Controlled Occupied Processes and Viscosity Solutions

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Abstract

We consider the optimal control of occupied processes which record all positions of the state process. Dynamic programming yields nonlinear equations on the space of positive measures. We develop the viscosity theory for this infinite dimensional parabolic *occupied* PDE by proving a comparison result between sub and supersolutions, and thus provide a characterization of the value function as the unique viscosity solution. Toward this proof, an extension of the celebrated Crandall-Ishii-Lions (second order) Lemma to this setting, as well as finite-dimensional approximations, is established. Examples including the occupied heat equation, and pricing PDEs of financial derivatives contingent on the occupation measure are also discussed.

Keywords: Stochastic optimal control, occupation flow, occupied PDEs, viscosity solutions

Mathematics Subject Classification: 49L12, 35K55, 35R15, 60J55, 93E20

1 Introduction

This paper introduces a class of path-dependent stochastic control problems involving the occupation measure and develops a viscosity theory for the associated dynamic programming equation. We consider the following control problem,

$$\inf_{\alpha \in \mathscr{A}} \mathbb{E}^{\mathbb{Q}}[\int_{0}^{T} \ell(\mathcal{O}_{t}^{\alpha}, X_{t}^{\alpha}, \alpha_{t}) dt + g(\mathcal{O}_{T}^{\alpha}, X_{T}^{\alpha})],$$
(1.1)

$$dX_t^{\alpha} = b(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t)dt + \sigma(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t)dW_t, \qquad (1.2)$$

where $\mathcal{O}_t^{\alpha} = \int_0^t \delta_{X_s^{\alpha}} ds$ is the standard time occupation flow of X^{α} . The latter is a measure-valued process that describes the cumulative time spent by X^{α} in arbitrary regions. Other clocks, possibly stochastic, may be used in the definition of \mathcal{O}^{α} ; see Section 3.1. The pair $(\mathcal{O}^{\alpha}, X^{\alpha})$, referred to as the occupied process [35, 36], induces a Markovian framework that lies strictly between the classical (path-independent) setting, and fully path-dependent models.

Our main result characterizes the value function associated to (1.1) as the unique viscosity solution to a parabolic *occupied PDE* which in standard time takes the form

$$-\partial_{\mathfrak{o}}u + \mathscr{H}(\mathfrak{o}, x, \nabla u, \nabla^2 u) = 0.$$
(1.3)

Here, the variable \mathfrak{o} represents the value of the occupation flow \mathcal{O}^{α} and lives in the linear space \mathscr{M} of finite Borel measures. Importantly, \mathfrak{o} generalizes the time variable in parabolic PDEs, and the *occupation derivative* $\partial_{\mathfrak{o}}$ in (1.3) replaces the usual time derivative. As we argue in Section 2.1, this differential is directly tied to the dynamics of the occupation flow and corresponds to a local projection of the linear derivative commonly used in mean-field games [10, 11, 25].

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Alternatively, one can formulate the control problem (1.1) using a pathwise setting and the tools from Dupire's functional Itô calculus [15]. The associated dynamic programming equation is a path-dependent PDE [17, 18, 19, 29, 37] which, in the canonical setting, reads

$$-\partial_t u + \mathscr{H}(t, \omega, \partial_\omega u, \partial_\omega^2 u) = 0, \tag{1.4}$$

where $u = u(t, \omega)$, $(t, \omega) \in \mathbb{R}_+ \times \Omega$ and $\Omega = \mathcal{C}([0, T]; \mathbb{R}^d)$ is the space of continuous paths on \mathbb{R}^d . The differential ∂_t denotes the functional time derivative in the sense of Dupire, while ∂_{ω} is the functional space derivative. Comparing equations (1.3) and (1.4), we observe that occupied PDEs correspond to the change of coordinates

$$\mathbb{R}_+ \times \Omega \ni (t, \omega) \longrightarrow (\mathcal{O}_t^{\alpha}(\omega), X_t^{\alpha}(\omega)) = (\mathfrak{o}, x) \in \mathscr{M} \times \mathbb{R}^d$$

In particular, the path is captured by the occupation flow, and the space variable becomes finitedimensional. Consequently, the space derivatives in (1.3) are the classical ones, and the path dependence only appears through a first-order term given by the occupation derivative. Many questions related to fully nonlinear parabolic path-dependent PDEs remains open, especially regarding the regularity of solutions. Additionally, the comparison principle in this context is highly involved and necessitate additional technical tools [18, 19, 38]. In contrast, we here provide a nearly classical proof of the comparison principle for occupied PDEs, covering a large class of path-dependent PDEs.

It is important to note that the occupation measure extracts aggregate features of the path and in particular erases its chronology. In fact, one can show the opposite, namely that for fixed $t \ge 0$, any chronology-invariant path functional $\omega \mapsto F(t, \omega)$ is necessarily function of the occupation measure [35, Theorem 2.1]. While some path-dependent control problems cannot be expressed as (1.1), e.g those involving time delays [23, 32], the present framework still includes a vast array of path-dependent problems, notably in finance [35].

It is now classical that the notion of viscosity solutions is the appropriate framework for nonlinear PDEs, delivering a complete theory including the existence and uniqueness of such type of solutions [13, 21]. In our context it is particularly needed, as the regularity of solutions has not been established and is not expected for non-linear equations. A central tool in the analysis of second-order nonlinear PDEs is the well-known Crandall-Ishii-Lions lemma [13] which we generalize to our context. This is achieved by projecting the problem to a finite-dimensional space using *cylindrical functions*, and the corresponding metric induced by the so-called cylindrical norm introduced in Section 2.2.

Classically, the comparison principle for viscosity solutions requires compactness assumptions on the domain so that upper/lower semicontinuous functions attain their maximum/minimum [13, 21]. Herein, we note that the space $\mathscr{M} \times \mathbb{R}^d$ is not compact with respect to the product topology. However, it is locally compact, and the existence of extrema is guaranteed by introducing *coercive approximations* as discussed in Section 4.1. The lack of compactness is typically reconciled with the Ekeland-Borwein-Preiss variational principle [16, 5]. While this powerful, yet complex tool may be applied to our context, it is actually not needed here due to the locally compact structure of $\mathscr{M} \times \mathbb{R}^d$; see again Section 4.1.

Related Literature. In a recent work, Béthencourt et al. [8] study Brownian particles controlled by their aggregate occupation flow with a central example related to the volume of their Wiener sausage. This highly intriguing model has the tractable linear-quadratic structure. Hence, the dynamic programming equation can be solved explicitly and does not necessitate the development of a viscosity theory. Our setting shares similarities with stochastic control problems in the Wasserstein space, including McKean-Vlasov optimal control [2, 4, 7, 12, 33, 34, 39] and controlled superprocesses [28]. Additionally, the local nature of the occupation derivative in (1.3) is reminiscent of mean field games with local coupling [9, 27], where the dynamics of the system depends on specific values of the marginal densities. Moreover, Bouchard and Tan [6] study regularity properties of linear parabolic PPDEs where the state variable is enlarged with an additive functional $\omega \mapsto \int_0^t \omega_u dA_u$ for some continuous function A of bounded variation. When A is non-decreasing, it can be regarded as a random time process (see Section 3) and their framework comes as a particular case of the present setting.

Infinite dimensional viscosity solutions have been discussed repeatedly in the literature and arise from different contexts, and the classical book from Fabbri, Gozzi, and Swiech [20] provides a comprehensive overview. The use of cylindrical functions as first introduced by Lasry and Lions [24, 26] for nonlinear PDEs in Hilbert spaces, that we extend to our setting. Relatedly,

[40] consider a finite-dimensional reduction of control problems in the Wasserstein space using interacting particle systems, and Bayraktar et al. [2] uses a supremum/infimum projection among all measures with equal barycenter. The cylindrical norm we introduce is also connected to the distance-like function in [7] pertaining to controlled McKean-Vlasov jump-diffusion processes.

Structure of the paper. The notations are outlined in Section 2, where the occupation derivative and cylindrical norm are introduced. Section 3.1 expands on the stochastic control problem, the corresponding viscosity theory, and summarizes our main results. Technical tools are presented in Section 4, while Section 5 focuses on the Crandall-Ishii-Lions Lemma. The comparison principle is stated and proved in Section 6. We finally discuss examples in Section 7 and provide postponed proofs in Appendices A and B.

