S}) \times \mathcal{A}_{state}$, let $\mathbb{P}^{t,\mu,\alpha}$ denote the probability measure on $\mathcal{F}_T$ determined recursively by: for $s = t, \cdots, T$,

$$(2.3) \quad \begin{aligned} \mathbb{P}^{t,\mu,\alpha} \circ X_t^{-1} = \mu, \quad & \mathbb{P}^{t,\mu,\alpha}(X_{s+1} = \tilde{x} | X_s = x) = q(s, x, \mu_s^\alpha, \alpha(s, x); \tilde{x}); \\ & \text{where} \quad \mu_s^\alpha := \mathbb{P}^{t,\mu,\alpha} \circ X_s^{-1}. \end{aligned}$$

We note that $\mu^\alpha := \{\mu_s^\alpha\}_{s \in \mathbb{T}_t}$ are uniquely determined and X is a Markov chain on $\mathbb{T}_t$ under $\mathbb{P}^{t,\mu,\alpha}$. We also note that μ^α depends on (t, μ) as well, but we omit it for notational simplicity. However, the distribution of $\{X_s\}_{s=0,\cdots,t-1}$ is not specified and is irrelevant, and $\{\alpha_s\}_{0 \leq s < t}$ is also irrelevant. Moreover, given $\{\mu_\cdot\} := \{\mu_s\}_{s \in \mathbb{T}_t}$,

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[4]We may allow the state space $\mathbb{S}_t$ to depend on time t and all the results in this paper will remain true.

[5]More precisely, the total variation distance is $\frac{1}{2} W_1$ for the W_1 in (2.1).

$x \in \mathbb{S}$, and $\tilde{\alpha} \in \mathcal{A}_{state}$, let $\mathbb{P}^{\{\mu_\cdot\};t,x,\tilde{\alpha}}$ denote the probability measure on $\mathcal{F}_T$ determined recursively by: for $s = t, \cdots, T - 1$,

(2.4)
$$\mathbb{P}^{\{\mu_\cdot\};t,x,\tilde{\alpha}}(X_t = x) = 1, \quad \mathbb{P}^{\{\mu_\cdot\};t,x,\tilde{\alpha}}(X_{s+1} = \bar{x}|X_s = \tilde{x}) = q(s, \tilde{x}, \mu_s, \tilde{\alpha}(s, \tilde{x}); \bar{x}).$$

As in the standard MFG literature, here we are assuming that the population uses the common control α while the individual player is allowed to use a different control $\tilde{\alpha}$.

We remark that, since we assume $q > 0$, then for any (t, μ) and α, $\mu_s^\alpha \in \mathcal{P}_0(\mathbb{S})$ for all $s > t$. For the convenience of presentation, in this section we shall restrict our discussion to the case $\mu \in \mathcal{P}_0(\mathbb{S})$. The general case that the initial measure μ is not fully supported can be treated fairly easily, as we will do in Section 6. The situation with degenerate q, however, is more subtle and we shall leave it for future research.

We finally introduce the cost functional for the MFG: for the $\mu^\alpha = \{\mu_\cdot^\alpha\}$ in (2.3),

(2.5)
$$J(t, \mu, \alpha; x, \tilde{\alpha}) := J(\mu^\alpha; t, x, \tilde{\alpha}), \qquad v(\{\mu_\cdot\}; s, x) := \inf_{\tilde{\alpha} \in \mathcal{A}_{state}} J(\{\mu_\cdot\}; s, x, \tilde{\alpha});$$

$$\text{where} \quad J(\{\mu_\cdot\}; s, x, \tilde{\alpha}) := \mathbb{E}^{\mathbb{P}^{\{\mu_\cdot\};s,x,\tilde{\alpha}}}\left[G(X_T, \mu_T) + \sum_{r=s}^{T-1} F(r, X_r, \mu_r, \tilde{\alpha}(r, X_r))\right].$$

Here, since $\mathbb{T}$ and $\mathbb{S}$ are finite, F and G are arbitrary measurable functions satisfying

$$\inf_{a \in \mathbb{A}} F(t, x, \mu, a) > -\infty \quad \text{for all } (t, x, \mu).$$

We remark that here $v(\{\mu_\cdot\}; \cdot, \cdot)$ is the value function of a standard stochastic control problem with parameter $\{\mu_\cdot\}$. In particular, in continuous time models, μ^α and $v(\mu^\alpha; \cdot, \cdot)$ will satisfy the Fokker-Planck equation and the HJB equation, respectively.

Definition 2.1. Given $(t, \mu) \in \mathbb{T} \times \mathcal{P}_0(\mathbb{S})$, we say $\alpha^* \in \mathcal{A}_{state}$ is a state dependent MFE at (t, μ), denoted as $\alpha^* \in \mathcal{M}_{state}(t, \mu)$, if

(2.6)
$$J(t, \mu, \alpha^*; x, \alpha^*) = v(\mu^{\alpha^*}; t, x), \quad \text{for all } x \in \mathbb{S}.$$

In this and the next section, we will use the following conditions.

Assumption 2.2.

 (i) $q \geq c_q$ for some constant $c_q > 0$;
 (ii) q is Lipschitz continuous in (μ, a), with a Lipschitz constant L_q;
 (iii) F, G are bounded by a constant C_0 and uniformly continuous in (μ, a), with a modulus of continuity function ρ.

2.2. **The raw set value** $\mathbb{V}_0$. We introduce the raw set value for the MFG over all state dependent MFEs:

(2.7)
$$\mathbb{V}_0(t, \mu) := \left\{ J(t, \mu, \alpha^*; \cdot, \alpha^*) : \alpha^* \in \mathcal{M}_{state}(t, \mu) \right\} \subset \mathbb{L}^0(\mathbb{S}; \mathbb{R}).$$

Here the elements of $\mathbb{V}_0(t, \mu)$ are functions from $\mathbb{S}$ to $\mathbb{R}$, which coincide with $\mathbb{R}^d$ by identifying $\varphi \in \mathbb{L}^0(\mathbb{S}; \mathbb{R})$ with $(\varphi(x) : x \in \mathbb{S}) \in \mathbb{R}^d$. We call $\mathbb{V}_0(t, \mu)$ the raw set value and we will introduce the set value $\mathbb{V}(t, \mu)$ of the MFG in the next subsection.

Next, for any $T_0 \in \mathbb{T}_t$, $\psi \in \mathbb{L}^0(\mathbb{S} \times \mathcal{P}_0(\mathbb{S}); \mathbb{R})$, we introduce the MFG on $\{t, \cdots, T_0\}$:

$$(2.8) \quad J(T_0, \psi; t, \mu, \alpha; x, \tilde{\alpha}) := \mathbb{E}^{\mathbb{P}^{\mu^\alpha; t, x, \tilde{\alpha}}} \left[\psi(X_{T_0}, \mu_{T_0}^\alpha) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^\alpha, \tilde{\alpha}(s, X_s)) \right].$$

In the obvious sense we define $\alpha^* \in \mathcal{M}_{state}(T_0, \psi; t, \mu)$ by: for any $x \in \mathbb{S}$,

$$(2.9) \quad J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) = v(T, \psi; \mu^{\alpha^*}; t, x) := \inf_{\tilde{\alpha} \in \mathcal{A}_{state}} J(T, \psi; t, \mu, \alpha^*; x, \tilde{\alpha}).$$

At below we will repeatedly use the following simple fact due to the tower property of conditional expectations:

$$(2.10) \quad J(t, \mu, \alpha; x, \tilde{\alpha}) = J(T_0, \psi; t, \mu, \alpha; x, \tilde{\alpha}), \quad \text{where} \quad \psi(y, \nu) := J(T_0, \nu, \alpha; y, \tilde{\alpha}).$$

The following time consistency of MFE is the essence of the DPP for the raw set value.

Proposition 2.3. *Fix $0 \le t < T_0 \le T$ and $\mu \in \mathcal{P}_0(\mathbb{S})$. For any $\alpha^*, \tilde{\alpha}^* \in \mathcal{A}_{state}$, denote $\hat{\alpha}^* := \alpha^* \oplus_{T_0} \tilde{\alpha}^*$ and $\psi(y, \nu) := J(T_0, \nu, \tilde{\alpha}^*; y, \tilde{\alpha}^*)$. Then $\hat{\alpha}^* \in \mathcal{M}_{state}(t, \mu)$ if and only if $\alpha^* \in \mathcal{M}_{state}(T_0, \psi; t, \mu)$ and $\tilde{\alpha}^* \in \mathcal{M}_{state}(T_0, \mu_{T_0}^{\alpha^*})$.*

Proof. (i) We first prove the if part. Let $\alpha^* \in \mathcal{M}_{state}(T_0, \psi; t, \mu)$ and $\tilde{\alpha}^* \in \mathcal{M}_{state}(T_0, \mu_{T_0}^{\alpha^*})$. For arbitrary $\alpha \in \mathcal{A}_{state}$ and $x \in \mathbb{S}$, by (2.10) we have

$$J(t, \mu, \hat{\alpha}^*; x, \alpha) = \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}} \left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \alpha) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right]$$

$$\ge \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}} \left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \tilde{\alpha}^*) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right]$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}} \left[\psi(X_{T_0}, \mu_{T_0}^{\alpha^*}) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right]$$

$$= J(T_0, \psi; t, \mu, \alpha^*; x, \alpha) \ge J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) = J(t, \mu, \hat{\alpha}^*; x, \hat{\alpha}^*),$$

where the first inequality is due to $\tilde{\alpha}^* \in \mathcal{M}_{state}(T_0, \mu_{T_0}^{\alpha^*})$ and the second inequality is due to $\alpha^* \in \mathcal{M}_{state}(T_0, \psi; t, \mu)$. Then $\hat{\alpha}^* \in \mathcal{M}_{state}(t, \mu)$.

(ii) We now prove the only if part. Let $\hat{\alpha}^* \in \mathcal{M}_{state}(t, \mu)$. For any $\alpha \in \mathcal{A}_{state}$, we have $\alpha \oplus_{T_0} \tilde{\alpha}^* \in \mathcal{A}_{state}$. Then, since $\hat{\alpha}^* \in \mathcal{M}_{state}(t, \mu)$, for any $x \in \mathbb{S}$, by (2.10) we have

$$J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) = J(t, \mu, \hat{\alpha}^*; x, \hat{\alpha}^*) \le J(t, \mu, \hat{\alpha}^*; x, \alpha \oplus_{T_0} \tilde{\alpha}^*)$$
$$= J(T, \psi; t, \mu, \alpha^*; x, \alpha).$$

This implies that $\alpha^* \in \mathcal{M}_{state}(T_0, \psi; t, \mu)$.

Moreover, note that $\alpha^* \oplus_{T_0} \alpha \in \mathcal{A}_{state}$ and again since $\hat{\alpha}^* \in \mathcal{M}_{state}(t, \mu)$, we have

$$\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha^*}} \left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \tilde{\alpha}^*) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha^*(s, X_s)) \right]$$

$$= J(t, \mu, \hat{\alpha}^*; x, \hat{\alpha}^*) \le J(t, \mu, \hat{\alpha}^*; x, \alpha^* \oplus_{T_0} \alpha)$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha^*}} \left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \alpha) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha^*(s, X_s)) \right].$$

This implies that, recalling the v in (2.5) and by the standard stochastic control theory,

$$\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*};t,x,\alpha^*}}\left[J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;X_{T_0},\tilde{\alpha}^*)\right] \leq \inf_{\alpha\in\mathcal{A}_{state}}\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*};t,x,\alpha^*}}\left[J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;X_{T_0},\alpha)\right]$$

$$(2.11) \qquad\qquad = \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*};t,x,\alpha^*}}\left[v(\mu^{\hat{\alpha}^*};T_0,X_{T_0})\right].$$

On the other hand, by definition $v(\mu^{\hat{\alpha}^*};T_0,\tilde{x}) \leq J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;\tilde{x},\tilde{\alpha}^*)$ for all $\tilde{x} \in \mathbb{S}$. Then

$$J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;X_{T_0},\tilde{\alpha}^*) = v(\mu^{\hat{\alpha}^*};T_0,X_{T_0}), \quad \mathbb{P}^{\mu^{\alpha^*};t,x,\alpha^*}\text{-a.s.}$$

Since $q > 0$, then clearly $\mathbb{P}^{\mu^{\alpha^*};t,x,\alpha^*}(X_{T_0} = \tilde{x}) > 0$ for all $\tilde{x} \in \mathbb{S}$. Thus $J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;\tilde{x},\tilde{\alpha}^*) = v(\mu^{\hat{\alpha}^*};T_0,\tilde{x})$, for all $\tilde{x} \in \mathbb{S}$. This implies that $\tilde{\alpha}^* \in \mathcal{M}_{state}(T_0,\mu_{T_0}^{\alpha^*})$. $\qquad\square$

We then have the following DPP.

Theorem 2.4. *For any $0 \leq t < T_0 \leq T$, and $\mu \in \mathcal{P}_0(\mathbb{S})$, we have*
(2.12)
$$\mathbb{V}_0(t,\mu) := \Big\{J(T_0,\psi;t,\mu,\alpha^*;\cdot,\alpha^*) : \text{for all } \psi \in \mathbb{L}^0(\mathbb{S}\times\mathcal{P}_0(\mathbb{S});\mathbb{R}) \text{ and } \alpha^* \in \mathcal{A}_{state}$$
$$\text{such that } \psi(\cdot,\mu_{T_0}^{\alpha^*}) \in \mathbb{V}_0(T_0,\mu_{T_0}^{\alpha^*}) \text{ and } \alpha^* \in \mathcal{M}_{state}(T_0,\psi;t,\mu)\Big\}.$$

Proof. Let $\tilde{\mathbb{V}}_0(t,\mu)$ denote the right side of (2.12). First, for any $J(T_0,\psi;t,\mu,\alpha^*;\cdot,\alpha^*) \in \hat{\mathbb{V}}_0(t,\mu)$ with desired ψ,α^* as in (2.12). Since $\psi(\cdot,\mu_{T_0}^{\alpha^*}) \in \mathbb{V}_0(T_0,\mu_{T_0}^{\alpha^*})$, there exists $\tilde{\alpha}^* \in \mathcal{M}_{state}(T_0,\mu_{T_0}^{\alpha^*})$ such that $\psi(\cdot,\mu_{T_0}^{\alpha^*}) = J(T_0,\mu_{T_0}^{\alpha^*},\tilde{\alpha}^*;\cdot,\tilde{\alpha}^*)$. By Proposition 2.3 we have $\hat{\alpha}^* := \alpha^* \oplus_{T_0} \tilde{\alpha}^* \in \mathcal{M}_{state}(t,\mu)$. Then, by (2.10), $J(T_0,\psi;t,\mu,\alpha^*;\cdot,\alpha^*) = J(t,\mu,\hat{\alpha}^*;\cdot,\hat{\alpha}^*) \in \mathbb{V}_0(t,\mu)$, and thus $\tilde{\mathbb{V}}_0(t,\mu) \subset \mathbb{V}_0(t,\mu)$.

On the other hand, let $J(t,\mu,\alpha^*;\cdot,\alpha^*) \in \mathbb{V}_0(t,\mu)$ with $\alpha^* \in \mathcal{M}_{state}(t,\mu)$. Introduce $\psi(x,\nu) := J(T_0,\nu,\alpha^*;x,\alpha^*)$. By Proposition 2.3 again we see that $\alpha^* \in \mathcal{M}_{state}(T_0,\psi;t,\mu)$ and $\alpha^* \in \mathcal{M}_{state}(T_0,\mu_{T_0}^{\alpha^*})$, and the latter implies further that $\psi(\cdot,\mu_{T_0}^{\alpha^*}) \in \mathbb{V}_0(T_0,\mu_{T_0}^{\alpha^*})$. Then by the definition of $\tilde{\mathbb{V}}_0(t,\mu)$ that $J(t,\mu,\alpha^*;\cdot,\alpha^*) = J(T_0,\psi;t,\mu,\alpha^*;\cdot,\alpha^*) \in \tilde{\mathbb{V}}_0(t,\mu)$. That is, $\mathbb{V}_0(t,\mu) \subset \tilde{\mathbb{V}}_0(t,\mu)$. $\qquad\square$

2.3. The set value $\mathbb{V}_{state}$. While Theorem 2.4 is elegant, the raw set value $\mathbb{V}_0(t,\mu)$ is very sensitive to small perturbations of the coefficients F,G and the variable μ. Moreover, in general it does not look possible to have the convergence of the raw set value of the corresponding N-player games to $\mathbb{V}_0(t,\mu)$. Therefore, in this subsection we shall modify $\mathbb{V}_0(t,\mu)$ and introduce the set value $\mathbb{V}_{state}(t,\mu)$ of the MFG as follows.

Definition 2.5.

(i) For any $(t,\mu) \in \mathbb{T} \times \mathcal{P}_0(\mathbb{S})$ and $\varepsilon > 0$, let $\mathcal{M}_{state}^\varepsilon(t,\mu)$ denote the set of $\alpha^* \in \mathcal{A}_{state}$ such that

$$(2.13) \qquad J(t,\mu,\alpha^*;x,\alpha^*) \leq v(\mu^{\alpha^*};t,x) + \varepsilon, \quad \text{for all } x \in \mathbb{S}.$$

(ii) The set value of the MFG at (t,μ) is defined as:

$$(2.14) \qquad\qquad \mathbb{V}_{state}(t,\mu) := \bigcap_{\varepsilon>0} \mathbb{V}_{state}^\varepsilon(t,\mu), \quad \text{where}$$

$$\mathbb{V}^\varepsilon_{state}(t,\mu)$$
$$:= \Big\{ \varphi \in \mathbb{L}^0(\mathbb{S};\mathbb{R}) : \|\varphi - J(t,\mu,\alpha^*;\cdot,\alpha^*)\|_\infty \le \varepsilon \text{ for some } \alpha^* \in \mathcal{M}^\varepsilon_{state}(t,\mu) \Big\}.$$

Recall (2.5), then (2.13) and (2.14) imply that

$$(2.15) \qquad 0 \le J(t,\mu,\alpha^*;x,\alpha^*) - v(\mu^{\alpha^*};t,x) \le \varepsilon, \quad \|\varphi - v(\mu^{\alpha^*};t,\cdot)\|_\infty \le 2\varepsilon.$$

So we may alternatively define $\mathbb{V}^\varepsilon_{state}(t,\mu)$ by using $\|\varphi - v(\mu^{\alpha^*};t,\cdot)\|_\infty \le \varepsilon$.

Remark 2.6.

(i) In the case that there is only one player, namely q,F,G do not depend on μ, $\mathbb{P}^{\mu^{\alpha^*};t,x,\alpha} = \mathbb{P}^{t,x,\alpha}$ does not depend on μ and α^*. Let

$$V(t,x) := \inf_{\alpha \in \mathcal{A}_{state}} \mathbb{E}^{\mathbb{P}^{t,x,\alpha}} \Big[G(X_T) + \sum_{s=t}^{T-1} F(s,X_s,\alpha(s,X_s)) \Big]$$

denote the value function of the standard stochastic control problem. One can easily see that, when there exists an optimal control α^*, $\mathbb{V}_0(t,\mu) = \mathbb{V}_{state}(t,\mu) = \{V(t,\cdot)\}$. However, when there is no optimal control, we still have $\mathbb{V}_{state}(t,\mu) = \{V(t,\cdot)\}$ but $\mathbb{V}_0(t,\mu) = \emptyset$. So the natural extension of the value function V is the set value $\mathbb{V}_{state}$, not $\mathbb{V}_0$.

(ii) We remark that $\bigcap_{\varepsilon>0} \mathcal{M}^\varepsilon_{state}(t,\mu) = \mathcal{M}_{state}(t,\mu)$, however, in general it is possible that $\mathbb{V}_{state}(t,\mu)$ is strictly larger than $\mathbb{V}_0(t,\mu)$. Indeed, $\mathbb{V}_{state}(t,\mu)$ can be even larger than the closure of $\mathbb{V}_0(t,\mu)$, where the latter is still empty when there is no optimal control.

Similarly, given T_0 and ψ, $\mathcal{M}^\varepsilon_{state}(T_0,\psi;t,\mu)$ denotes the set of $\alpha^* \in \mathcal{A}_{state}$ such that

$$(2.16) \qquad J(T_0,\psi;t,\mu,\alpha^*;x,\alpha^*) \le \inf_{\alpha \in \mathcal{A}_{state}} J(T_0,\psi;t,\mu,\alpha^*;x,\alpha) + \varepsilon, \quad \forall x \in \mathbb{S}.$$

The DPP remains true for $\mathbb{V}_{state}$ after appropriate modifications as follows.

Theorem 2.7. *Under Assumption* 2.2(i), *for any* $0 \le t < T_0 \le T$ *and* $\mu \in \mathcal{P}_0(\mathbb{S})$,

$$\mathbb{V}_{state}(t,\mu) := \bigcap_{\varepsilon>0} \Big\{ \varphi \in \mathbb{L}^0(\mathbb{S};\mathbb{R}) : \|\varphi - J(T_0,\psi;t,\mu,\alpha^*;\cdot,\alpha^*)\|_\infty \le \varepsilon$$
$$(2.17) \qquad \text{for some } \psi \in \mathbb{L}^0(\mathbb{S} \times \mathcal{P}_0(\mathbb{S});\mathbb{R}) \text{ and } \alpha^* \in \mathcal{A}_{state} \text{ such that}$$
$$\psi(\cdot,\mu^{\alpha^*}_{T_0}) \in \mathbb{V}^\varepsilon_{state}(T_0,\mu^{\alpha^*}_{T_0}), \alpha^* \in \mathcal{M}^\varepsilon_{state}(T_0,\psi;t,\mu) \Big\}.$$

This theorem can be proved by modifying the arguments in Theorem 2.4 and Proposition 2.3. However, since the proof is very similar to that of Theorem 4.2 below, except that the latter is in the more complicated path dependent setting, we thus postpone it to Appendix.

3. The N-player game with homogeneous equilibria

In this section we study the N-player game whose set value converges to $\mathbb{V}_{state}$.

3.1. The N-player game.

Set $\Omega^N := \mathbb{X}^N$ with canonical processes $\vec{X} = (X^1, \cdots, X^N)$, where X^i stands for the state process of player i. The empirical measure of $\vec{X}$ is denoted as: with the Dirac measure $\delta_{\cdot}$,

$$(3.1) \quad \mu_t^N := \mu_{\vec{X}_t}^N \quad \text{where} \quad \mu_{\vec{x}}^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i} \in \mathcal{P}(\mathbb{S}), \text{ for } \vec{x} = (x_1, \cdots, x_N) \in \mathbb{S}^N.$$

The player i will have control α^i. In the literature, a closed loop control α^i may depend on the full information $\vec{X}$. However, since we are talking about large N, in practice it may not be feasible for each player to observe all other players' states individually. Moreover, in the MFG setting the population state is characterized by its distribution, not by each player's individual state. So in this section we consider only symmetric controls, namely α^i depends on his/her own state X^i and on the others through the empirical measure μ^N.

In order to have the desired convergence, we introduce another parameter $L \geq 0$. Denote
$$(3.2)$$
$$\mathcal{A}_{state}^L := \Big\{ \alpha : \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \to \mathbb{A} : \big|\alpha(t, x, \mu) - \alpha(t, x, \nu)\big| \leq LW_1(\mu, \nu), \forall t, x, \mu, \nu \Big\},$$

and $\mathcal{A}_{state}^\infty := \bigcup_{L \geq 0} \mathcal{A}_{state}^L$. Given $t \in \mathbb{T}$, $\vec{x} \in \mathbb{S}^N$, and $\vec{\alpha} = (\alpha^1, \cdots, \alpha^N) \in (\mathcal{A}_{state}^\infty)^N$, let $\mathbb{P}^{t,\vec{x},\vec{\alpha}}$ denote the probability measure on $\mathcal{F}_T^{\vec{X}}$ determined recursively by: for $s = t, \cdots, T-1$,
$$(3.3)$$
$$\mathbb{P}^{t,\vec{x},\vec{\alpha}}(\vec{X}_t = \vec{x}) = 1, \mathbb{P}^{t,\vec{x},\vec{\alpha}}(\vec{X}_{s+1} = \vec{x}''|\vec{X}_s = \vec{x}') = \prod_{i=1}^N q(s, x_i', \mu_s^N, \alpha^i(s, x_i', \mu_s^N); x_i''),$$

and the cost function of player i is:

$$(3.4) \qquad J_i(t, \vec{x}, \vec{\alpha}) := \mathbb{E}^{\mathbb{P}^{t,\vec{x},\vec{\alpha}}}\Big[G(X_T^i, \mu_T^N) + \sum_{s=t}^{T-1} F(s, X_s^i, \mu_s^N, \alpha^i(s, X_s^i, \mu_s^N)) \Big].$$

Remark 3.1.

(i) It is obvious that $\mathcal{A}_{state}^0 = \mathcal{A}_{state}$ for the $\mathcal{A}_{state}$ in the previous subsection. For the MFG, there is no need to consider $\mathcal{A}_{state}^\infty$. Indeed, given $(t, \mu) \in \mathbb{T} \times \mathcal{P}_0(\mathbb{S})$, for any $\alpha \in \mathcal{A}_{state}^\infty$, let $\mathbb{P}^{t,\mu,\alpha}$ be defined as in (2.3): again denoting $\mu_s^\alpha := \mathbb{P}^{t,\mu,\alpha} \circ X_s^{-1}$,

$$\mathbb{P}^{t,\mu,\alpha} \circ X_t^{-1} = \mu, \quad \mathbb{P}^{t,\mu,\alpha}(X_{s+1} = \tilde{x}|X_s = x) = q(s, x, \mu_s^\alpha, \alpha(s, x, \mu_s^\alpha); \tilde{x}).$$

Introduce $\tilde{\alpha}(s, x) := \alpha(s, x, \mu_s^\alpha)$. Then $\tilde{\alpha} \in \mathcal{A}_{state}$ and one can easily verify that $\mu^{\tilde{\alpha}} = \mu^\alpha$. In particular, the set value $\mathbb{V}_{state}(t, \mu)$ will remain the same by allowing $\alpha \in \mathcal{A}_{state}^\infty$. For the N-player game, however, since μ^N is random, the dependence on μ^N makes the difference.

(ii) In the literature one typically uses $\mu_t^{N,-i} := \frac{1}{N-1}\sum_{j \neq i} \delta_{X_t^j}$, rather than μ_t^N, in (3.3) and (3.4). The convergence results in this section will remain true if we use $\mu^{N,-i}$ instead. However, we find it more convenient to use μ_t^N.

There is another crucial issue concerning the equilibria. Note that an MFE requires by definition that each player takes the same control α^*. To achieve the desired convergence, for the N-player game it is natural to consider only the homogeneous equilibria: $\alpha_1 = \cdots = \alpha_N$, which we will do in the rest of this section. We note that, for a homogeneous control α, the $\mathbb{P}^{t,\vec{x},\alpha} := \mathbb{P}^{t,\vec{x},(\alpha,\cdots,\alpha)}$ in (3.3) and $J_i(t,\vec{x},\alpha) := J_i(t,\vec{x},(\alpha,\cdots,\alpha))$ in (3.4) are also symmetric in $\vec{x}$, or say invariant in terms of its empirical measure:

$$(3.5) \qquad \mathbb{P}^{t,\vec{x},\alpha} = \mathbb{P}^{t,\mu_{\vec{x}}^N,\alpha}, \quad J_i(t,\vec{x},\alpha) = J^N(t,x_i,\mu_{\vec{x}}^N,\alpha).$$

Definition 3.2. For any $\varepsilon > 0, L \geq 0$, we say $\alpha^* \in \mathcal{A}_{state}^L$ is a homogeneous state dependent (ε, L)-equilibrium of the N-player game at $(t,\vec{x})$, denoted as $\alpha^* \in \mathcal{M}_{state}^{N,\varepsilon,L}(t,\vec{x})$, if:
$$(3.6)$$
$$J_i(t,\vec{x},\alpha^*) \leq v_i^{N,L}(t,\vec{x},\alpha^*) := \inf_{\tilde{\alpha}\in\mathcal{A}_{state}^L} J_i(t,\vec{x},(\alpha^*,\tilde{\alpha})_i) + \varepsilon, \quad i = 1,\cdots,N,$$

where $(\alpha,\tilde{\alpha})_i$ denote the vector $\vec{\alpha}$ such that $\alpha^i = \tilde{\alpha}$ and $\alpha^j = \alpha$ for all $j \neq i$.

In light of (3.5), clearly $\mathcal{M}_{state}^{N,\varepsilon,L}(t,\vec{x})$ is law invariant: $\mathcal{M}_{state}^{N,\varepsilon,L}(t,\vec{x}) = \mathcal{M}_{state}^{N,\varepsilon,L}(t,\vec{x}')$ whenever $\mu_{\vec{x}}^N = \mu_{\vec{x}'}^N$. Thus, by abusing the notation, we may denote $\mathcal{M}_{state}^{N,\varepsilon,L}(t,\vec{x}) = \mathcal{M}_{state}^{N,\varepsilon,L}(t,\mu_{\vec{x}}^N)$ and call α^* a homogeneous state dependent (ε,L)-equilibrium at $(t,\mu_{\vec{x}}^N)$.

Note again that $q > 0$, then similar to Subsection 2.1, for convenience in this section we restrict to only those $\vec{x}$ such that $\mu_{\vec{x}}^N$ has full support, and we denote

$$(3.7) \qquad \mathbb{S}_0^N := \left\{ \vec{x} \in \mathbb{S}^N : \mu_{\vec{x}}^N \in \mathcal{P}_0(\mathbb{S}) \right\}, \quad \mathcal{P}_N(\mathbb{S}) := \left\{ \mu_{\vec{x}}^N : \vec{x} \in \mathbb{S}_0^N \right\} \subset \mathcal{P}_0(\mathbb{S}).$$

We now define the set value of the homogeneous N-player game: recalling (3.5),
$$(3.8)$$
$$\mathbb{V}_{state}^N(t,\mu) := \bigcap_{\varepsilon>0} \mathbb{V}_{state}^{N,\varepsilon}(t,\mu) := \bigcap_{\varepsilon>0} \bigcup_{L\geq 0} \mathbb{V}_{state}^{N,\varepsilon,L}(t,\mu), \forall (t,\mu) \in \mathbb{T} \times \mathcal{P}_N(\mathbb{S}), \quad \text{where}$$

$$\mathbb{V}_{state}^{N,\varepsilon,L}(t,\mu) := \left\{ \varphi \in \mathbb{L}^0(\mathbb{S};\mathbb{R}) : \exists \alpha^* \in \mathcal{M}_{state}^{N,\varepsilon,L}(t,\mu) \text{ s.t. } \|\varphi - J^N(t,\cdot,\mu,\alpha^*)\|_\infty \leq \varepsilon \right\}.$$

Remark 3.3. Note that we require $\tilde{\alpha} \in \mathcal{A}_{state}^L$ in (3.6) for the same L, so $\bigcup_{L\geq 0} \mathbb{V}_{state}^{N,\varepsilon,L}(t,\mu)$ at above is in general different from $\mathbb{V}_{state}^{N,\varepsilon,\infty}(t,\mu)$, which is defined in an obvious way by requiring $\alpha^*, \tilde{\alpha} \in \mathcal{A}_{state}^\infty$ in (3.6). See also Remark 3.8(ii).

3.2. Convergence of the empirical measures.

Theorem 3.4. *Let Assumption 2.2(ii) hold. Then, for any $L \geq 0$, there exists a constant C_L, which depends only on T, d, L_q, and L such that, for any $t \in \mathbb{T}$, $\vec{x} \in \mathbb{S}_0^N$, $\mu \in \mathcal{P}_0(\mathbb{S})$, $\alpha, \tilde{\alpha} \in \mathcal{A}_{state}^L$, and $s \geq t$, $i = 1,\cdots,N$,*

$$(3.9) \qquad \mathbb{E}^{\mathbb{P}^{t,\vec{x},(\alpha,\tilde{\alpha})_i}}\left[\mathcal{W}_1(\mu_s^N, \mu_s^\alpha)\right] \leq C_L \theta_N, \quad \text{where} \quad \theta_N := W_1(\mu_{\vec{x}}^N, \mu) + \frac{1}{\sqrt{N}};$$

$$(3.10) \qquad \mathcal{W}_1\left(\mathbb{P}^{t,\vec{x},(\alpha,\tilde{\alpha})_i} \circ (X_s^i)^{-1}, \mathbb{P}^{\mu^\alpha;t,x_i,\tilde{\alpha}} \circ X_s^{-1}\right) \leq C_L \theta_N.$$

Proof. We first recall Remark 3.1 and extend all the notations in Subsection 2.1 to those $\alpha \in \mathcal{A}_{state}^L$ in the obvious sense. Fix t, i and denote $\mathbb{P}^N := \mathbb{P}^{t,\vec{x},(\alpha,\tilde{\alpha})_i}$.