2 Notations

The *ambient space* is the *d*-dimensional Euclidean space \mathbb{R}^d and \mathbb{S}^d is the set of all *d* by *d* symmetric matrices. For $\Gamma \in \mathbb{S}^d$, $\operatorname{tr}(\Gamma)$ is the trace of Γ and *I* is the identity matrix. We let $\mathscr{C} := \mathscr{C}(\mathbb{R}^d)$ be the set of continuous functions on \mathbb{R}^d and $\mathscr{C}_b := \mathscr{C}_b(\mathbb{R}^d)$ be the bounded ones. $\mathscr{M} = \mathscr{M}(\mathbb{R}^d)$ is the set of all signed measures on \mathbb{R}^d endowed with the weak $\sigma(\mathscr{C}_b(\mathbb{R}^d), \mathscr{M})$ topology and \mathscr{M}_+ are the positive ones. We denote the total variation of $\mathfrak{o} \in \mathscr{M}$ by $|\mathfrak{o}|$ and for $\mathfrak{o} \in \mathscr{M}_+$, $|\mathfrak{o}| = \mathfrak{o}(\mathbb{R}^d)$.

For $x \in \mathbb{R}^d$, set $q(x) := \sqrt{1+|x|^2}$ and $\mathscr{C}_q := \{g \in \mathscr{C} : |g| \leq c q \text{ for some } c > 0\}$. The Wasserstein space $\mathscr{M}_q := \mathscr{M}_q(\mathbb{R}^d) = \{\mathfrak{o} \in \mathscr{M}_+ : \int q(x)\mathfrak{o}(\mathrm{d} x) < \infty\}$, is endowed with the weak $\sigma(\mathscr{C}_b(\mathbb{R}^d), \mathscr{M})$ topology, despite with this topology \mathscr{M}_q is not complete. Given $\mathfrak{o} \in \mathscr{M}_q$, $f \in \mathscr{C}_q$, we can define their pairing as

$$\mathfrak{o}(f) := \int f(x)\mathfrak{o}(\mathrm{d}x).$$

Let $\mathscr{D} := \mathscr{M}_q \times \mathbb{R}^d$ be the state space endowed with the product topology. For a finite horizon T > 0, we set

$$\mathscr{M}_T := \{ \mathfrak{o} \in \mathscr{M}_q : |\mathfrak{o}| \le T \}, \quad \mathscr{D}_T := \{ (\mathfrak{o}, x) \in \mathscr{D} : |\mathfrak{o}| \le T \},$$

with the interior $\mathscr{D}_T := \{(\mathfrak{o}, x) \in \mathscr{D} : |\mathfrak{o}| < T\}$, and boundary $\partial \mathscr{D}_T := \{(\mathfrak{o}, x) \in \mathscr{D} : |\mathfrak{o}| = T\}$. The function space $\mathscr{U}(\mathscr{D})$ is the set of upper semi-continuous functions on \mathscr{D} , and $\mathscr{U}_b(\mathscr{D})$ are the ones that are bounded from above. Analogously, $\mathscr{L}(\mathscr{D})$ is the set of lower semi-continuous functions on \mathscr{D} , and $\mathscr{L}_b(\mathscr{D})$ are the ones that are bounded from below.

2.1 Occupation Derivative

A function $\varphi : \mathscr{D} \to \mathbb{R}$ is called *locally differentiable at* $(\mathfrak{o}, x) \in \mathscr{D}$ if

$$\partial_{\mathfrak{o}}\varphi(\mathfrak{o},x) := \lim_{h \downarrow 0} \frac{\varphi(\mathfrak{o} + h\delta_x, x) - \varphi(\mathfrak{o}, x)}{h} \quad \text{exists and is finite.}$$
(2.1)

We say that φ is *locally differentiable* if (2.1) holds for all $(\mathfrak{o}, x) \in \mathscr{D}$ and call $\partial_{\mathfrak{o}}\varphi : \mathscr{D} \to \mathbb{R}$ the *occupation derivative of* φ [35]. The local nature of the occupation derivative comes from the dynamics of the occupation flow given in Section 3. We also introduce the *linear derivative* of φ [11] and denote it by $\delta_{\mathfrak{o}}\varphi(\mathfrak{o}, x) \in \mathscr{C}(\mathbb{R}^d)$. We recall for convenience that $\delta_{\mathfrak{o}}\varphi$ is characterized by the identity

$$\varphi(\mathfrak{o}', x) - \varphi(\mathfrak{o}, x) = \int_0^1 \int_{\mathbb{R}^d} \delta_{\mathfrak{o}} \varphi(\mathfrak{o} + \eta(\mathfrak{o}' - \mathfrak{o}), x)(y)(\mathfrak{o}' - \mathfrak{o})(dy) d\eta, \quad \forall \mathfrak{o}, \mathfrak{o}' \in \mathscr{M}.$$
(2.2)

If $\partial_{\mathfrak{o}}\varphi$ is continuous in the product topology and satisfies $|\partial_{\mathfrak{o}}\varphi(\mathfrak{o}, x)| \leq C(1+|x|^2)$ for some constant C > 0, then the linear derivative of φ exists and relates to the occupation derivative through

$$\partial_{\mathfrak{o}}\varphi(\mathfrak{o},x) = \delta_{\mathfrak{o}}\varphi(\mathfrak{o},x)(x), \quad \forall (\mathfrak{o},x) \in \mathscr{D}.$$

$$(2.3)$$

See for instance [30, Theorem 2.1]. In other words, $\partial_{\mathfrak{o}}\varphi: \mathscr{D} \to \mathbb{R}$ is a local projection of the linear derivative. Finally, $\varphi \in \mathscr{C}(\mathscr{D})$ is said to be in $\mathscr{C}^{1,2}(\mathscr{D})$ if it has derivatives $\partial_{\mathfrak{o}}\varphi$, $\nabla\varphi$, $\nabla^{2}\varphi$ that are continuous.

2.2 Cylindrical Norm

Let $(f_k)_{k \in \mathbb{N}} \subset \mathscr{C}_b^1(\mathbb{R}^d)$ be a separating family of \mathscr{M} . Namely, $\mathfrak{o}(f_k) = \mathfrak{o}'(f_k)$ for all $k \in \mathbb{N}$ implies that $\mathfrak{o} = \mathfrak{o}'$; see, e.g., [28, Section 2.1]. Suppose that f_0 is a scalar multiple of $\tilde{f}_0 \equiv 1$ and by normalizing, we assume that

$$\sum_{k} \|f_{k}\|_{\mathscr{C}^{1}}^{2} \leq 1, \quad \text{where} \quad \|f\|_{\mathscr{C}^{1}} := \|f\|_{\infty} + \|\nabla f\|_{\infty}.$$
(2.4)

Using this family, we introduce the *cylindrical* and the *parabolic* norms, given respectively by

$$\varrho(\mathfrak{o}) := \left(\sum_{k} |\mathfrak{o}(f_k)|^2\right)^{\frac{1}{2}}, \qquad \rho(\mathfrak{o}, x) := \sqrt{\varrho^2(\mathfrak{o}) + |x|^2}, \qquad \mathfrak{o} \in \mathscr{M}, \ x \in \mathbb{R}^d.$$

In particular, $\varrho(\mathfrak{o}) = 0$ if and only if $\mathfrak{o} = 0$, since $(f_k)_{k \in \mathbb{N}}$ is a separating family of \mathscr{M} . As $|\mathfrak{o}(f_k)| \leq ||f_k||_{\infty} |\mathfrak{o}| \leq ||f_k||_{\mathscr{C}^1} |\mathfrak{o}|$, we conclude that $\varrho(\mathfrak{o}) \leq |\mathfrak{o}|$. Hence, the metric induced by ϱ is weaker than the total variation distance. In fact, it is easily seen that on \mathscr{M}_T , ϱ metrizes the weak topology. Moreover, for $\mathfrak{o} \in \mathscr{M}_+$, we directly calculate that

$$\partial_{\mathfrak{o}}\rho^{2}(o,x) = \delta_{\mathfrak{o}}\varrho^{2}(\mathfrak{o})(x) = 2\sum_{k}\mathfrak{o}(f_{k})f_{k}(x).$$
(2.5)

3 Stochastic Control Problem

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{Q})$ satisfying the usual conditions. Let X be a continuous semi-martingale, and $\Lambda_t \geq 0$ be a *strictly increasing* continuous adapted process. Then, the *occupation flow of* X with random time Λ is defined by,

$$\mathcal{O} = (\mathcal{O}_t)_{t \ge 0}, \quad \mathcal{O}_t(B) := \int_0^t \mathbb{1}_B(X_s) \mathrm{d}\Lambda_s, \quad B \in \mathscr{B}(\mathbb{R}^d).$$
(3.1)

Clearly for each t > 0, the support of \mathcal{O}_t is included in the ball of radius $\sup_{s \in [0,t]} |X_s|$ centered at the origin. Therefore, $\mathcal{O}_t \in \mathcal{M}_q$ and $(\mathcal{O}_t, X_t) \in \mathcal{D}$. We call the pair (\mathcal{O}, X) an occupied process. For all the properties of these processes we refer the reader to [35], and the references therein.