Step 1. We first prove (3.9) for $s = t+1$. Note that $X_{t+1}^1, \cdots, X_{t+1}^N$ are independent under $\mathbb{P}^N$. By (2.1), we have

$$\mathbb{E}^{\mathbb{P}^N}\big[W_1(\mu_{t+1}^N, \mu_{t+1}^\alpha)\big] = \sum_{\tilde{x} \in \mathbb{S}} \mathbb{E}^{\mathbb{P}^N}\big[|\mu_{t+1}^N(\tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})|\big]$$

$$\leq \sum_{\tilde{x} \in \mathbb{S}} \Big(\mathbb{E}^{\mathbb{P}^N}\big[|\mu_{t+1}^N(\tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})|^2\big]\Big)^{\frac{1}{2}}$$

$$(3.11) \quad = \sum_{\tilde{x} \in \mathbb{S}} \Big[Var^{\mathbb{P}^N}\big[\mu_{t+1}^N(\tilde{x})\big] + \big(\mathbb{E}^{\mathbb{P}^N}\big[\mu_{t+1}^N(\tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})\big]\big)^2\Big]^{\frac{1}{2}}$$

$$= \sum_{\tilde{x} \in \mathbb{S}} \Big[\frac{1}{N^2} \sum_{j=1}^N Var^{\mathbb{P}^N}\big[\mathbf{1}_{\{X_{t+1}^j = \tilde{x}\}}\big] + \big(\frac{1}{N} \sum_{j=1}^N \mathbb{P}^N(X_{t+1}^j = \tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})\big)^2\Big]^{\frac{1}{2}}$$

$$\leq \frac{C}{\sqrt{N}} + \sum_{\tilde{x} \in \mathbb{S}} \big|\frac{1}{N} \sum_{j=1}^N \mathbb{P}^N(X_{t+1}^j = \tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})\big|.$$

Note that, by the desired Lipschitz continuity of q in μ and that $|\mathbb{S}| = d$ is finite,

$$\big|\frac{1}{N} \sum_{j=1}^N \mathbb{P}^N(X_{t+1}^j = \tilde{x}) - \mu_{t+1}^\alpha(\tilde{x})\big|$$

$$= \Big|\frac{1}{N} \sum_{x \in \mathbb{S}} \Big[\sum_{j \neq i} q(t, x, \mu_{\vec{x}}^N, \alpha(t, x, \mu_{\vec{x}}^N); \tilde{x})\mathbf{1}_{\{x_j = x\}} + q(t, x, \mu_{\vec{x}}^N, \tilde{\alpha}(t, x, \mu_{\vec{x}}^N); \tilde{x})\mathbf{1}_{\{x_i = x\}}\Big]$$

$$- \sum_{x \in \mathbb{S}} q(t, x, \mu, \alpha(t, x, \mu); \tilde{x})\mu(x)\Big|$$

$$\leq \Big|\frac{1}{N} \sum_{x \in \mathbb{S}} \sum_{j=1}^N q(t, x, \mu_{\vec{x}}^N, \alpha(t, x, \mu_{\vec{x}}^N); \tilde{x})\mathbf{1}_{\{x_j = x\}} - \sum_{x \in \mathbb{S}} q(t, x, \mu, \alpha(t, x, \mu); \tilde{x})\mu(x)\Big|$$

$$+ \frac{1}{N} \sum_{x \in \mathbb{S}} \big|q(t, x, \mu_{\vec{x}}^N, \alpha(t, x, \mu_{\vec{x}}^N); \tilde{x}) - q(t, x, \mu_{\vec{x}}^N, \tilde{\alpha}(t, x, \mu_{\vec{x}}^N); \tilde{x})\big|\mathbf{1}_{\{x_i = x\}}$$

$$\leq \big|\sum_{x \in \mathbb{S}} q(t, x, \mu_{\vec{x}}^N, \alpha(t, x, \mu_{\vec{x}}^N); \tilde{x})\mu_{\vec{x}}^N(x) - \sum_{x \in \mathbb{S}} q(t, x, \mu, \alpha(t, x, \mu); \tilde{x})\mu(x)\big| + \frac{1}{N}$$

$$\leq \sum_{x \in \mathbb{S}} \Big[|\mu_{\vec{x}}^N(x) - \mu(x)| + C_L W_1(\mu_{\vec{x}}^N, \mu)\mu(x)\Big] + \frac{1}{N} \leq C_L \theta_N.$$

Then, $\mathbb{E}^{\mathbb{P}^N}\big[\mathcal{W}_1(\mu_{t+1}^N, \mu_{t+1}^\alpha)\big] \leq \frac{C}{\sqrt{N}} + C_L \theta_N \leq C_L \theta_N.$

Step 2. We next prove (3.9) by induction. For any $s = t, \cdots, T-1$, by Step 1 we have

$$\mathbb{E}^{\mathbb{P}^N}\big[\mathcal{W}_1(\mu_{s+1}^N, \mu_{s+1}^\alpha)\big|\mathcal{F}_s^{\vec{X}}\big] \leq C_L\Big[W_1(\mu_s^N, \mu_s^\alpha) + \frac{1}{\sqrt{N}}\Big], \quad \mathbb{P}^N\text{-a.s.}$$

Then

$$\mathbb{E}^{\mathbb{P}^N}\big[\mathcal{W}_1(\mu_{s+1}^N, \mu_{s+1}^\alpha)\big] = \mathbb{E}^{\mathbb{P}^N}\Big[\mathbb{E}^{\mathbb{P}^N}\big[\mathcal{W}_1(\mu_{s+1}^N, \mu_{s+1}^\alpha)\big|\vec{X}_s^N\big]\Big]$$

$$\leq C_L \mathbb{E}^{\mathbb{P}^N}\big[\mathcal{W}_1(\mu_s^N, \mu_s^\alpha)\big] + \frac{C_L}{\sqrt{N}}.$$

Since T is finite, by induction we obtain (3.9) immediately.

Step 3. We now prove (3.10). Denote

$$\kappa_s := W_1\Big(\mathbb{P}^N \circ (X_s^i)^{-1}, \mathbb{P}^i \circ X_s^{-1}\Big) \quad \text{where} \quad \mathbb{P}^i := \mathbb{P}^{\mu^\alpha; t, x_i, \tilde{\alpha}}.$$

Then $\kappa_t = 0$, and for $s = t, \cdots, T - 1$,

$$
\begin{aligned}
\kappa_{s+1} &= \sum_{\tilde{x} \in \mathbb{S}} \big| \mathbb{P}^N(X_{s+1}^i = \tilde{x}) - \mathbb{P}^i(X_{s+1} = \tilde{x}) \big| \\
&= \sum_{\tilde{x} \in \mathbb{S}} \Big| \mathbb{E}^{\mathbb{P}^N}\big[q(s, X_s^i, \mu_s^N, \tilde{\alpha}(s, X_s^i, \mu_s^N); \tilde{x}) \big] - \mathbb{E}^{\mathbb{P}^i}\big[q(s, X_s, \mu_s^\alpha, \tilde{\alpha}(s, X_s, \mu_s^\alpha); \tilde{x}) \big] \Big| \\
&\leq \sum_{\tilde{x} \in \mathbb{S}} \Big| \mathbb{E}^{\mathbb{P}^N}\big[q(s, X_s^i, \mu_s^N, \tilde{\alpha}(s, X_s^i, \mu_s^N); \tilde{x}) \big] - \mathbb{E}^{\mathbb{P}^N}\big[q(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha); \tilde{x}) \big] \Big| \\
&\quad + \sum_{\tilde{x} \in \mathbb{S}} \Big| \mathbb{E}^{\mathbb{P}^N}\big[q(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha); \tilde{x}) \big] - \mathbb{E}^{\mathbb{P}^i}\big[q(s, X_s, \mu_s^\alpha, \tilde{\alpha}(s, X_s, \mu_s^\alpha); \tilde{x}) \big] \Big| \\
&\leq C_L \mathbb{E}^{\mathbb{P}^N}\big[W_1(\mu_s^N, \mu_s^\alpha) \big] \\
&\quad + \sum_{x, \tilde{x} \in \mathbb{S}} q(s, x, \mu_s^\alpha, \tilde{\alpha}(s, x, \mu_s^\alpha); \tilde{x}) \big| \mathbb{P}^N(X_s^i = x) - \mathbb{P}^i(X_s = x) \big| \\
&\leq C_L \theta_N + \kappa_s,
\end{aligned}
$$

where the last inequality thanks to (3.9). Now by induction one can easily prove (3.10). $\qquad \square$

3.3. Convergence of the set values.

We first study the convergence of the cost functions. Recall the θ_N in (3.9) and the functions v in (2.5) and $v_i^{N,L}$ in (3.6).

Theorem 3.5. *Let Assumption* 2.2(ii) *and* (iii) *hold. For any* $L \geq 0$, *there exists a modulus of continuity function* ρ_L, *which depends only on* T, d, L_q, C_0, ρ, *and* L *such that, for any* $t \in \mathbb{T}$, $\mu_{\vec{x}}^N \in \mathcal{P}_N(\mathbb{S})$, $\mu \in \mathcal{P}_0(\mathbb{S})$, *and any* $\alpha, \tilde{\alpha} \in \mathcal{A}_{state}^L$, $i = 1, \cdots, N$,

$$(3.12) \quad \big| J_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha}) \big| + \big| v_i^{N,L}(t, \vec{x}, \alpha) - v(\mu^\alpha; t, x_i) \big| \leq \rho_L(\theta_N).$$

Proof. Clearly the uniform estimates for J implies that for v, so we shall only prove the former one. Recall (3.4), (2.5), and the notations $\mathbb{P}^N$, $\mathbb{P}^i$ in the proof of Theorem 3.4. Then

$$
\Big| J_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha}) \Big| \leq I_T + \sum_{s=t}^{T-1} I_s, \quad \text{where}
$$

$$I_T := \Big| \mathbb{E}^{\mathbb{P}^N}\big[G(X_T^i, \mu_T^N) \big] - \mathbb{E}^{\mathbb{P}^i}\big[G(X_T, \mu_T^\alpha) \big] \Big|;$$

$$I_s := \Big| \mathbb{E}^{\mathbb{P}^N}\big[F(s, X_s^i, \mu_s^N, \tilde{\alpha}(s, X_s^i, \mu_s^N)) \big] - \mathbb{E}^{\mathbb{P}^i}\big[F(s, X_s, \mu_s^\alpha, \tilde{\alpha}(s, X_s, \mu_s^\alpha)) \big] \Big|, \quad s < T.$$

Note that, for $s < T$, by (3.10),

$$
\begin{aligned}
I_s &\leq \left| \mathbb{E}^{\mathbb{P}^N} \left[F(s, X_s^i, \mu_s^N, \tilde{\alpha}(s, X_s^i, \mu_s^N)) \right] - \mathbb{E}^{\mathbb{P}^N} \left[F(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha)) \right] \right| \\
&\quad + \left| \mathbb{E}^{\mathbb{P}^N} \left[F(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha)) \right] - \mathbb{E}^{\mathbb{P}^i} \left[F(s, X_s, \mu_s^\alpha, \tilde{\alpha}(s, X_s, \mu_s^\alpha)) \right] \right| \\
&\leq \mathbb{E}^{\mathbb{P}^N} \left[\rho \big(C_L W_1(\mu_s^N, \mu_s^\alpha) \big) \right] \\
&\quad + \sum_{x \in \mathbb{S}} \left| F(s, x, \mu_s^\alpha, \tilde{\alpha}(s, x, \mu_s^\alpha)) \right| \left| \mathbb{P}^N(X_s^i = x) - \mathbb{P}^i(X_s = x) \right| \\
&\leq \mathbb{E}^{\mathbb{P}^N} \left[\rho \big(C_L W_1(\mu_s^N, \mu_s^\alpha) \big) \right] + C_L \theta_N.
\end{aligned}
$$

Similarly we have the estimate for I_T, and thus

$$
\left| J_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha}) \right| \leq \sum_{s=t}^{T} \mathbb{E}^{\mathbb{P}^N} \left[\rho \big(C_L W_1(\mu_s^N, \mu_s^\alpha) \big) \right] + C_L \theta_N.
$$

This, together with (3.9), implies (3.12) for some appropriately defined modulus of continuity function ρ_L. $\qquad\square$

Our main result of this section is the following convergence of the set values. Recall, for a sequence of sets $\{E_N\}_{N \geq 1}$, $\overline{\lim}_{N \to \infty} E_N := \bigcap_{n \geq 1} \bigcup_{N \geq n} E_N$, $\underline{\lim}_{N \to \infty} E_N := \bigcup_{n \geq 1} \bigcap_{N \geq n} E_N$.

Theorem 3.6. *Let Assumption 2.2(ii), (iii) hold and $\mu_{\vec{x}}^N \in \mathcal{P}_N(\mathbb{S}) \to \mu \in \mathcal{P}_0(\mathbb{S})$. Then*

$$
(3.13) \qquad \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \overline{\lim_{N \to \infty}} \, \mathbb{V}_{state}^{N, \varepsilon, L}(t, \mu_{\vec{x}}^N) \subset \mathbb{V}_{state}(t, \mu) \subset \bigcap_{\varepsilon > 0} \underline{\lim_{N \to \infty}} \, \mathbb{V}_{state}^{N, \varepsilon, 0}(t, \mu_{\vec{x}}^N).
$$

In particular, since $\underline{\lim}_{N \to \infty} \mathbb{V}_{state}^{N, \varepsilon, 0}(t, \mu_{\vec{x}}^N) \subset \bigcup_{L \geq 0} \overline{\lim}_{N \to \infty} \mathbb{V}_{state}^{N, \varepsilon, L}(t, \mu_{\vec{x}}^N)$, actually equalities hold.

Note that $\vec{x} \in \mathbb{S}_0^N$ obviously depends on N, so more rigorously we should write $\vec{x}^N$ in the above statements. For notational simplicity we omit this N here. We also remark that at above we are not able to switch the order of $\overline{\lim}_{N \to \infty}$ and $\bigcap_{\varepsilon > 0} \bigcup_{L \geq 0}$ in the left side, or the order of $\underline{\lim}_{N \to \infty}$ and $\bigcap_{\varepsilon > 0}$ in the right side.

Proof. (i) We first prove the right inclusion in (3.13). Fix $\varphi \in \mathbb{V}_{state}(t, \mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{2}$. Note that $\mathcal{A}_{state} = \mathcal{A}_{state}^0$. By (2.14), there exists $\alpha^* \in \mathcal{M}_{state}^{\varepsilon_1}(t, \mu)$ such that $\|\varphi - J(t, \mu, \alpha^*; \cdot, \alpha^*)\|_\infty \leq \varepsilon_1$. Recall (2.13), we have

$$
J(t, \mu, \alpha^*; x, \alpha^*) \leq v(\mu^{\alpha^*}; t, x) + \varepsilon_1, \quad \text{for all } x \in \mathbb{S}.
$$

For any $\alpha \in \mathcal{A}_{state}^0 = \mathcal{A}_{state}$, by Theorem 3.5 we have

$$J_i(t, \vec{x}, \alpha^*) \leq J(t, \mu, \alpha^*; x_i, \alpha^*) + \rho_0(\theta_N)$$
$$\leq v(\mu^{\alpha^*}; t, x) + \varepsilon_1 + \rho_0(\theta_N) \leq v_i^{N,L}(t, \vec{x}, \alpha^*) + \varepsilon_1 + 2\rho_0(\theta_N).$$

Choose N large enough such that $\rho_0(\theta_N) \leq \frac{\varepsilon}{4}$, then $J_i(t, \vec{x}, \alpha^*) \leq v_i^{N,L}(t, \vec{x}, \alpha^*) + \varepsilon$. This implies that $\alpha^* \in \mathcal{M}_{\varepsilon,0}^N(t, \mu_{\vec{x}}^N)$. Moreover,

$$\|\varphi - J^N(t, \cdot, \mu_{\vec{x}}^N, \alpha^*)\|_\infty \leq \varepsilon_1 + \sup_i \left| J_i(t, \vec{x}, \alpha^*) - J(t, \mu, \alpha^*; x_i, \alpha^*) \right|$$
$$\leq \varepsilon_1 + \rho_0(\theta_N) \leq \varepsilon_1 + \frac{\varepsilon}{4} \leq \varepsilon.$$

Then $\varphi \in \mathbb{V}_{state}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N)$ for all N large enough. That is, $\varphi \in \underline{\lim}_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N)$. Since $\varphi \in \mathbb{V}_{state}(t, \mu)$ and $\varepsilon > 0$ are arbitrary, we obtain the right inclusion in (3.13).

(ii) We next show the left inclusion in (3.13). Fix $\varphi \in \bigcap_{\varepsilon>0} \bigcup_{L\geq0} \overline{\lim}_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon,L}(t, \mu_{\vec{x}}^N)$ and $\varepsilon > 0$. Then, for $\varepsilon_1 := \frac{\varepsilon}{2} > 0$, there exist $L_\varepsilon > 0$ and an infinite sequence $\{N_k\}_{k\geq1}$ such that $\varphi \in \mathbb{V}_{state}^{N_k,\varepsilon_1,L_\varepsilon}(t, \mu_{\vec{x}}^{N_k})$ for all $k \geq 1$. Recall (3.8), for each $k \geq 1$ there exists $\alpha^k \in \mathcal{M}_{state}^{N_k,\varepsilon_1,L_\varepsilon}(t, \mu_{\vec{x}}^{N_k})$ such that $\|\varphi - J^N(t, \cdot, \mu_{\vec{x}}^{N_k}, \alpha^k)\|_\infty \leq \varepsilon_1$. By Definition 3.2, we have $J_i(t, \vec{x}, \alpha^k) \leq v_i^{N_k,L_\varepsilon}(t, \vec{x}, \alpha^k) + \varepsilon_1$. Similar to (i), by Theorem 3.5 we have

$$J(t, \mu, \alpha^k; x_i, \alpha^k) \leq v(\mu^{\alpha^k}; t, x_i) + \varepsilon_1 + 2\rho_{L_\varepsilon}(\theta_{N_k}) \leq v(\mu^{\alpha^k}; t, x_i) + \varepsilon,$$

for k large enough. That is, $\alpha^k \in \mathcal{M}_{state}^\varepsilon(t, \mu)$. Similar to (i) again, for k large enough we have $\|\varphi - J(t, \mu, \alpha^k; \cdot, \alpha^k)\|_\infty \leq \varepsilon$. Then $\varphi \in \mathbb{V}_{state}^\varepsilon(t, \mu)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \mathbb{V}_{state}(t, \mu)$, and hence derive the left inclusion in (3.13). $\qquad\square$

Remark 3.7.

(i) From Theorem 3.6(i) we see that, for any $\alpha^* \in \mathcal{M}_{state}^{\frac{\varepsilon}{2}}(t, \mu)$, we have $\alpha^* \in \mathcal{M}_{state}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N)$ when N is large enough. Moreover, by (3.9) we have the desired estimate for the approximate equilibrium measure $\mathbb{E}^{\mathbb{P}^{t,\vec{x},\alpha^*}}\left[W_1(\mu_s^N, \mu_s^{\alpha^*})\right] \leq C_L\theta_N$. This verifies the standard result in the literature that an approximate MFE is an approximate equilibrium of the N-player game.

(ii) From Theorem 3.6(ii) we see that, for any $\alpha^k \in \mathcal{M}_{state}^{N_k,\frac{\varepsilon}{2},L_\varepsilon}(t, \mu_{\vec{x}}^{N_k})$, we have $\alpha^k \in \mathcal{M}_{state}^\varepsilon(t, \mu)$ when k is large enough, and we again have the estimate for the approximate equilibrium measure $\mathbb{E}^{\mathbb{P}^{t,\vec{x},\alpha^k}}\left[W_1(\mu_s^{N_k}, \mu_s^{\alpha^k})\right] \leq C_L\theta_{N_k}$. This is in the spirit that any limit point of the N-player equilibrium measures is an MFE measure.

Remark 3.8.

(i) We should point out that the key to obtain the convergence here is to consider homogeneous equilibria for the N-player games. If we use heterogeneous equilibria for the N-player games, it turns out that we will have the desired convergence when we consider relaxed controls for the MFG, as we will do in the next two sections.

(ii) Another feature of our convergence result is the uniform Lipschitz continuity requirement on the admissible controls. Indeed, the left inclusion in (3.13) would fail in general if we replace $\bigcap_{\varepsilon>0} \bigcup_{L\geq0} \overline{\lim}_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon,L}(t, \mu_{\vec{x}}^N)$ with $\bigcap_{\varepsilon>0} \overline{\lim}_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon,\infty}(t, \mu_{\vec{x}}^N)$

or with $\bigcap_{\varepsilon>0} \overline{\lim_{N\to\infty}} \mathbb{V}^{N,\varepsilon}_{state}(t, \mu^N_{\bar{x}})$, where $\mathbb{V}^{N,\varepsilon,\infty}_{state}$ is defined in Remark 3.3 and $\mathbb{V}^{N,\varepsilon}_{state}$ is defined similarly, by requiring $\alpha^*, \tilde{\alpha} : \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \to \mathbb{A}$ in (3.6) to be measurable only. See Example 7.2. We refer to [22, 32, 33] for some related convergence analysis without such regularity requirement.

(iii) We note that the above regularity requirement on the admissible controls is also crucial for numerical computations of set values, as well as for practical implementation of the equilibria, although these issues are not studied in the present paper.

4. MEAN FIELD GAMES ON FINITE SPACE WITH RELAXED CONTROLS

In this section we study MFG with relaxed controls, or say mixed strategies. Besides its independent interest, our main motivation is to characterize the limit of N-player games with heterogeneous equilibria. We shall still consider the finite space in Section 2, however, for the purpose of generality in this section we consider path dependent setting.

4.1. The relaxed set value with path dependent controls.
We start with some notations for the path dependent setting. For $\mathbf{x} = (\mathbf{x}_t)_{0\le t\le T} \in \mathbb{X}$, denote by $\mathbf{x}_{t\wedge.} = (\mathbf{x}_0, \cdots, \mathbf{x}_t, \mathbf{x}_t, \cdots, \mathbf{x}_t)$ the path stopping at t and $\mathbb{X}_t := \{\mathbf{x}_{t\wedge.} : \mathbf{x} \in \mathbb{X}\} \subset \mathbb{X}$. For $\mathbf{x}, \tilde{\mathbf{x}} \in \mathbb{X}$, we say $\mathbf{x} =_t \tilde{\mathbf{x}}$ if $\mathbf{x}_{t\wedge.} = \tilde{\mathbf{x}}_{t\wedge.}$. Denote $\mathbb{X}^{t,\mathbf{x}} := \{\tilde{\mathbf{x}} \in \mathbb{X} : \tilde{\mathbf{x}} =_t \mathbf{x}\}$ and $\mathbb{X}^{t,\mathbf{x}}_s := \mathbb{X}^{t,\mathbf{x}} \cap \mathbb{X}_s$, for $s \ge t$. Introduce the concatenation $\mathbf{x} \oplus_t \tilde{\mathbf{x}} \in \mathbb{X}$ by

$$(\mathbf{x} \oplus_t \tilde{\mathbf{x}})_s := \mathbf{x}_s \mathbf{1}_{\{s\le t\}} + \tilde{\mathbf{x}}_s \mathbf{1}_{\{s>t\}}, \quad \text{and} \quad (\mathbf{x} \oplus_t x)_s := \mathbf{x}_s \mathbf{1}_{\{s\le t\}} + x \mathbf{1}_{\{s>t\}}, x \in \mathbb{S}.$$

For each $t \in \mathbb{T}$, let $\mathcal{P}(\mathbb{X}_t)$ denote the set of probability measures on $(\Omega, \mathcal{F}^X_t)$, equipped with

$$W_1(\mu, \nu) := \sum_{\mathbf{x}\in\mathbb{X}_t} |\mu(\mathbf{x}) - \nu(\mathbf{x})|, \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{X}_t),$$

and $\mathcal{P}_0(\mathbb{X}_t)$ the subset of $\mu \in \mathcal{P}(\mathbb{X}_t)$ with full support $\mathbb{X}_t$. Again this is just for convenience of presentation. For a measure $\mu \in \mathcal{P}(\mathbb{X}) = \mathcal{P}(\mathbb{X}_T)$, denote $\mu_{t\wedge.} := \mu \circ X^{-1}_{t\wedge.} \in \mathcal{P}(\mathbb{X}_t)$. We remark that, by abusing the notation μ, here $\mu_{t\wedge.}$ denote the joint law of the stopped process $X_{t\wedge.}$, while in Section 2 $\{\mu.\}$ denote the family of marginal laws.

For a path dependent function φ on $\mathbb{T} \times \mathbb{X} \times \mathcal{P}(\mathbb{X})$, we say φ is adapted if $\varphi(t, \mathbf{x}, \mu) = \varphi(t, \mathbf{x}_{t\wedge.}, \mu_{t\wedge.})$. Throughout this section, all the path dependent functions are required to be adapted. In particular, the data of the game $q : \mathbb{T} \times \mathbb{X} \times \mathcal{P}(\mathbb{X}) \times \mathbb{A} \times \mathbb{S} \to (0, 1)$, $F : \mathbb{T} \times \mathbb{X} \times \mathcal{P}(\mathbb{X}) \times \mathbb{A} \to \mathbb{R}$, and $G : \mathbb{X} \times \mathcal{P}(\mathbb{X}) \to \mathbb{R}$ are path dependent with q, F adapted. By adapting to the path dependent setting, we shall still assume Assumption 2.2.

Let $\mathcal{A}_{relax}$ denote the set of path dependent adapted relaxed controls $\gamma : \mathbb{T} \times \mathbb{X} \to \mathcal{P}(\mathbb{A})$. Given $t \in \mathbb{T}$, $\mu \in \mathcal{P}(\mathbb{X}_t)$, $\gamma \in \mathcal{A}_{relax}$, and $\mathbf{x} \in \mathbb{X}_t$, $\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}$, $\tilde{\gamma} \in \mathcal{A}_{relax}$, we

introduce:
(4.1)
$$\mathbb{P}^{t,\mu,\gamma} \circ X_{t\wedge.}^{-1} = \mu, \quad \mathbb{P}^{t,\mu,\gamma}(X_{s+1} = \tilde{x}|X =_s \mathbf{x}) = \int_{\mathbb{A}} q(s, \mathbf{x}, \mu^\gamma, a; \tilde{x})\gamma(s, \mathbf{x}; da);$$

$$\text{where} \quad \mu_{s\wedge.}^\gamma := \mathbb{P}^{t,\mu,\gamma} \circ X_{s\wedge.}^{-1}, \quad s \geq t;$$

$$\mathbb{P}^{\mu^\gamma;t,\mathbf{x},\tilde{\gamma}}(X =_t \mathbf{x}) = 1, \quad \mathbb{P}^{\mu^\gamma;t,\mathbf{x},\tilde{\gamma}}(X_{s+1} =_s \bar{x}|X =_s \tilde{\mathbf{x}}) = \int_{\mathbb{A}} q(s, \tilde{\mathbf{x}}, \mu^\gamma, a; \bar{x})\tilde{\gamma}(s, \tilde{\mathbf{x}}; da);$$

$$J(\mu^\gamma; s, \tilde{\mathbf{x}}, \tilde{\gamma}) := \mathbb{E}^{\mathbb{P}^{\mu^\gamma;t,\mathbf{x},\tilde{\gamma}}}\Big[G(X, \mu^\gamma) + \sum_{r=s}^{T-1} \int_{\mathbb{A}} F(r, X, \mu^\gamma, a)\tilde{\gamma}(r, X, da)\Big|X =_s \tilde{\mathbf{x}}\Big];$$

$$J(t, \mu, \gamma; \mathbf{x}, \tilde{\gamma}) := J(\mu^\gamma; t, \mathbf{x}, \tilde{\gamma}), \quad v(\mu^\gamma; s, \tilde{\mathbf{x}}) := \inf_{\tilde{\gamma}\in\mathcal{A}_{relax}} J(\mu^\gamma; s, \tilde{\mathbf{x}}, \tilde{\gamma}).$$

Definition 4.1.

(i) For any $t \in \mathbb{T}$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, and $\varepsilon > 0$, let $\mathcal{M}_{relax}^\varepsilon(t, \mu)$ denote the set of relaxed ε-MFE $\gamma^* \in \mathcal{A}_{relax}$ such that

$$(4.2) \qquad J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) \leq v(\mu^{\gamma^*}; t, \mathbf{x}) + \varepsilon, \quad \text{for all } \mathbf{x} \in \mathbb{X}_t.$$

(ii) The relaxed set value of the MFG at (t, μ) is defined as:

$$(4.3) \qquad \mathbb{V}_{relax}(t, \mu) := \bigcap_{\varepsilon > 0} \mathbb{V}_{relax}^\varepsilon(t, \mu), \quad \text{where } \|\varphi\|_{\mathbb{X}_t} := \sup_{\mathbf{x}\in\mathbb{X}_t} |\varphi(\mathbf{x})|, \text{ and}$$

$$\mathbb{V}_{relax}^\varepsilon(t, \mu)$$
$$:= \Big\{\varphi \in \mathbb{L}^0(\mathbb{X}_t; \mathbb{R}) : \exists \gamma^* \in \mathcal{M}_{relax}^\varepsilon(t, \mu) \text{ s.t. } \|\varphi - J(t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon\Big\}.$$

Similarly, given T_0 and $\psi : \mathbb{X}_{T_0} \times \mathcal{P}(\mathbb{X}_{T_0}) \to \mathbb{R}$, as in (2.8) define
(4.4)
$$J(T_0, \psi; t, \mu, \gamma; \mathbf{x}, \tilde{\gamma}) := \mathbb{E}^{\mathbb{P}^{\mu^\gamma;t,\mathbf{x},\tilde{\gamma}}}\Big[\psi(X_{T_0\wedge.}, \mu_{T_0\wedge.}^\gamma) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^\gamma, a)\tilde{\gamma}(s, X, da)\Big],$$

and let $\mathcal{M}_{relax}^\varepsilon(T_0, \psi; t, \mu)$ denote the set of $\gamma^* \in \mathcal{A}_{relax}$ such that, $\forall \mathbf{x} \in \mathbb{X}_t$,
(4.5)
$$J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma^*) \leq v(T, \psi; \mu^\gamma; s, \mathbf{x}) := \inf_{\gamma\in\mathcal{A}_{relax}} J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma) + \varepsilon.$$

Note that the tower property in (2.10) remains true for relaxed controls:

$$(4.6) \quad J(t, \mu, \gamma; \mathbf{x}, \tilde{\gamma}) = J(T_0, \psi; t, \mu, \gamma; \mathbf{x}, \tilde{\gamma}), \quad \text{where} \quad \psi(\mathbf{y}, \nu) := J(T_0, \nu, \gamma; \mathbf{y}, \tilde{\gamma}).$$

The DPP for $\mathbb{V}_{relax}$ takes the following form.

Theorem 4.2. *Under Assumption* 2.2(i), *for any* $t \in \mathbb{T}$, $T_0 \in \mathbb{T}_t$, *and* $\mu \in \mathcal{P}_0(\mathbb{X}_t)$,

$$\mathbb{V}_{relax}(t, \mu) = \bigcap_{\varepsilon > 0} \Big\{\varphi \in \mathbb{L}^0(\mathbb{X}_t; \mathbb{R}) : \|\varphi - J(T_0, \psi; t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon$$

$$(4.7) \qquad \text{for some } \psi \in \mathbb{L}^0(\mathbb{X}_{T_0} \times \mathcal{P}_0(\mathbb{X}_{T_0}); \mathbb{R}) \text{ and } \gamma^* \in \mathcal{A}_{relax} \text{ such that}$$

$$\psi(\cdot, \mu_{T_0\wedge.}^{\gamma^*}) \in \mathbb{V}_{relax}^\varepsilon(T_0, \mu_{T_0\wedge.}^{\gamma^*}), \gamma^* \in \mathcal{M}_{relax}^\varepsilon(T_0, \psi; t, \mu)\Big\}.$$

Proof. We shall follow the arguments in Theorem 2.4, in particular, we shall extend Proposition 2.3. Let $\tilde{\mathbb{V}}_{relax}(t, \mu) = \bigcap_{\varepsilon>0} \tilde{\mathbb{V}}_{relax}^\varepsilon(t, \mu)$ denote the right side of (4.2).

(i) We first prove $\tilde{\mathbb{V}}_{relax}(t,\mu) \subset \mathbb{V}_{relax}(t,\mu)$. Fix $\varphi \in \tilde{\mathbb{V}}_{relax}(t,\mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{4}$. Since $\varphi \in \tilde{\mathbb{V}}^{\varepsilon_1}_{relax}(t,\mu)$, then

$$\|\varphi - J(T_0, \psi; t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon_1 \quad \text{for some desirable } \psi, \gamma^* \text{ as in (4.2).}$$

Since $\psi(\cdot, \mu^{\gamma^*}_{T_0 \wedge \cdot}) \in \mathbb{V}^{\varepsilon_1}_{relax}(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot})$, there exists $\tilde{\gamma}^* \in \mathcal{M}^{\varepsilon_1}_{relax}(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot})$ such that

$$\|\psi(\cdot, \mu^{\gamma^*}_{T_0 \wedge \cdot}) - J(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}, \tilde{\gamma}^*; \cdot, \tilde{\gamma}^*)\|_{\mathbb{X}_{T_0}} \leq \varepsilon_1.$$

As in (2.2) denote $\hat{\gamma}^* := \gamma^* \oplus_{T_0} \tilde{\gamma}^* := \gamma^* \mathbf{1}_{\{s < T_0\}} + \tilde{\gamma}^* \mathbf{1}_{\{s \geq T_0\}} \in \mathcal{A}_{relax}$. Then, for any $\mathbf{x} \in \mathbb{X}_t$ and $\gamma \in \mathcal{A}_{relax}$, similarly to Proposition 2.3(i) we have

$$J(t, \mu, \hat{\gamma}^*; \mathbf{x}, \gamma)$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma} \left[J(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}, \tilde{\gamma}^*; X_{T_0 \wedge \cdot}, \gamma) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^{\gamma^*}, a) \gamma(s, X, da) \right]$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma} \left[J(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}, \tilde{\gamma}^*; X_{T_0 \wedge \cdot}, \tilde{\gamma}^*) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^{\gamma^*}, a) \gamma(s, X, da) \right] - \varepsilon_1$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma} \left[\psi(X_{T_0 \wedge \cdot}, \mu^{\gamma^*}_{T_0 \wedge \cdot}) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^{\gamma^*}, a) \gamma(s, X, da) \right] - 2\varepsilon_1$$

$$= J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma) - 2\varepsilon_1 \geq J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma^*) - 3\varepsilon_1$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma^*} \left[\psi(X_{T_0 \wedge \cdot}, \mu^{\gamma^*}_{T_0 \wedge \cdot}) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^{\gamma^*}, a) \gamma^*(s, X, da) \right] - 3\varepsilon_1$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma^*} \left[J(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}, \tilde{\gamma}^*; X_{T_0 \wedge \cdot}, \tilde{\gamma}^*) + \sum_{s=t}^{T_0-1} \int_{\mathbb{A}} F(s, X, \mu^{\gamma^*}, a) \gamma^*(s, X, da) \right] - 4\varepsilon_1$$

$$= J(t, \mu, \hat{\gamma}^*; \mathbf{x}, \hat{\gamma}^*) - 4\varepsilon_1 = J(t, \mu, \hat{\gamma}^*; \mathbf{x}, \hat{\gamma}^*) - \varepsilon.$$

That is, $\hat{\gamma}^* \in \mathcal{M}^{\varepsilon}_{relax}(t, \mu)$. Moreover, note that, by (4.6),

$$\|\varphi - J(t, \mu, \hat{\gamma}^*; \cdot, \hat{\gamma}^*)\|_{\mathbb{X}_t} \leq \varepsilon_1 + \|J(T_0, \psi; t, \mu, \gamma^*; \cdot, \gamma^*) - J(t, \mu, \hat{\gamma}^*; \cdot, \hat{\gamma}^*)\|_{\mathbb{X}_t}$$

$$= \varepsilon_1 + \sup_{\mathbf{x} \in \mathbb{X}_t} \left| \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*}}; t, \mathbf{x}, \gamma^*} \left[\psi(X_{T_0 \wedge \cdot}, \mu^{\gamma^*}_{T_0 \wedge \cdot}) - J(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}, \tilde{\gamma}^*; X_{T_0 \wedge \cdot}, \tilde{\gamma}^*) \right] \right| \leq 2\varepsilon_1 < \varepsilon.$$

Then $\varphi \in \mathbb{V}^{\varepsilon}_{relax}(t, \mu)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \mathbb{V}_{relax}(t, \mu)$.

(ii) We now prove the opposite inclusion. Fix $\varphi \in \mathbb{V}_{relax}(t, \mu)$ and $\varepsilon > 0$. Let $\varepsilon_2 > 0$ be a small number which will be specified later. Since $\varphi \in \mathbb{V}^{\varepsilon_2}_{relax}(t, \mu)$, then

$$\|\varphi - J(t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon_2 \quad \text{for some } \gamma^* \in \mathcal{M}^{\varepsilon_2}_{relax}(t, \mu).$$

Introduce $\psi(\mathbf{y}, \nu) := J(T_0, \nu, \gamma^*; \mathbf{y}, \gamma^*)$ and recall (4.6). Then

$$\|\varphi - J(T_0, \psi; t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} = \|\varphi(\mathbf{x}) - J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon_2.$$

Moreover, since $\gamma^* \in \mathcal{M}^{\varepsilon_2}_{relax}(t, \mu)$, for any $\gamma \in \mathcal{A}_{relax}$ and $\mathbf{x} \in \mathbb{X}_t$, we have

$$J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma^*) = J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*)$$

$$\leq J(t, \mu, \gamma^*; \mathbf{x}, \gamma \oplus_{T_0} \gamma^*) + \varepsilon_2 = J(T_0, \psi; t, \mu, \gamma^*; \mathbf{x}, \gamma) + \varepsilon_2.$$

This implies that $\gamma^* \in \mathcal{M}^{\varepsilon_2}_{relax}(T_0, \psi; t, \mu)$. We claim further that

$$(4.8) \qquad \psi(\cdot, \mu^{\gamma^*}_{T_0 \wedge \cdot}) \in \mathbb{V}^{C\varepsilon_2}_{relax}(T_0, \mu^{\gamma^*}_{T_0 \wedge \cdot}),$$

for some constant $C \geq 1$. Then by (4.2) we see that $\varphi \in \tilde{\mathbb{V}}_{relax}^{C\varepsilon_2}(t,\mu) \subset \tilde{\mathbb{V}}_{relax}^{\varepsilon}(t,\mu)$ by setting $\varepsilon_2 \leq \frac{\varepsilon}{C}$. Since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \tilde{\mathbb{V}}_{relax}(t,\mu)$.

To see (4.8), recalling (4.1), for any $\gamma \in \mathcal{A}_{relax}$ we have

$$\mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}}\left[J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; X_{T_0\wedge\cdot}, \gamma^*)\right] - \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}}\left[J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; X_{T_0\wedge\cdot}, \gamma)\right]$$
$$= J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) - J(t, \mu, \gamma^*; \mathbf{x}, \gamma^* \oplus_{T_0} \gamma) \leq \varepsilon_2.$$

Then, by taking infimum over $\gamma \in \mathcal{A}_{relax}$, it follows from the standard control theory that

$$\mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};\mathbf{x},\gamma^*}}\left[J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; X_{T_0\wedge\cdot}, \gamma^*)\right] \leq \mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}}\left[v(\mu^{\gamma^*}; T_0, X_{T_0\wedge\cdot})\right] + \varepsilon_2,$$
$$\forall \mathbf{x} \in \mathbb{X}_t.$$

On the other hand, it is obvious that $v(\mu^{\gamma^*}; T_0, \tilde{\mathbf{x}}) \leq J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; \tilde{\mathbf{x}}, \gamma^*)$ for all $\tilde{\mathbf{x}} \in \mathbb{X}_{T_0}$. Moreover, since $q \geq c_q$, clearly $\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}(X =_{T_0} \tilde{\mathbf{x}}) \geq c_q^{T_0-t}$, for any $\tilde{\mathbf{x}} \in \mathbb{X}_{T_0}^{t,\mathbf{x}}$. Thus,

$$0 \leq J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; \tilde{\mathbf{x}}, \gamma^*) - v(\mu^{\gamma^*}; T_0, \tilde{\mathbf{x}})$$
$$\leq C\mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}}\left[\left[J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; X_{T_0\wedge\cdot}, \gamma^*) - v(\mu^{\gamma^*}; T_0, X_{T_0\wedge\cdot})\right]\mathbf{1}_{\{X=_{T_0}\tilde{\mathbf{x}}\}}\right]$$
$$\leq C\mathbb{E}^{\mathbb{P}^{\mu^{\gamma^*};t,\mathbf{x},\gamma^*}}\left[J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; X_{T_0\wedge\cdot}, \gamma^*) - v(\mu^{\gamma^*}; T_0, X_{T_0\wedge\cdot})\right] \leq C\varepsilon_2,$$

where $C := c_q^{t-T_0}$. This implies that $\gamma^* \in \mathcal{M}_{relax}^{C\varepsilon_2}(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*})$. Then (4.8) follows directly from $\psi(\cdot, \mu_{T_0\wedge\cdot}^{\gamma^*}) = J(T_0, \mu_{T_0\wedge\cdot}^{\gamma^*}, \gamma^*; \cdot, \gamma^*)$, and hence $\varphi \in \tilde{\mathbb{V}}_{relax}(t,\mu)$. $\square$

Remark 4.3. Consider the setting that q, F, G are state dependent, as in Section 2. There is a very subtle issue between state dependence and path dependence of the controls.

(i) For a standard non-zero sum game problems where the players may have different cost functions F_i, G_i, if one uses state dependent controls, in general the set value does not satisfy DPP. See a counterexample in [24]. However, with path dependent controls the set value of the game satisfies the DPP.

(ii) In Section 2, since all players have the same cost function, as we saw the set value with state dependent controls satisfies DPP. If we consider path dependent controls $\alpha \in \mathcal{A}_{path}$, the set value will also satisfy DPP. However, the set values in these two settings are in general not equal, see Example 7.1 in Appendix for a counterexample.

(iii) For relaxed controls, again restricting to state dependent q, F, G, it turns out that state dependent and path dependent controls lead to the same set value, see Theorem 7.6 in Appendix. The main reason is that the convex combination of relaxed controls remains a relaxed control, while the controls α in Section 2 does not share this property.

4.2. An alternative formulation of the relaxed mean field game.

In this subsection we provide an alternative formulation for the MFG with relaxed controls. This new formulation is motivated from the heterogeneous controls for the N-player games, and thus is crucial for the convergence result in the next section.

Let $\mathcal{A}_{path}$ denote the set of adapted path dependent controls $\alpha : \mathbb{T} \times \mathbb{X} \to \mathbb{A}$, and for each $t \in \mathbb{T}$, $\mathcal{A}_{path}^t = \{(\alpha(t, \cdot), \cdots, \alpha(T-1, \cdot)) : \alpha \in \mathcal{A}_{path}\}$. Denote $\Xi_t := \mathcal{P}(\mathbb{X}_t \times \mathcal{A}_{path}^t)$, and for each $\Lambda \in \Xi_t$, define recursively: for $s \geq t$, $\mathbf{x} \in \mathbb{X}_t$, and $\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}$,
(4.9)

$$\mu_{t\wedge\cdot}^\Lambda(\mathbf{x}) := \Lambda(\mathbf{x}, \mathcal{A}_{path}^t), \quad \mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}}) := \int_{\mathcal{A}_{path}^t} \prod_{r=t}^{s-1} q(r, \tilde{\mathbf{x}}, \mu^\Lambda, \alpha(r, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}_{r+1}) \Lambda(\mathbf{x}, d\alpha).$$

Here, noting that $\alpha \in \mathcal{A}_{path}^t$ can be equivalently expressed as $\{\alpha(s, \tilde{\mathbf{x}}) : t \leq s \leq T-1, \tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}\}$, we are using the following interpretation on $d\alpha$: for any $\varphi : \mathcal{A}_{path}^t \to \mathbb{R}$,

$$(4.10) \qquad \int_{\mathcal{A}_{path}^t} \varphi(\alpha) d\alpha := \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} \varphi(\{\alpha(s, \tilde{\mathbf{x}})\}) \prod_{s=t}^{T-1} \prod_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} d\alpha(s, \tilde{\mathbf{x}}).$$

Next, for $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, denote $\Xi_t(\mu) := \{\Lambda \in \Xi_t : \mu_{t\wedge\cdot}^\Lambda = \mu\}$. Moreover, recall (4.1),
(4.11)

$$J(t, \Lambda; \mathbf{x}, \alpha) := J(\mu^\Lambda; t, \mathbf{x}, \alpha), \quad v(t, \Lambda; \mathbf{x}) := v(\mu^\Lambda; t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{X}_t, \alpha \in \mathcal{A}_{path}^t.$$

To simplify the notations, we introduce:

$$(4.12) \qquad Q_s^t(\{\mu.\}; \tilde{\mathbf{x}}, \alpha) := \prod_{r=t}^{s-1} q(r, \tilde{\mathbf{x}}, \mu, \alpha(r, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}_{r+1}).$$

In particular, $Q_t^t(\{\mu.\}; \mathbf{x}, \alpha) = 1$. Then we have, for any $\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}$,

$$(4.13) \quad \mu_s^\Lambda(\tilde{\mathbf{x}}) := \int_{\mathcal{A}_{path}^t} Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}, \alpha) \Lambda(\mathbf{x}, d\alpha), \quad \mathbb{P}^{\mu^\Lambda; t, \mathbf{x}, \alpha}(X =_s \tilde{\mathbf{x}}) = Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}, \alpha).$$

Definition 4.4. For any $t \in \mathbb{T}$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, and $\varepsilon > 0$, we call $\Lambda^* \in \Xi_t(\mu)$ a global ε-MFE at (t, μ), denoted as $\Lambda^* \in \mathcal{M}_{global}^\varepsilon(t, \mu)$, if

$$(4.14) \qquad \int_{\mathcal{A}_{path}^t} [J(t, \Lambda^*; \mathbf{x}, \alpha) - v(t, \Lambda^*; \mathbf{x})] \Lambda^*(\mathbf{x}, d\alpha) \leq \varepsilon, \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

Note that the above α is global in time, so we call Λ^* a global equilibrium. Moreover, since there are infinitely many $\alpha \in \mathcal{A}_{path}^t$, it is hard to require $J(t, \Lambda^*; \mathbf{x}, \alpha) - v(t, \Lambda^*; \mathbf{x}) \leq \varepsilon$ for each $\alpha \in \mathcal{A}_{path}^t$, we thus use the above $\mathbb{L}^1$-type of optimality condition. For the $\mathbf{x}$ part, however, since there are only finitely many $\mathbf{x}$ and each of them has positive probability, we may require the optimality for each $\mathbf{x}$.

The main result of this subsection is the following equivalence result.

Theorem 4.5. *For any $t \in \mathbb{T}$ and $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, we have*
(4.15)

$$\mathbb{V}_{relax}(t, \mu) = \mathbb{V}_{global}(t, \mu) := \bigcap_{\varepsilon > 0} \mathbb{V}_{global}^\varepsilon(t, \mu), \quad where$$

$$\mathbb{V}_{global}^\varepsilon(t, \mu) := \left\{ \varphi \in \mathbb{L}^0(\mathbb{X}_t, \mathbb{R}) : \exists \Lambda^* \in \mathcal{M}_{global}^\varepsilon(t, \mu) \text{ s.t. } \|\varphi - v(t, \Lambda^*; \cdot)\|_{\mathbb{X}_t} \leq \varepsilon \right\}.$$

We shall prove the mutual inclusion of the two sides separately. First, given (t, Λ), we construct a relaxed control as follows: for any $t \in \mathbb{T}$, $\mathbf{x} \in \mathbb{X}_t$, and $s \geq t$, $\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}$,

$$(4.16) \qquad \gamma^\Lambda(s, \tilde{\mathbf{x}}, da) := \frac{1}{\mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \delta_{\alpha(s, \tilde{\mathbf{x}})}(da) \Lambda(\mathbf{x}, d\alpha).$$

On the opposite direction, given $t \in \mathbb{T}$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, $\gamma \in \mathcal{A}_{relax}$, recalling (4.10) we construct

$$(4.17) \qquad \Lambda^\gamma(\mathbf{x}, d\alpha) := \mu(\mathbf{x}) \prod_{s=t}^{T-1} \prod_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} \gamma(s, \tilde{\mathbf{x}}, d\alpha(s, \tilde{\mathbf{x}})), \quad \forall \mathbf{x} \in \mathbb{X}_t, \alpha \in \mathcal{A}_{path}^t.$$

In particular, the following calculation implies $\Lambda^\gamma \in \Xi_t(\mu)$:

$$\Lambda^\gamma(\mathbf{x}, \mathcal{A}_{path}^t) = \mu(\mathbf{x}) \prod_{s=t}^{T-1} \prod_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} \gamma(s, \tilde{\mathbf{x}}, \mathbb{A}) = \mu(\mathbf{x}) \prod_{s=t}^{T-1} \prod_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} 1 = \mu(\mathbf{x}).$$

Lemma 4.6. *For any* $t \in \mathbb{T}$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, *and* $\Lambda \in \Xi_t(\mu)$, $\gamma \in \mathcal{A}_{relax}$, *we have* $\mu^{\gamma^\Lambda} = \mu^\Lambda$ *and* $\mu^{\Lambda^\gamma} = \mu^\gamma$. *Moreover,*

$$(4.18) \qquad J(t, \mu, \gamma^\Lambda; \mathbf{x}, \gamma^\Lambda) = \frac{1}{\mu(\mathbf{x})} \int_{\mathcal{A}_{path}^t} J(t, \Lambda; \mathbf{x}, \alpha) \Lambda(\mathbf{x}, d\alpha), \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

Proof. We first prove $\mu_{s\wedge\cdot}^{\gamma^\Lambda} = \mu_{s\wedge\cdot}^\Lambda$ by induction. The case $s = t$ follows from the definitions. Assume it holds for all $r \leq s$. For $s + 1$ and $\tilde{\mathbf{x}} \in \mathbb{X}_{s+1}^{t,\mathbf{x}}$, by Fubini Theorem we have

$$\frac{\mu_{(s+1)\wedge\cdot}^{\gamma^\Lambda}(\tilde{\mathbf{x}})}{\mu_{s\wedge\cdot}^{\gamma^\Lambda}(\tilde{\mathbf{x}}_{s\wedge\cdot})} = \int_{\mathbb{A}} q(s, \tilde{\mathbf{x}}, \mu^{\gamma^\Lambda}, a; \tilde{\mathbf{x}}_{s+1}) \gamma^\Lambda(s, \tilde{\mathbf{x}}, da)$$

$$= \int_{\mathbb{A}} q(s, \tilde{\mathbf{x}}, \mu^{\gamma^\Lambda}, a; \tilde{\mathbf{x}}_{s+1}) \frac{1}{\mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \delta_{\alpha(s,\tilde{\mathbf{x}})}(da) \Lambda(\mathbf{x}, d\alpha)$$

$$= \frac{1}{\mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} q(s, \tilde{\mathbf{x}}, \mu^\Lambda, \alpha(s, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}_{s+1}) Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \Lambda(\mathbf{x}, d\alpha)$$

$$= \frac{1}{\mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} Q_{s+1}^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \Lambda(\mathbf{x}, d\alpha) = \frac{\mu_{(s+1)\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})}{\mu_{s\wedge\cdot}^\Lambda(\tilde{\mathbf{x}})}.$$

Then $\mu_{(s+1)\wedge\cdot}^{\gamma^\Lambda} = \mu_{(s+1)\wedge\cdot}^\Lambda$, and we complete the induction argument.

We next prove $\mu_{s\wedge\cdot}^{\Lambda^\gamma} = \mu_{s\wedge\cdot}^\gamma$ by induction. Again the case $s = t$ is obvious. Assume it holds for all $r < s$. Now for s, recalling (4.10) we have

$$\mu_{s\wedge\cdot}^{\Lambda^\gamma}(\tilde{\mathbf{x}}) = \int_{\mathcal{A}_{path}^t} \Big[\prod_{r=t}^{s-1} q(r, \tilde{\mathbf{x}}, \mu^\gamma, \alpha(r, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}_{r+1}) \Big] \Big[\mu(\mathbf{x}) \prod_{r=t}^{T-1} \prod_{\bar{\mathbf{x}} \in \mathbb{X}_r^{t,\mathbf{x}}} \gamma(r, \bar{\mathbf{x}}, d\alpha(r, \bar{\mathbf{x}})) \Big]$$

$$= \mu(\mathbf{x}) \Big[\prod_{r=t}^{s-1} \int_{\mathbb{A}} q(r, \tilde{\mathbf{x}}, \mu^\gamma, \alpha(r, \tilde{\mathbf{x}}); \tilde{\mathbf{x}}_{r+1}) \gamma(r, \tilde{\mathbf{x}}, d\alpha(r, \bar{\mathbf{x}})) \Big]$$

$$\times \Big[\prod_{r=t}^{s-1} \prod_{\bar{\mathbf{x}} \in \mathbb{X}_r^{t,\mathbf{x}} \setminus \{\tilde{\mathbf{x}}\}} \gamma(r, \bar{\mathbf{x}}, \mathbb{A}) \Big] \times \Big[\prod_{r=s}^{T-1} \prod_{\bar{\mathbf{x}} \in \mathbb{X}_r^{t,\mathbf{x}}} \gamma(r, \bar{\mathbf{x}}, \mathbb{A}) \Big]$$

$$= \mu(\mathbf{x}) \prod_{r=t}^{s-1} \int_{\mathbb{A}} q(r, \tilde{\mathbf{x}}, \mu^\gamma, a; \tilde{\mathbf{x}}_{r+1}) \gamma(r, \tilde{\mathbf{x}}, da) = \mu_{s\wedge\cdot}^\gamma(\tilde{\mathbf{x}}).$$

We finally prove (4.18). For each $s \geq t$ and $\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}$, by Fubini Theorem again we have

$$\int_{\mathbb{A}} F(s, \tilde{\mathbf{x}}, \mu^\Lambda, a) \gamma^\Lambda(s, \tilde{\mathbf{x}}, da)$$

$$= \int_{\mathbb{A}} \frac{F(s, \tilde{\mathbf{x}}, \mu^\Lambda, a)}{\mu_{s \wedge \cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \delta_{\alpha(s,\tilde{\mathbf{x}})}(da) \Lambda(\mathbf{x}, d\alpha)$$

$$= \frac{1}{\mu_{s \wedge \cdot}^\Lambda(\tilde{\mathbf{x}})} \int_{\mathcal{A}_{path}^t} F(s, \tilde{\mathbf{x}}, \mu^\Lambda, \alpha(s,\tilde{\mathbf{x}})) Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \Lambda(\mathbf{x}, d\alpha).$$

By (4.1) we have $\mathbb{P}^{\mu^\Lambda; t, \mathbf{x}, \gamma^\Lambda}(X =_s \tilde{\mathbf{x}}) = \frac{\mu_{s \wedge \cdot}^\Lambda(\tilde{\mathbf{x}})}{\mu(\mathbf{x})}$. Thus

$$J(t, \mu, \gamma^\Lambda; \mathbf{x}, \gamma^\Lambda)$$

$$= \frac{1}{\mu(\mathbf{x})} \Big[\sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}} G(\tilde{\mathbf{x}}, \mu^\Lambda) \mu_{T \wedge \cdot}^\Lambda(\tilde{\mathbf{x}}) + \sum_{s=t}^{T-1} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} \mu_{s \wedge \cdot}^\Lambda(\tilde{\mathbf{x}}) \int_{\mathbb{A}} F(s, \tilde{\mathbf{x}}, \mu^\Lambda, a) \gamma^\Lambda(s, \tilde{\mathbf{x}}, da) \Big]$$

$$= \frac{1}{\mu(\mathbf{x})} \int_{\mathcal{A}_{path}^t} \Big[\sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}} G(\tilde{\mathbf{x}}, \mu^\Lambda) Q_T^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha)$$

$$+ \sum_{s=t}^{T-1} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}_s^{t,\mathbf{x}}} F(s, \tilde{\mathbf{x}}, \mu^\Lambda, \alpha(s,\tilde{\mathbf{x}})) Q_s^t(\mu^\Lambda; \tilde{\mathbf{x}}; \alpha) \Big] \Lambda(\mathbf{x}, d\alpha).$$

This implies (4.18) immediately. $\qquad \square$

Remark 4.7. We can actually show that $\gamma^{(\Lambda^\gamma)} = \gamma$ for all $\gamma \in \mathcal{A}_{relax}$, see Appendix. However, it is not clear that we would have $\Lambda^{(\gamma^\Lambda)} = \Lambda$ for all $\Lambda \in \Xi_t(\mu)$.

Proof of Theorem 4.5. Since $\mu \in \mathcal{P}_0(\mathbb{X}_t)$ has full support, then $c_\mu := \inf_{\mathbf{x} \in \mathbb{X}_t} \mu(\mathbf{x}) > 0$.

(i) We first prove $\mathbb{V}_{global}(t, \mu) \subset \mathbb{V}_{relax}(t, \mu)$. Fix $\varphi \in \mathbb{V}_{global}(t, \mu)$ and $\varepsilon > 0$. Let $\varepsilon_1 > 0$ be a small number which will be specified later. Since $\varphi \in \mathbb{V}_{global}^{\varepsilon_1}(t, \mu)$, there exists $\Lambda^* \in \mathcal{M}_{global}^{\varepsilon_1}(t, \mu)$ such that $\|\varphi - v(t, \Lambda^*; \cdot)\|_{\mathbb{X}_t} \leq \varepsilon_1$. Set $\gamma^* := \gamma^{\Lambda^*}$. For any $\mathbf{x} \in \mathbb{X}_t$, since $\mu^{\gamma^*} = \mu^{\Lambda^*}$, by (4.1), (4.11) we have $v(\mu^{\gamma^*}; t, \mathbf{x}, \gamma^*) = v(t, \Lambda^*; \mathbf{x})$, and, by (4.18), (4.14),

$$J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) - v(t, \Lambda^*; \mathbf{x})$$

$$= \frac{1}{\mu(\mathbf{x})} \int_{\mathcal{A}_{path}^t} [J(t, \Lambda^*; \mathbf{x}, \alpha) - v(t, \Lambda^*; \mathbf{x})] \Lambda^*(\mathbf{x}, d\alpha) \leq \frac{\varepsilon_1}{c_\mu} \leq \varepsilon,$$

provided $\varepsilon_1 > 0$ is small enough. This implies $\gamma^* \in \mathcal{M}_{relax}^\varepsilon(t, \mu)$.

Moreover, it is clear now that, for any $\mathbf{x} \in \mathbb{X}_t$ and for a possibly smaller ε_1,

$$\big| \varphi(\mathbf{x}) - J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) \big| \leq \varepsilon_1 + \big| v(t, \Lambda^*; \mathbf{x}) - J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) \big| \leq \varepsilon_1 + \frac{\varepsilon_1}{c_\mu} \leq \varepsilon.$$

Then $\varphi \in \mathbb{V}_{relax}^\varepsilon(t, \mu)$, and since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \mathbb{V}_{relax}(t, \mu)$.

(ii) We next prove $\mathbb{V}_{relax}(t, \mu) \subset \mathbb{V}_{global}(t, \mu)$. Fix $\varphi \in \mathbb{V}_{relax}(t, \mu)$, $\varepsilon > 0$, and set $\varepsilon_2 := \frac{\varepsilon}{2}$. Since $\varphi \in \mathbb{V}_{relax}^{\varepsilon_2}(t, \mu)$, there exists $\gamma^* \in \mathcal{M}_{relax}^{\varepsilon_2}(t, \mu)$ such that $\|\varphi - J(t, \mu, \gamma^*; \cdot, \gamma^*)\|_{\mathbb{X}_t} \leq \varepsilon_2$. Set $\Lambda^* := \Lambda^{\gamma^*}$, then $\mu^{\Lambda^*} = \mu^{\gamma^*}$. Since $\gamma^* \in \mathcal{M}_{relax}^{\varepsilon_2}(t, \mu)$, we have

$$|\varphi(\mathbf{x}) - v(t, \Lambda^*; \mathbf{x})| = |\varphi(\mathbf{x}) - v(\mu^{\gamma^*}; t, \mathbf{x})| \leq 2\varepsilon_2 \leq \varepsilon, \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

Moreover, note that, by (4.18) again,

$$(4.19) \quad \int_{\mathcal{A}_{path}^t} [J(t, \Lambda^*; \mathbf{x}, \alpha) - v(t, \Lambda^*; \mathbf{x})] \Lambda^*(\mathbf{x}, d\alpha)$$
$$= \mu(\mathbf{x})[J(t, \mu, \gamma^*; \mathbf{x}, \gamma^*) - v(t, \Lambda^*; \mathbf{x})] \leq \mu(\mathbf{x})\varepsilon_2 \leq \varepsilon_2 \leq \varepsilon.$$

This implies $\varphi \in \mathbb{V}_{global}^\varepsilon(t, \mu)$, and hence by the arbitrariness of ε, $\varphi \in \mathbb{V}_{global}(t, \mu)$.
□

5. The N-player game with heterogeneous equilibria

In this section we drop the requirement $\alpha^1 = \cdots = \alpha^N$ for the N-player game, and show that the corresponding set value converges to $\mathbb{V}_{relax}$, which in general is strictly larger than $\mathbb{V}_{state}$. We note that we shall still use the pure strategies, rather than mixed strategies, for the N-player game. Moreover, since we used path dependent controls in Section 4, we shall also use path dependent controls here.

5.1. The N-player game.
Let Ω^N and $\vec{X}$ be as in Section 3, and denote
$$(5.1)$$
$$\mu_{t\wedge}^N := \mu_{t,\vec{X}_{t\wedge}}^N, \quad \text{where} \quad \mu_{t,\vec{x}}^N := \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{x}^i} \in \mathcal{P}(\mathbb{X}_t), \vec{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^N) \in \mathbb{X}_t^N.$$

Similarly to (3.7), for the convenience of the presentation we introduce
$$(5.2) \quad \mathbb{X}_{0,t}^N := \left\{ \vec{x} \in \mathbb{X}_t^N : \text{supp}(\mu_{t,\vec{x}}^N) = \mathbb{X}_t \right\}, \quad \mathcal{P}_N(\mathbb{X}_t) := \left\{ \mu_{t,\vec{x}}^N : \vec{x} \in \mathbb{X}_{0,t}^N \right\}.$$

We shall consider path dependent symmetric controls: $\mathcal{A}_{path}^{t,\infty} := \bigcup_{L \geq 0} \mathcal{A}_{path}^{t,L}$, where

$$\mathcal{A}_{path}^{t,L} := \Big\{ \alpha : \{t, \cdots, T-1\} \times \mathbb{X} \times \mathcal{P}(\mathbb{X}) \to \mathbb{A} \Big| \alpha \text{ is adapted and}$$

uniformly Lipschitz continuous in μ (under W_1) with Lipschitz constant $L \Big\}$.

Given $t \in \mathbb{T}$, $\vec{x} \in \mathbb{X}_{0,t}^N$, and $\vec{\alpha} = (\alpha^1, \cdots, \alpha^N) \in (\mathcal{A}_{path}^{t,\infty})^N$, introduce, for $s \geq t$,

$$\mathbb{P}^{t,\vec{x},\vec{\alpha}}(\vec{X} =_t \vec{x}) = 1, \mathbb{P}^{t,\vec{x},\vec{\alpha}}(\vec{X}_{s+1} = \vec{x}'' | \vec{X} =_s \vec{x}') = \prod_{i=1}^N q(s, \mathbf{x}'^i, \mu^N, \alpha^i(s, \mathbf{x}'^i, \mu^N); x_i''),$$

$$(5.3) \quad J_i(t, \vec{x}, \vec{\alpha}) := \mathbb{E}^{\mathbb{P}^{t,\vec{x},\vec{\alpha}}}\Big[G(X^i, \mu^N) + \sum_{s=t}^{T-1} F(s, X^i, \mu^N, \alpha^i(s, X^i, \mu^N)) \Big];$$

$$v_i^{N,L}(t, \vec{x}, \vec{\alpha}) := \inf_{\tilde{\alpha} \in \mathcal{A}_{path}^{t,L}} J_i(t, \vec{x}, \vec{\alpha}^{-i}, \tilde{\alpha}), \quad i = 1, \cdots, N.$$

Here $(\vec{\alpha}^{-i}, \tilde{\alpha})$ is the vector obtained by replacing α^i in $\vec{\alpha}$ with $\tilde{\alpha}$.

Definition 5.1. For any $\varepsilon > 0, L \geq 0$, we say $\vec{\alpha} \in (\mathcal{A}_{path}^{t,L})^N$ is an (ε, L)-equilibrium of the N-player game at $(t, \vec{x})$, denoted as $\vec{\alpha} \in \mathcal{M}_{hetero}^{N,\varepsilon,L}(t, \vec{x})$, if:

$$(5.4) \quad \frac{1}{N}\sum_{i=1}^N \big[J_i(t, \vec{x}, \vec{\alpha}) - v_i^{N,L}(t, \vec{x}, \vec{\alpha}) \big] \leq \varepsilon.$$

Here, since there are N players and we will send $N \to \infty$, similar to (4.14) we do not require the optimality for each player. In fact, by (5.4) one can easily show that

$$(5.5) \qquad \frac{1}{N}\Big|\Big\{i = 1, \cdots, N : J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) \geq \sqrt{\varepsilon}\Big\}\Big| \leq \sqrt{\varepsilon}.$$

This is exactly the $(\sqrt{\varepsilon}, \sqrt{\varepsilon})$-equilibrium in [11].

We then define the set value of the N-player game with heterogeneous equilibria:

$$\mathbb{V}_{hetero}^N(t, \vec{\mathbf{x}}) := \bigcap_{\varepsilon > 0} \mathbb{V}_{hetero}^{N,\varepsilon}(t, \vec{\mathbf{x}}) := \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \vec{\mathbf{x}}),$$

$$(5.6) \qquad \text{where} \quad \mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \vec{\mathbf{x}}) := \Big\{\varphi \in \mathbb{L}^0(\mathbb{X}_t; \mathbb{R}) : \exists \vec{\alpha} \in \mathcal{M}_{hetero}^{N,\varepsilon,L}(t, \vec{\mathbf{x}}) \text{ such that}$$

$$\max_{\mathbf{x} \in \mathbb{X}_t} \min_{\{i : \mathbf{x}^i = \mathbf{x}\}} \big|\varphi(\mathbf{x}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})\big| \leq \varepsilon\Big\}.$$

Remark 5.2.