Two important examples of Λ are the followings:

- Standard time occupation flow corresponds to the choice $\Lambda_t = t$.
- Let $\Lambda_t = \operatorname{tr}(\langle X \rangle_t)$, where $\langle X \rangle = (\langle X^i, X^j \rangle)_{i,j}$ is the covariation matrix of X. Then, we call \mathcal{O} the occupation flow of X, which generates the local times [31, 35].

3.1 Controlled Occupied Processes

Let A be a Borel subset of a Euclidean space and the set \mathscr{A} of *admissible controls* are all predictable process $\alpha = (\alpha_t)_{t\geq 0}$ taking values in A and $\mathbb{Q} \times \text{Lebesgue square integrable}$. For fixed $\alpha \in \mathscr{A}$, we let $(\mathcal{O}^{\alpha}, X^{\alpha})$ be the strong solution of the controlled *occupied stochastic differential equation* (OSDE),

$$d\mathcal{O}_t^{\alpha} = \lambda(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t) \,\delta_{X_t^{\alpha}} dt, \tag{3.2}$$

$$dX_t^{\alpha} = b(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t)dt + \sigma(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t)dW_t,$$
(3.3)

with given functions $\lambda : \mathscr{D} \times A \to \mathbb{R}_+$, $b : \mathscr{D} \times A \to \mathbb{R}^d$, $\sigma : \mathscr{D} \times A \to \mathbb{R}^{d \times d}$ and initial condition $(\mathcal{O}_0^{\alpha}, X_0^{\alpha}) = (\mathfrak{o}, x) \in \mathscr{D}$. First equation implies that \mathcal{O}^{α} is the occupation flow with random time $\Lambda_t^{\alpha} = |\mathfrak{o}| + \int_0^t \lambda(\mathcal{O}_s^{\alpha}, X_s^{\alpha}, \alpha_s) \mathrm{d}s$. If we let $\lambda = \|\sigma\|_F^2 = \mathrm{tr}(\sigma\sigma^{\top})$ be the squared Frobenius norm of σ , then \mathcal{O}^{α} is the occupation flow of the process X^{α} . The choice $\lambda \equiv 1$ corresponds to the standard time occupation flow of X^{α} .

Following [35], we make the following definition.

Definition 3.1. For a normed vector space $(E, |\cdot|)$, we say a function $\varphi : \mathscr{D}_T \times A \mapsto E$ satisfies the growth and Lipschitz conditions with constant c_* , if the followings hold for every $(\mathfrak{o}, x, a), (\mathfrak{o}', x', a)$ in the set $\mathscr{D}_T \times A$:

$$|\varphi(\mathfrak{o}, x, a)| \le c_*(1 + \rho(\mathfrak{o}, x)), \tag{3.4}$$

$$|\varphi(\mathfrak{o}, x, a) - \varphi(\mathfrak{o}', x', a)| \le c_* \rho(\mathfrak{o} - \mathfrak{o}', x - x').$$
(3.5)

Lemma 3.2. Suppose that the coefficients λ, b, σ satisfy the growth and Lipschitz conditions with a constant c_* with the Euclidean and Frobenius norm for λ, b and σ , respectively. Then, there is a unique strong solution of OSDE (3.2)–(3.3) for any initial condition $(\mathfrak{o}, x) \in \mathscr{D}_T$.

Proof. This result is proved in [36, Theorem 4.2.11] under the growth and Lipschitz conditions with $\rho_{\rm BL}(\mathfrak{o}, x) := \sqrt{|\mathfrak{o}|_{\rm BL} + |x|^2}$, where $|\cdot|_{\rm BL}$ is the bounded Lipschitz norm

$$|\mathfrak{o}|_{\rm BL} := \sup\left\{\mathfrak{o}(f) : \max(\|f\|_{\infty}, [f]_{\rm Lip}) \le 1\right\}, \quad [f]_{\rm Lip} := \sup_{x \ne x'} \frac{|f(x) - f(x')|}{|x - x'|}$$

We now verify that ρ_{BL} dominates ρ . Indeed, define $\tilde{f}_k = f_k / ||f_k||_{\mathscr{C}^1} \in \mathscr{C}_b^1(\mathbb{R}^d)$ with (f_k) given in Section 2.2. Then $\max(||\tilde{f}_k||_{\infty}, [\tilde{f}_k]_{\text{Lip}}) \leq ||\tilde{f}_k||_{\mathscr{C}^1} \leq 1$, hence

$$\varrho(\mathfrak{o})^2 = \sum_k |\mathfrak{o}(f_k)|^2 \le \sum_k ||f_k||_{\mathscr{C}^1}^2 |\mathfrak{o}(\tilde{f}_k)|^2 \le |o|_{\mathrm{BL}}^2 \sum_k ||f_k||_{\mathscr{C}^1}^2 \le |o|_{\mathrm{BL}}^2.$$

Then it is clear that $\rho^2(\mathfrak{o}, x) \leq \rho_{BL}^2(\mathfrak{o}, x)$. Thus, the conditions of [36, Theorem 4.2.11] hold and the existence of a strong solution follows.

The following Itô formula is a generalization of Theorem 3.3 in [35]. See also [8, Proposition 9].

Proposition 3.3. (Itô Formula) Consider an occupied process (\mathcal{O}, X) with dynamics

$$d\mathcal{O}_t = \lambda_t \delta_{X_t} dt, \quad dX_t = b_t dt + \sigma_t dW_t,$$

where λ, b, σ are given adapted processes taking values in $\mathbb{R}_+, \mathbb{R}^d, \mathbb{R}^{d \times d}$, respectively, such that λ and b are locally integrable and σ is locally square integrable. If $v \in \mathcal{C}^{1,2}(\mathcal{D})$, then for all $t \geq 0$,

$$dv(\mathcal{O}_t, X_t) = \left(\lambda_t \partial_{\mathfrak{o}} v + b_t \cdot \nabla v + \frac{1}{2} \operatorname{tr}(\sigma_t \sigma_t^\top \nabla^2 v)\right) (\mathcal{O}_t, X_t) dt + \nabla v(\mathcal{O}_t, X_t) \cdot \sigma_t dW_t.$$
(3.6)

Proof. See Appendix A.

Evidently, the above formula applies to controlled occupied processes solving (3.2)-(3.3) by setting $\varphi_t = \varphi(\mathcal{O}_t^{\alpha}, X_t^{\alpha}, \alpha_t), \varphi \in \{\lambda, b, \sigma\}.$

3.2 Value function

Given $\ell : \mathscr{D}_T \times A \to \mathbb{R}, g : \partial \mathscr{D}_T \to \mathbb{R}$, and the shorthand notation $\mathbb{E}^{\mathbb{Q}}_{\mathfrak{o},x}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot \mid (\mathcal{O}^{\alpha}_0, X^{\alpha}_0) = (\mathfrak{o}, x)]$, consider the control problem of minimizing

$$J(\mathfrak{o}, x, \alpha) := \mathbb{E}^{\mathbb{Q}}_{\mathfrak{o}, x} [\int_{0}^{\tau^{\alpha}} \ell(\mathcal{O}^{\alpha}_{t}, X^{\alpha}_{t}, \alpha_{t}) dt + g(\mathcal{O}^{\alpha}_{\tau^{\alpha}}, X^{\alpha}_{\tau^{\alpha}})],$$
(3.7)

over all $\alpha \in \mathscr{A}$, with the exit time $\tau^{\alpha} = \inf\{t \ge 0 : (\mathcal{O}_t^{\alpha}, X_t^{\alpha}) \notin \mathscr{D}_T\} = \inf\{t \ge 0 : \Lambda_t^{\alpha} \ge T\}$. The value function is given by,

$$v(\mathbf{o}, x) := \inf_{\alpha \in \mathscr{A}} J(\mathbf{o}, x, \alpha), \qquad (\mathbf{o}, x) \in \mathscr{D}_T.$$
(3.8)

In what follows, we make the following standing assumption.

Assumption 3.4. There exists $c_* \geq 1$ such that λ , b, σ , ℓ , and g all satisfy growth and Lipschitz conditions with c_* . Moreover,

$$\lambda(\mathfrak{o}, x, a) \ge 1/c_*, \qquad \forall \ (\mathfrak{o}, x, a) \in \mathscr{D}_T \times A.$$
(3.9)

We remark that we require the nondegeneracy of λ , but not that of σ . In particular, condition (3.9) gives a lower bound on the total mass process, namely

$$\Lambda_t^{\alpha} = |\mathcal{O}_0^{\alpha}| + \int_0^t \lambda(\mathcal{O}_s^{\alpha}, X_s^{\alpha}, \alpha_s) \, ds \ge t/c_*, \qquad t \ge 0$$

Hence the exit time in (3.7) satisfies $\tau^{\alpha} \leq c_*T$, ensuring that the objective function J is finite. If $\lambda \equiv 1$ (standard time occupation flow), then (3.9) holds with $c_* = 1$ and we have $\tau^{\alpha} \equiv T$. If $\lambda = \|\sigma\|_F^2$ (occupation flow), then (3.9) can be seen as a weak ellipticity condition on σ .

Proposition 3.5. Under Assumption 3.4, the value function is locally 1/2-Hölder continuous with respect to ρ , that is, for all $\delta > 0$ there exists $\hat{c} > 0$ that depends on c_* , T, and δ such that

$$\rho(\mathfrak{o} - \mathfrak{o}', x - x') \le \delta \implies |v(\mathfrak{o}, x) - v(\mathfrak{o}', x')| \le \hat{c} \ \rho(\mathfrak{o} - \mathfrak{o}', x - x')^{1/2}.$$
(3.10)

Proof. See Appendix B.

Remark 3.6. In standard stochastic control theory, typically the value function is 1/2-Hölder continuous in t, but is Lipschitz continuous in x. Here, however, v is only 1/2-Hölder continuous in x in general. The reason is that the exit time τ^{α} may depend on x. For example, let d = 1, $b \equiv 0$, $\sigma \equiv 1$, $\ell \equiv 0$, and $\lambda = \lambda(x) \ge 1$, g = g(x) are Lipschitz continuous in x with Lipschitz constant C = 1. Then the problem does not involve the control α . Given $(\mathbf{o}, x) \in \mathscr{D}_T$, the exit time $\tau_{\mathbf{o},x} = \inf\{t \ge 0: \int_0^t \lambda(x+W_s)ds \ge T - |\mathbf{o}|\}$, and $v(\mathbf{o}, x) = \mathbb{E}^{\mathbb{Q}}[g(x+W_{\tau_{\mathbf{o},x}})]$. Thus

$$|v(\mathfrak{o}, x) - v(\mathfrak{o}, x')| \le |x - x'| + \mathbb{E}^{\mathbb{Q}}\left[|W_{\tau_{\mathfrak{o}, x}} - W_{\tau_{\mathfrak{o}, x'}}|\right]$$

From the Burkholder-Davis-Gundy inequality [31, Chapter IV], we can see that the last term above is of order $\mathbb{E}^{\mathbb{Q}}\left[|\tau_{\mathfrak{o},x}-\tau_{\mathfrak{o},x'}|^{1/2}\right] \sim |x-x'|^{1/2}$.

3.3 Viscosity Solutions and Uniqueness

For $(\mathfrak{o}, x) \in \mathscr{D}_T$, $\zeta := (\theta, \Delta, \Gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, the Hamiltonian is given by,

$$\mathscr{H}(\mathfrak{o}, x, \zeta) = -\inf_{a \in A} \left(\lambda(\mathfrak{o}, x, a) \ \theta + b(\mathfrak{o}, x, a) \cdot \Delta + \frac{1}{2} \operatorname{tr}((\sigma \sigma^{\top})(\mathfrak{o}, x, a)\Gamma) + \ell(\mathfrak{o}, x, a) \right).$$
(3.11)

Then, by [21] the dynamic programming equation associated to (3.7) is given by

$$\begin{cases} \mathscr{H}(\mathfrak{o}, x, \partial_{\mathfrak{o}} u, \nabla u, \nabla^{2} u) = 0 & \text{on } \dot{\mathscr{D}}_{T}, \\ u = g, & \text{on } \partial \mathscr{D}_{T}, \end{cases}$$
(3.12a)
(3.12b)

where $\partial_{\mathfrak{o}} u = \partial_{\mathfrak{o}} u(\mathfrak{o}, x)$ is the occupation derivative. As announced in the Introduction, equation (3.12a) is linear in $\partial_{\mathfrak{o}} u$ when $\lambda \equiv 1$ (standard time) and admits the more familiar expression,

 $-\partial_{\mathfrak{o}} u + \mathscr{H}_{x}(\mathfrak{o}, x, \nabla u, \nabla^{2} u) = 0, \quad \mathscr{H}_{x}(\mathfrak{o}, x, \Delta, \Gamma) = \mathscr{H}(\mathfrak{o}, x, 0, \Delta, \Gamma).$

In general, however, the control may influence the rate λ leading to nonlinearities in the occupation derivative as well.

Next, we introduce suitable families of test functions towards a viscosity theory. For T > 0, $u \in \mathscr{U}(\mathscr{D}_T), w \in \mathscr{L}(\mathscr{D}_T)$, and $(\mathfrak{o}, x) \in \mathscr{D}_T$, following [13, 21] we set

$$\mathfrak{S}_{+} u(\mathfrak{o}, x) = \mathfrak{S}_{+}^{T} u(\mathfrak{o}, x) := \{ \phi \in \mathscr{C}^{1,2}(\mathscr{D}_{T}) : 0 = (u - \phi)(\mathfrak{o}, x) = \max_{\mathscr{D}_{T}} (u - \phi) \},\$$
$$\mathfrak{S}_{-} w(\mathfrak{o}, x) = \mathfrak{S}_{-}^{T} w(\mathfrak{o}, x) := \{ \phi \in \mathscr{C}^{1,2}(\mathscr{D}_{T}) : 0 = (w - \phi)(\mathfrak{o}, x) = \min_{\mathscr{D}_{T}} (w - \phi) \}.$$

The definition of viscosity solutions is classical [13, 21].

Definition 3.7. (Viscosity solutions) We say that

• $w \in \mathscr{L}(\mathscr{D}_T)$ is a viscosity supersolution of $\mathscr{H} \geq 0$, if

$$\mathscr{H}(\mathfrak{o}, x, \partial_{\mathfrak{o}}\phi, \nabla\phi, \nabla^{2}\phi) \geq 0, \qquad \forall \phi \in \mathfrak{S}_{-} w(\mathfrak{o}, x), \quad (\mathfrak{o}, x) \in \mathscr{D}_{T}.$$
(3.13)

• $u \in \mathscr{U}(\mathscr{D}_T)$ is a viscosity subsolution of $\mathscr{H} \leq 0$, if

$$\mathscr{H}(\mathfrak{o}, x, \partial_{\mathfrak{o}}\phi, \nabla\phi, \nabla^{2}\phi) \leq 0, \qquad \forall \phi \in \mathfrak{S}_{+} u(\mathfrak{o}, x), \quad (\mathfrak{o}, x) \in \mathring{\mathscr{D}}_{T}.$$
(3.14)

• $v \in \mathscr{C}(\mathscr{D}_T)$ a viscosity solution $\mathscr{H} = 0$, if it is both a viscosity supersolution of $\mathscr{H} \ge 0$ and a subsolution of $\mathscr{H} \le 0$.

Remark 3.8. Following the classical theory, in the definition of a supersolution we can take any function (not even measurable) and then consider its upper semicontinuous envelope. Similarly for subsolutions, we could consider the lower semicontinuous envelope.

The main result of this paper is the following comparison principle.

Theorem 3.9. (Comparison) Suppose that Assumption 3.4 holds, $u \in \mathscr{U}_b(\mathscr{D}_T)$ is a viscosity subsolution of $\mathscr{H} \leq 0$, $w \in \mathscr{L}_b(\mathscr{D}_T)$ is a viscosity supersolution of $\mathscr{H} \geq 0$, and $u \leq w$ on $\partial \mathscr{D}_T$. Then, $u \leq w$ on \mathscr{D}_T .

We shall devote the next three sections to the proof of this theorem. We now provide the characterization of the value function v.

Theorem 3.10. Suppose that Assumption 3.4 holds. Then, the value function v is the unique viscosity solution of the dynamic programming equation (3.12a)–(3.12b).

Proof. Given the regularity in Proposition 3.5 and using the techniques developed in [21], it is classical that the value function v is a viscosity solution. Moreover, the uniqueness of viscosity solution is a direct consequence of Theorem 3.9.

4 Technical tools

In this section, we outline several concepts that we utilize.

4.1 Coercivity and coercive approximations

We use the notion of coercivity to construct extrema of functions on a topological space \mathscr{X} .

Definition 4.1. We say that a function $u \in \mathscr{U}(\mathscr{X})$ is *upper coercive* if all hypographs of u are sequentially compact, i.e., for every $c \in \mathbb{R}$, every sequence $\{\xi_n\}_n$ in \mathscr{X} with $u(\xi_n) \geq c$ has a limit point $\xi^* \in \mathscr{X}$. $\mathscr{U}_e(\mathscr{X})$ denotes the set of upper coercive functions.

Similarly, we say that a function $w \in \mathscr{L}(\mathscr{X})$ is *lower coercive* if every epigraph of w is sequentially compact, and $\mathscr{L}_{e}(\mathscr{X})$ denotes the set of lower coercive functions.