(i) An alternative definition of $\mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \vec{\mathbf{x}})$ is to require φ satisfying

$$(5.7) \qquad \max_{i=1,\cdots,N} \big|\varphi(\mathbf{x}^i) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})\big| = \max_{\mathbf{x} \in \mathbb{X}_t} \max_{\{i : \mathbf{x}^i = \mathbf{x}\}} \big|\varphi(\mathbf{x}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})\big| \leq \varepsilon.$$

Indeed, the convergence result Theorem 5.3 remains true if we use (5.7). However, in general it is possible that $\mathbf{x}^i = \mathbf{x}^j$ but $v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) \neq v_j^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})$. Then, by fixing N and sending $\varepsilon \to 0$, under (5.7) we would have $\mathbb{V}_{hetero}^N(t, \vec{\mathbf{x}}) := \bigcap_{\varepsilon > 0} \mathbb{V}_{hetero}^{N,\varepsilon}(t, \vec{\mathbf{x}}) = \emptyset$.

(ii) In the homogeneous case, $v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) = v_j^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})$ whenever $\mathbf{x}^i = \mathbf{x}^j$, so we don't have this issue in (3.8).

(iii) Note that $\mu_{t,\vec{\mathbf{x}}}^N = \mu_{t,\vec{\mathbf{x}}'}^N$ if and only if $\vec{\mathbf{x}}$ is a permutation of $\vec{\mathbf{x}}'$, and one can easily verify that $v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) = v_{\pi(i)}^{N,L}(t, (\mathbf{x}_{\pi(1)}, \cdots, \mathbf{x}_{\pi(N)}), (\alpha_{\pi(1)}, \cdots, \alpha_{\pi(N)}))$ for any permutation π on $\{1, \cdots, N\}$, . Then, similar to the homogenous case, $\mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \vec{\mathbf{x}})$ is invariant in $\mu_{t,\vec{\mathbf{x}}}^N$ and we will denote it as $\mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \mu_{t,\vec{\mathbf{x}}}^N)$.

The following convergence result of the set value is in the same spirit of Theorem 3.6.

Theorem 5.3. *Let Assumption* 2.2 *hold and* $\mu_{t,\vec{\mathbf{x}}}^N \in \mathcal{P}_N(\mathbb{X}_t) \to \mu \in \mathcal{P}_0(\mathbb{X}_t)$ *under* W_1. *Then*

$$(5.8) \qquad \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \overline{\lim_{N \to \infty}} \mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \mu_{t,\vec{\mathbf{x}}}^N) \subset \mathbb{V}_{relax}(t, \mu) \subset \bigcap_{\varepsilon > 0} \underline{\lim_{N \to \infty}} \mathbb{V}_{hetero}^{N,\varepsilon,0}(t, \mu_{t,\vec{\mathbf{x}}}^N).$$

In particular, since $\underline{\lim}_{N \to \infty} \mathbb{V}_{hetero}^{N,\varepsilon,0}(t, \mu_{t,\vec{\mathbf{x}}}^N) \subset \bigcup_{L \geq 0} \overline{\lim}_{N \to \infty} \mathbb{V}_{hetero}^{N,\varepsilon,L}(t, \mu_{t,\vec{\mathbf{x}}}^N)$, *actually equalities hold.*

Unlike Theorem 3.6, here the N-player game and the MFG take different types of controls $\vec{\alpha}$ and γ, respectively. The key for the convergence is the global formulation in Subsection 4.2 for MFG. Indeed, given $t \in \mathbb{T}$, $\vec{\mathbf{x}} \in \mathbb{X}_{0,t}^N$, and $\vec{\alpha} \in (\mathcal{A}_{path}^{t,L})^N$, the N-player game is naturally related to the following $\Lambda^N \in \mathcal{P}(\mathbb{X}_t \times \mathcal{A}_{path}^{t,L})$:

$$(5.9)$$
$$\Lambda^N(\mathbf{x}, d\alpha) := \frac{1}{N} \sum_{i \in I(\mathbf{x})} \delta_{\alpha_i}(d\alpha), \text{ where } I(\mathbf{x}) := \big\{i = 1, \cdots, N : \mathbf{x}^i = \mathbf{x}\big\}, \mathbf{x} \in \mathbb{X}_t.$$

By the symmetry of the problem, there exists a function J^N, independent of i, such that

(5.10) $$J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) = J^N(\Lambda^N; t, \mathbf{x}^i, \alpha_i), \quad i = 1, \cdots, N.$$

We shall use this and Theorem 4.5 to prove Theorem 5.3 in the rest of this section. We also make the following obvious observation:

(5.11) $$\Lambda^N(\mathbf{x}, \mathcal{A}^t_{path}) = \frac{|I(\mathbf{x})|}{N} = \mu^N_{t,\vec{\mathbf{x}}}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

Remark 5.4.

(i) In this section we are using symmetric controls and we obtain the convergence in Theorem 5.3. If we use full information controls $\alpha_i(t, \vec{X})$, as observed in [32] in terms of the equilibrium measure, one may expect the limit set value will be strictly larger than $\mathbb{V}_{relax}$. It will be interesting to find an appropriate notion of MFE so that the corresponding MFG set value will be equal to the above limit, in the sense of Theorem 5.3.

(ii) While the convergence in Theorem 5.3 is about set values, the proofs in the rest of this section confirm the convergence of the approximate equilibria as well, exactly in the same manner as in Remark 3.7.

5.2. **From N-player games to mean field games.** In this subsection we prove the left inclusion in (5.8). Notice that the Λ^N in (5.9) is defined on $\mathcal{A}^{t,L}_{path}$, rather than $\mathcal{A}^t_{path} = \mathcal{A}^{t,0}_{path}$. For this purpose, recall (4.12) and introduce

(5.12)
$$\nu^N_{t\wedge\cdot}(\mathbf{x}) := \mu^N_{t,\vec{\mathbf{x}}}(\mathbf{x}), \nu^N_{s\wedge\cdot}(\tilde{\mathbf{x}}) := \frac{1}{N} \sum_{i \in I(\mathbf{x})} Q^t_s(\nu^N; \tilde{\mathbf{x}}, \alpha_i(\cdot, \cdot, \nu^N)), \mathbf{x} \in \mathbb{X}_t, \tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_s, s \geq t;$$

$$\bar{\Lambda}^N(\mathbf{x}, d\alpha) := \frac{\mu(\mathbf{x})}{|I(\mathbf{x})|} \sum_{i \in I(\mathbf{x})} \delta_{\bar{\alpha}_i}(d\alpha), \quad \text{where} \quad \bar{\alpha}_i(s, \tilde{\mathbf{x}}) := \alpha_i(s, \tilde{\mathbf{x}}, \nu^N).$$

Then it is obvious that $\bar{\alpha}_i \in \mathcal{A}^t_{path}$ and $\bar{\Lambda}^N \in \Xi_t(\mu)$. Moreover, when $\mu = \mu^N_{t,\vec{\mathbf{x}}}$, by (4.13) and (5.11) it is straightforward to verify by induction that $\mu^{\bar{\Lambda}^N} = \nu^N$.

Theorem 5.5. *Let Assumption* 2.2(ii) *hold. Then, for any $L \geq 0$, there exists a constant C_L, depending only on T, d, L_q, and L such that, for any $t \in \mathbb{T}$, $\vec{\mathbf{x}} \in \mathbb{X}^N_{0,t}$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, $\vec{\alpha} \in (\mathcal{A}^{t,L}_{path})^N, \tilde{\alpha} \in \mathcal{A}^{t,L}_{path}$, and for the $\nu^N, \bar{\Lambda}^N$ defined in (5.12), we have*
(5.13)
$$\max_{1 \leq i \leq N} \max_{t \leq s \leq T} \mathbb{E}^{\mathbb{P}^{t,\vec{\mathbf{x}},(\vec{\alpha}^{-i},\tilde{\alpha})}} \big[\mathcal{W}_1(\mu^N_{s\wedge\cdot}, \mu^{\bar{\Lambda}^N}_{s\wedge\cdot}) \big] \leq C_L \theta_N, \theta_N := W_1(\mu^N_{t,\vec{\mathbf{x}}}, \mu) + \frac{1}{\sqrt{N}}.$$

Proof. Fix i and denote $\tilde{\alpha}_j := \alpha_j$ for $j \neq i$, and $\tilde{\alpha}_i := \tilde{\alpha}_i$. We first show that

(5.14) $$\kappa_s := \mathbb{E}^{\mathbb{P}^N} \big[\mathcal{W}_1(\mu^N_{s\wedge\cdot}, \nu^N_{s\wedge\cdot}) \big] \leq \frac{C_L}{\sqrt{N}}, \quad \text{where} \quad \mathbb{P}^N := \mathbb{P}^{t,\vec{x},(\vec{\alpha}^{-i},\tilde{\alpha})}.$$

Indeed, for $s \geq t$, by the conditional independence of $\{X^j_{s+1}\}_{1 \leq j \leq N}$ under $\mathbb{P}^N$, conditional on $\mathcal{F}_s$, it follows from the same arguments as in (3.11) that

$$\kappa_{s+1} = \mathbb{E}^{\mathbb{P}^N} \Big[\mathbb{E}^{\mathbb{P}^N}_{\mathcal{F}_s} \big[\mathcal{W}_1(\mu^N_{(s+1)\wedge\cdot}, \nu^N_{(s+1)\wedge\cdot}) \big] \Big]$$

$$\leq \frac{C}{\sqrt{N}} + C \sum_{\mathbf{x} \in \mathbb{X}_{s+1}} \mathbb{E}^{\mathbb{P}^N} \Big[\Big| \frac{1}{N} \sum_{j=1}^{N} \mathbb{P}^N(X^j =_{s+1} \mathbf{x} | \mathcal{F}_s) - \nu^N_{(s+1)\wedge\cdot}(\mathbf{x}) \Big| \Big].$$

Note that,

$$\Big| \frac{1}{N} \sum_{j=1}^{N} \mathbb{P}^N(X^j =_{s+1} \mathbf{x} | \mathcal{F}_s) - \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\{X^j =_s \mathbf{x}\}} q(s, \mathbf{x}, \nu^N, \alpha_j(s, \mathbf{x}, \nu^N); \mathbf{x}_{s+1}) \Big|$$

$$= \Big| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\{X^j =_s \mathbf{x}\}} \big[q(s, \mathbf{x}, \mu^N, \tilde{\alpha}_j(s, \mathbf{x}, \mu^N); \mathbf{x}_{s+1}) - q(s, \mathbf{x}, \nu^N, \alpha_j(s, \mathbf{x}, \nu^N); \mathbf{x}_{s+1}) \big] \Big|$$

$$\leq C_L W_1(\mu^N_{s\wedge\cdot}, \nu^N_{s\wedge\cdot}) + \frac{1}{N} = C_L \kappa_s + \frac{1}{N},$$

where in the last inequality, the first term is due to the sum over all $j \neq i$. Then

$$\kappa_{s+1} \leq C_L \kappa_s + \frac{C}{\sqrt{N}} + \mathbb{E}^{\mathbb{P}^N} \Big[\sum_{\mathbf{x} \in \mathbb{X}_{s+1}} \Big| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\{X^j =_s \mathbf{x}\}} q(s, \mathbf{x}, \nu^N, \alpha_j(s, \mathbf{x}, \nu^N); \mathbf{x}_{s+1})$$

$$- \frac{1}{N} \sum_{j \in I(\mathbf{x}_{t\wedge\cdot})} Q^t_s(\nu^N; \mathbf{x}, \bar{\alpha}_j) q(s, \mathbf{x}, \nu^N, \alpha_j(s, \mathbf{x}, \nu^N); \mathbf{x}_{s+1}) \Big| \Big]$$

$$= C_L \kappa_s + \frac{C}{\sqrt{N}} + \mathbb{E}^{\mathbb{P}^N} \Big[\sum_{\mathbf{x} \in \mathbb{X}_s} \Big| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{\{X^j =_s \mathbf{x}\}} - \frac{1}{N} \sum_{j \in I(\mathbf{x}_{t\wedge\cdot})} Q^t_s(\nu^N; \mathbf{x}, \bar{\alpha}_j) \Big| \Big]$$

$$= C_L \kappa_s + \frac{C}{\sqrt{N}} + \mathbb{E}^{\mathbb{P}^N} \Big[\sum_{\mathbf{x} \in \mathbb{X}_s} | \mu^N_{s\wedge\cdot}(\mathbf{x}) - \nu^N_{s\wedge\cdot}(\mathbf{x}) | \Big] \leq C_L \kappa_s + \frac{C}{\sqrt{N}}.$$

It is obvious that $\kappa_t = 0$. Then by induction we obtain (5.14).

Next, denote $\bar{\kappa}_s := W_1(\nu^N_{s\wedge\cdot}, \mu^{\bar{\Lambda}^N}_{s\wedge\cdot})$. For $s \geq t$, by (5.12), (4.13), and (4.12), we have

$$\bar{\kappa}_{s+1} = \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} | \nu^N_{(s+1)\wedge\cdot}(\tilde{\mathbf{x}}) - \mu^{\bar{\Lambda}^N}_{(s+1)\wedge\cdot}(\tilde{\mathbf{x}}) |$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \Big| \frac{1}{N} \sum_{j \in I(\mathbf{x})} Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \bar{\alpha}_j) - \frac{\mu(\mathbf{x})}{|I(\mathbf{x})|} \sum_{j \in I(\mathbf{x})} Q^t_{s+1}(\mu^{\bar{\Lambda}^N}; \tilde{\mathbf{x}}, \bar{\alpha}_j) \Big|$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \Big[\frac{1}{N} \sum_{j \in I(\mathbf{x})} | Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \bar{\alpha}_j) - Q^t_{s+1}(\mu^{\bar{\Lambda}^N}; \tilde{\mathbf{x}}, \bar{\alpha}_j) |$$

$$+ \Big| \frac{1}{N} - \frac{\mu(\mathbf{x})}{|I(\mathbf{x})|} \Big| \sum_{j \in I(\mathbf{x})} Q^t_{s+1}(\mu^{\bar{\Lambda}^N}; \tilde{\mathbf{x}}, \bar{\alpha}_j) \Big]$$

$$\leq C \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \Big[\frac{1}{N} \sum_{j \in I(\mathbf{x})} \sum_{r=t}^{s} W_1(\nu^N_{r\wedge\cdot}, \mu^{\bar{\Lambda}^N}_{r\wedge\cdot}) + \Big| \frac{1}{N} - \frac{\mu(\mathbf{x})}{|I(\mathbf{x})|} \Big| |I(\mathbf{x})| \Big]$$

$$\leq C \sum_{r=t}^{s} \bar{\kappa}_r + C \sum_{\mathbf{x} \in \mathbb{X}_t} \left| \mu_{t,\vec{\mathbf{x}}}^N(\mathbf{x}) - \mu(\mathbf{x}) \right| \leq C \sum_{r=t}^{s} \bar{\kappa}_r.$$

Obviously $\bar{k}_t = W_1(\mu_{t,\vec{\mathbf{x}}}^N, \mu)$. Then by induction we have $\sup_{t \leq s \leq T} \bar{\kappa}_s \leq CW_1(\mu_{t,\vec{\mathbf{x}}}^N, \mu)$. This, together with (5.14), implies (5.13) immediately. $\square$

Theorem 5.6. *For the setting in Theorem 5.5 and assuming further Assumption 2.2(iii), there exists a modulus of continuity function ρ_L, depending on T, d, L_q, C_0, ρ, L, such that*
$$(5.15)$$
$$\left| J_i(t, \vec{x}, (\vec{\alpha}^{-i}, \tilde{\alpha})) - J(t, \bar{\Lambda}^N; \mathbf{x}^i, \tilde{\alpha}(\cdot, \nu^N)) \right| + \left| v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) - v(\mu^{\bar{\Lambda}^N}; t, \mathbf{x}^i) \right| \leq \rho_L(\theta_N).$$

Moreover, assume $\vec{\alpha} \in \mathcal{M}_{hetero}^{N, \varepsilon_1, L}(t, \vec{\mathbf{x}})$ for some $\varepsilon_1 > 0$, then

$$(5.16) \quad \int_{\mathcal{A}_{path}^t} [J(t, \bar{\Lambda}^N; \mathbf{x}, \alpha) - v(t, \bar{\Lambda}^N; \mathbf{x})] \bar{\Lambda}^N(\mathbf{x}, d\alpha) \leq \varepsilon_1 + 2\rho_L(\theta_N), \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

In particular, if $\varepsilon_1 + 2\rho_L(\theta_N) \leq \varepsilon$, then $\bar{\Lambda}^N \in \mathcal{M}_{global}^{\varepsilon}(t, \mu)$.

Proof. First, given Theorem 5.5, (5.15) follows from the arguments in Theorem 3.5. Then, for $\vec{\alpha} \in \mathcal{M}_{hetero}^{N, \varepsilon_1, L}(t, \vec{\mathbf{x}})$ and $\mathbf{x} \in \mathbb{X}_t$, by (5.4) we have

$$\int_{\mathcal{A}_{path}^t} [J(t, \bar{\Lambda}^N; \mathbf{x}, \alpha) - v(t, \bar{\Lambda}^N; \mathbf{x})] \bar{\Lambda}^N(\mathbf{x}, d\alpha)$$
$$= \frac{1}{N} \sum_{i \in I(\mathbf{x})} \left[J(t, \bar{\Lambda}^N; \mathbf{x}, \bar{\alpha}_i) - v(t, \bar{\Lambda}^N; \mathbf{x}) \right]$$
$$\leq \frac{1}{N} \sum_{i \in I(\mathbf{x})} \left[\left| J(t, \bar{\Lambda}^N; \mathbf{x}^i, \bar{\alpha}_i) - J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) \right| + \left[J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) \right] \right.$$
$$\left. + \left| v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) - v(\mu^{\bar{\Lambda}^N}; t, \mathbf{x}^i) \right| \right]$$
$$\leq \rho_L(\theta_N) + \varepsilon_1 + \rho_L(\theta_N) = \varepsilon_1 + 2\rho_L(\theta_N).$$
$$\square$$

Proof of Theorem 5.3: the left inclusion. We first fix an arbitrary function $\varphi \in \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \overline{\lim}_{N \to \infty} \mathbb{V}_{hetero}^{N, \varepsilon, L}(t, \mu_{t,\vec{\mathbf{x}}}^N)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{2}$. Then there exists $L_\varepsilon \geq 0$ and a sequence $N_k \to \infty$ (possibly depending on ε) such that $\varphi \in \mathbb{V}_{hetero}^{N_k, \varepsilon_1, L_{\varepsilon_1}}(t, \mu_{t,\vec{\mathbf{x}}}^{N_k})$, for all $k \geq 1$. Now choose k large enough so that $2\rho_{L_\varepsilon}(\theta_{N_k}) \leq \varepsilon_1$. By (5.1) there exists $\vec{\alpha} \in \mathcal{M}_{hetero}^{N_k, \varepsilon_1, L_\varepsilon}(t, \vec{\mathbf{x}})$ such that $\max_{\mathbf{x} \in \mathbb{X}_t} \min_{i \in I(\mathbf{x})} |\varphi(\mathbf{x}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha})| \leq \varepsilon_1$. By Theorem 5.6 we see that $\bar{\Lambda}^{N_k} \in \mathcal{M}_{global}^{\varepsilon}(t, \mu)$ and, by (5.15),

$$\|\varphi - v(\mu^{\bar{\Lambda}^N}; t, \cdot)\|_{\mathbb{X}_t}$$
$$\leq \max_{\mathbf{x} \in \mathbb{X}_t} \min_{i \in I(\mathbf{x})} \left[\left| \varphi(\mathbf{x}) - v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) \right| + \left| v_i^{N,L}(t, \vec{\mathbf{x}}, \vec{\alpha}) - v(\mu^{\bar{\Lambda}^N}; t, \mathbf{x}) \right| \right]$$
$$\leq \varepsilon_1 + \rho_{L_\varepsilon}(\theta_N) \leq \varepsilon.$$

Then $\varphi \in \mathbb{V}_{global}^{\varepsilon}(t, \mu)$. Since $\varepsilon > 0$ is arbitrary, by Theorem 4.5 we get $\varphi \in \mathbb{V}_{relax}(t, \mu)$. $\square$

5.3. **From mean field games to N-player games.** We now turn to the right inclusion in (5.8). Fix $t \in \mathbb{T}$, $\vec{x} \in \mathbb{X}_{0,t}^N$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, and $\gamma \in \mathcal{A}_{relax}$. Our goal is to construct a desired $\vec{\alpha} \in (\mathcal{A}_{path}^{t,0})^N$. However, since $\vec{\alpha}$, or equivalently the corresponding Λ^N, is discrete, we need to discretize γ first. We note that it is slightly easier to discretize γ than a general $\Lambda \in \Xi_t(\mu)$.

First, given $\varepsilon > 0$, there exists a partition $\mathbb{A} = \cup_{k=0}^{n_\varepsilon} A_k$ with n_ε depending on ε (and γ) such that, for some arbitrarily fixed $a_k \in A_k$, $k = 0, \cdots, n_\varepsilon$,
(5.17)
$$\gamma(s, \mathbf{x}, A_0) \le \varepsilon, \forall s \in \mathbb{T}_t, \mathbf{x} \in \mathbb{X}_s, \quad \text{and} \quad |a - a_k| \le \varepsilon, \forall a \in A_k, k = 1, \cdots, n_\varepsilon.$$

Denote by $\mathcal{A}_{path}^{t,\varepsilon}$ the subset of $\alpha \in \mathcal{A}_{path}^{t,0}$ taking values in $\mathbb{A}_\varepsilon := \{a_k : k = 0, \cdots, n_\varepsilon\}$. Define

$$\gamma^\varepsilon(s, \mathbf{x}, da) := \sum_{k=0}^{n_\varepsilon} \gamma(s, \mathbf{x}, A_k) \delta_{a_k}(da). \tag{5.18}$$

Recall (4.17), we see that $\mathrm{supp}(\Lambda^{\gamma^\varepsilon}(\mathbf{x}, d\alpha)) = \mathcal{A}_{path}^{t,\varepsilon} \subset \mathcal{A}_{path}^{t,0}$ for all $\mathbf{x} \in \mathbb{X}_t$.

Next, recall (5.11) that $N\mu_{t,\vec{x}}^N(\mathbf{x}) = |I(\mathbf{x})|$ is a positive integer for all $\mathbf{x} \in \mathbb{X}_t$. Let $\Lambda_{t,\vec{x}}^\varepsilon \in \mathcal{P}(\mathbb{X}_t \times \mathcal{A}_{path}^{t,\varepsilon})$ be a modification of $\Lambda^{\gamma^\varepsilon}$ such that,
(5.19)
$$\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \mathcal{A}_{path}^{t,\varepsilon}) = \mu_{t,\vec{x}}^N(\mathbf{x}) \text{ and } N\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \alpha) \text{ is an integer;}$$
$$|\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \alpha) - \Lambda^{\gamma^\varepsilon}(\mathbf{x}, \alpha)| \le \tfrac{1}{N} + |\mu_{t,\vec{x}}^N(\mathbf{x}) - \mu(\mathbf{x})|; \quad \forall (\mathbf{x}, \alpha) \in \mathbb{X}_t \times \mathcal{A}_{path}^{t,\varepsilon}.$$

Note that, since $\mathcal{A}_{path}^{t,\varepsilon}$ is finite, such a construction is easy.

We now construct $\vec{\alpha} \in (\mathcal{A}_{path}^{t,\varepsilon})^N$, which relies on γ^ε and hence on ε. Note that

$$\sum_{\alpha \in \mathcal{A}_{path}^{t,\varepsilon}} [N\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \alpha)] = N\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \mathcal{A}_{path}^{t,\varepsilon}) = N\mu_{t,\vec{x}}^N(\mathbf{x}) = |I(\mathbf{x})|,$$

and each $N\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \alpha)$ is an integer. Let $I(\mathbf{x}) = \cup_{\alpha \in \mathcal{A}_{path}^{t,\varepsilon}} I(\mathbf{x}, \alpha)$ be a partition of $I(\mathbf{x})$ such that $|I(\mathbf{x}, \alpha)| = N\Lambda_{t,\vec{x}}^\varepsilon(\mathbf{x}, \alpha)$. We then set

$$\alpha_i := \alpha, \quad i \in I(\mathbf{x}, \alpha), \quad (\mathbf{x}, \alpha) \in \mathbb{X}_t \times \mathcal{A}_{path}^{t,\varepsilon}. \tag{5.20}$$

Let Λ^N be the one defined by (5.9) corresponding to this $\vec{\alpha}$. It is clear that $\Lambda^N = \Lambda_{t,\vec{x}}^\varepsilon$.

Theorem 5.7.

(i) *Let Assumption 2.2(ii) hold. Then there exists a constant C, depending only on T, d, L_q, such that, for any $t \in \mathbb{T}$, $\vec{x} \in \mathbb{X}_{0,t}^N$, $\mu \in \mathcal{P}_0(\mathbb{X}_t)$, $\gamma \in \mathcal{A}_{relax}$, $\varepsilon > 0$, and for the $\vec{\alpha} \in (\mathcal{A}_{path}^{t,\varepsilon})^N$ constructed above, we have, for the θ_N in (5.13) and for any $\tilde{\alpha} \in \mathcal{A}_{path}^{t,0}$,*

$$\max_{1 \le i \le N} \max_{t \le s \le T} \mathbb{E}^{\mathbb{P}^{t,\vec{x},(\vec{\alpha}^{-i}, \tilde{\alpha})}} [W_1(\mu_{s\wedge \cdot}^N, \mu_{s\wedge \cdot}^\gamma)] \le C\varepsilon + C_\varepsilon \theta_N, \tag{5.21}$$

where C_ε may depend on ε as well.

(ii) *Assume further Assumption 2.2(iii), then there exists a modulus of continuity function ρ_0, depending only on T, d, L_q, C_0, and ρ, such that,*
(5.22)
$$\left| J_i(t, \vec{x}, (\vec{\alpha}^{-i}, \tilde{\alpha})) - J(\mu^\gamma; t, \mathbf{x}^i, \tilde{\alpha}) \right| + \left| v_i^{N,0}(t, \vec{x}, \vec{\alpha}) - v(\mu^\gamma; t, \mathbf{x}^i) \right| \le \rho_0 (C\varepsilon + C_\varepsilon \theta_N).$$

Moreover, assume $\gamma \in \mathcal{M}^\varepsilon_{relax}(t, \mu)$, *then*

$$(5.23) \qquad \frac{1}{N} \sum_{i=1}^{N} \left[J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) - v_i^{N,0}(t, \vec{\mathbf{x}}, \vec{\alpha}) \right] \leq \varepsilon + 2\rho_0 \left(C\varepsilon + C_\varepsilon \theta_N \right), \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

In particular, this means that $\vec{\alpha} \in \mathcal{M}^{N,\tilde{\varepsilon},0}_{hetero}(t, \vec{\mathbf{x}})$ *with* $\tilde{\varepsilon} := \varepsilon + 2\rho_0 \left(C\varepsilon + C_\varepsilon \theta_N \right)$.

Proof. (i) We first show by induction that

$$(5.24) \qquad \kappa_s := W_1 \left(\mu^\gamma_{s\wedge\cdot}, \mu^{\gamma^\varepsilon}_{s\wedge\cdot} \right) \leq C\varepsilon, \quad s = t, \cdots, T.$$

Indeed, it is obvious that $\kappa_t = 0$. For $s \geq t$, by (4.1), (5.17), and (5.18), we have

$$\kappa_{s+1} = \sum_{\mathbf{x} \in \mathbb{X}_{s+1}} \left| \mu^\gamma_{(s+1)\wedge\cdot}(\mathbf{x}) - \mu^{\gamma^\varepsilon}_{(s+1)\wedge\cdot}(\mathbf{x}) \right|$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_s, x \in \mathbb{S}} \left| \mu^\gamma_{s\wedge\cdot}(\mathbf{x}) \int_{\mathbb{A}} q(s, \mathbf{x}, \mu^\gamma, a; x) \gamma(s, \mathbf{x}, da) \right.$$

$$\left. - \mu^{\gamma^\varepsilon}_{s\wedge\cdot}(\mathbf{x}) \int_{\mathbb{A}} q(s, \mathbf{x}, \mu^{\gamma^\varepsilon}, a; x) \gamma^\varepsilon(s, \mathbf{x}, da) \right|$$

$$\leq \sum_{\mathbf{x} \in \mathbb{X}_s, x \in \mathbb{S}} \left[\left| \mu^\gamma_{s\wedge\cdot}(\mathbf{x}) - \mu^{\gamma^\varepsilon}_{s\wedge\cdot}(\mathbf{x}) \right| \right.$$

$$+ \sum_{k=1}^{n_\varepsilon} \int_{A_k} \left| q(s, \mathbf{x}, \mu^\gamma, a; x) - q(s, \mathbf{x}, \mu^{\gamma^\varepsilon}, a_k; x) \right| \gamma(s, \mathbf{x}, da)$$

$$\left. + \int_{A_0} q(s, \mathbf{x}, \mu^\gamma, a; x) \gamma(s, \mathbf{x}, da) + \int_{A_0} q(s, \mathbf{x}, \mu^{\gamma^\varepsilon}, a; x) \gamma^\varepsilon(s, \mathbf{x}, da) \right]$$

$$\leq C\kappa_s + C\varepsilon.$$

Then by induction we have (5.24).

We next show by induction that, recalling (5.12),

$$(5.25) \qquad \bar{\kappa}_s := W_1 \left(\nu^N_{s\wedge\cdot}, \mu^{\gamma^\varepsilon}_{s\wedge\cdot} \right) \leq C_\varepsilon \theta_N, \quad s = t, \cdots, T.$$

Indeed, $\bar{\kappa}_t = W_1(\mu^N_{t,\vec{\mathbf{x}}}, \mu)$. For $s \geq t$, noting that $\alpha_i \in \mathcal{A}^{t,\varepsilon}_{path} \subset \mathcal{A}^{t,0}_{path}$ and recalling from Lemma 4.6 that $\mu^{\Lambda^{\gamma^\varepsilon}} = \mu^{\gamma^\varepsilon}$, then by (5.12) and (4.13) that

$$\bar{\kappa}_{s+1} = W_1 \left(\nu^N_{s+1\wedge\cdot}, \mu^{\Lambda^{\gamma^\varepsilon}}_{(s+1)\wedge\cdot} \right)$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \left| \frac{1}{N} \sum_{\alpha \in \mathcal{A}^{t,\varepsilon}_{path}} \sum_{i \in I(\mathbf{x},\alpha)} Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \alpha) - \int_{\mathcal{A}^t_{path}} Q^t_{s+1}(\mu^{\gamma^\varepsilon}; \tilde{\mathbf{x}}, \alpha) \Lambda^{\gamma^\varepsilon}(\mathbf{x}, d\alpha) \right|$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \left| \sum_{\alpha \in \mathcal{A}^{t,\varepsilon}_{path}} \left[\Lambda^\varepsilon_{t,\vec{\mathbf{x}}}(\mathbf{x}, \alpha) Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \alpha) - \Lambda^{\gamma^\varepsilon}(\mathbf{x}, \alpha) Q^t_{s+1}(\mu^{\gamma^\varepsilon}; \tilde{\mathbf{x}}, \alpha) \right] \right|$$

$$\leq \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}^{t,\mathbf{x}}_{s+1}} \sum_{\alpha \in \mathcal{A}^{t,\varepsilon}_{path}} \left[\left| \Lambda^\varepsilon_{t,\vec{\mathbf{x}}}(\mathbf{x}, \alpha) - \Lambda^{\gamma^\varepsilon}(\mathbf{x}, \alpha) \right| Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \alpha) \right.$$

$$\left. + \Lambda^{\gamma^\varepsilon}(\mathbf{x}, \alpha) \left| Q^t_{s+1}(\nu^N; \tilde{\mathbf{x}}, \alpha) - Q^t_{s+1}(\mu^{\gamma^\varepsilon}; \tilde{\mathbf{x}}, \alpha) \right| \right].$$

Then, by (5.19) and noting that $C_\varepsilon := |\mathcal{A}_{path}^{t,\varepsilon}|$ is independent of N, we have

$$\bar{\kappa}_{s+1} \leq \sum_{\mathbf{x} \in \mathbb{X}_t} \sum_{\tilde{\mathbf{x}} \in \mathbb{X}_{s+1}^{t,\mathbf{x}}} \sum_{\alpha \in \mathcal{A}_{path}^{t,\varepsilon}} \left[\theta_N Q_{s+1}^t(\nu^N; \tilde{\mathbf{x}}, \alpha) + C\Lambda^{\gamma^\varepsilon}(\mathbf{x}, \alpha) \sum_{r=t}^s W_1\big(\nu_{r\wedge\cdot}^N, \mu_{r\wedge\cdot}^{\gamma^\varepsilon}\big) \right]$$

$$\leq C_\varepsilon \theta_N + C \sum_{r=t}^s \bar{\kappa}_r.$$

This implies (5.25) immediately.

Finally, combining (5.24), (5.25), and (5.13), we obtain (5.21).

(ii) First, similar to (5.15), by (5.21) we have (5.22) following from the arguments in Theorem 3.5. Next, for $\gamma \in \mathcal{M}_{relax}^\varepsilon(t, \mu)$, by (4.19) we have $\Lambda^\gamma \in \mathcal{M}_{global}^\varepsilon(t, \mu)$. Then (5.23) follows from similar arguments as those for (5.16). $\qquad \square$

Proof of Theorem 5.3: *the right inclusion.* Fix $\varphi \in \mathbb{V}_{relax}(t, \mu)$ and $\varepsilon > 0$. Let $\varepsilon_1 > 0$ be a small number which will be specified later. There exists $\gamma \in \mathcal{M}_{relax}^{\varepsilon_1}(t, \mu)$ such that $\|\varphi - J(t, \mu, \gamma; \cdot, \gamma)\|_{\mathbb{X}_t} \leq \varepsilon_1$. Let γ^{ε_1} and $\vec{\alpha}$ be constructed as above. By (5.23) we have

$$\frac{1}{N} \sum_{i=1}^N \big[J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) - v_i^{N,0}(t, \vec{\mathbf{x}}, \vec{\alpha}) \big] \leq \varepsilon_1 + 2\rho_0\big(C\varepsilon_1 + C_{\varepsilon_1}\theta_N\big), \quad \forall \mathbf{x} \in \mathbb{X}_t.$$

Choose ε_1 small enough such that $\varepsilon_1 + 2\rho_0(C\varepsilon_1 + \varepsilon_1) < \varepsilon$. Then, for all N large enough such that $\theta_N \leq \frac{\varepsilon_1}{C_{\varepsilon_1}}$, we have $\frac{1}{N} \sum_{i=1}^N \big[J_i(t, \vec{\mathbf{x}}, \vec{\alpha}) - v_i^{N,0}(t, \vec{\mathbf{x}}, \vec{\alpha}) \big] \leq \varepsilon$. That is, $\vec{\alpha} \in \mathbb{V}_{hetero}^{N,\varepsilon,0}(t, \mu_{t,\vec{\mathbf{x}}}^N)$ for all N large enough. Then, following the same arguments as those in the proof for the left inclusion, we can easily get $\varphi \in \mathbb{V}_{hetero}^{N,\varepsilon,0}(t, \mu_{t,\vec{\mathbf{x}}}^N)$ for all N large enough, and thus $\varphi \in \underline{\lim}_{N \to \infty} \mathbb{V}_{hetero}^{N,\varepsilon,0}(t, \mu_{t,\vec{\mathbf{x}}}^N)$. Since $\varepsilon > 0$ is arbitrary, we get the desired inclusion. $\qquad \square$

6. A CONTINUOUS TIME MODEL WITH CONTROLLED DIFFUSIONS

In this section we study a continuous time model where the state process is a controlled diffusion with closed loop drift controls. In this case the laws of the controlled state process are all equivalent. The volatility control case involves mutually singular measures (corresponding to degenerate q in the discrete setting) and is much more challenging. We shall leave that for future research. To ensure the convergence, we consider state dependent homogeneous controls for the N-player games, as we did in Section 3.

6.1. **The mean field game and the dynamic programming principle.** Let $T > 0$ be a fixed terminal time, $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ a filtered probability space where $\mathcal{F}_0$ is atomless; B a d-dimensional Brownian motion; and the set $\mathbb{A} \subset \mathbb{R}^{d_0}$ a Borel measurable set. The state process X will also take values in $\mathbb{R}^d$. Its law lies in the space $\mathcal{P}_2 := \mathcal{P}_2(\mathbb{R}^d)$ equipped with the 2-Wasserstein distance W_2. We remark that in the finite state space case W_1 and W_2 are equivalent, while in continuous models they are not. In fact, at below we shall require W_1-regularity, which is stronger than the W_2-regularity, and obtain W_1-convergence, which is weaker than the W_2-convergence. This is not surprising in the mean field literature, see, e.g. [38].

The main advantage of the W_1-distance is the following well known representation, see e.g. [13]: for any $\mu, \tilde{\mu} \in \mathcal{P}_1(\mathbb{R}^d)$,

(6.1)
$$W_1(\mu, \tilde{\mu}) = \sup \Big\{ \int_{\mathbb{R}^d} \varphi(x)[\mu(dx) - \tilde{\mu}(dx)] : \varphi \in C_{Lip}(\mathbb{R}^d) \text{ s.t. } |\varphi(x) - \varphi(\tilde{x})| \le |x - \tilde{x}| \Big\}.$$

Here $C_{Lip}(\mathbb{R}^d)$ denote the set of uniformly Lipschitz continuous functions $\varphi : \mathbb{R}^d \to \mathbb{R}$. Moreover, for each $(t, \mu) \in [0, T] \times \mathcal{P}_2$, let $\mathbb{L}^2(t, \mu)$ denote the set of $\mathcal{F}_t$-measurable random variables ξ whose law (under $\mathbb{P}$) $\mathcal{L}_\xi = \mu$.

We consider coefficients $(b, f) : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \times \mathbb{A} \to (\mathbb{R}^d, \mathbb{R})$ and $g : \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{R}$. Throughout this section, the following assumptions will always be in force.

Assumption 6.1.

    (i) b, f, g are Borel measurable in t and bounded by C_0 (for simplicity);

    (ii) b, f, g are uniformly Lipschitz continuous in (x, μ, a) with a Lipschitz constant L_0, where the Lipschitz continuity in μ is under W_1.

Let $\mathcal{A}_{cont}$ denote the set of admissible controls $\alpha : [0, T] \times \mathbb{R}^d \to \mathbb{A}$ which is measurable in t and Lipschitz continuous in x, with the Lipschitz constant L_α possibly depending on α. Given $(t, \mu) \in [0, T] \times \mathcal{P}_2$, $\xi \in \mathbb{L}^2(t, \mu)$, and $\alpha \in \mathcal{A}_{cont}$, consider the McKean-Vlasov SDE:

(6.2) $\quad X_s^{t,\xi,\alpha} = \xi + \int_t^s b(r, X_r^{t,\xi,\alpha}, \mu_r^\alpha, \alpha(r, X_r^{t,\xi,\alpha}))dr + B_s - B_t, \quad \mu_s^\alpha := \mathcal{L}_{X_s^{t,\xi,\alpha}}.$

By the required Lipschitz continuity, the above SDE is wellposed, and it is obvious that $\mu_t^\alpha = \mu$ and μ_s^α does not depend on the choice of $\xi \in \mathbb{L}^2(t, \mu)$. Then, when only the law is involved, by abusing the notations we may also denote $X^{t,\xi,\alpha}$ as $X^{t,\mu,\alpha}$.

Next, for any $x \in \mathbb{R}^d$, and $\tilde{\alpha} \in \mathcal{A}_{cont}$, we introduce

$$J(t, \mu, \alpha; x, \tilde{\alpha}) := J(\mu^\alpha; t, x, \tilde{\alpha}), \quad v(\mu^\alpha; s, x) := \inf_{\tilde{\alpha} \in \mathcal{A}_{cont}} J(\mu^\alpha; s, x, \tilde{\alpha}), s \ge t, \quad \text{where}$$

(6.3) $\quad X_r^{\mu^\alpha; s, x, \tilde{\alpha}} = x + \int_s^r b(l, X_l^{\mu^\alpha; s, x, \tilde{\alpha}}, \mu_l^\alpha, \tilde{\alpha}(l, X_l^{\mu^\alpha; s, x, \tilde{\alpha}}))dl + B_r - B_s, r \ge s;$

$$J(\mu^\alpha; s, x, \tilde{\alpha}) := \mathbb{E}\Big[g(X_T^{\mu^\alpha; s, x, \tilde{\alpha}}, \mu_T^\alpha) + \int_s^T f(r, X_r^{\mu^\alpha; s, x, \tilde{\alpha}}, \mu_r^\alpha, \tilde{\alpha}(r, X_r^{\mu^\alpha; s, x, \tilde{\alpha}}))dr \Big].$$

Here we abuse the notations by using the same notations as in the discrete setting. Clearly $u(s, x) := J(\mu^\alpha; s, x, \tilde{\alpha})$ and $v(s, x) := v(\mu^\alpha; s, x)$ satisfy the following linear PDE and standard HJB equation on $[t, T] \times \mathbb{R}^d$, respectively, with parameter μ^α:

(6.4)
$$\partial_s u(s, x) + \frac{1}{2} \operatorname{tr}\big(\partial_{xx} u(s, x)\big) + b(s, x, \mu_s^\alpha, \tilde{\alpha}(s, x)) \cdot \partial_x u(s, x) + f(s, x, \mu_s^\alpha, \tilde{\alpha}(s, x)) = 0;$$

$$\partial_t v(s, x) + \frac{1}{2} \operatorname{tr}\big(\partial_{xx} v(s, x)\big) + \inf_{a \in \mathbb{A}} \big[b(s, x, \mu_s^\alpha, a) \cdot \partial_x v(s, x) + f(s, x, \mu_s^\alpha, a) \big] = 0;$$

$$u(T, x) = v(T, x) = g(x, \mu_T^\alpha).$$

Definition 6.2. Fix $(t, \mu) \in [0, T] \times \mathcal{P}_2$. For any $\varepsilon > 0$, we say $\alpha^* \in \mathcal{A}_{cont}$ is an ε-MFE at (t, μ), denoted as $\alpha^* \in \mathcal{M}_{cont}^\varepsilon(t, \mu)$, if

(6.5) $$\int_{\mathbb{R}^d} \big[J(t, \mu, \alpha^*; x, \alpha^*) - v(\mu^{\alpha^*}; t, x) \big] \mu(dx) \le \varepsilon.$$

Remark 6.3. Similar to (5.4) and (5.5), here we do not require α^* to be optimal for every player x. In fact, alternatively, we may replace (6.5) with

$$(6.6) \qquad \mu\Big\{ x \ : \ |J(t,\mu,\alpha^*;x,\alpha^*) - v(\mu^{\alpha^*};t,x)| > \varepsilon \Big\} < \varepsilon.$$

The intuition is that, since there are infinitely many players, we shall tolerate that a small portion of players may not be happy for the α^*, as in [11], and their possible deviation from α^* won't change the equilibrium measure μ^{α^*} significantly. We note that, although (6.6) and (6.5) are not equivalent for fixed ε, they define the same set value in (6.1) below, and the proofs are slightly easier by using (6.5).

However, if we require the ε-optimality for μ-a.e. x, namely the probability in the left side of (6.6) becomes 0, then the set value will be different and may not satisfy the DPP. Such difference would disappear in the discrete model though.

To define the set value, we need the following simple but crucial regularity result, whose proof is postponed to Appendix.

Lemma 6.4. *Let Assumption* 6.1 *hold. There exists a constant $C > 0$, depending only on T, d, C_0, L_0, such that, for any $t, \mu, \alpha, \tilde{\alpha}$ and $s \geq t$,*

$$(6.7) \ \ \big|J(\mu^\alpha;\tilde{\alpha},s,x) - J(\mu^\alpha;\tilde{\alpha},s,\tilde{x})\big| + \big|v(\mu^\alpha;s,x) - v(\mu^\alpha;s,\tilde{x})\big| \leq C|x - \tilde{x}|, \quad \forall x, \tilde{x}.$$

We then define the set value of the mean field game:

$$\mathbb{V}_{cont}(t,\mu) := \bigcap_{\varepsilon>0} \mathbb{V}^\varepsilon_{cont}(t,\mu), \quad \text{where}$$

$$(6.8) \qquad \mathbb{V}^\varepsilon_{cont}(t,\mu) := \Big\{ \varphi \in C_{Lip}(\mathbb{R}^d) : \ \text{there exists } \alpha^* \in \mathcal{M}^\varepsilon_{cont}(t,\mu) \text{ such that}$$

$$\int_{\mathbb{R}^d} \big|\varphi(x) - J(t,\mu,\alpha^*;x,\alpha^*)\big|\mu(dx) \leq \varepsilon \Big\}.$$

In particular, since $J(t,\mu,\alpha^*;x,\alpha^*) \geq v(\mu^{\alpha^*};t,x)$, then by (6.7) and (6.5) we see that both $J(t,\mu,\alpha^*;\cdot,\alpha^*)$ and $v(\mu^{\alpha^*};t,\cdot)$ belong to $\mathbb{V}_{cont}(t,\mu)$. Moreover, again due to (6.5), we may replace the inequality in the last line of (6.1) with $\int_{\mathbb{R}^d} \big|\varphi(x) - v(\mu^{\alpha^*};t,x)\big|\mu(dx) \leq \varepsilon$.

Similarly, given T_0 and $\psi \in C_{Lip}(\mathbb{R}^d)$, we may define the functions $J(T_0,\psi;t,\mu,\alpha;x,\tilde{\alpha})$, $J(T_0,\psi;\mu^\alpha;s,x,\tilde{\alpha})$, $v(T_0,\psi;\mu^\alpha;s,x)$, as well as the sets $\mathcal{M}^\varepsilon_{cont}(T_0,\psi;t,\mu)$, $\mathbb{V}^\varepsilon_{cont}(T_0,\psi;t,\mu)$, $\mathbb{V}_{cont}(T_0,\psi;t,\mu)$ in the obvious sense. In particular, we have the following tower property:

$$(6.9) \quad \begin{aligned} J(t,\mu,\alpha;x,\tilde{\alpha}) &= J(T_0,\psi;t,\mu,\alpha;x,\tilde{\alpha}), \quad \text{where} \quad \psi(x) := J(T_0,\mu^\alpha_{T_0},\alpha;x,\tilde{\alpha}); \\ v(\mu^\alpha;t,x) &= v(T_0,\tilde{\psi};\mu^\alpha;t,x), \qquad \text{where} \quad \tilde{\psi}(x) := v(\mu^\alpha;T_0,x). \end{aligned}$$

We now establish the DPP for $\mathbb{V}_{cont}(t,\mu)$.

Theorem 6.5. *Let Assumption* 6.1 *hold. For any $0 \leq t \leq T_0 \leq T$ and $\mu \in \mathcal{P}_2$, it holds*

$$\mathbb{V}_{cont}(t,\mu) = \tilde{\mathbb{V}}_{cont}(t,\mu) := \bigcap_{\varepsilon>0} \tilde{\mathbb{V}}^\varepsilon_{cont}(t,\mu), \quad \text{where}$$

(6.10)

$$\tilde{\mathbb{V}}^\varepsilon_{cont}(t,\mu) := \Big\{ \varphi \in C_{Lip}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\varphi(x) - J(T_0,\psi;t,\mu,\alpha^*;x,\alpha^*)|\mu(dx) \leq \varepsilon,$$

$$\text{for some } (\psi,\alpha^*) \text{ satisfying: } \psi \in \mathbb{V}^\varepsilon_{cont}(T_0,\mu^{\alpha^*}_{T_0}), \alpha^* \in \mathcal{M}^\varepsilon_{cont}(T_0,\psi;t,\mu) \Big\}.$$

Proof. (i) We first prove $\mathbb{V}_{cont}(t,\mu) \subset \tilde{\mathbb{V}}_{cont}(t,\mu)$. Fix $\varphi \in \mathbb{V}_{cont}(t,\mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{2}$. Since $\varphi \in \mathbb{V}_{cont}^{\varepsilon_1}(t,\mu)$, there exists $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon_1}(t,\mu)$ satisfying (6.1) for ε_1. Denote

$$\psi(x) := J(T_0, \mu_{T_0}^{\alpha^*}, \alpha^*; x, \alpha^*), \quad \tilde{\psi}(x) := v(\mu^{\alpha^*}; T_0, x).$$

By (6.9) we have $J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) = J(t, \mu, \alpha^*; x, \alpha^*)$ and thus

$$\int_{\mathbb{R}^d} \big| \varphi(x) - J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) \big| \mu(dx) \leq \varepsilon_1 \leq \varepsilon.$$

We shall show that $\psi \in \mathbb{V}_{cont}^{\varepsilon}(T_0, \mu_{T_0}^{\alpha^*})$ and $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon}(T_0, \psi; t, \mu)$. Then $\varphi \in \tilde{\mathbb{V}}_{cont}^{\varepsilon}(t,\mu)$, and therefore, since $\varepsilon > 0$ is arbitrary, we have $\varphi \in \hat{\mathbb{V}}(t,\mu)$.

Step 1. In this step we show that
(6.11)
$$\int_{\mathbb{R}^d} \big[J(T_0, \mu_{T_0}^{\alpha^*}, \alpha^*; x, \alpha^*) - v(\mu^{\alpha^*}; T_0, x) \big] \mu_{T_0}^{\alpha^*}(dx) = \int_{\mathbb{R}^d} [\psi(x) - \tilde{\psi}(x)] \mu_{T_0}^{\alpha^*}(dx) \leq \varepsilon_1.$$

Then $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon}(T_0, \mu_{T_0}^{\alpha^*})$, which, together with the regularity of ψ from Lemma 6.4, implies immediately that $\psi \in \mathbb{V}_{cont}^{\varepsilon}(T_0, \mu_{T_0}^{\alpha^*})$.

To see this, we recall (6.2) with $\xi \in \mathbb{L}^2(t,\mu)$. Since $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon_1}(t,\mu)$, by (6.9) we have

$$\begin{aligned}
\varepsilon_1 &\geq \mathbb{E}\Big[J(t, \mu, \alpha^*; \xi, \alpha^*) - v(\mu^{\alpha^*}; t, \xi) \Big] \\
&= \mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - v(T_0, \tilde{\psi}; \mu^{\alpha^*}; t, \xi) \Big] \\
&\geq \mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - J(T_0, \tilde{\psi}; t, \mu, \alpha^*; \xi, \alpha^*) \Big] \\
&= \mathbb{E}\Big[\psi(X_{T_0}^{t,\xi,\alpha^*}) - \tilde{\psi}(X_{T_0}^{t,\xi,\alpha^*}) \Big].
\end{aligned}$$

Note that $\mathcal{L}_{X_{T_0}^{t,\xi,\alpha^*}} = \mu_{T_0}^{\alpha^*}$, then this is exactly (6.11).

Step 2. It remains to show that $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon}(T_0, \psi; t, \mu)$. By the definition of v and its regularity from Lemma 6.4, there exists $\tilde{\alpha}^* \in \mathcal{A}_{cont}$ such that

$$J(T_0, \psi; t, \mu, \alpha^*; x, \tilde{\alpha}^*) \leq v(T_0, \psi; \mu^{\alpha^*}; t, x) + \varepsilon_1, \quad \forall x \in \mathbb{R}^d.$$

Then, denoting $\hat{\alpha}^* := \tilde{\alpha}^* \oplus_{T_0} \alpha^* \in \mathcal{A}_{cont}$, by (6.9) again we have

$$\begin{aligned}
&\mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - v(T_0, \psi; \mu^{\alpha^*}; t, \xi) \Big] \\
&\leq \mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - J(T_0, \psi; t, \mu, \alpha^*; \xi, \tilde{\alpha}^*) \Big] + \varepsilon_1 \\
&= \mathbb{E}\Big[J(t, \mu, \alpha^*; \xi, \alpha^*) - J(t, \mu, \alpha^*; \xi, \hat{\alpha}^*) \Big] + \varepsilon_1 \\
&\leq \mathbb{E}\Big[J(t, \mu, \alpha^*; \xi, \alpha^*) - v(\mu^{\alpha^*}; t, \xi) \Big] + \varepsilon_1 \leq \varepsilon_1 + \varepsilon_1 = \varepsilon.
\end{aligned}$$

This means $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon}(T_0, \psi; t, \mu)$.

(ii) We next prove $\tilde{\mathbb{V}}_{cont}(t,\mu) \subset \mathbb{V}_{cont}(t,\mu)$. Fix $\varphi \in \tilde{\mathbb{V}}_{cont}(t,\mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{4}$. Since $\varphi \in \tilde{\mathbb{V}}_{cont}^{\varepsilon_1}(t,\mu)$, there exist (ψ, α^*) satisfying the desired properties in (6.5) for ε_1. In particular, since $\psi \in \mathbb{V}_{cont}^{\varepsilon_1}(T_0, \mu_{T_0}^{\alpha^*})$, there exists desired $\tilde{\alpha}^* \in \mathcal{M}_{cont}^{\varepsilon_1}(T_0, \mu_{T_0}^{\alpha^*})$ required in (6.1) for ε_1. Denote $\hat{\alpha}^* := \alpha^* \oplus_{T_0} \tilde{\alpha}^* \in \mathcal{A}_{cont}$ and

$$\hat{\psi}(x) := J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; x, \tilde{\alpha}^*), \quad \check{\psi}(x) := v(\mu^{\hat{\alpha}^*}; T_0, x).$$

By (6.5),

$$\mathbb{E}\Big[\big|J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - J(T_0, \hat{\psi}; t, \mu, \alpha^*; \xi, \alpha^*)\big|\Big]$$

(6.12)
$$= \mathbb{E}\Big[\big|\psi(X_{T_0}^{\mu^{\alpha^*}; t, \xi, \alpha^*}) - \hat{\psi}(X_{T_0}^{\mu^{\alpha^*}; t, \xi, \alpha^*})\big|\Big]$$

$$= \int_{\mathbb{R}^d} \big|\psi(x) - J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; x, \tilde{\alpha}^*)\big| \mu_{T_0}^{\alpha^*}(dx) \le \varepsilon_1.$$

Then, since $\varphi \in \tilde{\mathbb{V}}_{cont}^{\varepsilon_1}(t, \mu)$ with corresponding (ψ, α^*), by (6.9) and (6.12) we have

$$\mathbb{E}\Big[\big|\varphi(\xi) - J(t, \mu, \hat{\alpha}^*; \xi, \hat{\alpha}^*)\big|\Big] \le \mathbb{E}\Big[\big|\varphi(\xi) - J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*)\big|\Big] + \varepsilon_1 \le 2\varepsilon_1 \le \varepsilon,$$

where $\xi \in \mathbb{L}^2(t, \mu)$. We claim further that $\hat{\alpha}^* \in \mathcal{M}_{cont}^{\varepsilon}(t, \mu)$. Then $\varphi \in \mathbb{V}_{cont}^{\varepsilon}(t, \mu)$, and thus $\varphi \in \mathbb{V}_{cont}(t, \mu)$, since $\varepsilon > 0$ is arbitrary.

To see the claim, since $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon_1}(T_0, \psi; t, \mu)$, $\tilde{\alpha}^* \in \mathcal{M}_{cont}^{\varepsilon_1}(T_0, \mu_{T_0}^{\alpha^*})$, by (6.9) we have

$$\mathbb{E}\Big[J(t, \mu, \hat{\alpha}^*; \xi, \hat{\alpha}^*) - v(\mu^{\hat{\alpha}^*}; t, \xi)\Big]$$

$$= \mathbb{E}\Big[J(T_0, \hat{\psi}; t, \mu, \alpha^*; \xi, \alpha^*) - v(T_0, \tilde{\psi}; \mu^{\alpha^*}; t, \xi)\Big]$$

$$\le \mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \alpha^*) - v(T_0, \tilde{\psi}; \mu^{\alpha^*}; t, \xi)\Big] + \varepsilon_1$$

$$\le \mathbb{E}\Big[v(T_0, \psi; \mu^{\alpha^*}; t, \xi) - v(T_0, \tilde{\psi}; \mu^{\alpha^*}; t, \xi)\Big] + 2\varepsilon_1$$

$$\le \sup_{\tilde{\alpha} \in \mathcal{A}_{cont}} \mathbb{E}\Big[J(T_0, \psi; t, \mu, \alpha^*; \xi, \tilde{\alpha}) - J(T_0, \tilde{\psi}; t, \mu, \alpha^*; \xi, \tilde{\alpha})\Big] + 2\varepsilon_1$$

$$= \mathbb{E}\big[\psi(X_{T_0}^{t, \xi, \alpha^*}) - \tilde{\psi}(X_{T_0}^{t, \xi, \alpha^*})\big] + 2\varepsilon_1 \le \mathbb{E}\big[\hat{\psi}(X_{T_0}^{t, \xi, \alpha^*}) - \tilde{\psi}(X_{T_0}^{t, \xi, \alpha^*})\big] + 3\varepsilon_1$$

$$\le \varepsilon_1 + 3\varepsilon_1 = \varepsilon.$$

This means $\hat{\alpha}^* \in \mathcal{M}_{cont}^{\varepsilon}(t, \mu)$, and hence completes the proof. $\qquad\square$

Remark 6.6.

(i) Our set value $\mathbb{V}_{cont}(t, \mu)$ is defined for each (t, μ) with elements in $C_{Lip}(\mathbb{R}^d)$, instead of $\mathbb{V}(t, x, \mu) \subset \mathbb{R}$ for each (t, x, μ). This is consistent with (2.7) in the discrete model, and is due to the fact that an ε-MFE α^* in Definition 6.2 depends on (t, μ), but is common for all initial states x. Indeed, if we define $\mathbb{V}_{cont}(t, x, \mu)$ in an obvious manner, it will not satisfy the DPP.

(ii) The above observation is also consistent with the fact that the following master equation is local in (t, μ), but non-local in x due to the term $\partial_x V(t, \tilde{x}, \mu)$:
(6.13)

$$\partial_t V(t, x, \mu) + \frac{1}{2} \operatorname{tr}(\partial_{xx} V) + H(x, \mu, \partial_x V)$$

$$+ \int_{\mathbb{R}^d} \Big[\frac{1}{2} \operatorname{tr}(\partial_{\tilde{x}\mu} V(t, x, \mu, \tilde{x})) + \partial_p H(\tilde{x}, \mu, \partial_x V(t, \tilde{x}, \mu)) \partial_\mu V(t, x, \mu, \tilde{x})\Big] \mu(d\tilde{x}) = 0.$$

Under appropriate conditions, in particular under certain monotonicity conditions, the above master equation has a unique solution and we have $\mathbb{V}_{cont}(t, \mu) = \{\mathcal{V}(t, \mu)\}$ is a singleton, where $\mathcal{V}(t, \mu)(x) := V(t, x, \mu)$ is a function of x. In this way, we may also view (6.13) as a first order ODE on the space $C^2(\mathbb{R}^d)$ (the regularity in x is a

lot easier to obtain):

$$\partial_t \mathcal{V}(t,\mu) + \mathcal{H}(\mu, \mathcal{V}(t,\mu)) + \mathcal{M}(\mu, \mathcal{V}(t,\mu), \partial_\mu \mathcal{V}(t,\mu)) = 0,$$

(6.14)   where $\mathcal{H}(\mu, v(\cdot))(x) := \dfrac{1}{2}\operatorname{tr}(\partial_{xx}v(x)) + H(x, \mu, \partial_x v(x)),$

$$\mathcal{M}(\mu, v(\cdot), \tilde{v}(\cdot, \cdot))(x) := \int_{\mathbb{R}^d}\big[\dfrac{1}{2}\operatorname{tr}(\partial_{\tilde{x}}\tilde{v}(x, \tilde{x})) + \partial_p H(\tilde{x}, \mu, \partial_x v(\tilde{x}))\tilde{v}(x, \tilde{x})\big]\mu(d\tilde{x}).$$

It could be interesting to explore master equations from this perspective as well.

6.2. Convergence of the N-player game.

By enlarging the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, if necessary, we let $B^1, \cdots, B^N$ be independent d-dimensional Brownian motions on it. Set $\mathcal{A}_{cont}^\infty := \cup_{L \geq 0}\mathcal{A}_{cont}^L$, where, for each $L \geq 0$, $\mathcal{A}_{cont}^L$ denotes the set of admissible controls $\alpha : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2 \to \mathbb{A}$ such that

$$|\alpha(t, x, \mu) - \alpha(t, \tilde{x}, \tilde{\mu})| \leq L_\alpha|x - \tilde{x}| + LW_1(\mu, \tilde{\mu}).$$

Here the Lipschitz constant L_α may depend on α, hence the Lipschitz continuity in x is not uniform in α. We emphasize that the Lipschitz continuity in μ is under W_1, rather than W_2, so that we can use the representation (6.1). Note that $\mathcal{A}_{cont} = \mathcal{A}_{cont}^0$, and by Remark 3.1(i), all the results in the previous subsection remain true if we replace $\mathcal{A}_{cont}$ with $\mathcal{A}_{cont}^\infty$.

Given $t \in [0, T]$, $\vec{x} = (x_1, \cdots, x_N) \in \mathbb{R}^{dN}$ and $\vec{\alpha} = (\alpha_1, \cdots, \alpha_N) \in (\mathcal{A}_{cont}^L)^N$, consider

(6.15)

$$X_s^{t,\vec{x},\vec{\alpha};i} = x_i + \int_t^s b\big(r, X_r^{t,\vec{x},\vec{\alpha};i}, \mu_r^{t,\vec{x},\vec{\alpha}}, \alpha_i(r, X_r^{t,\vec{x},\vec{\alpha};i}, \mu_r^{t,\vec{x},\vec{\alpha}})\big)dr + B_s^i - B_t^i, i = 1, \cdots, N;$$

$$\text{where}\quad \mu_s^{t,\vec{x},\vec{\alpha}} := \dfrac{1}{N}\sum_{i=1}^N \delta_{X_s^{t,\vec{x},\vec{\alpha};i}};$$

$$J_i(t, \vec{x}, \vec{\alpha}) := \mathbb{E}\Big[g(X_T^{t,\vec{x},\vec{\alpha};i}, \mu_T^{t,\vec{x},\vec{\alpha}}) + \int_t^T f\big(s, X_s^{t,\vec{x},\vec{\alpha};i}, \mu_s^{t,\vec{x},\vec{\alpha}}, \alpha_i(s, X_s^{t,\vec{x},\vec{\alpha};i}, \mu_s^{t,\vec{x},\vec{\alpha}})\big)ds\Big],$$

$$v_i^{N,L}(t, \vec{x}, \vec{\alpha}) := \inf_{\tilde{\alpha} \in \mathcal{A}_{cont}^L} J_i(t, \vec{x}, (\vec{\alpha}^{-i}, \tilde{\alpha})).$$

In light of Lemma 6.4, the following regularity result is interesting in its own right. However, since it will not be used for our main result, we postpone its proof to Appendix.

Proposition 6.7. *Let Assumption* 6.1 *hold. For any $L \geq 0$, there exists a constant $C_L > 0$, depending only on T, d, C_0, L_0, and L, such that, for any $(t, \vec{x}) \in [0, T] \times \mathbb{R}^{dN}$, $\bar{x}, \tilde{x} \in \mathbb{R}^d$, and $\vec{\alpha} \in (\mathcal{A}_{cont}^L)^N$, we have*

(6.16) $\big|v_i^{N,L}\big(t, (\vec{x}^{-i}, \bar{x}), \vec{\alpha}\big) - v_i^{N,L}\big(t, (\vec{x}^{-i}, \tilde{x}), \vec{\alpha}\big)\big| \leq C_L|\bar{x} - \tilde{x}|, \quad i = 1, \cdots, N.$

Given $\alpha \in \mathcal{A}_{cont}^L$, by viewing it as the homogeneous control $(\alpha, \cdots, \alpha)$, we may use the simplified notations $X^{t,\vec{x},\alpha;i}$, $\mu^{t,\vec{x},\alpha}$, $J_i(t, \vec{x}, \alpha)$, and $v_i^{N,L}(t, \vec{x}, \alpha)$ in the obvious sense.

Definition 6.8.

(i) For $(t, \vec{x}) \in [0, T] \times \mathbb{R}^{dN}$, $\varepsilon > 0$, $L \geq 0$, we call $\alpha^* \in \mathcal{A}^L_{cont}$ a homogeneous (ε, L)-equilibrium of the N-player game at $(t, \vec{x})$, denoted as $\alpha^* \in \mathcal{M}^{N,\varepsilon,L}_{cont}(t, \vec{x})$, if

$$(6.17) \qquad \frac{1}{N} \sum_{i=1}^{N} \left[J_i(t, \vec{x}, \alpha^*) - v_i^{N,L}(t, \vec{x}, \alpha^*) \right] \leq \varepsilon.$$

(ii) The set value for the N-player game is defined as:

$$(6.18) \qquad \mathbb{V}^N_{cont}(t, \vec{x}) := \bigcap_{\varepsilon > 0} \mathbb{V}^{N,\varepsilon}_{cont}(t, \vec{x}) := \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \mathbb{V}^{N,\varepsilon,L}_{cont}(t, \vec{x}), \quad \text{where}$$

$$\mathbb{V}^{N,\varepsilon,L}_{cont}(t, \vec{x})$$

$$:= \left\{ \varphi \in C_{Lip}(\mathbb{R}^d) : \exists \alpha^* \in \mathcal{M}^{N,\varepsilon,L}_{cont}(t, \vec{x}) \text{ s.t. } \frac{1}{N} \sum_{i=1}^{N} |\varphi(x_i) - J_i(t, \vec{x}, \alpha^*)| \leq \varepsilon \right\}.$$

We remark that, although $\mathbb{V}^{N,\varepsilon,L}_{cont}(t, \vec{x})$ involves only the values $\{\varphi(x_i)\}_{1 \leq i \leq N}$, for the convenience of the convergence analysis we consider its elements as $\varphi \in C_{Lip}(\mathbb{R}^d)$.

Remark 6.9.

(i) Recall (3.1). By the required symmetry, obviously there exist functions $J^N, v^{N,L} : [0, T] \times \mathcal{P}_2 \times \mathcal{A}^L_{cont} \times \mathbb{R}^d \to \mathbb{R}$ such that
$$(6.19)$$
$$J_i(t, \vec{x}, \alpha) = J^N(t, \mu_{\vec{x}}^N, \alpha; x_i), \quad v_i^{N,L}(t, \vec{x}, \alpha) = v^{N,L}(t, \mu_{\vec{x}}^N, \alpha; x_i), \quad i = 1, \cdots, N.$$
Moreover, $\mathbb{V}^N_{cont}(t, \vec{x})$ is invariant in $\mu_{\vec{x}}^N$ and thus can be denoted as $\mathbb{V}^N_{cont}(t, \mu_{\vec{x}}^N)$.