A direct consequence of the definition is the following.

Lemma 4.2. Any $u \in \mathscr{U}_e(\mathscr{X})$ is bounded from above and achieves its maximum on any closed subset of \mathscr{X} . Analogously, any $w \in \mathscr{L}_e(\mathscr{X})$ is bounded from below and achieves its minimum on any closed subset of \mathscr{X} .

Proof. Consider an upper coercive function $u \in \mathscr{U}(\mathscr{X})$. Towards a contraposition, suppose that for each positive integer n, there is $\xi_n \in \mathscr{X}$ such that $u(\xi_n) \ge n$. Since $\{u \ge 1\}$ is compact by hypothesis, on a subsequence n_k , ξ_{n_k} is convergent. Let ξ_* be the limit point. Since u is upper semicontinuous and real-valued, $\infty > u(\xi_*) \ge \lim_k u(\xi_{n_k}) = \infty$. Hence, u is bounded from above. Then, the upper semicontinuity and the upper coercivity implies that the maximum of u on any closed set is achieved.

Recall from Section 2 that $q(x) = \sqrt{1 + |x|^2}$ and set

$$\vartheta(\mathfrak{o}, x) := \mathfrak{o}(q) + q(x), \qquad (\mathfrak{o}, x) \in \mathscr{D} = \mathscr{M}_q \times \mathbb{R}^d.$$

$$(4.1)$$

It is classical that the function $\vartheta \in \mathscr{L}_b(\mathscr{D})$, and for any constant c > 0, the sub-level set $\{(\mathfrak{o}, x) \in \mathscr{D} : \vartheta(\mathfrak{o}, x) \leq c\}$ is compact. Then,

$$\mathscr{D} = \bigcup_{m \ge 1} \mathscr{D}^m, \qquad \mathscr{D}^m := \{(\mathfrak{o}, x) \in \mathscr{D} : \vartheta(\mathfrak{o}, x) \le m\}.$$

Although \mathscr{D} is not compact, each \mathscr{D}^m is compact giving it a locally compact structure. Additionally, $\vartheta \in \mathscr{C}^{1,2}(\mathscr{D})$ and the derivatives are given by,

$$\partial_{\mathfrak{o}}\vartheta(\mathfrak{o},x) = q(x), \qquad \nabla\vartheta(\mathfrak{o},x) = \frac{x}{q(x)}, \qquad \nabla^{2}\vartheta(\mathfrak{o},x) = \frac{1}{q(x)} \Big(I - \frac{x \otimes x}{q(x)^{2}}\Big). \tag{4.2}$$

For $w \in \mathscr{L}(\mathscr{D})$, $u \in \mathscr{U}(\mathscr{D})$, and $\gamma > 0$, we define *coercive approximations* by,

$$w_{\gamma} := w + \gamma \vartheta, \qquad u^{\gamma} := u - \gamma \vartheta.$$
 (4.3)

Lemma 4.3. For any $w \in \mathscr{L}_b(\mathscr{D})$, $w_{\gamma} \in \mathscr{L}_e(\mathscr{D})$, and for any $u \in \mathscr{U}_b(\mathscr{D})$, $u^{\gamma} \in \mathscr{U}_e(\mathscr{D})$.

Proof. As $\vartheta \in \mathscr{L}_b(\mathscr{D})$, it is clear that $w_{\gamma} \in \mathscr{L}_b(\mathscr{D})$. Additionally, as w is bounded from below and lower semicontinuous, for any constant c, $\{w_{\gamma} \leq c\}$ is a closed subset of \mathscr{D}^{m_*} with $m_* = (c - \inf_{\mathscr{D}} w)/\gamma$. Since \mathscr{D}^m is compact for every m, any closed subset of it is also compact. Hence, w_{γ} is lower coercive. The proof for u is essentially identical.

4.2 Semijets

Following definition is classical in the theory of viscosity solutions [13, 21].

Definition 4.4. For $(\mathfrak{o}, x) \in \mathscr{D}_T$, the parabolic superjet of $u \in \mathscr{U}(\mathscr{D}_T)$ at (\mathfrak{o}, x) is given by,

$$\mathscr{P}^{1,2}_+u(\mathfrak{o},x) := \{ (\partial_{\mathfrak{o}}\phi(\mathfrak{o},x), \nabla\phi(\mathfrak{o},x), \nabla^2\phi(\mathfrak{o},x)) \, : \, \phi \in \mathfrak{S}_+ \, u(\mathfrak{o},x) \}.$$

The parabolic subjet of $w \in \mathscr{L}(\mathscr{D}_T)$ at $(\mathfrak{o}, x) \in \mathscr{D}_T$ is given by,

$$\mathscr{P}^{1,2}_{-}w(\mathfrak{o},x) := \{ (\partial_{\mathfrak{o}}\phi(\mathfrak{o},x), \nabla\phi(\mathfrak{o},x), \nabla^{2}\phi(\mathfrak{o},x)) : \phi \in \mathfrak{S}_{-}w(\mathfrak{o},x) \}.$$

Above sets are subsets of $\mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, and their closures are defined by,

$$\overline{\mathscr{P}}^{1,2}_{\pm}v(\mathfrak{o},x) := \{ \lim_{n \to \infty} \zeta_n : \zeta_n \in \mathscr{P}^{1,2}_{\pm}v(\mathfrak{o}_n, x_n), \ (\mathfrak{o}_n, x_n, v(\mathfrak{o}_n, x_n)) \to (\mathfrak{o}, x, v(\mathfrak{o}, x)) \}.$$

The following equivalent definition of viscosity solutions follows directly from the continuity of the Hamiltonian and the definitions. Indeed, $u \in \mathscr{U}(\mathscr{D}_T)$ satisfies (3.14) if and only if

$$\mathscr{H}(\mathfrak{o}, x, \zeta) \ge 0, \qquad \forall \ \zeta \in \overline{\mathscr{P}}_{-}^{1,2}u(\mathfrak{o}, x), \ (\mathfrak{o}, x) \in \mathring{\mathscr{D}}_{T}.$$

$$(4.4)$$

Similarly, $w \in \mathscr{L}(\mathscr{D}_T)$ satisfies (3.13) if and only if

$$\mathscr{H}(\mathfrak{o}, x, \zeta) \leq 0, \qquad \forall \zeta \in \overline{\mathscr{P}}^{1,2}_+ w(\mathfrak{o}, x), \ (\mathfrak{o}, x) \in \mathring{\mathscr{D}}_T.$$
 (4.5)

The following is a direct consequence of the definitions.

Lemma 4.5. Suppose that $\zeta_n \in \overline{\mathscr{P}}^{1,2}_{\pm}v(\mathfrak{o}_n, x_n)$, and $(\zeta_n, \mathfrak{o}_n, x_n, v(\mathfrak{o}_n, x_n)) \to (\zeta, \mathfrak{o}, x, v(\mathfrak{o}, x))$ as n tends to infinity. Then, $\zeta \in \overline{\mathscr{P}}^{1,2}_{\pm}v(\mathfrak{o}, x)$.

4.3 Finite-dimensional Projections

We use a regularization technique similar to the one introduced by Lasry and Lions [24] and related finite-dimensional projections by Lions [26] to prove the uniqueness of viscosity solutions in Hilbert spaces. Recall the separating class $(f_k)_{k\in\mathbb{N}}\subset \mathscr{C}_b^1(\mathbb{R}^d)$ of subsection 2.2. For a positive integer K, we define the finite-dimensional projections by

$$\pi_K(\mathfrak{o}) := (\mathfrak{o}(f_1), \dots, \mathfrak{o}(f_K)) \in \mathscr{R}_K := \pi_K(\mathscr{M}_T).$$

The range \mathscr{R}_K depends on T and is a compact subset of \mathbb{R}^K . Indeed, as $|\pi_K(\mathfrak{o})| \leq \varrho(\mathfrak{o}) \leq |\mathfrak{o}| \leq T$, \mathscr{R}_K is contained in $\{z \in \mathbb{R}^K : |z| \leq T\}$. All the projected objects we define in this subsection, like \mathscr{R}_K , may depend on T, but we suppress this dependence.