(ii) The required inequalities in Definition 6.8 are equivalent to:

$$\int_{\mathbb{R}^d} [J^N - v^{N,L}](t, \mu_{\vec{x}}^N, \alpha^*; x) \mu_{\vec{x}}^N(dx) \leq \varepsilon, \quad \int_{\mathbb{R}^d} \left[\varphi(x) - J^N(t, \mu_{\vec{x}}^N, \alpha^*; x) \right] \mu_{\vec{x}}^N(dx) \leq \varepsilon.$$

We now turn to the convergence, starting with the convergence of the equilibrium measures. Recall the vector $(\alpha, \tilde{\alpha})_i$ introduced in (3.6).

Theorem 6.10. *Let Assumption* 6.1 *hold. For any* $L \geq 0$, *there exists a constant* $C_L > 0$, *depending only on* T, d, C_0, L_0, *and* L, *such that, for any* $t \in [0, T]$, $\vec{x} \in \mathbb{R}^{dN}$, $\mu \in \mathcal{P}_2$, $\alpha, \tilde{\alpha} \in \mathcal{A}^L_{cont}$, *and* $i = 1, \cdots, N$,

$$(6.20) \qquad \sup_{t \leq s \leq T} \mathbb{E}\left[W_1(\mu_s^{t, \vec{x}, (\alpha, \tilde{\alpha})_i}, \mu_s^\alpha) \right] \leq C_L \theta_N,$$

$$\text{where} \quad \theta_N := W_1(\mu_{\vec{x}}^N, \mu) + N^{-\frac{1}{d \vee 3}} \|\vec{x}\|_2 + N^{-1}, \quad \|\vec{x}\|_2^2 := \frac{1}{N} \sum_{i=1}^{N} |x_i|^2.$$

Proof. Recall (6.15) and introduce, for $j = 1, \cdots, N$,
$$(6.21)$$
$$\tilde{X}_s^j = x_j + \int_t^s b(r, \tilde{X}_r^j, \mu_r^\alpha, \alpha(r, \tilde{X}_r^j, \mu_r^\alpha)) dr + B_s^j - B_t^j, \quad \tilde{\mu}_s^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\tilde{X}_s^j};$$

$$\tilde{X}_s = \tilde{\xi} + \int_t^s b(r, \tilde{X}_r, \mu_r^\alpha, \alpha(r, \tilde{X}_r, \mu_r^\alpha)) dr + B_s - B_t, \text{ where } \tilde{\xi} \in \mathbb{L}^2(\mathcal{F}_0; \mu_{\vec{x}}^N).$$

Note that $\tilde{X}^1, \cdots, \tilde{X}^N$ are independent. We proceed the rest of the proof in two steps.

Step 1. In this step we estimate $\mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big]$. First, by [38, Lemma 8.4] we have

$$\mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mathcal{L}_{\tilde{X}_s})\big] \leq CN^{-\frac{1}{d \vee 3}} \|\vec{x}\|_2.$$

Next, fix an φ in (6.1) and let $u = u_\varphi$ denote the solution to the following PDE on $[t, s]$:

$$(6.22) \qquad \partial_r u + \frac{1}{2} \operatorname{tr}\big(\partial_{xx} u\big) + b(r, x, \mu_s^\alpha, \alpha(r, x, \mu_r^\alpha)) \cdot \partial_x u = 0, \quad u(s, x) = \varphi(x).$$

Applying Lemma 6.4 with $\tilde{\alpha}(r, x) := \alpha(r, x, \mu_r^\alpha)$ and $f = 0$, we see that u is uniformly Lipschitz continuous in x, with a Lipschitz constant C independent of φ and L. Thus,

$$\mathbb{E}\big[\varphi(\tilde{X}_s) - \varphi(X_s^\alpha)\big] = \mathbb{E}\big[u(t, \tilde{\xi}) - u(t, \xi)\big] \leq C\mathbb{E}[|\tilde{\xi} - \xi|].$$

Since $\mathcal{F}_0$ is atomless, we may choose $\xi, \tilde{\xi}$ such that $\mathbb{E}[|\tilde{\xi} - \xi|] = W_1(\mu_{\vec{x}}^N, \mu)$, then (6.1) implies $W_1(\mathcal{L}_{\tilde{X}_s}, \mu_s^\alpha) \leq CW_1(\mu_{\vec{x}}^N, \mu)$. Put together, we have

$$(6.23) \qquad \mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big] \leq CW_1(\mu_{\vec{x}}^N, \mu) + CN^{-\frac{1}{d \vee 3}} \|\vec{x}\|_2 \leq C\theta_N, \quad t \leq s \leq T.$$

Step 2. We next estimate $\mathbb{E}\big[W_1(\mu_s^{t, \vec{x}, (\alpha, \tilde{\alpha})_i}, \mu_s^\alpha)\big]$. Denote $\alpha_i := \tilde{\alpha}$, $\alpha_j := \alpha$ for $j \neq i$, and

$$\beta_s^j := b(s, \tilde{X}_s^j, \tilde{\mu}_s^N, \alpha_j(s, \tilde{X}_s^j, \tilde{\mu}_s^N)) - b(s, \tilde{X}_s^j, \mu_s^\alpha, \alpha(s, \tilde{X}_s^j, \mu_s^\alpha)), \quad 1 \leq j \leq N$$

$$M_s := \prod_{j=1}^N M_s^j, \quad M_s^j := \exp\Big(\int_t^s \beta_r^j dB_r^j - \frac{1}{2} \int_t^s |\beta_r^j|^2 dr\Big).$$

Then, by the Girsanov theorem we have

$$(6.24)$$
$$\begin{aligned}
\mathbb{E}\big[W_1(\mu_s^{t, \vec{x}, (\alpha, \tilde{\alpha})_i}, \mu_s^\alpha)\big] &= \mathbb{E}\big[M_s W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big] \\
&= \mathbb{E}\big[[M_s - 1] W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big] + \mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big] \\
&= \sum_{j=1}^N \mathbb{E}\Big[\int_t^s M_r \beta_r^j dB_r^j W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\Big] + \mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big].
\end{aligned}$$

By the martingale representation theorem, we have

$$(6.25) \qquad W_1(\tilde{\mu}_s^N, \mu_s^\alpha) = \mathbb{E}\big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\big] + \sum_{j=1}^N \int_t^s Z_r^j dB_r^j.$$

Note that $\tilde{X}^j$ are independent. Consider the following linear PDE on $[t, s] \times \mathbb{R}^{dN}$:
$$(6.26)$$
$$\partial_r u(r, \vec{x}') + \frac{1}{2} \sum_{j=1}^N \operatorname{tr}\big(\partial_{x_j x_j} u(r, \vec{x}')\big) + \sum_{j=1}^N b(r, x_j', \mu_s^\alpha, \alpha(r, x_j', \mu_r^\alpha)) \cdot \partial_{x_j} u(r, \vec{x}') = 0,$$
$$u(s, \vec{x}') = W_1(\mu_{\vec{x}'}^N, \mu_s^\alpha).$$

By standard BSDE theory, see e.g. [43, Chapter 5], we have $Z_r^j = \partial_{x_j} u(r, \vec{X}_r^{t, \vec{x}})$, where $X_r^{t, \vec{x}, j} := x_j + B_r^j - B_t^j$. Note that the terminal condition $u(s, \vec{x}')$ is Lipschitz continuous in x_j' with Lipschitz constant $\frac{1}{N}$. Then, similarly to (6.22), by

Lemma 6.4 we see that $|Z^j| \leq |\partial_{x_j} u| \leq \frac{C}{N}$ for some constant C independent of α and L. Thus, by (6.24) and (6.25),

$$\mathbb{E}\Big[W_1(\mu_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i}, \mu_s^\alpha) - W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\Big] = \sum_{j=1}^N \mathbb{E}\Big[\int_t^s M_r \beta_r^j \cdot Z_r^j dr\Big]$$

$$\leq \frac{C}{N} \sum_{j=1}^N \mathbb{E}\Big[\int_t^s M_r |\beta_r^j| dr\Big].$$

Note that $|\beta^i| \leq C$ and, for $j \neq i$, $|\beta_r^j| \leq C_L W_1(\tilde{\mu}_r^N, \mu_r^\alpha)$. Then, by (6.23),

$$\mathbb{E}\Big[W_1(\mu_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i}, \mu_s^\alpha)\Big] \leq \mathbb{E}\Big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\Big] + \frac{C}{N}\mathbb{E}\Big[\int_t^s M_r |\beta_r^i| dr + \sum_{j \neq i} \int_t^s M_r |\beta_r^j| dr\Big]$$

$$\leq \mathbb{E}\Big[W_1(\tilde{\mu}_s^N, \mu_s^\alpha)\Big] + \frac{C}{N} + \frac{C_L}{N} \sum_{j \neq i} \mathbb{E}\Big[\int_t^s M_r W_1(\tilde{\mu}_r^N, \mu_r^\alpha) dr\Big] = \frac{C}{N} + C_L \theta_N \leq C_L \theta_N.$$

$\square$

Theorem 6.11. *For the setting in Theorem* 6.10, *we have*

$$(6.27) \quad \Big|J_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha})\Big| + \Big|v_i^{N,L}(t, \vec{x}, \alpha) - v(\mu^\alpha; t, x_i)\Big| \leq C_L \theta_N^{\frac{1}{4}}.$$

Proof. Fix i. First, by taking supremum over $\tilde{\alpha} \in \mathcal{A}_{cont}^L$, the uniform estimate for J implies that for v immediately. So it suffices to prove the former estimate.

For this purpose, recall (6.15) and denote

$$\tilde{J}_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i)$$

$$:= \mathbb{E}^{\mathbb{P}}\Big[g(X_T^{t,\vec{x},(\alpha,\tilde{\alpha})_i;i}, \mu_T^\alpha) + \int_t^T f(s, X_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i;i}, \mu_s^\alpha, \tilde{\alpha}(s, X_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i;i}, \mu_s^\alpha)) ds\Big].$$

Then one can easily see that, by applying Theorem 6.10,
(6.28)
$$\Big|J_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - \tilde{J}_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i)\Big| \leq C_L \sup_{t \leq s \leq T} \mathbb{E}\Big[W_1(\mu_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i}, \mu_s^\alpha)\Big] \leq C_L \theta_N.$$

Next, denote

$$X_s^i := x_i + B_s^i - B_t^i, \quad \tilde{\mu}_s^{N,i} := \frac{1}{N}\Big[\sum_{j \neq i} \delta_{X_s^{t,\vec{x},(\alpha,\tilde{\alpha})_i;j}} + \delta_{X_s^i}\Big];$$

$$\beta_s := b(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha)), \quad M_s := \exp\Big(\int_t^s \beta_r dB_r^i - \frac{1}{2}\int_t^s |\beta_r|^2 dr\Big);$$

$$\tilde{\beta}_s := b(s, X_s^i, \tilde{\mu}_s^{N,i}, \tilde{\alpha}(s, X_s^i, \tilde{\mu}_s^{N,i})), \quad \tilde{M}_s := \exp\Big(\int_t^s \tilde{\beta}_r dB_r^i - \frac{1}{2}\int_t^s |\tilde{\beta}_r|^2 dr\Big).$$

By (6.1) and (6.15), it follows from the Girsanov theorem again that

$$\Big|\tilde{J}_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha})\Big|$$

$$(6.29) \quad = \Big|\mathbb{E}\Big[\big[\tilde{M}_T - M_T\big]\big[g(X_T^i, \mu_T^\alpha) + \int_t^T f(s, X_s^i, \mu_s^\alpha, \tilde{\alpha}(s, X_s^i, \mu_s^\alpha) ds\big]\Big]\Big|$$

$$\leq C\mathbb{E}\big[|\tilde{M}_T - M_T|\big].$$

Denote $\Delta M_s := \tilde{M}_s - M_s$, $\Delta \beta_s := \tilde{\beta}_s - \beta_s$. Then, since b is bounded,

$$\mathbb{E}[|\Delta M_s|^2] = \mathbb{E}\Big[\Big(\int_t^s [\tilde{M}_r \tilde{\beta}_r - M_r \beta_r] dB_r^i\Big)^2\Big] = \mathbb{E}\Big[\int_t^s [\tilde{M}_r \tilde{\beta}_r - M_r \beta_r]^2 dr\Big]$$

$$\leq C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C\mathbb{E}\Big[\int_t^s |\tilde{M}_r|^2 |\Delta \beta_r|^2 dr\Big]$$

$$\leq C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C\mathbb{E}\Big[\int_t^s \tilde{M}_r^{\frac{3}{2}} \tilde{M}_r^{\frac{1}{2}} |\Delta \beta_r|^{\frac{1}{2}} dr\Big]$$

$$\leq C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C\Big(\mathbb{E}\Big[\int_t^s \tilde{M}_r |\Delta \beta_r| dr\Big]\Big)^{\frac{1}{2}}$$

$$\leq C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C_L\Big(\mathbb{E}\Big[\int_t^s \tilde{M}_r W_1(\tilde{\mu}_r^{N,i}, \mu_r^\alpha) dr\Big]\Big)^{\frac{1}{2}}$$

$$= C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C_L\Big(\mathbb{E}\Big[\int_t^s W_1(\mu_r^{t,\vec{x},(\alpha,\tilde{\alpha})_i}, \mu_r^\alpha) dr\Big]\Big)^{\frac{1}{2}}$$

$$\leq C \int_t^s \mathbb{E}[|\Delta M_r|^2] dr + C_L \theta_N^{\frac{1}{2}},$$

where the last inequality thanks to Theorem 6.10. Then, by the Gröwnwall inequality we obtain $\mathbb{E}[|\Delta M_s|^2] \leq C_L \theta_N^{\frac{1}{2}}$, and thus (6.29) implies

$$\Big|\tilde{J}_i(t, \vec{x}, (\alpha, \tilde{\alpha})_i) - J(t, \mu, \alpha; x_i, \tilde{\alpha})\Big| \leq C_L \theta_N^{\frac{1}{4}}.$$

This, together with (6.28), implies the estimate for J in (6.27) immediately. $\square$

Theorem 6.12. *Let Assumption* 6.1 *hold. Assume further that* $\lim_{N\to\infty} W_1(\mu_{\vec{x}}^N, \mu) = 0$, *and there exists a constant* $C > 0$ *such that*[6] $\|\vec{x}\|_2 \leq C$ *for all* N. *Then*

$$(6.30) \qquad \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \overline{\lim_{N\to\infty}} \, \mathbb{V}_{cont}^{N,\varepsilon,L}(t, \mu_{\vec{x}}^N) \subset \mathbb{V}_{cont}(t, \mu) \subset \bigcap_{\varepsilon > 0} \lim_{N\to\infty} \mathbb{V}_{cont}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N).$$

In particular, since $\lim_{N\to\infty} \mathbb{V}_{cont}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N) \subset \bigcup_{L \geq 0} \overline{\lim_{N\to\infty}} \, \mathbb{V}_{cont}^{N,\varepsilon,L}(t, \mu_{\vec{x}}^N)$, *actually equalities hold.*

Proof. (i) We first prove the right inclusion in (6.30). Fix $\varphi \in \mathbb{V}_{cont}(t, \mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{2}$. By (6.1) and (6.5), there exists $\alpha^* \in \mathcal{M}_{cont}^{\varepsilon_1}(t, \mu)$ such that

$$\int_{\mathbb{R}^d} [J(t, \mu, \alpha^*; x, \alpha^*) - v(\mu^{\alpha^*}; t, x)] \mu(dx) \leq \varepsilon_1, \quad \int_{\mathbb{R}^d} |\varphi(x) - J(t, \mu, \alpha^*; x, \alpha^*)| \mu(dx) \leq \varepsilon_1.$$

Recall Lemma 6.4 and note that $\varphi \in C_{Lip}(\mathbb{R}^d)$, then by (6.1) we have

$$\int_{\mathbb{R}^d} [J(t, \mu, \alpha^*; x, \alpha^*) - v(\mu^{\alpha^*}; t, x)] \mu_{\vec{x}}^N(dx) \leq \varepsilon_1 + CW_1(\mu_{\vec{x}}^N, \mu),$$

$$\int_{\mathbb{R}^d} |\varphi(x) - J(t, \mu, \alpha^*; x, \alpha^*)| \mu_{\vec{x}}^N(dx) \leq \varepsilon_1 + C_\varphi W_1(\mu_{\vec{x}}^N, \mu),$$

---

[6]Note again that $\vec{x}$ depends on N. Also, the conditions here are slightly weaker than $\lim_{N\to\infty} W_2(\mu_{\vec{x}}^N, \mu) = 0$.

where C_φ may depend on the Lipschitz constant of φ. Moreover, by (6.27) we have

$$
\frac{1}{N} \sum_{i=1}^{N} \left[J_i(t, \vec{x}, \alpha^*) - v_i^{N,L}(t, \vec{x}, \alpha^*) \right]
$$

$$
\leq \frac{1}{N} \sum_{i=1}^{N} \left[J(t, \mu, \alpha^*; x_i, \alpha^*) - v(\mu^{\alpha^*}; t, x_i) \right] + C_L \theta_N^{\frac{1}{4}}
$$

$$
= \int_{\mathbb{R}^d} \left[J(t, \mu, \alpha^*; x, \alpha^*) - v(\mu^{\alpha^*}; t, x) \right] \mu_{\vec{x}}^N(dx) + C_L \theta_N^{\frac{1}{4}} \leq \varepsilon_1 + C_L \theta_N^{\frac{1}{4}};
$$

$$
\frac{1}{N} \sum_{i=1}^{N} |\varphi(x_i) - J_i(t, \vec{x}, \alpha^*)| \leq \frac{1}{N} \sum_{i=1}^{N} |\varphi(x_i) - J(t, \mu, \alpha^*; x_i, \alpha^*)| + C_L \theta_N^{\frac{1}{4}}
$$

$$
= \int_{\mathbb{R}^d} |\varphi(x) - J(t, \mu, \alpha^*; x, \alpha^*)| \mu_{\vec{x}}^N(dx) + C_L \theta_N^{\frac{1}{4}} \leq \varepsilon_1 + C_{L,\varphi} \theta_N^{\frac{1}{4}}.
$$

We emphasize again that $\|\vec{x}\|_2 \leq C$ is independent of N. Then, by choosing N large enough such that $C_L \theta_N^{\frac{1}{4}} \leq \varepsilon_1$, $C_{L,\varphi} \theta_N^{\frac{1}{4}} \leq \varepsilon_1$, we obtain

$$
\frac{1}{N} \sum_{i=1}^{N} \left[J_i(t, \vec{x}, \alpha^*) - v_i^{N,L}(t, \vec{x}, \alpha^*) \right] \leq \varepsilon; \quad \frac{1}{N} \sum_{i=1}^{N} |\varphi(x_i) - J_i(t, \vec{x}, \alpha^*)| \leq \varepsilon.
$$

This implies that $\alpha^* \in \mathcal{M}_{cont}^{N,\varepsilon,0}(t, \vec{x})$ and $\varphi \in \mathbb{V}_{cont}^{N,\varepsilon,0}(t, \mu_{\vec{x}}^N)$, for all N large enough. That is, $\varphi \in \underline{\lim}_{N \to \infty} \mathbb{V}_{cont}^{N,\varepsilon,0}(t, \vec{x})$ for any $\varepsilon > 0$.

(ii) We next show the left inclusion in (6.30). Fix $\varphi \in \bigcap_{\varepsilon > 0} \bigcup_{L \geq 0} \overline{\lim}_{N \to \infty} \mathbb{V}_{cont}^{N,\varepsilon,L}(t, \mu_{\vec{x}}^N)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{2}$. There exist $L_\varepsilon \geq 0$ and an infinite sequence $\{N_k\}_{k \geq 1}$ such that $\varphi \in \mathbb{V}_{cont}^{N_k,\varepsilon_1,L_\varepsilon}(t, \mu_{\vec{x}}^N)$ for all $k \geq 1$. Recall (6.17) and (6.18), there exists $\alpha^k \in \mathcal{A}_{cont}^{L_\varepsilon}$ such that

$$
\frac{1}{N_k} \sum_{i=1}^{N_k} \left[J_i(t, \vec{x}, \alpha^k) - v_i^{N_k,L_\varepsilon}(t, \vec{x}, \alpha^k) \right] \leq \varepsilon_1; \quad \frac{1}{N_k} \sum_{i=1}^{N_k} |\varphi(x_i) - J_i(t, \vec{x}, \alpha^k)| \leq \varepsilon_1.
$$

Note that L_ε is fixed, in particular it is independent of k. In light of Remark 3.1(i) and denote $\tilde{\alpha}^k(s, x) := \alpha^k(s, x, \mu^{\alpha^k})$, then $\mu^{\tilde{\alpha}^k} = \mu^{\alpha^k}$. Similarly to (i), by (6.27) we have

$$
\int_{\mathbb{R}^d} \left[J(t, \mu, \tilde{\alpha}^k; x, \tilde{\alpha}^k) - v(\mu^{\alpha^k}; t, x) \right] \mu_{\vec{x}}^{N_k}(dx) \leq \varepsilon_1 + C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}},
$$

$$
\int_{\mathbb{R}^d} |\varphi(x) - J(t, \mu, \tilde{\alpha}^k; x, \tilde{\alpha}^k)| \mu_{\vec{x}}^N(dx) \leq \varepsilon_1 + C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}}.
$$

Then, by Lemma 6.4 and (6.1) we have

$$
\int_{\mathbb{R}^d} \left[J(t, \mu, \tilde{\alpha}^k; x, \tilde{\alpha}^k) - v(\mu^{\alpha^k}; t, x) \right] \mu(dx) \leq \varepsilon_1 + C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}} + C W_1(\mu_{\vec{x}}^{N_k}, \mu),
$$

$$
\int_{\mathbb{R}^d} |\varphi(x) - J(t, \mu, \tilde{\alpha}^k; x, \tilde{\alpha}^k)| \mu(dx) \leq \varepsilon_1 + C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}} + C_\varphi W_1(\mu_{\vec{x}}^{N_k}, \mu).
$$

Now choose k large enough (possibly depending on ε and φ) such that

$$
C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}} + C W_1(\mu_{\vec{x}}^{N_k}, \mu) \leq \varepsilon_1, \quad C_{L_\varepsilon} \theta_{N_k}^{\frac{1}{4}} + C_\varphi W_1(\mu_{\vec{x}}^{N_k}, \mu) \leq \varepsilon_1.
$$

Then we have

$$\int_{\mathbb{R}^d}\big[J(t,\mu,\tilde{\alpha}^k;x,\tilde{\alpha}^k)-v(\mu^{\alpha^k};t,x)\big]\mu(dx)\leq\varepsilon,\quad\int_{\mathbb{R}^d}\big|\varphi(x)-J(t,\mu,\tilde{\alpha}^k;x,\tilde{\alpha}^k)\big|\mu(dx)\leq\varepsilon.$$

This implies that $\tilde{\alpha}^k\in\mathcal{M}^{\varepsilon}_{cont}(t,\mu)$ and $\varphi\in\mathbb{V}^{\varepsilon}_{cont}(t,\mu)$. Since $\varepsilon>0$ is arbitrary, we obtain $\varphi\in\mathbb{V}_{cont}(t,\mu)$. $\qquad\square$

7. Appendix

7.1. **Some examples.** In this subsection we first construct an example in discrete setting such that $\mathbb{V}_0\subset\mathbb{V}_{state}\subset\mathbb{V}_{path}\subset\mathbb{V}_{relax}$ with all the inclusions strict, where $\mathbb{V}_{path}$ are defined in an obvious way. In particular, $\mathbb{V}_0$ is empty.

Example 7.1. Set $T=2$, $\mathbb{S}=\{\underline{x},\overline{x}\}$, $\mathbb{A}=(\frac{1}{3},\frac{2}{3})$, and

$$q(0,x,\mu,a;\underline{x})=q(0,x,\mu,a;\overline{x})\equiv\frac{1}{2},\quad q(1,x,\mu,a;\underline{x})=a,\quad q(1,x,\mu,a;\overline{x})=1-a;$$

$$F(0,x,\mu,a)=0,\quad F(1,x,\mu,a)=F_1(a):=a[1-a],\quad G(x,\mu)=\mu(\underline{x}).$$

Then for any $\mu\in\mathcal{P}_0(\mathbb{S})$, we have $\mathbb{V}=\{(y,y):y\in\hat{\mathbb{V}}\}$ for $\mathbb{V}=\mathbb{V}_0,\mathbb{V}_{state},\mathbb{V}_{path}$, $\mathbb{V}_{relax}$, and

$$\hat{\mathbb{V}}_0(0,\mu)=\emptyset,\qquad\hat{\mathbb{V}}_{state}(0,\mu)=\Big\{\frac{5}{9},\frac{13}{18},\frac{8}{9}\Big\},$$

(7.1) $\qquad\hat{\mathbb{V}}_{path}(0,\mu):=\Big\{\underline{\lambda}\,\mu(\underline{x})+\overline{\lambda}\,\mu(\overline{x})+\frac{2}{9}:\quad\underline{\lambda},\overline{\lambda}\in\Big\{\frac{1}{3},\frac{1}{2},\frac{2}{3}\Big\}\Big\},$

$$\hat{\mathbb{V}}_{relax}(0,\mu):=\Big\{\underline{\lambda}\,\mu(\underline{x})+\overline{\lambda}\,\mu(\overline{x})+\frac{2}{9}:\quad\underline{\lambda},\overline{\lambda}\in\Big[\frac{1}{3},\frac{2}{3}\Big]\Big\}.$$

Proof. Since $|\mathbb{S}|=2$, for any $\mu\in\mathcal{P}_0(\mathbb{S})$ clearly it suffices to specify $\mu(\underline{x})$.

(i) We first compute $\mathbb{V}_0(0,\mu)$. For any $\alpha,\tilde{\alpha}\in\mathcal{A}_{state}$, it is straightforward to compute:

$$\mu_1^{\alpha}(\underline{x})=\sum_{x_0\in\mathbb{S}}\mu(x_0)q(0,x_0,\mu,\alpha(0,x_0);\underline{x})=\sum_{x_0\in\mathbb{S}}\mu(x_0)\frac{1}{2}=\frac{1}{2};$$

(7.2) $\qquad\mu_2^{\alpha}(\underline{x})=\sum_{x_1\in\mathbb{S}}\mu_1^{\alpha}(x_1)q(1,x_1,\mu_1^{\alpha},\alpha(1,x_1);\underline{x})=\frac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1);$

$$\mathbb{P}^{\mu^{\alpha};0,x_0,\tilde{\alpha}}(X_1=\underline{x})=q(0,x_0,\mu,\tilde{\alpha}(0,x_0);\underline{x})=\frac{1}{2}.$$

Then

$$J(0,\mu,\alpha;x_0,\tilde{\alpha})=\mathbb{E}^{\mathbb{P}^{\mu^{\alpha};0,x_0,\tilde{\alpha}}}\Big[G(X_2,\mu_2^{\alpha})+\sum_{t=0,1}F(t,X_t,\mu_t^{\alpha},\tilde{\alpha}(t,X_t))\Big]$$

$$=\mu_2^{\alpha}(\underline{x})+\mathbb{E}^{\mathbb{P}^{\mu^{\alpha};0,x_0,\tilde{\alpha}}}\Big[F_1(\tilde{\alpha}(1,X_1))\Big]$$

(7.3) $\qquad=\dfrac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1)+\dfrac{1}{2}\sum_{x_1\in\mathbb{S}}F_1(\tilde{\alpha}(1,x_1)).$

Given α, we see that $\inf_{\tilde{\alpha}}J(0,\mu,\alpha;x_0,\tilde{\alpha})=\frac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1)+\frac{2}{9}$, and the minimum is achieved when $\tilde{\alpha}(1,x_1)=\frac{1}{3},\frac{2}{3}$, $\forall x_1\in\mathbb{S}$, which are not included in $\mathbb{A}$. Thus $\mathcal{M}_{state}(0,\mu)=\emptyset$, and hence $\mathbb{V}_0(0,\mu)=\emptyset$.

(ii) We next compute $\mathbb{V}_{state}(0,\mu)$. Fix $\varepsilon > 0$ small. By (2.13) and (7.3) it is clear that

$$(7.4) \qquad \alpha^\varepsilon \in \mathcal{M}_{state}^\varepsilon(0,\mu) \quad \text{if and only if} \quad \frac{1}{2}\sum_{x_1 \in \mathbb{S}} F_1(\alpha^\varepsilon(1,x_1)) \le \frac{2}{9} + \varepsilon,$$

and in this case, for any $x_0 \in \mathbb{S}$, by (7.3) again we have

$$J(0,\mu,\alpha^\varepsilon;x_0,\alpha^\varepsilon) = J_0(\alpha^\varepsilon) := \frac{1}{2}\sum_{x_1 \in \mathbb{S}} \tilde{F}_1(\alpha^\varepsilon(1,x_1)),$$

$$\text{where} \quad \tilde{F}_1(a) := a + F_1(a) = a[2-a].$$

In particular, this implies that $\mathbb{V}_{state}^\varepsilon(0,\mu) = \left\{(y,y) : y \in \hat{\mathbb{V}}_{state}^\varepsilon(0,\mu)\right\}$ where

$$\hat{\mathbb{V}}_{state}^\varepsilon(0,\mu) := \left\{J_0(\alpha^\varepsilon) : \alpha^\varepsilon \in \mathcal{M}_{state}^\varepsilon(0,\mu)\right\}.$$

Recall again that $\inf_{a \in \mathbb{A}} F_1(a) = \frac{2}{9}$. By (7.4), $\alpha^\varepsilon \in \mathcal{M}_{state}^\varepsilon(0,\mu)$ if and only if there exists a function $\chi_\varepsilon : \mathbb{S} \to \mathbb{R}$ such that $F_1(\alpha^\varepsilon(1,x_1)) = \frac{2}{9} + \chi_\varepsilon(x_1)$ for all $x_1 \in \mathbb{S}$, and

$$(7.5) \qquad \chi_\varepsilon(\underline{x}), \chi_\varepsilon(\overline{x}) > 0, \quad \chi_\varepsilon(\underline{x}) + \chi_\varepsilon(\overline{x}) \le 2\varepsilon.$$

This implies that

$$\alpha^\varepsilon(1,x_1) = \frac{1}{3} + \hat{\chi}_\varepsilon(x_1) \text{ or } \frac{2}{3} - \hat{\chi}_\varepsilon(x_1), \quad \text{where} \quad \hat{\chi}_\varepsilon(x_1) := \frac{6\chi_\varepsilon(x_1)}{1+\sqrt{1-36\chi_\varepsilon(x_1)}}.$$

Note that $\tilde{F}_1$ is strictly increasing for $a \in \mathbb{A}$. Then, by (2.14) we have, for $\varepsilon > 0$ small,

$$\hat{\mathbb{V}}_{state}^\varepsilon(0,\mu) = \bigcup_{\chi_\varepsilon}\bigcup_{i=1}^4 (y_i - \varepsilon, y_i + \varepsilon),$$

$$y_1 := \frac{1}{2}\Big[\tilde{F}_1\big(\frac{1}{3}+\chi_\varepsilon(\underline{x})\big) + \tilde{F}_1\big(\frac{1}{3}+\chi_\varepsilon(\overline{x})\big)\Big],$$

$$y_2 := \frac{1}{2}\Big[\tilde{F}_1\big(\frac{1}{3}+\chi_\varepsilon(\underline{x})\big) + \tilde{F}_1\big(\frac{2}{3}-\chi_\varepsilon(\overline{x})\big)\Big],$$

$$y_3 := \frac{1}{2}\Big[\tilde{F}_1\big(\frac{2}{3}-\chi_\varepsilon(\underline{x})\big) + \tilde{F}_1\big(\frac{1}{3}+\chi_\varepsilon(\overline{x})\big)\Big],$$

$$y_4 := \frac{1}{2}\Big[\tilde{F}_1\big(\frac{2}{3}-\chi_\varepsilon(\underline{x})\big) + \tilde{F}_1\big(\frac{2}{3}-\chi_\varepsilon(\overline{x})\big)\Big],$$

where the first union is over all χ_ε satisfying (7.5). Note that $0 < \chi_\varepsilon(\underline{x}), \chi_\varepsilon(\overline{x}) < 2\varepsilon$. Then by (2.14) it is obvious that $\mathbb{V}_{state}(0,\mu) = \left\{(y,y) : y \in \hat{\mathbb{V}}_{state}(0,\mu)\right\}$ and

$$\hat{\mathbb{V}}_{state}(0,\mu) = \left\{\tilde{F}_1\big(\frac{1}{3}\big), \frac{1}{2}\big[\tilde{F}_1\big(\frac{1}{3}\big)+\tilde{F}_1\big(\frac{2}{3}\big)\big], \tilde{F}_1\big(\frac{2}{3}\big)\right\} = \left\{\frac{5}{9}, \frac{13}{18}, \frac{8}{9}\right\}.$$

(iii) We now compute $\mathbb{V}_{path}(0, \mu)$. For any $\alpha, \tilde{\alpha} \in \mathcal{A}_{path}$, we still have $\mu_1^\alpha(\underline{x}) = \frac{1}{2}$ and $\mathbb{P}^{\mu^\alpha; 0, x_0, \tilde{\alpha}}(X_1 = \underline{x}) = \frac{1}{2}$, for all $x_0 \in \mathbb{S}$. Moreover,

$$\mu_2^\alpha(\underline{x}) = \sum_{x_0, x_1 \in \mathbb{S}} \mu(x_0) q(0, x_0, \mu, \alpha(0, x_0); x_1) q(1, x_1, \mu_1^\alpha, \alpha(1, x_0, x_1); \underline{x})$$

$$= \frac{1}{2} \sum_{x_0, x_1 \in \mathbb{S}} \mu(x_0) \alpha(1, x_0, x_1);$$

$$J(0, \mu, \alpha; x_0, \tilde{\alpha}) = \mathbb{E}^{\mathbb{P}^{\mu^\alpha; 0, x_0, \tilde{\alpha}}} \Big[G(X_2, \mu_2^\alpha) + F(1, X_1, \mu_1^\alpha, \tilde{\alpha}(1, X_0, X_1)) \Big]$$

$$= \mu_2^\alpha(\underline{x}) + \mathbb{E}^{\mathbb{P}^{\mu^\alpha; 0, x_0, \tilde{\alpha}}} \Big[F_1(\tilde{\alpha}(1, X_0, X_1)) \Big]$$

$$(7.6) \qquad = \sum_{\tilde{x}_0 \in \mathbb{S}} \mu(\tilde{x}_0) \times \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \alpha(1, \tilde{x}_0, x_1) + \frac{1}{2} \sum_{x_1 \in \mathbb{S}} F_1(\tilde{\alpha}(1, x_0, x_1)).$$

Similarly to (7.4),

$$\alpha^\varepsilon \in \mathcal{M}_{path}^\varepsilon(0, \mu) \quad \text{if and only if} \quad \frac{1}{2} \sum_{x_1 \in \mathbb{S}} F_1(\alpha^\varepsilon(1, x_0, x_1)) \le \frac{2}{9} + \varepsilon, \forall x_0 \in \mathbb{S}.$$

Furthermore, by abusing the notation χ_ε, the above is equivalent to that there exists $\chi_\varepsilon : \mathbb{S} \times \mathbb{S} \to \mathbb{A}$ such that, by denoting $\hat{\chi}_\varepsilon(x_0, x_1) := \frac{6\chi_\varepsilon(x_0, x_1)}{1 + \sqrt{1 - 36\chi_\varepsilon(x_0, x_1)}}$,

$$\chi_\varepsilon(x_0, x_1) > 0, \forall x_0, x_1 \in \mathbb{S}, \quad \text{and} \quad \chi_\varepsilon(x_0, \underline{x}) + \chi_\varepsilon(x_0, \overline{x}) \le 2\varepsilon, \forall x_0 \in \mathbb{S};$$

$$\alpha^\varepsilon(1, x_0, x_1) = \frac{1}{3} + \hat{\chi}_\varepsilon(x_0, x_1) \text{ or } \frac{2}{3} - \hat{\chi}_\varepsilon(x_0, x_1).$$

Following the same arguments as in (ii), we can easily see that $\mathbb{V}_{path}(0, \mu)$ consists of pairs $\big(J(0, \mu, \alpha^*; \underline{x}, \alpha^*), J(0, \mu, \alpha^*; \overline{x}, \alpha^*) \big)$ for all $\alpha^* : \mathbb{S}^2 \to \{\frac{1}{3}, \frac{2}{3}\}$. Note that $F_1(\frac{1}{3}) = F_1(\frac{2}{3}) = \frac{2}{9}$, and $\frac{1}{2} \sum_{x_1 \in \mathbb{S}} \alpha^*(1, \tilde{x}_0, x_1)$ takes 3 possible values: $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. Then by (7.6) we have

$$(7.7) \qquad J(0, \mu, \alpha^*; x_0, \alpha^*) = \underline{\lambda} \, \mu(\underline{x}) + \overline{\lambda} \, \mu(\overline{x}) + \frac{2}{9}, \quad \text{where} \quad \underline{\lambda}, \overline{\lambda} \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}.$$

Again this is independent of x_0. Then $\mathbb{V}_{path}(0, \mu) = \Big\{ (y, y) : y \in \hat{\mathbb{V}}_{path}(0, \mu) \Big\}$ and

$$\hat{\mathbb{V}}_{path}(0, \mu) := \Big\{ \underline{\lambda} \, \mu(\underline{x}) + \overline{\lambda} \, \mu(\overline{x}) + \frac{2}{9} : \quad \underline{\lambda}, \overline{\lambda} \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\} \Big\}.$$

In particular, we see that $\hat{\mathbb{V}}_{state}(0, \mu)$ consists of the elements of $\hat{\mathbb{V}}_{path}(0, \mu)$ with $\underline{\lambda} = \overline{\lambda}$, and $\hat{\mathbb{V}}_{path}(0, \mu) = \hat{\mathbb{V}}_{state}(0, \mu)$ when $\mu(\underline{x}) = \mu(\overline{x})$.

(iv) Finally we compute $\mathbb{V}_{relax}(0, \mu)$. Fix $\gamma, \tilde{\gamma} \in \mathcal{A}_{relax}$, it is straightforward to compute:

$$\mu_1^\gamma(\underline{x}) = \sum_{x_0 \in \mathbb{S}} \mu(x_0) \int_{\mathbb{A}} q(0, x_0, \mu, a; \underline{x}) \gamma(0, x_0; da) = \sum_{x_0 \in \mathbb{S}} \mu(x_0) \times \frac{1}{2} = \frac{1}{2};$$

$$\mathbb{P}^{\mu^\gamma; 0, x_0, \tilde{\gamma}}(X_1 = \underline{x}) = \int_{\mathbb{A}} q(0, x_0, \mu, a; \underline{x}) \tilde{\gamma}(0, x_0; da) = \frac{1}{2};$$

$$\mu_2^\gamma(\underline{x})$$

$$= \sum_{x_0, x_1 \in \mathbb{S}} \mu(x_0) \int_{\mathbb{A}^2} q(0, x_0, \mu, a_0; x_1) q(1, x_1, \mu_1^\gamma, a_1; \underline{x}) \gamma(0, x_0; da_0) \gamma(1, x_0, x_1; da_1)$$

$$= \frac{1}{2} \sum_{x_0, x_1 \in \mathbb{S}} \mu(x_0) \int_{\mathbb{A}} a \gamma(1, x_0, x_1; da);$$

$$J(0, \mu, \gamma; x_0, \tilde{\gamma}) = \mathbb{E}^{\mathbb{P}^{\mu^\gamma; 0, x_0, \tilde{\gamma}}} \Big[G(X_2, \mu_2^\gamma) + \sum_{t=0,1} \int_{\mathbb{A}} F(t, X_t, \mu_t^\gamma, a) \tilde{\gamma}(t, X; da) \Big]$$

$$= \mu_2^\gamma(\underline{x}) + \mathbb{E}^{\mathbb{P}^{\mu^\gamma; 0, x_0, \tilde{\gamma}}} \Big[\int_{\mathbb{A}} F_1(a) \tilde{\gamma}(1, X; da) \Big]$$

$$= \frac{1}{2} \sum_{\tilde{x}_0, x_1 \in \mathbb{S}} \mu(\tilde{x}_0) \int_{\mathbb{A}} a \gamma(1, \tilde{x}_0, x_1; da) + \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} F_1(a) \tilde{\gamma}(1, x_0, x_1; da).$$

Similarly to (7.4),
(7.8)
$$\gamma^\varepsilon \in \mathcal{M}_{relax}^\varepsilon(0, \mu) \quad \text{if and only if} \quad \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} F_1(a) \gamma^\varepsilon(1, x_0, x_1; da) \le \frac{2}{9} + \varepsilon, \forall x_0 \in \mathbb{S},$$

and in this case, for any $x_0 \in \mathbb{S}$,

$$(7.9) \quad J(0, \mu, \gamma^\varepsilon; x_0, \gamma^\varepsilon)$$
$$= \frac{1}{2} \sum_{\tilde{x}_0, x_1 \in \mathbb{S}} \mu(\tilde{x}_0) \int_{\mathbb{A}} a \gamma^\varepsilon(1, \tilde{x}_0, x_1; da) + \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} F_1(a) \gamma^\varepsilon(1, x_0, x_1; da).$$

Let $\hat{\mathcal{M}}_{relax}$ denote the set of $\gamma^* : \mathbb{S}^2 \to \mathcal{P}(\{\frac{1}{3}, \frac{2}{3}\})$ and set

$$(7.10) \qquad \hat{J}(\gamma^*) := \frac{1}{2} \sum_{x_0, x_1 \in \mathbb{S}} \mu(x_0) \Big[\frac{1}{3} \gamma^*(x_0, x_1; \frac{1}{3}) + \frac{2}{3} \gamma^*(x_0, x_1; \frac{2}{3}) \Big] + \frac{2}{9}.$$

We claim that, for any $\gamma^\varepsilon \in \mathcal{M}_{relax}^\varepsilon(0, \mu)$, there exists $\hat{\gamma}^\varepsilon \in \hat{\mathcal{M}}_{relax}$ such that

$$(7.11) \qquad \Big| J(0, \mu, \gamma^\varepsilon; x_0, \gamma^\varepsilon) - \hat{J}(\hat{\gamma}^\varepsilon) \Big| \le C\sqrt{\varepsilon}.$$

On the other hand, for any $\gamma^* \in \hat{\mathcal{M}}_{relax}$, denote

$$(7.12) \qquad A_1^\varepsilon := (\frac{1}{3}, \frac{1}{3} + \sqrt{\varepsilon}], \quad A_2^\varepsilon := [\frac{2}{3} - \sqrt{\varepsilon}, \frac{2}{3}), \quad A_3^\varepsilon := \mathbb{A} \backslash (A_1^\varepsilon \cup A_2^\varepsilon),$$

and set $\gamma^\varepsilon \in \mathcal{A}_{relax}$ such that

$$\gamma^\varepsilon(1, x_0, x_1; da) := \frac{1}{2\sqrt{\varepsilon}} \Big[\gamma^*(x_0, x_1; \frac{1}{3}) \mathbf{1}_{A_1^\varepsilon}(a) + \gamma^*(x_0, x_1; \frac{2}{3}) \mathbf{1}_{A_2^\varepsilon}(a) \Big] da.$$

Note that $F_1(a) \leq (\frac{1}{3} + \sqrt{\varepsilon})(\frac{2}{3} - \sqrt{\varepsilon}) = \frac{2}{9} + \frac{\sqrt{\varepsilon}}{3} - \varepsilon$, $\gamma^\varepsilon(1, x_0, x_1; da)$-a.s. Then it is clear that $\gamma^\varepsilon \in \mathcal{M}_{relax}^{\frac{\sqrt{\varepsilon}}{3} - \varepsilon}$. Moreover, one can easily verify that

$$\left| J(0, \mu, \gamma^\varepsilon; x_0, \gamma^\varepsilon) - \hat{J}(\hat{\gamma}^\varepsilon) \right|$$

$$\leq \sum_{i=1}^{2} \frac{1}{2} \sum_{\tilde{x}_0, x_1 \in \mathbb{S}} \mu(\tilde{x}_0) \gamma^*(x_0, x_1; \frac{i}{3}) \left| \frac{1}{\sqrt{\varepsilon}} \int_{A_i^\varepsilon} a\, da - \frac{i}{3} \right| + \frac{\sqrt{\varepsilon}}{3} - \varepsilon \leq C\sqrt{\varepsilon}.$$

This, together with (7.11) and (4.3), implies that $\mathbb{V}_{relax}(0, \mu) = \Big\{ (y, y) : y \in \hat{\mathbb{V}}_{relax}(0, \mu) \Big\}$ and, by denoting $\underline{\lambda} := \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \left[\frac{1}{3} \gamma^*(\underline{x}, x_1; \frac{1}{3}) + \frac{2}{3} \gamma^*(\underline{x}, x_1; \frac{2}{3}) \right] \in [\frac{1}{3}, \frac{2}{3}]$ and similarly for $\overline{\lambda}$,

$$\hat{\mathbb{V}}_{relax}(0, \mu) := \left\{ \hat{J}(\gamma^*) : \gamma^* \in \hat{\mathcal{M}}_{relax} \right\} = \left\{ \underline{\lambda}\, \mu(\underline{x}) + \overline{\lambda}\, \mu(\overline{x}) + \frac{2}{9} : \quad \underline{\lambda}, \overline{\lambda} \in [\frac{1}{3}, \frac{2}{3}] \right\}.$$

It remains to prove (7.11). Let γ^ε satisfies (7.8). Then, for any $x_0 \in \mathbb{S}$, we have

$$\varepsilon \geq \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} F_1(a) \gamma^\varepsilon(1, x_0, x_1; da) - \frac{2}{9} = \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} (a - \frac{1}{3})(\frac{2}{3} - a) \gamma^\varepsilon(1, x_0, x_1; da)$$

$$\geq \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{A_3^\varepsilon} (a - \frac{1}{3})(\frac{2}{3} - a) \gamma^\varepsilon(1, x_0, x_1; da)$$

$$\geq \sqrt{\varepsilon}(\frac{1}{3} - \sqrt{\varepsilon}) \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{A_3^\varepsilon} \gamma^\varepsilon(1, x_0, x_1; da).$$

Thus

$$\int_{A_3^\varepsilon} \gamma^\varepsilon(1, x_0, x_1; da) \leq C\sqrt{\varepsilon}, \quad \forall x_0, x_1 \in \mathbb{S}.$$

Recall (7.12) and set $\hat{\gamma}^\varepsilon \in \hat{\mathcal{M}}_{relax}$ by:

$$\hat{\gamma}^\varepsilon(x_0, x_1; \frac{1}{3}) := \frac{\gamma^\varepsilon(1, x_0, x_1; A_1^\varepsilon)}{\sum_{i=1}^{2} \gamma^\varepsilon(1, x_0, x_1; A_i^\varepsilon)}, \quad \hat{\gamma}^\varepsilon(x_0, x_1; \frac{2}{3}) := \frac{\gamma^\varepsilon(1, x_0, x_1; A_2^\varepsilon)}{\sum_{i=1}^{2} \gamma^\varepsilon(1, x_0, x_1; A_i^\varepsilon)}.$$

Then $F_1(a) = \frac{2}{9}$, $\hat{\gamma}^\varepsilon(x_0, x_1; da)$-a.s., and thus

$$\left| J(0, \mu, \gamma^\varepsilon; x_0, \gamma^\varepsilon) - \hat{J}(\hat{\gamma}^\varepsilon) \right|$$

$$\leq \sum_{i=1}^{2} \frac{1}{2} \sum_{\tilde{x}_0, x_1 \in \mathbb{S}} \mu(\tilde{x}_0) \left| \int_{\mathbb{A}_i^\varepsilon} a \gamma^\varepsilon(1, \tilde{x}_0, x_1; da) - \frac{i}{3} \hat{\gamma}^\varepsilon(\tilde{x}_0, x_1; A_i^\varepsilon) \right|$$

$$+ \frac{1}{2} \sum_{\tilde{x}_0, x_1 \in \mathbb{S}} \mu(\tilde{x}_0) \int_{\mathbb{A}_3^\varepsilon} a \gamma^\varepsilon(1, \tilde{x}_0, x_1; da) + \left| \frac{1}{2} \sum_{x_1 \in \mathbb{S}} \int_{\mathbb{A}} F_1(a) \gamma^\varepsilon(1, x_0, x_1; da) - \frac{2}{9} \right|$$

$$\leq C \sum_{i=1}^{2} \left| \gamma^\varepsilon(1, \tilde{x}_0, x_1; A_i^\varepsilon) - \hat{\gamma}^\varepsilon(\tilde{x}_0, x_1; A_i^\varepsilon) \right| + C\sqrt{\varepsilon}$$

$$\leq C \frac{1 - \sum_{i=1}^{2} \gamma^\varepsilon(1, \tilde{x}_0, x_1; A_i^\varepsilon)}{\sum_{i=1}^{2} \gamma^\varepsilon(1, \tilde{x}_0, x_1; A_i^\varepsilon)} + C\sqrt{\varepsilon} \leq \frac{C\sqrt{\varepsilon}}{1 - C\sqrt{\varepsilon}} + C\sqrt{\varepsilon} \leq C\sqrt{\varepsilon}.$$

This proves (7.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

Our next example shows that the left inclusion in (3.13) fails if we remove the L-Lipschitz continuity requirement, as mentioned in Remark 3.8(ii). This justifies our uniform regularity requirement on the admissible controls in order to have the desired convergence as in Theorem 3.6. Recall $\mathbb{V}^{N,\varepsilon}$ and $\mathbb{V}^{N,\varepsilon,\infty}$ in Remark 3.8(ii).

Example 7.2. Let $T,\mathbb{S},q$ be as in Example 7.1, and

$$\mathbb{A} = [\frac{1}{3}, \frac{2}{3}], \quad F \equiv 0, \quad G(\underline{x}, \mu) = \frac{20}{9} - 5\mu(\underline{x}), \quad G(\overline{x}, \mu) = \frac{20}{9} - 3\mu(\overline{x}).$$

Then, for any $\mu \in \mathcal{P}_0(\mathbb{S})$ and $\mu_{\vec{x}}^N \in \mathcal{P}_N(\mathbb{S})$ with $\mu_{\vec{x}}^N \to \mu$, $(0,0)$ is in $\bigcap_{\varepsilon>0} \varliminf_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon,\infty}(0, \mu_{\vec{x}}^N)$ and $\bigcap_{\varepsilon>0} \varliminf_{N\to\infty} \mathbb{V}_{state}^{N,\varepsilon}(0, \mu_{\vec{x}}^N)$, but not in $\mathbb{V}_{state}(0, \mu)$.

Proof. (i) We first compute $\mathbb{V}_{state}(0, \mu)$. For $\alpha, \tilde{\alpha} \in \mathcal{A}_{state}$ (which do not depend on μ), similarly to (7.1) we have

$$(7.13) \qquad \mu_1^\alpha(\underline{x}) = \frac{1}{2}, \quad \mu_2^\alpha(\underline{x}) = \frac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1), \quad \mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}(X_1 = \underline{x}) = \frac{1}{2},$$

$$\mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}(X_2 = \underline{x}) = \sum_{x_1\in\mathbb{S}}\mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}(X_1 = x_1)q(1,x_1,\mu_1^\alpha,\tilde{\alpha}(1,x_1);\underline{x})$$

$$= \frac{1}{2}\sum_{x_1\in\mathbb{S}}\tilde{\alpha}(1,x_1).$$

Then

$$J(0,\mu,\alpha;x_0,\tilde{\alpha}) = \mathbb{E}^{\mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}}[G(X_2,\mu_2^\alpha)]$$

$$= \frac{20}{9} - 5\mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}(X_2 = \underline{x})\mu_2^\alpha(\underline{x}) - 3\mathbb{P}^{\mu^\alpha;0,x_0,\tilde{\alpha}}(X_2 = \overline{x})\mu_2^\alpha(\overline{x})$$

$$= \frac{20}{9} - \frac{5}{2}\sum_{x_1\in\mathbb{S}}\tilde{\alpha}(1,x_1)\times\frac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) - 3\Big[1 - \frac{1}{2}\sum_{x_1\in\mathbb{S}}\tilde{\alpha}(1,x_1)\Big]\Big[1 - \frac{1}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1)\Big]$$

$$= \frac{1}{2}\Big[3 - 4\sum_{x_1\in\mathbb{S}}\alpha(1,x_1)\Big]\sum_{x_1\in\mathbb{S}}\tilde{\alpha}(1,x_1) + \frac{3}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) - \frac{7}{9}.$$

Note that, when $\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) > \frac{3}{4}$, $\inf_{\tilde{\alpha}\in\mathcal{A}_{state}} J(0,\mu,\alpha;x_0,\tilde{\alpha})$ is achieved at $\tilde{\alpha} \equiv \frac{2}{3}$. Since $\sum_{x_1\in\mathbb{S}}\frac{2}{3} = \frac{4}{3} > \frac{3}{4}$, then $\alpha \equiv \frac{2}{3}$ is an equilibrium with

$$J(0,\mu,\frac{2}{3};x_0,\frac{2}{3}) = \frac{1}{2}\Big[3 - 4\sum_{x_1\in\mathbb{S}}\frac{2}{3}\Big]\sum_{x_1\in\mathbb{S}}\frac{2}{3} + \frac{3}{2}\sum_{x_1\in\mathbb{S}}\frac{2}{3} - \frac{7}{9} = -\frac{1}{3}, \quad \forall x_0 \in \mathbb{S}.$$

Similarly, when $\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) < \frac{3}{4}$, $\inf_{\tilde{\alpha}\in\mathcal{A}_{state}} J(0,\mu,\alpha;x_0,\tilde{\alpha})$ is achieved at $\tilde{\alpha} \equiv \frac{1}{3}$. Since $\sum_{x_1\in\mathbb{S}}\frac{1}{3} = \frac{2}{3} < \frac{3}{4}$, then $\alpha \equiv \frac{1}{3}$ is also an equilibrium with

$$J(0,\mu,\frac{1}{3};x_0,\frac{1}{3}) = \frac{1}{2}\Big[3 - 4\sum_{x_1\in\mathbb{S}}\frac{1}{3}\Big]\sum_{x_1\in\mathbb{S}}\frac{1}{3} + \frac{3}{2}\sum_{x_1\in\mathbb{S}}\frac{1}{3} - \frac{7}{9} = \frac{1}{3}, \quad \forall x_0 \in \mathbb{S}.$$

Moreover, when $\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) = \frac{3}{4}$, then all $\tilde{\alpha}$, including $\tilde{\alpha} = \alpha$, are minimizers of J, and thus such α is an equilibrium. In this case

$$J(0,\mu,\alpha;x_0,\alpha) = \frac{3}{2}\sum_{x_1\in\mathbb{S}}\alpha(1,x_1) - \frac{7}{9} = \frac{3}{2}\times\frac{3}{4} - \frac{7}{9} = \frac{25}{72}, \quad \forall x_0 \in \mathbb{S}.$$

Put all cases together, we have $\mathbb{V}_{state}(0, \mu) = \big\{(-\frac{1}{3}, -\frac{1}{3}), (\frac{1}{3}, \frac{1}{3}), (\frac{25}{72}, \frac{25}{72})\big\}$.

(ii) We next show that $(0,0) \in \bigcap_{\varepsilon>0} \overline{\lim}_{N\to\infty} \mathbb{V}^{N,\varepsilon}_{state}(0, \mu^N_{\vec{x}})$. Set

$$\alpha(t,x,\mu) := \alpha(\mu) := \frac{1}{3}\mathbf{1}_{\{\mu(\underline{x}) \leq \frac{1}{2}\}} + \frac{2}{3}\mathbf{1}_{\{\mu(\underline{x}) > \frac{1}{2}\}},$$

$$E^N_1 := \{\mu^N_1(\underline{x}) \leq \frac{1}{2}\}, E^N_2 := \{\mu^N_1(\underline{x}) > \frac{1}{2}\},$$

where α does not depend on (t,x). Then, for any $\tilde{\alpha} : \mathbb{T} \times \mathbb{S} \times \mathcal{P}(\mathbb{S}) \to \mathbb{A}$, recalling the setting in Subsection 3.1 and denoting $\mathbb{P}^i := \mathbb{P}^{0,\vec{x},(\alpha,\tilde{\alpha})_i}$, we have

$$J_i(0, \vec{x}, (\alpha, \tilde{\alpha})_i) = \mathbb{E}^{\mathbb{P}^i}\left[G(X^i_2, \mu^N_2)\right] = \frac{20}{9} - \mathbb{E}^{\mathbb{P}^i}\left[5\mu^N_2(\underline{x})\mathbf{1}_{\{X^i_2=\underline{x}\}} + 3\mu^N_2(\overline{x})\mathbf{1}_{\{X^i_2=\overline{x}\}}\right]$$

$$= \frac{20}{9} - \frac{1}{N}\mathbb{E}^{\mathbb{P}^i}\left[5\mathbf{1}_{\{X^i_2=\underline{x}\}} + 3\mathbf{1}_{\{X^i_2=\overline{x}\}}\right] - \frac{1}{N}\sum_{j\neq i}\mathbb{E}^{\mathbb{P}^i}\left[5\mathbf{1}_{\{X^j_2=X^i_2=\underline{x}\}} + 3\mathbf{1}_{\{X^j_2=X^i_2=\overline{x}\}}\right]$$

$$= \frac{20}{9} - \frac{1}{N}\sum_{j\neq i}\mathbb{E}^{\mathbb{P}^i}\left[5\alpha(\mu^N_1)\tilde{\alpha}(1, X^i_1, \mu^N_1) + 3[1-\alpha(\mu^N_1)][1-\tilde{\alpha}(1, X^i_1, \mu^N_1)]\right] + O\left(\frac{1}{N}\right)$$

$$= \frac{20}{9} - \mathbb{E}^{\mathbb{P}^i}\left[\left[2 - \frac{1}{3}\tilde{\alpha}(1, X^i_1, \mu^N_1)\right]\mathbf{1}_{E^N_1} + \left[1 + \frac{7}{3}\tilde{\alpha}(1, X^i_1, \mu^N_1)\right]\mathbf{1}_{E^N_2}\right] + O\left(\frac{1}{N}\right).$$

Notice that, under each $\mathbb{P}^i$, $X^1_1, \cdots, X^N_1$ are i.i.d. with $\mathbb{P}^i(X^j_1 = \underline{x}) = \mathbb{P}^i(X^j_1 = \overline{x}) = \frac{1}{2}$. Thus we may use a common $\bar{\mathbb{P}}$, under which $\vec{X}_1$ has the above distribution, such that

(7.14) $J_i(0, \vec{x}, (\alpha, \tilde{\alpha})_i)$
$$= \frac{20}{9} - \mathbb{E}^{\bar{\mathbb{P}}}\left[\left[2 - \frac{1}{3}\tilde{\alpha}(1, X^i_1, \mu^N_1)\right]\mathbf{1}_{E^N_1} + \left[1 + \frac{7}{3}\tilde{\alpha}(1, X^i_1, \mu^N_1)\right]\mathbf{1}_{E^N_2}\right] + O\left(\frac{1}{N}\right).$$

If we ignore the term $O\left(\frac{1}{N}\right)$, clearly $\tilde{\alpha} = \alpha$ is the minimizer of the above J_i. Then for fixed $\varepsilon > 0$ and for N large enough, α is an ε-minimizer for all i, and thus α is an ε-equilibrium. Note that $N\mu^N_1(\underline{x}) = \sum_{i=1}^N \mathbf{1}_{\{X^i_1=\underline{x}\}}$ has distribution Binomial$(N, \frac{1}{2})$ under $\bar{\mathbb{P}}$. Then $\bar{\mathbb{P}}(E^N_1) = \frac{1}{2}$ when N is odd, and

$$\frac{1}{2} \leq \bar{\mathbb{P}}(E^N_1) \leq \frac{1}{2} + \bar{\mathbb{P}}(N\mu^N_1(\underline{x}) = \frac{N}{2}) = \frac{1}{2} + \frac{1}{2^N}\binom{N}{\frac{N}{2}} = \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right),$$

when N is even. Thus

$$J_i(0, \vec{x}, \alpha) = \frac{20}{9} - \frac{17}{9}\bar{\mathbb{P}}(E^N_1) - \frac{23}{9}\bar{\mathbb{P}}(E^N_2) + O\left(\frac{1}{N}\right)$$

$$= \frac{20}{9} - \frac{1}{2}\left[\frac{17}{9} + \frac{23}{9}\right] + O\left(\frac{1}{\sqrt{N}}\right) = O\left(\frac{1}{\sqrt{N}}\right).$$

Since $\mu^N_{\vec{x}} \to \mu \in \mathcal{P}_0(\mathbb{S})$, we have $\mu^N_{\vec{x}} \in \mathcal{P}_0(\mathbb{S})$ for N large enough. Then, in light of (3.5),

$$J_N(0, x_0, \mu^N_{\vec{x}}, \alpha) = O\left(\frac{1}{\sqrt{N}}\right), \quad \forall x_0 \in \mathbb{S}.$$

This implies that $(0,0) \in \bigcap_{\varepsilon>0} \overline{\lim}_{N\to\infty} \mathbb{V}^{N,\varepsilon}_{state}(0, \mu^N_{\vec{x}})$.

(iii) We finally show that $(0,0) \in \bigcap_{\varepsilon>0} \overline{\lim}_{N\to\infty} \mathbb{V}^{N,\varepsilon,\infty}_{state}(0,\mu^N_{\vec{x}})$. Set

$$\alpha^N(t,x,\mu) := \frac{1}{3}\mathbf{1}_{\{\mu(\underline{x})\le p_N\}} + \frac{2}{3}\mathbf{1}_{\{\mu(\underline{x})\ge q_N\}} + \Big[\frac{1}{3} + \frac{N}{3}\big(\mu(\underline{x}) - p_N\big)\Big]\mathbf{1}_{\{p_N<\mu(\underline{x})<q_N\}},$$

where $\quad p_N := \frac{1}{2} - \frac{1}{2N}, \quad q_N := \frac{1}{2} + \frac{1}{2N}$

$$\tilde{E}^N_1 := \{\mu^N_1(\underline{x}) \le p_N\}, \quad \tilde{E}^N_2 := \{\mu^N_1(\underline{x}) \ge q_N\}, \quad \tilde{E}^N_3 := \{p_N < \mu^N_1(\underline{x}) < q_N\}.$$

Then clearly $\alpha^N \in \mathcal{A}^\infty_{state}$. For any $\tilde{\alpha} \in \mathcal{A}^\infty_{state}$, similarly to (7.14) we have

$$J_i(0,\vec{x},(\alpha^N,\tilde{\alpha})_i) = \frac{20}{9} - \mathbb{E}^{\bar{\mathbb{P}}}\Big[\big[2 - \frac{1}{3}\tilde{\alpha}(1,X^i_1,\mu^N_1)\big]\mathbf{1}_{\tilde{E}^N_1} + \big[1 + \frac{7}{3}\tilde{\alpha}(1,X^i_1,\mu^N_1)\big]\mathbf{1}_{\tilde{E}^N_2}$$

$$- \big[5\alpha(\mu^N_1)\tilde{\alpha}(1,X^i_1,\mu^N_1) + 3[1 - \alpha(\mu^N_1)][1 - \tilde{\alpha}(1,X^i_1,\mu^N_1)]\big]\mathbf{1}_{\tilde{E}^N_3}\Big] + O\big(\frac{1}{N}\big).$$

Again, fix $\varepsilon > 0$ and consider N large enough. On $\tilde{E}^N_1 \cup \tilde{E}^N_2$, it is optimal to choose $\tilde{\alpha} = \alpha^N$, up to the error $O\big(\frac{1}{N}\big)$. Then

$$J_i(0,\vec{x},(\alpha^N,\tilde{\alpha})_i) - J_i(0,\vec{x},\alpha^N) \le C\bar{\mathbb{P}}(\tilde{E}^N_3) + O\big(\frac{1}{N}\big).$$

When N is odd, $\tilde{E}^N_3 = \emptyset$ and thus $\bar{\mathbb{P}}(\tilde{E}^N_3) = 0$. When N is even,

$$\bar{\mathbb{P}}(\tilde{E}^N_3) = \bar{\mathbb{P}}(\mu^N_1(\underline{x}) = \frac{1}{2}) = \frac{1}{2^N}\binom{N}{\frac{N}{2}} = O\big(\frac{1}{\sqrt{N}}\big).$$

So in both cases, we have

$$J_i(0,\vec{x},(\alpha^N,\tilde{\alpha})_i) - J_i(0,\vec{x},\alpha^N) \le O\big(\frac{1}{\sqrt{N}}\big).$$

That is, $\alpha^N \in \mathcal{M}^{N,\varepsilon,\infty}_{state}(0,\mu^N_{\vec{x}})$ for N large enough. Thus $J_N(0,\cdot,\mu^N_{\vec{x}},\alpha^N) \in \mathbb{V}^{N,\varepsilon,\infty}_{state}(0,\mu^N_{\vec{x}})$. Then by similar arguments as in (ii) we see that $(0,0) \in \bigcap_{\varepsilon>0} \overline{\lim}_{N\to\infty} \mathbb{V}^{N,\varepsilon,\infty}_{state}(0,\mu^N_{\vec{x}})$. $\qquad\square$

Remark 7.3. Consider the setting in Example 7.2(ii). Denote $\mathbb{P}^\alpha = \mathbb{P}^{0,\vec{x},\alpha}$, we have

$$\mathbb{E}^{\mathbb{P}^\alpha}[\mu^N_2(\underline{x})] = \frac{1}{N}\sum_{i=1}^N \mathbb{P}^\alpha(X^i_2 = \underline{x}) = \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\bar{\mathbb{P}}}[\alpha(\mu^N_1)]$$

$$= \frac{1}{3}\bar{\mathbb{P}}\big(\mu^N_1(\underline{x}) \le \frac{1}{2}\big) + \frac{2}{3}\bar{\mathbb{P}}\big(\mu^N_1(\underline{x}) > \frac{1}{2}\big) = \frac{1}{2};$$

$$\mathbb{E}^{\mathbb{P}^\alpha}[|\mu^N_2(\underline{x})|^2] = \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{P}^\alpha(X^i_2 = X^j_2 = \underline{x})$$

$$= \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}^{\bar{\mathbb{P}}}[\alpha(\mu^N_1)] + \frac{1}{N^2}\sum_{i\ne j}\mathbb{E}^{\bar{\mathbb{P}}}[|\alpha(\mu^N_1)|^2]$$

$$= \frac{1}{9}\bar{\mathbb{P}}\big(\mu^N_1(\underline{x}) \le \frac{1}{2}\big) + \frac{4}{9}\bar{\mathbb{P}}\big(\mu^N_1(\underline{x}) > \frac{1}{2}\big) + O\big(\frac{1}{N}\big) = \frac{5}{18} + O\big(\frac{1}{N}\big);$$

$$Var^{\mathbb{P}^\alpha}(\mu^N_2(\underline{x})) = \frac{5}{18} + O\big(\frac{1}{N}\big) - \big(\frac{1}{2}\big)^2 = \frac{1}{36} + O\big(\frac{1}{N}\big).$$

Then we see that the random measure μ^N_2 under $\mathbb{P}^\alpha$, which is an $O\big(\frac{1}{\sqrt{N}}\big)$-equilibrium measure of the N-player problem, does not converge to a deterministic measure. This explains why [32] introduced the weak mean field equilibrium when considering

the convergence issue for all measurable controls. However, we shall emphasize again that, as pointed out in Remark 3.8(iii), measurable controls/equilibria are not desirable for numerical or practical purpose.