We project a given $v: \mathscr{D}_T \to \mathbb{R}$ onto the projected state space $\mathscr{R}_K \times \mathbb{R}^d$ as follows,

$$\Pi^{K}(v)(z,x) := \sup\{v(\mathbf{o},x) : \pi_{K}(\mathbf{o}) = z\},$$

$$\Pi_{K}(v)(z,x) := \inf\{v(\mathbf{o},x) : \pi_{K}(\mathbf{o}) = z\}, \qquad (z,x) \in \mathscr{R}_{K} \times \mathbb{R}^{d}.$$

Let $\mathscr{M}^{K}(v, z, x)$ be the maximizers of the first expression, and $\mathscr{M}_{K}(v, z, x)$ be the minimizers of the second expression. Although these sets might be empty, in view of Lemma 4.2, $\mathscr{M}^{K}(u, z, x)$ is non-empty when $u \in \mathscr{U}_{e}(\mathscr{D})$, and $\mathscr{M}_{K}(w, z, x)$ is non-empty if $w \in \mathscr{L}_{e}(\mathscr{D})$. We make the simple yet crucial observation that

$$\Pi_{K}(v)(\pi_{K}(\mathfrak{o}), x) \leq v(\mathfrak{o}, x) \leq \Pi^{K}(v)(\pi_{K}(\mathfrak{o}), x), \quad \forall (\mathfrak{o}, x) \in \mathscr{D}.$$

$$(4.6)$$

Lemma 4.6. For any K, $u \in \mathscr{U}_e(\mathscr{D}_T)$, and $w \in \mathscr{L}_e(\mathscr{D}_T)$, we have $\Pi^K(u) \in \mathscr{U}_e(\mathscr{R}_K \times \mathbb{R}^d)$ and $\Pi_K(w) \in \mathscr{L}_e(\mathscr{R}_K \times \mathbb{R}^d)$.

Proof. Set $w_K := \prod_K(w)$ and consider a sequence (z_n, x_n) with $w_K(z_n, x_n) \leq c$ for some constant c. Choose $\mathfrak{o}_n \in \mathscr{M}_K(w, z_n, x_n)$. Then, $w(\mathfrak{o}_n, x_n) = w_K(z_n, x_n) \leq c$, and by the coercivity of w, there is $(\mathfrak{o}_*, x_*) \in \mathscr{D}$ and a subsequence, denoted by n again, such that $\lim_n (\mathfrak{o}_n, x_n) = (\mathfrak{o}_*, x_*)$. Then, $\lim_n z_n = \lim_n \pi_K(\mathfrak{o}_n) = \pi_K(\mathfrak{o}_*) =: z_*$ which follows from the weak convergence of \mathfrak{o}_n to \mathfrak{o}_* . Hence, (z_*, x_*) is a limit point of the sequence $\{(z_n, x_n)\}_n$, proving the compactness of epigraphs of w_K in the space $\mathscr{R}_K \times \mathbb{R}^d$.

Consider a sequence $\{(z_n, x_n)\}_n$ satisfying $\lim_n(z_n, x_n, w_K(z_n, x_n)) = (z_*, x_*, w_*)$. To prove the lower semicontinuity of w_K , we need to show that $w_* \ge w_K(z_*, x_*)$. Indeed, by the same argument as above, any sequence $\mathfrak{o}_n \in \mathscr{M}_K(w, z_n, x_n)$ has a limit point \mathfrak{o}_* . In light of $\pi_K(\mathfrak{o}_*) = z_*$ and (4.6), $w_K(z_*, x_*) \leq w(\mathfrak{o}_*, x_*)$. We now use the lower semicontinuity of w to arrive at

$$w_* = \lim_n w_K(z_n, x_n) = \liminf_n w(\mathfrak{o}_n, x_n) \ge w(\mathfrak{o}_*, x_*) \ge w_K(z_*, x_*).$$

The sub and super differentials and the sub and superjets in finite dimensional spaces are classical. Indeed, for any $\tilde{v} : \mathscr{R}_K \times \mathbb{R}^d \to \mathbb{R}$, we follow [13, 14] and define the parabolic sub and superjets $\mathscr{P}^{1,2}_{\pm} \tilde{v}(z,x), \overline{\mathscr{P}}^{1,2}_{\pm} \tilde{v}(z,x)$ as subsets of $\mathbb{R}^K \times \mathbb{R}^d \times \mathbb{S}_d$ using the Taylor expansion restricted to the set $\mathscr{R}^K \times \mathbb{R}^d$. These sets depend on the domain $\mathscr{R}_K \times \mathbb{R}^d$ of \tilde{v} and they have the following representation (see [13, Page 11]),

$$\begin{aligned} \mathscr{P}^{1,2}_+\tilde{v}(z,x) &= \{ (\nabla_z \phi(z,x), \nabla_x \phi(z,x), \nabla_x^2 \phi(z,x)) \, : \, \phi \in \mathfrak{S}_+ \, \tilde{v}(z,x) \} \\ \mathscr{P}^{1,2}_-\tilde{v}(z,x) &= \{ (\nabla_z \phi(z,x), \nabla_x \phi(z,x), \nabla_x^2 \phi(z,x)) \, : \, \phi \in \mathfrak{S}_- \, \tilde{v}(z,x) \}, \end{aligned}$$

where for $(z, x) \in \mathscr{R}_K \times \mathbb{R}^d$,

$$\begin{split} \mathfrak{S}_+ \, \tilde{v}(z,x) &:= \{ \phi \in \mathscr{C}^{1,2}(\mathbb{R}^K \times \mathbb{R}^d) \, : \, 0 = (\tilde{v} - \phi)(z,x) = \max_{\mathscr{R}_K \times \mathbb{R}^d} (\tilde{v} - \phi) \}, \\ \mathfrak{S}_- \, \tilde{v}(z,x) &:= \{ \phi \in \mathscr{C}^{1,2}(\mathbb{R}^K \times \mathbb{R}^d) \, : \, 0 = (\tilde{v} - \phi)(z,x) = \min_{\mathscr{R}_K \times \mathbb{R}^d} (\tilde{v} - \phi) \}. \end{split}$$

We remark that here we do not require (z, x) to be an interior point of $\mathscr{R}_K \times \mathbb{R}^d$, see Remark 5.2 below. We map the sub and superjets of \tilde{v} into smaller sets as follows:

$$\overline{\mathscr{P}}_{K,\pm}^{1,2}\tilde{v}(z,x) := \{ (\tilde{\theta} \cdot (f_1(x), \dots, f_K(x)), \Delta, \Gamma) : (\tilde{\theta}, \Delta, \Gamma) \in \overline{\mathscr{P}}_{\pm}^{1,2}\tilde{v}(z,x) \},\$$

and $\mathscr{P}^{1,2}_{K,\pm}\tilde{v}(z,x)$ is defined analogously. Then, for any $v:\mathscr{D}_T\to\mathbb{R}, \,\tilde{v}:\mathscr{R}_K\times\mathbb{R}^d\to\mathbb{R},$

$$\mathscr{P}^{1,2}_{\pm}v(\mathfrak{o},x), \ \overline{\mathscr{P}}^{1,2}_{\pm}v(\mathfrak{o},x), \ \mathscr{P}^{1,2}_{K,\pm}\tilde{v}(z,x), \ \overline{\mathscr{P}}^{1,2}_{K,\pm}\tilde{v}(z,x) \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d.$$

Using the projection maps Π_K, Π^K , we connect the semijets in $\mathscr{R}_K \times \mathbb{R}^d$ with those in \mathscr{D}_T . Lemma 4.7. For any $(z, x) \in \mathscr{R}_K \times \mathbb{R}^d$ and $w \in \mathscr{L}(\mathscr{D}_T)$,

$$\mathscr{P}_{K,-}^{1,2}w_K(z,x)\subset \mathscr{P}_{-}^{1,2}w(\mathfrak{o},x),\quad \forall \mathfrak{o}\in \mathscr{M}_K(w,z,x),$$

where $w_K := \Pi_K(w)$. If additionally w is lower coercive, then

$$\overline{\mathscr{P}}_{K,-}^{1,2}w_K(z,x)\subset \bigcup_{\mathfrak{o}\in\mathscr{M}_K(w,z,x)}\overline{\mathscr{P}}_{-}^{1,2}w(\mathfrak{o},x).$$

Similarly, for any $(z, x) \in \mathscr{R}_K \times \mathbb{R}^d$ and $u \in \mathscr{U}(\mathscr{D}_T)$ with $u^K := \Pi^K(u)$,

$$\mathscr{P}^{1,2}_{K,+}u^{K}(z,x)\subset \mathscr{P}^{1,2}_{+}u(\mathfrak{o},x), \quad \forall \mathfrak{o}\in \mathscr{M}^{K}(u,z,x).$$

If $u \in \mathscr{U}_e(\mathscr{D})$, then

$$\overline{\mathscr{P}}_{K,+}^{1,2}u^K(z,x)\subset \bigcup_{\mathfrak{o}\in\mathscr{M}^K(u,z,x)}\overline{\mathscr{P}}_+^{1,2}u(\mathfrak{o},x).$$