7.2. **The subtle path dependence issue in Remark 4.3.** In this subsection we elaborate Remark 4.3(ii) and (iii). Throughout the subsection, q, F, G are state dependent as in Section 2. As we saw in Example 7.1, in general $\mathbb{V}_{state} \neq \mathbb{V}_{path}$, confirming Remark 4.3(ii). We now turn to Remark 4.3(iii) for relaxed controls. For simplicity we verify it only for raw set values. The equality for set values follow similar ideas but with more involved approximations, as we saw in Example 7.1(iv). Let $\mathcal{A}_{relax}$ be the path dependent ones in Section 4, and $\mathcal{A}_{relax}^{state}$ denote the subset taking the form $\gamma(t, x, da)$. We emphasize again that here we are considering state dependent q, F, G. Fix $t = 0$ and $\mu \in \mathcal{P}_0(\mathbb{S})$.

Lemma 7.4. *For any $\gamma \in \mathcal{A}_{relax}$, define*
(7.15)
$$\tilde{\gamma}(s, x, da) := \frac{1}{\mu_s^\gamma(x)} \sum_{\mathbf{x} \in \mathbb{X}_s : \mathbf{x}_s = x} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}) \gamma(s, \mathbf{x}, da), \quad \text{where } \mu_s^\gamma(x) := \sum_{\mathbf{x} \in \mathbb{X}_s : \mathbf{x}_s = x} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}).$$

Then $\tilde{\gamma} \in \mathcal{A}_{relax}^{state}$ and $\mu_s^{\tilde{\gamma}} = \mu_s^\gamma$.

Proof. First it is obvious that
$$\tilde{\gamma}(s, x, \mathbb{A}) = \frac{1}{\mu_s^\gamma(x)} \sum_{\mathbf{x} \in \mathbb{X}_s : \mathbf{x}_s = x} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}) \gamma(s, \mathbf{x}, \mathbb{A}) = \frac{1}{\mu_s^\gamma(x)} \sum_{\mathbf{x} \in \mathbb{X}_s : \mathbf{x}_s = x} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}) = 1,$$

so $\tilde{\gamma} \in \mathcal{A}_{relax}^{state}$. Next, by definition $\mu_0^{\tilde{\gamma}} = \mu = \mu_0^\gamma$. Assume $\mu_s^{\tilde{\gamma}} = \mu_s^\gamma$, then for $s + 1$,

$$\mu_{s+1}^{\tilde{\gamma}}(x) = \sum_{\tilde{x} \in \mathbb{S}} \mu_s^{\tilde{\gamma}}(\tilde{x}) \int_{\mathbb{A}} q(s, \tilde{x}, \mu_s^{\tilde{\gamma}}, a; x) \tilde{\gamma}(s, \tilde{x}, da)$$

$$= \sum_{\tilde{x} \in \mathbb{S}} \mu_s^\gamma(\tilde{x}) \int_{\mathbb{A}} q(s, \tilde{x}, \mu_s^\gamma, a; x) \frac{1}{\mu_s^\gamma(\tilde{x})} \sum_{\mathbf{x} \in \mathbb{X}_s : \mathbf{x}_s = \tilde{x}} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}) \gamma(s, \mathbf{x}, da)$$

$$= \sum_{\mathbf{x} \in \mathbb{X}_s} \mu_{s\wedge \cdot}^\gamma(\mathbf{x}) \int_{\mathbb{A}} q(s, \mathbf{x}_s, \mu_s^\gamma, a; x) \gamma(s, \mathbf{x}, da) = \mu_{s+1}^\gamma(x).$$

This completes the induction argument. $\qquad\square$

Lemma 7.5. *If $\gamma^* \in \mathcal{A}_{relax}$ is a relaxed MFE at $(0, \mu)$, then the corresponding $\tilde{\gamma}^* \in \mathcal{A}_{relax}^{state}$ is a state dependent relaxed MFE at $(0, \mu)$. Moreover, in this case we have*
(7.16)
$$J(0, \mu, \gamma^*; x, \gamma^*) = J(0, \mu, \tilde{\gamma}^*; x, \tilde{\gamma}^*).$$

Proof. First, by Lemma 7.4 it is straightforward to verify that
$$\int_{\mathbb{S}} J(0, \mu, \gamma; x, \gamma) \mu(dx) = \int_{\mathbb{S}} J(0, \mu, \tilde{\gamma}; x, \tilde{\gamma}) \mu(dx).$$

On the other hand, since $\gamma^* \in \mathcal{A}_{relax}$, by the standard control theory we have
(7.17)
$$\inf_{\gamma \in \mathcal{A}_{relax}} J(0, \mu, \gamma^*; x, \gamma) = v(\mu^{\gamma^*}; 0, x) = v(\mu^{\tilde{\gamma}^*}; 0, x) = \inf_{\gamma' \in \mathcal{A}_{relax}^{state}} J(0, \mu, \tilde{\gamma}^*; x, \gamma').$$

Then
$$\int_{\mathbb{S}} J(0, \mu, \tilde{\gamma}^*; x, \tilde{\gamma}^*) \mu(dx) = \int_{\mathbb{S}} J(0, \mu, \gamma^*; x, \gamma^*) \mu(dx) = \int_{\mathbb{S}} v(\mu^{\tilde{\gamma}^*}; 0, x) \mu(dx).$$

Since $J(0, \mu, \tilde{\gamma}^*; x, \tilde{\gamma}^*) \geq v(\mu^{\tilde{\gamma}^*}; 0, x)$ and $\mathrm{supp}(\mu) = \mathbb{S}$, then $J(0, \mu, \tilde{\gamma}^*; x, \tilde{\gamma}^*) = v(\mu^{\tilde{\gamma}^*}; 0, x)$ for all $x \in \mathbb{S}$. This implies that $\tilde{\gamma}^* \in \mathcal{A}_{relax}^{state}$ is a state dependent relaxed MFE at $(0, \mu)$, and consequently (7.17) leads to (7.16). $\qquad\square$

Theorem 7.6. *The MFGs with state dependent relaxed controls and path dependent relaxed controls have the same relaxed raw set value.*

Proof. By Lemma 7.5, clearly the path dependent raw set value is included in the state dependent raw set value. On the other hand, for any state dependent relaxed control $\hat{\gamma}^* \in \mathcal{A}_{relax}^{state}$, we may still view $\gamma^* := \hat{\gamma}^*$ as a path dependent relaxed control,[7] and it is straightforward to verify that the $\tilde{\gamma}^* \in \mathcal{A}_{relax}^{state}$ corresponding to γ^* is equal to $\hat{\gamma}^*$. Then, following the arguments in Lemma 7.5, in particular (7.17), one can easily show that $J(0, \mu, \gamma^*; x, \gamma^*) = v(\mu^{\gamma^*}; 0, x)$ and thus γ^* is also an MFE among $\mathcal{A}_{relax}$. Therefore, $J(0, \mu, \gamma^*; \cdot, \gamma^*)$ belong to the path dependent raw set value as well. $\qquad\square$

7.3. Some technical proofs.

Proof of Theorem 2.7. Let $\tilde{\mathbb{V}}_{state}(t, \mu) = \bigcap_{\varepsilon > 0} \tilde{\mathbb{V}}_{state}^{\varepsilon}(t, \mu)$ denote the right side of (2.17) in the obvious sense. We shall follow the arguments in Theorem 2.4.

(i) We first prove $\tilde{\mathbb{V}}_{state}(t, \mu) \subset \mathbb{V}_{state}(t, \mu)$. Fix $\varphi \in \tilde{\mathbb{V}}_{state}(t, \mu)$, $\varepsilon > 0$, and set $\varepsilon_1 := \frac{\varepsilon}{4}$. Since $\varphi \in \tilde{\mathbb{V}}_{state}^{\varepsilon_1}(t, \mu)$, there exist desirable ψ and $\alpha^* \in \mathcal{M}_{state}^{\varepsilon_1}(T_0, \psi; t, \mu)$ as in (2.17), and the property $\psi(\cdot, \mu_{T_0}^{\alpha^*}) \in \mathbb{V}_{state}^{\varepsilon_1}(T_0, \mu_{T_0}^{\alpha^*})$ implies further that there exists $\tilde{\alpha}^* \in \mathcal{M}_{state}^{\varepsilon_1}(T_0, \mu_{T_0}^{\alpha^*})$ such that

$$\|\varphi - J(T_0, \psi; t, \mu, \alpha^*; \cdot, \alpha^*)\|_\infty \leq \varepsilon_1, \quad \|\psi(\cdot, \mu_{T_0}^{\alpha^*}) - J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; \cdot, \tilde{\alpha}^*)\|_\infty \leq \varepsilon_1.$$

Denote $\hat{\alpha}^* := \alpha^* \oplus_{T_0} \tilde{\alpha}^* \in \mathcal{A}_{state}$. Then, for any $\alpha \in \mathcal{A}_{state}$ and $x \in \mathbb{S}$, similar to the arguments in Proposition 2.3(i), we have

$$J(t, \mu, \hat{\alpha}^*; x, \alpha) = \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}}\left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \alpha) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right]$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}}\left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \tilde{\alpha}^*) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right] - \varepsilon_1$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha}}\left[\psi(X_{T_0}, \mu_{T_0}^{\alpha^*}) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha(s, X_s)) \right] - 2\varepsilon_1$$

$$= J(T_0, \psi; t, \mu, \alpha^*; x, \alpha) - 2\varepsilon_1 \geq J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) - 3\varepsilon_1$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha^*}}\left[\psi(X_{T_0}, \mu_{T_0}^{\alpha^*}) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha^*(s, X_s)) \right] - 3\varepsilon_1$$

$$\geq \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha^*}}\left[J(T_0, \mu_{T_0}^{\alpha^*}, \tilde{\alpha}^*; X_{T_0}, \tilde{\alpha}^*) + \sum_{s=t}^{T_0-1} F(s, X_s, \mu_s^{\alpha^*}, \alpha^*(s, X_s)) \right] - 4\varepsilon_1$$

$$= J(t, \mu, \hat{\alpha}^*; x, \hat{\alpha}^*) - \varepsilon.$$

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[7]While it is trivial that $\mathcal{A}_{relax}^{state} \subset \mathcal{A}_{relax}^{path} := \mathcal{A}_{relax}$, as stated here, in general it is not trivial that $\mathcal{M}_{relax}^{state} \subset \mathcal{M}_{relax}^{path}$, because for the latter one has to compare with other path dependent relaxed controls, which is a stronger requirement than that for $\mathcal{M}_{relax}^{state}$. The rest of the proof is exactly to prove $\mathcal{M}_{relax}^{state} \subset \mathcal{M}_{relax}^{path}$.

That is, $\hat{\alpha}^* \in \mathcal{M}^\varepsilon_{state}(t, \mu)$. Moreover, note that

$$\|\varphi - J(t, \mu, \hat{\alpha}^*; \cdot, \hat{\alpha}^*)\|_\infty \leq \varepsilon_1 + \|J(T_0, \psi; t, \mu, \alpha^*; \cdot, \alpha^*) - J(t, \mu, \hat{\alpha}^*; \cdot, \hat{\alpha}^*)\|_\infty$$

$$= \varepsilon_1 + \sup_{x \in \mathbb{S}} \left| \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[\psi(X_{T_0}, \mu^{\alpha^*}_{T_0}) - J(T_0, \mu^{\alpha^*}_{T_0}, \tilde{\alpha}^*; X_{T_0}, \tilde{\alpha}^*) \right] \right| \leq 2\varepsilon_1 \leq \varepsilon.$$

Then $\varphi \in \mathbb{V}^\varepsilon_{state}(t, \mu)$. Since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \mathbb{V}_{state}(t, \mu)$.

(ii) We now prove the opposite inclusion. Fix $\varphi \in \mathbb{V}_{state}(t, \mu)$ and $\varepsilon > 0$. Let $\varepsilon_1 > 0$ be a small number which will be specified later. Since $\varphi \in \mathbb{V}^{\varepsilon_1}_{state}(t, \mu)$, there exists $\alpha^* \in \mathcal{M}^{\varepsilon_1}_{state}(t, \mu)$ such that $\|\varphi - J(t, \mu, \alpha^*; \cdot, \alpha^*)\|_\infty \leq \varepsilon_1$. Introduce $\psi(x, \nu) := J(T_0, \nu, \alpha^*; x, \alpha^*)$. By (2.10) we have

$$\|\varphi - J(T_0, \psi; t, \mu, \alpha^*; \cdot, \alpha^*)\|_\infty = \|\varphi - J(t, \mu, \alpha^*; \cdot, \alpha^*)\|_\infty \leq \varepsilon_1.$$

Moreover, since $\alpha^* \in \mathcal{M}^{\varepsilon_1}_{state}(t, \mu)$, for any $\alpha \in \mathcal{A}_{state}$ and $x \in \mathbb{S}$, we have

$$J(T_0, \psi; t, \mu, \alpha^*; x, \alpha^*) = J(t, \mu, \alpha^*; x, \alpha^*)$$
$$\leq J(t, \mu, \alpha^*; x, \alpha \oplus_{T_0} \alpha^*) + \varepsilon_1 = J(T, \psi; t, \mu, \alpha^*; x, \alpha) + \varepsilon_1.$$

This implies that $\alpha^* \in \mathcal{M}^{\varepsilon_1}_{state}(T_0, \psi; t, \mu)$. We claim further that

$$(7.18) \qquad \psi(\cdot, \mu^{\alpha^*}_{T_0}) \in \mathbb{V}_{C\varepsilon_1}(T_0, \mu^{\alpha^*}_{T_0}),$$

for some constant $C \geq 1$. Then by (2.17) we see that $\varphi \in \tilde{\mathbb{V}}^{C\varepsilon_1}_{state}(t, \mu) \subset \tilde{\mathbb{V}}^\varepsilon_{state}(t, \mu)$ by setting $\varepsilon_1 \leq \frac{\varepsilon}{C}$. Since $\varepsilon > 0$ is arbitrary, we obtain $\varphi \in \tilde{\mathbb{V}}_{state}(t, \mu)$.

To show (7.18), we follow the arguments in Proposition 2.3(ii). Recall v in (2.5) and the standard DPP (2.11) for v, for any $x \in \mathbb{S}$ we have

$$\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; X_{T_0}, \alpha^*) \right]$$

$$\leq \inf_{\alpha \in \mathcal{A}_{state}} \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; X_{T_0}, \alpha) \right] + \varepsilon_1$$

$$= \mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[v(\mu^{\alpha^*}; T_0, X_{T_0}) \right] + \varepsilon_1.$$

It is obvious that $v(\mu^{\alpha^*}; T_0, \cdot) \leq J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; \cdot, \alpha^*)$. Moreover, since $q \geq c_q$, clearly $\mathbb{P}^{\mu^{\alpha^*}; t, x, \alpha^*}(X_{T_0} = \tilde{x}) \geq c_0^{T_0 - t}$, for any $\tilde{x} \in \mathbb{S}$. Thus, for $C := c_0^{t - T_0}$,

$$0 \leq J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; \tilde{x}, \alpha^*) - v(\mu^{\alpha^*}; T_0, \tilde{x})$$

$$\leq C\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[\left[J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; X_{T_0}, \alpha^*) - v(\mu^{\alpha^*}; T_0, X_{T_0}) \right] \mathbf{1}_{\{X_{T_0} = \tilde{x}\}} \right]$$

$$\leq C\mathbb{E}^{\mathbb{P}^{\mu^{\alpha^*}}; t, x, \alpha^*} \left[\left[J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; X_{T_0}, \alpha^*) - v(\mu^{\alpha^*}; T_0, X_{T_0}) \right] \right] \leq C\varepsilon_1.$$

This implies that $\alpha^* \in \mathcal{M}^{C\varepsilon_1}_{state}(T_0, \mu^{\alpha^*}_{T_0})$. Since $\psi(\cdot, \mu^{\alpha^*}_{T_0}) = J(T_0, \mu^{\alpha^*}_{T_0}, \alpha^*; \cdot, \alpha^*)$, we obtain (7.18) immediately, and hence $\varphi \in \tilde{\mathbb{V}}_{state}(t, \mu)$. $\square$

Proof of the claim in Remark 4.7. By (4.16) and (4.17) we have

$$\gamma^{(\Lambda^\gamma)}(s, \tilde{\mathbf{x}}, da) := \frac{1}{\mu^\gamma_{s\wedge\cdot}(\tilde{\mathbf{x}})} \int_{\mathcal{A}^t_{path}} Q^t_s(\mu^\gamma; \tilde{\mathbf{x}}; \alpha) \delta_{\alpha(s,\tilde{\mathbf{x}})}(da) \Lambda^\gamma(\mathbf{x}, d\alpha)$$

$$= \frac{1}{\mu^\gamma_{s\wedge\cdot}(\tilde{\mathbf{x}})} \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} \Big[\prod_{r=t}^{s-1} q(r, \tilde{\mathbf{x}}, \mu^\gamma, \alpha(r, \tilde{\mathbf{x}}); \mathbf{x}_{r+1})\Big] \times \delta_{\alpha(s,\tilde{\mathbf{x}})}(da) \times$$

$$\Big[\mu(\mathbf{x}) \prod_{r=t}^{T-1} \prod_{\bar{\mathbf{x}}\in\mathbb{X}^{t,\mathbf{x}}_s} \gamma(r, \bar{\mathbf{x}}, d\alpha(r, \bar{\mathbf{x}}))\Big]$$

$$= \frac{\mu(\mathbf{x})}{\mu^\gamma_{s\wedge\cdot}(\tilde{\mathbf{x}})} \int_{\mathbb{A}} \cdots \int_{\mathbb{A}} \Big[\prod_{r=t}^{s-1} q(r, \tilde{\mathbf{x}}, \mu^\gamma, \alpha(r, \tilde{\mathbf{x}}); \mathbf{x}_{r+1})\gamma(r, \tilde{\mathbf{x}}, d\alpha(r, \tilde{\mathbf{x}}))\Big] \times$$

$$\Big[\delta_{\alpha(s,\tilde{\mathbf{x}})}(da)\gamma(s, \tilde{\mathbf{x}}, d\alpha(s, \tilde{\mathbf{x}})) \prod_{\bar{\mathbf{x}}\in\mathbb{X}^{t,\mathbf{x}}_s\setminus\{\tilde{\mathbf{x}}\}} \gamma(s, \bar{\mathbf{x}}, d\alpha(s, \bar{\mathbf{x}}))\Big] \times$$

$$\Big[\prod_{r=t}^{s-1} \prod_{\bar{\mathbf{x}}\in\mathbb{X}^{t,\mathbf{x}}_s\setminus\{\tilde{\mathbf{x}}\}} \gamma(r, \bar{\mathbf{x}}, d\alpha(r, \bar{\mathbf{x}}))\Big]\Big[\prod_{r=s}^{T-1} \prod_{\bar{\mathbf{x}}\in\mathbb{X}^{t,\mathbf{x}}_s} \gamma(r, \bar{\mathbf{x}}, d\alpha(r, \bar{\mathbf{x}}))\Big]$$

$$= \frac{\mu(\mathbf{x})}{\mu^\gamma_{s\wedge\cdot}(\tilde{\mathbf{x}})} \Big[\prod_{r=t}^{s-1} \int_{\mathbb{A}} q(r, \tilde{\mathbf{x}}, \mu^\gamma, \bar{a}; \mathbf{x}_{r+1})\gamma(r, \tilde{\mathbf{x}}, d\bar{a})\Big] \times \big[\gamma(s, \tilde{\mathbf{x}}, da)\big]$$

$$= \frac{\mu(\mathbf{x})}{\mu^\gamma_{s\wedge\cdot}(\tilde{\mathbf{x}})} Q^t_s(\mu^\gamma; \tilde{\mathbf{x}}, \gamma)\gamma(s, \tilde{\mathbf{x}}, da) = \gamma(s, \tilde{\mathbf{x}}, da).$$

That is, $\gamma^{(\Lambda^\gamma)} = \gamma$. $\qquad\square$

Proof of Lemma 6.4. Clearly the uniform estimate for $J(\mu^\alpha; \cdot)$ implies that for $v(\mu^\alpha; \cdot)$, so we shall only prove the former one. Fix $(t, \mu) \in [0, T] \times \mathcal{P}_2$ and $\alpha, \tilde{\alpha} \in \mathcal{A}_{cont}$, and denote $u(s, x) := J(\mu^\alpha; \tilde{\alpha}, s, x)$. By standard PDE theory u is a classical solution to the linear PDE in (6.4) and we have the following formula: denoting $X^{s,x}_r := x + B_r - B_s$,

$$\partial_x u(s, x) = \mathbb{E}^{\mathbb{P}}\Big[[g(X^{s,x}_T, \mu^\alpha_T) - g(x, \mu^\alpha_T)]\frac{B_T - B_s}{T - s}$$

$$+ \int_s^T \big[b(r, X^{s,x}_t, \mu^\alpha_r, \tilde{\alpha}(r, X^{s,x}_r)) \cdot \partial_x u(r, X^{s,x}_r)$$

$$+ f(r, X^{s,x}_t, \mu^\alpha_r, \tilde{\alpha}(r, X^{s,x}_r))\big]\frac{B_r - B_s}{r - s} dr\Big].$$

Then, by the Lipschitz continuity of g and the boundedness of b and f,

$$|\partial_x u(s, x)| \leq \mathbb{E}\Big[L_0\frac{|B_T - B_s|^2}{T - s} + C_0 \int_s^T \big[|\partial_x u(r, X^{s,x}_r)| + 1\big]\frac{|B_r - B_s|}{r - s} dr\Big]$$

$$\leq C + C_0\mathbb{E}\Big[\int_s^T |\partial_x u(r, X^{s,x}_r)|\frac{|B_r - B_s|}{r - s} dr\Big].$$

Denote $K_s := e^{\lambda s} \sup_x |\partial_x u(s,x)|$, $\bar{K} := \sup_{t \le s \le T} K_s$, for some constant $\lambda > 0$. Then

$$K_s \le Ce^{\lambda s} + C_0 \int_s^T \frac{K_r e^{-\lambda(r-s)}}{\sqrt{r-s}} dr \le Ce^{\lambda s} + C_0 \bar{K} \int_s^T \frac{e^{-\lambda(r-s)}}{\sqrt{r-s}} dr$$

$$\le Ce^{\lambda s} + C_0 \bar{K} \int_s^\infty \frac{e^{-\lambda(r-s)}}{\sqrt{r-s}} dr = Ce^{\lambda s} + C_0 \bar{K} \int_0^\infty \frac{e^{-\lambda r}}{\sqrt{r}} dr = Ce^{\lambda s} + \frac{C_0}{\sqrt{\pi\lambda}} \bar{K}.$$

Thus $\bar{K} \le \frac{C_0}{\sqrt{\pi\lambda}} \bar{K} + Ce^{\lambda T}$. Set $\lambda := \frac{4C_0^2}{\pi}$ so that $\frac{C_0}{\sqrt{\pi\lambda}} = \frac{1}{2}$, we obtain $\bar{K} \le C_1 := 2Ce^{\lambda T}$, which implies the desired estimate immediately. $\qquad\square$

Proof of Proposition 6.7. Fix $(t, \vec{x}, \vec{\alpha}, \bar{x}, \tilde{x})$ and i. For any $\tilde{\alpha} \in \mathcal{A}_{cont}^L$, introduce $\bar{\alpha}(s, x, \mu) := \tilde{\alpha}(s, x - \bar{x} + \tilde{x}, \mu)$, and denote

$$\bar{X}_s^i := \bar{x} + B_s^i - B_t^i, \quad X_s^j := x_j + B_s^j - B_t^j, \quad j \ne i, \quad;$$

$$\bar{\mu}_s^N := \frac{1}{N}\big[\delta_{\bar{X}_s^i} + \sum_{j \ne i} \delta_{X_s^j}\big], \bar{M}_s^j := \exp\Big(\int_t^s \bar{b}_r^j dB_r^j - \frac{1}{2}\int_t^s |\bar{b}_r^j|^2 dr\Big), j \ge 1, \text{ where}$$

$$\bar{b}_s^i := b(s, \bar{X}_s^i, \bar{\mu}_s^N, \bar{\alpha}(s, \bar{X}_s^i, \bar{\mu}_s^N)), \quad \bar{b}_s^j := b(s, X_s^j, \bar{\mu}_s^N, \alpha_j(s, X_s^j, \bar{\mu}_s^N)), j \ne i.$$

By the Girsanov Theorem we have

$$J_i(t, (\vec{x}^{-i}, \bar{x}), (\vec{\alpha}^{-i}, \bar{\alpha}))$$
$$= \mathbb{E}\Big[\big[\prod_{j=1}^N \bar{M}_T^j\big]\big[g(\bar{X}_T^i, \bar{\mu}_T^N) + \int_t^T f(s, \bar{X}_s^i, \bar{\mu}_s^N, \bar{\alpha}(s, \bar{X}_s^i, \bar{\mu}_s^N))]ds\Big].$$

Similarly define $\tilde{X}^i$, $\tilde{\mu}^N$, $\tilde{M}^j$, $\tilde{b}^i$, $\tilde{b}^j$ corresponding to $(\tilde{x}, \tilde{\alpha})$ in the obvious sense. Then we have a similar expression as above and $\bar{\alpha}(s, \bar{X}_s^i, \mu) = \tilde{\alpha}(s, \tilde{X}_s^i, \mu)$. Therefore,

$$v_i^{N,L}\big(t, (\vec{x}^{-i}, \bar{x}), \vec{\alpha}\big) - J_i(t, (\vec{x}^{-i}, \tilde{x}), (\vec{\alpha}^{-i}, \tilde{\alpha}))$$
(7.19)
$$\le J_i(t, (\vec{x}^{-i}, \bar{x}), (\vec{\alpha}^{-i}, \bar{\alpha})) - J_i(t, (\vec{x}^{-i}, \tilde{x}), (\vec{\alpha}^{-i}, \tilde{\alpha})) \le C \sum_{j=1}^N K_T^j + K_0,$$

where

$$K_s^j := \mathbb{E}\Big[\big[\prod_{k<j} \bar{M}_s^k\big]\big[\prod_{k>j} \tilde{M}_s^k\big]|\bar{M}_s^j - \tilde{M}_s^j|\Big], \quad j \ge 1;$$

$$K_0 := \mathbb{E}\Big[\prod_{j=1}^N \bar{M}_T^j\big[|g(\bar{X}_T^i, \bar{\mu}_T^N) - g(\tilde{X}_T^i, \tilde{\mu}_T^N)|$$
$$+ \int_t^T |f(s, \bar{X}_s^i, \bar{\mu}_s^N, \bar{\alpha}(s, \bar{X}_s^i, \bar{\mu}_s^N)) - f(s, \tilde{X}_s^i, \tilde{\mu}_s^N, \tilde{\alpha}(s, \tilde{X}_s^i, \tilde{\mu}_s^N))|ds\Big].$$

Denote $\Delta x := \bar{x} - \tilde{x}$. Note that

(7.20)
$$\bar{X}_s^i - \tilde{X}_s^i = \Delta x, \quad W_1(\bar{\mu}_s^N, \tilde{\mu}_s^N) \le \frac{|\Delta x|}{N},$$

$$\big|\bar{\alpha}(s, \bar{X}_s^i, \bar{\mu}_s^N) - \tilde{\alpha}(s, \tilde{X}_s^i, \tilde{\mu}_s^N)\big| = \big|\tilde{\alpha}(s, \tilde{X}_s^i, \bar{\mu}_s^N) - \tilde{\alpha}(s, \tilde{X}_s^i, \tilde{\mu}_s^N)\big| \le \frac{L}{N}|\Delta x|.$$

By the required Lipschitz continuity, we have

$$(7.21) \qquad K_0 \leq C\mathbb{E}^{\mathbb{P}}\Big[\prod_{j=1}^{N}\bar{M}_T^j\big[[1+\frac{1}{N}]|\Delta x| + \int_t^T[1+\frac{L}{N}]|\Delta x|ds]\Big] \leq C|\Delta x|.$$

Next, introduce

$$\Gamma_s^j := \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_s^k\big]\big[\prod_{k>j}\tilde{M}_s^k\big]|\bar{M}_s^j|^2\Big], \quad \Delta\Gamma_s^j := \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_s^k\big]\big[\prod_{k>j}\tilde{M}_s^k\big]|\bar{M}_s^j - \tilde{M}_s^j|^2\Big].$$

Note that $B^1, \cdots, B^N$ are independent. By applying the Itô formula, we have

$$\Gamma_s^j = 1 + \int_t^s \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_r^k\big]\big[\prod_{k>j}\tilde{M}_r^k\big]|\bar{M}_r^j\bar{b}_r^j|^2\Big]dr \leq 1 + C\int_t^s \Gamma_r^j dr.$$

Then $\Gamma_s^j \leq C$. Thus, by applying the Itô formula again we have

$$\Delta\Gamma_s^j = \int_t^s \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_r^k\big]\big[\prod_{k>j}\tilde{M}_r^k\big][\bar{M}_r^j\bar{b}_r^j - \tilde{M}_r^j\tilde{b}_r^j]^2\Big]dr$$

$$\leq C\int_t^s \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_r^k\big]\big[\prod_{k>j}\tilde{M}_r^k\big][|\bar{M}_r^j - \tilde{M}_r^j| + \bar{M}_r^j|\bar{b}_r^j - \tilde{b}_r^j|]^2\Big]dr$$

$$\leq C\int_t^s \Delta\Gamma_r^j dr + C\int_t^s \mathbb{E}\Big[\big[\prod_{k<j}\bar{M}_r^k\big]\big[\prod_{k>j}\tilde{M}_r^k\big][\bar{M}_r^j|\bar{b}_r^j - \tilde{b}_r^j|]^2\Big]dr.$$

Note that, by (7.20),

$$|\bar{b}_r^i - \tilde{b}_r^i| = \Big|b(s, \bar{X}_s^i, \bar{\mu}_s^N, \tilde{\alpha}(s, \tilde{X}_s^i, \bar{\mu}_s^N)) - b(s, \tilde{X}_s^i, \tilde{\mu}_s^N, \tilde{\alpha}(s, \tilde{X}_s^i, \tilde{\mu}_s^N))\Big| \leq C_L|\Delta x|$$

$$|\bar{b}_r^j - \tilde{b}_r^j| \leq \frac{C_L}{N}|\Delta x|, \quad j \neq i.$$

Then, since $\Gamma_s^j \leq C$,

$$\Delta\Gamma_s^i \leq C\int_t^s \Delta\Gamma_r^i dr + C_L|\Delta x|^2, \quad \Delta\Gamma_s^j \leq C\int_t^s \Delta\Gamma_r^j dr + \frac{C_L}{N^2}|\Delta x|^2, j \neq i,$$

and thus

$$(7.22) \qquad \Delta\Gamma_s^i \leq C_L|\Delta x|^2, \quad K_s^i \leq \frac{|\Delta x|}{2} + \frac{\Delta\Gamma_s^i}{2|\Delta x|} \leq C_L|\Delta x|;$$

$$\Delta\Gamma_s^j \leq \frac{C_L}{N^2}|\Delta x|^2, \quad K_s^j \leq \frac{|\Delta x|}{2N} + \frac{N\Delta\Gamma_s^j}{2|\Delta x|} \leq \frac{C_L}{N}|\Delta x|, \quad j \neq i.$$

Then, by (7.19), (7.21) and (7.22) we have

$$v_i^{N,L}\big(t, (\vec{x}^{-i}, \bar{x}), \vec{\alpha}\big) - J_i(t, (\vec{x}^{-i}, \tilde{x}), (\vec{\alpha}^{-i}, \tilde{\alpha})) \leq K_0 + CK_s^i + C\sum_{j \neq i}K_s^j$$

$$\leq C|\Delta x| + C_L|\Delta x| + C_L\sum_{j \neq i}\frac{|\Delta x|}{N} \leq C_L|\Delta x|.$$

Since $\tilde{\alpha} \in \mathcal{A}^L$ is arbitrary, we obtain $v_i^{N,L}\big(t, (\vec{x}^{-i}, \bar{x}), \vec{\alpha}\big) - v_i^{N,L}\big(t, (\vec{x}^{-i}, \tilde{x}), \vec{\alpha}\big) \leq C_L|\Delta x|$. Similarly we have $v_i^{N,L}\big(t, (\vec{x}^{-i}, \tilde{x}), \vec{\alpha}\big) - v\big(t, (\vec{x}^{-i}, \bar{x}), \vec{\alpha}\big) \leq C_L|\Delta x|$, and hence (6.16). $\qquad\square$

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