Proof. Fix $(z, x) \in \mathscr{R}_K \times \mathbb{R}^d$ and $(\theta, \Delta, \Gamma) \in \mathscr{P}^{1,2}_{K, -} w_K(z, x)$. Then by definition, there exists $\varphi \in \mathfrak{S}_+ w_K(z, x)$ such that

$$\theta = \nabla_z \varphi(z, x) \cdot (f_1(x), \dots, f_K(x)), \ \Delta = \nabla_x \varphi(z, x), \ \Gamma = \nabla_x^2 \varphi(z, x).$$

Since $\varphi \in \mathfrak{S}_+ w_K(z, x)$, we have $\varphi \leq w_K$. Set also

$$\phi(\mathfrak{o}, x) := \varphi(\pi_K(\mathfrak{o}), x), \quad (\mathfrak{o}, x) \in \mathscr{D}.$$

Then, for any $(\mathfrak{o}', x') \in \mathscr{D}$, using (4.6) and $\varphi \leq w_K$ we arrive at

$$w(\mathfrak{o}', x') \ge w_K(\pi_K(\mathfrak{o}'), x') \ge \varphi(\pi_K(\mathfrak{o}'), x') = \phi(\mathfrak{o}', x').$$

Additionally for any $\mathfrak{o} \in \mathcal{M}_K(w, z, x), \pi_K(\mathfrak{o}) = z$ and

$$w(\mathbf{o}, x) = w_K(z, x) = \varphi(z, x) = \phi(\mathbf{o}, x).$$

Summarizing, we have shown that,

$$\mathfrak{O} = (w - \phi)(\mathfrak{o}, x) \le (w - \phi)(\mathfrak{o}', x'), \qquad \forall (\mathfrak{o}', x') \in \mathscr{D}_T.$$

Hence, $\phi \in \mathfrak{S}_+ w(\mathfrak{o}, x)$. Moreover,

$$\delta_{\mathfrak{o}}\phi(\mathfrak{o},x)(\cdot) = \nabla_{z}\varphi(\pi_{K}(\mathfrak{o}),x) \cdot (f_{1}(\cdot),\ldots,f_{K}(\cdot)),$$

$$\implies \quad \partial_{\mathfrak{o}}\phi(\mathfrak{o},x) = \delta_{\mathfrak{o}}\phi(\mathfrak{o},x)(x) = \nabla_{z}\varphi(\pi_{K}(\mathfrak{o}),x) \cdot (f_{1}(x),\ldots,f_{K}(x)) = \theta.$$

Since $\nabla_x \phi(\mathfrak{o}, x) = \nabla_x \varphi(z, x) = \Delta$ and $\nabla_x^2 \phi(\mathfrak{o}, x) = \nabla_x^2 \varphi(z, x) = \Gamma$, we conclude that

$$(\theta, \Delta, \Gamma) \in \mathscr{P}^{1,2}_{-}w(\mathfrak{o}, x).$$

Now suppose that $\zeta = (\theta, \Delta, \Gamma) \in \overline{\mathscr{P}}_{K,-}^{1,2} w_K(z, x)$. Then, there are $(x_n, z_n) \to (x, z)$ and $\zeta_n \to \zeta$ such that $\zeta_n \in \mathscr{P}_{K,-}^{1,2} w_K(z_n, x_n)$ and $w_K(z_n, x_n) \to w_K(z, x)$. Since $w \in \mathscr{L}_e(\mathscr{D})$, $\mathscr{M}_K(w, z_n, x_n)$ is non-empty and by the above result there is $\mathfrak{o}_n \in \mathscr{M}_K(w, z_n, x_n)$ such that $\zeta_n \in \mathscr{P}_+^{1,2} w(\mathfrak{o}_n, x_n)$. In particular, since $w_K(z_n, x_n) \to w_K(z, x)$, the sequence $w_K(z_n, x_n)$ is uniformly bounded from above. Set $m := \sup_n w_K(z_n, x_n)$. As $w(\mathfrak{o}_n, x_n) = w_K(z_n, x_n)$, $(\mathfrak{o}_n, x_n) \in \{w \leq m\}$. Then by the lower coercivity of w, there is $(\mathfrak{o}, x) \in \mathscr{D}$ and a subsequence, denoted by n again, such that $(\mathfrak{o}_n, x_n) \to (\mathfrak{o}, x)$.

We claim that $\mathfrak{o} \in \mathscr{M}_K(w, z, x)$ and $\zeta \in \overline{\mathscr{P}}^{1,2}_{-}w(\mathfrak{o}, x)$. Indeed,

$$\pi_K(\mathfrak{o}) = \lim_n \pi_K(\mathfrak{o}_n) = \lim_n z_n = z, \quad \Longrightarrow \quad w(\mathfrak{o}, x) \ge w_K(\pi_K(\mathfrak{o}), x) = w_K(z, x).$$

Moreover, since $\lim_{n \to \infty} w_K(z_n, x_n) = w_K(z, x)$ and $w \in \mathscr{L}(\mathscr{D}_T)$,

$$w_K(z,x) \le w(\mathfrak{o},x) \le \liminf w(\mathfrak{o}_n,x_n) = \lim w_K(z_n,x_n) = w_K(z,x).$$

Hence, $\lim_{n} w(\mathfrak{o}_n, x_n) = w(\mathfrak{o}, x) = w_K(z, x)$, implying that $\mathfrak{o} \in \mathscr{M}_K(w, z, x)$. Since the sequence $(\zeta_n, \mathfrak{o}_n, x_n, w(\mathfrak{o}_n, x_n))$ converges to $(\zeta, \mathfrak{o}, x, w(\mathfrak{o}, x))$ and $\zeta_n \in \mathscr{P}_{K,-}^{1,2} w_K(z_n, x_n)$, we conclude that $\zeta \in \overline{\mathscr{P}}_{-}^{1,2} w(\mathfrak{o}, x)$, proving the second statement for w.

The proof for u is essentially identical.

We write $\mathbf{x} = (\underline{x}, \overline{x})$, $\mathbf{o} = (\underline{o}, \overline{o})$, $\pi_K(\mathbf{o}) = (\pi_K(\underline{o}), \pi_K(\overline{o}))$, and for a given $\Phi : \mathscr{D}_T^2 \to \mathbb{R}$, we also write $\Phi(\mathbf{o}, \mathbf{x}) = \Phi(\underline{o}, \underline{x}, \overline{o}, \overline{x})$. Let I be the identity matrix in \mathbb{S}^d . For $\Gamma_1, \Gamma_2 \in \mathbb{S}^d$, introduce the block diagonal matrices in \mathbb{S}^{2d} by,

diag
$$(\Gamma_1, \Gamma_2) = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Then, $\mathbf{G} \mathbf{x} \cdot \mathbf{x} = |\underline{x} - \overline{x}|^2$, and $\mathbf{I} = \text{diag}(I, I)$ is the identity matrix in \mathbb{S}^{2d} .

Lemma 5.1. (Crandall-Ishii-Lions Lemma) Suppose that $u \in \mathscr{U}_e(\mathscr{D}_T)$, $w \in \mathscr{L}_e(\mathscr{D}_T)$, $\varepsilon > 0$, and $(\mathbf{o}^*, \mathbf{x}^*) = (\underline{\mathbf{o}}^*, \overline{\mathbf{o}}^*, \underline{x}^*, \overline{x}^*)$ is a maximizer of

$$\Phi: \mathscr{D}_T^2 \to \mathbb{R}, \quad \Phi(\mathbf{o}, \mathbf{x}) = u(\underline{\mathbf{o}}, \underline{x}) - w(\overline{\mathbf{o}}, \overline{x}) - \frac{1}{2\varepsilon} \rho(\underline{\mathbf{o}} - \overline{\mathbf{o}}, \underline{x} - \overline{x})^2.$$

Then, there exist $\underline{\Gamma}, \overline{\Gamma} \in \mathbb{S}^d$ such that

$$(\Theta(\underline{x}^*), \Delta, \underline{\Gamma}) \in \overline{\mathscr{P}}^{1,2}_+ u(\underline{\mathfrak{o}}^*, \underline{x}^*), \qquad (\Theta(\overline{x}^*), \Delta, \overline{\Gamma}) \in \overline{\mathscr{P}}^{1,2}_- w(\overline{\mathfrak{o}}^*, \overline{x}^*), \tag{5.1}$$

$$-\frac{3}{\varepsilon}\mathbf{I} \le \operatorname{diag}(\underline{\Gamma}, -\overline{\Gamma}) \le \frac{3}{\varepsilon}\mathbf{G},\tag{5.2}$$

where $\Delta = (\underline{x} - \overline{x})/\varepsilon$, and $\Theta \in \mathscr{C}_b(\mathbb{R}^d)$ is given by

$$\Theta(x) = \delta_{\mathfrak{o}} \Psi(\underline{\mathfrak{o}}^*)(x) = \frac{1}{\varepsilon} \sum_{k} (\underline{\mathfrak{o}}^* - \overline{\mathfrak{o}}^*)(f_k) f_k(x), \quad \Psi(\mathfrak{o}) := \frac{1}{2\varepsilon} \varrho^2 (\mathfrak{o} - \overline{\mathfrak{o}}^*).$$
(5.3)

Proof. We proceed in several steps. We first assume that $(\mathbf{o}^*, \mathbf{x}^*)$ is a strict maximizer, and consider the general case at the final step.

Step 1 (Projection). Set $u^K := \Pi^K(u), w_K := \Pi_K(w)$, and

$$\Phi_{K}(\mathbf{z},\mathbf{x}) := u^{K}(\underline{z},\underline{x}) - w_{K}(\overline{z},\overline{x}) - \frac{1}{2\varepsilon} \left(|\underline{z} - \overline{z}|^{2} + |\underline{x} - \overline{x}|^{2} \right).$$

By Lemma 4.6, $u^K \in \mathscr{U}_e(\mathscr{R}_K \times \mathbb{R}^d)$ and $w_K \in \mathscr{L}_e(\mathscr{R}_K \times \mathbb{R}^d)$. Therefore, Φ_K achieves its maximum at, say, $(\mathbf{z}_K, \mathbf{x}_K)$. Additionally, by (4.6),

$$\Phi(\mathbf{o}, \mathbf{x}) \le \Phi_K(\pi_K(\mathbf{o}), \mathbf{x}) \le \Phi_K(\mathbf{z}_K, \mathbf{x}_K), \qquad \forall (\mathbf{o}, \mathbf{x}) \in \mathscr{D}_T^2.$$
(5.4)

Moreover, for any $\mathbf{o} \in \mathscr{E}_K(\mathbf{z}, \mathbf{x}) := \{ \mathbf{o} = (\underline{\mathbf{o}}, \overline{\mathbf{o}}) \in \mathscr{M}_T^2 : \underline{\mathbf{o}} \in \mathscr{M}^K(u, \underline{z}, \underline{x}), \overline{\mathbf{o}} \in \mathscr{M}_K(w, \overline{z}, \overline{x}) \},\$

$$0 \le \Phi_K(\mathbf{z}, \mathbf{x}) - \Phi(\mathbf{o}, \mathbf{x}) = \frac{1}{2\varepsilon} (\varrho^2(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}) - |\pi_K(\underline{\mathfrak{o}}) - \pi_K(\overline{\mathfrak{o}})|^2)$$
$$= \frac{1}{2\varepsilon} \sum_{k>K} |(\underline{\mathfrak{o}} - \overline{\mathfrak{o}})(f_k)|^2 \le \frac{1}{2\varepsilon} \sum_{k>K} ||f_k||_{\infty}^2 |\underline{\mathfrak{o}} - \overline{\mathfrak{o}}|^2.$$

Since $|\mathfrak{o}| \leq T$ for all $\mathfrak{o} \in \mathscr{M}_T$ and $\sum_k ||f_k||_{\infty}^2 \leq 1$, we conclude that

$$\lim_{K \to \infty} \sup\{ |\Phi(\mathbf{o}, \mathbf{x}) - \Phi_K(\mathbf{z}, \mathbf{x})| : \mathbf{o} \in \mathscr{E}_K(\mathbf{z}, \mathbf{x}), (\mathbf{z}, \mathbf{x}) \in \mathscr{R}_K \times \mathbb{R}^d \} = 0.$$
(5.5)

Step 2 (Finite dimensional Crandall-Ishii-Lions). By the classical Crandall-Ishii-Lions Lemma on the closed set $\mathscr{R}_K \times \mathbb{R}^d$ [13, Theorem 3.2], [14], there exists $\underline{\Gamma}_K, \overline{\Gamma}_K \in \mathbb{S}^d$ such that

$$(\underline{\theta}_{K}, \Delta, \underline{\Gamma}_{K}) \in \overline{\mathscr{P}}_{+}^{1,2} u^{K}(\underline{z}_{K}, \underline{x}_{K}), \quad (\overline{\theta}_{K}, \Delta, \overline{\Gamma}_{K}) \in \overline{\mathscr{P}}_{-}^{1,2} w_{K}(\overline{z}_{K}, \overline{x}_{K}), \tag{5.6}$$

$$-\frac{3}{\varepsilon}\mathbf{I} \le \operatorname{diag}(\underline{\Gamma}_{K}, -\overline{\Gamma}_{K}) \le \frac{3}{\varepsilon}\mathbf{G},\tag{5.7}$$

where $\Delta = (\underline{x}_K - \overline{x}_K)/\varepsilon$, and

$$\underline{\theta}_{K} = \frac{1}{\varepsilon} (\underline{z}_{K} - \overline{z}_{K}) \cdot (f_{1}(\underline{x}_{K}), \dots, f_{K}(\underline{x}_{K})), \quad \overline{\theta}_{K} = \frac{1}{\varepsilon} (\underline{z}_{K} - \overline{z}_{K}) \cdot (f_{1}(\overline{x}_{K}), \dots, f_{K}(\overline{x}_{K})).$$

Moreover, by (5.6) and Lemma 4.7, there exist $\mathbf{o}_K = (\underline{\mathbf{o}}_K, \overline{\mathbf{o}}_K) \in \mathscr{E}_K(\mathbf{z}_K, \mathbf{x}_K)$. such that

$$(\underline{\theta}_{K}, \Delta, \underline{\Gamma}_{K}) \in \overline{\mathscr{P}}_{+}^{1,2} u(\underline{\mathfrak{o}}_{K}, \underline{x}_{K}), \qquad (\overline{\theta}_{K}, \Delta, \overline{\Gamma}_{K}) \in \overline{\mathscr{P}}_{-}^{1,2} w(\overline{\mathfrak{o}}_{K}, \overline{x}_{K}).$$
(5.8)

Step 3 (Convergence of $(\mathbf{o}_K, \mathbf{x}_K)$). We claim that $\lim_K (\mathbf{o}_K, \mathbf{x}_K) = (\mathbf{o}^*, \mathbf{x}^*)$. Indeed, by coercivity, arguing as before we conclude that on a subsequence, denoted by K again, $\lim_K (\mathbf{o}_K, \mathbf{x}_K) =: (\hat{\mathbf{o}}, \hat{\mathbf{x}})$ exists. Since $\mathbf{o}_K \in \mathscr{E}_K(\mathbf{z}_K, \mathbf{x}_K)$, by (5.5),

$$\lim_{K\to\infty} |\Phi(\mathbf{o}_K,\mathbf{x}_K) - \Phi_K(\mathbf{z}_K,\mathbf{x}_K)| = 0.$$

Moreover by (5.4), $\Phi_K(\mathbf{z}_K, \mathbf{x}_K) \geq \Phi(\mathbf{o}^*, \mathbf{x}^*)$. Combining all these inequalities, we arrive at

$$\Phi(\mathbf{o}^*, \mathbf{x}^*) \ge \Phi(\hat{\mathbf{o}}, \hat{\mathbf{x}}) \ge \limsup_{K} \Phi(\mathbf{o}_K, \mathbf{x}_K) = \limsup_{K} \Phi_K(\mathbf{z}_K, \mathbf{x}_K) \ge \Phi(\mathbf{o}^*, \mathbf{x}^*),$$

where in the second inequality, we used the upper semi-continuity of Φ . As $(\mathbf{o}^*, \mathbf{x}^*)$ is the strict maximizer of Φ , we conclude that $(\hat{\mathbf{o}}, \hat{\mathbf{x}}) = (\mathbf{o}^*, \mathbf{x}^*)$ and all above inequalities are equalities. In particular,

$$\lim_{K \to \infty} u(\underline{\mathfrak{o}}_K, x_K) = u(\underline{\mathfrak{o}}^*, x^*), \quad \text{and} \quad \lim_{K \to \infty} w(\overline{\mathfrak{o}}_K, x_K) = w(\overline{\mathfrak{o}}^*, x^*).$$
(5.9)

Step 4 (Passage to limit). As $\mathbf{z}_K = \pi_K(\mathbf{o}_K)$, we have

$$\underline{\theta}_{K} = \frac{1}{\varepsilon} \sum_{k \leq K} (\underline{\mathfrak{o}}^{*} - \overline{\mathfrak{o}}^{*})(f_{k}) f_{k}(\underline{x}_{K}), \qquad \overline{\theta}_{K} = \frac{1}{\varepsilon} \sum_{k \leq K} (\underline{\mathfrak{o}}^{*} - \overline{\mathfrak{o}}^{*})(f_{k}) f_{k}(\overline{x}_{K}),$$

and recalling Θ of (5.3), this implies that $\lim_{K\to\infty}(\underline{\theta}_K, \overline{\theta}_K) = (\Theta(\underline{x}^*), \Theta(\overline{x}^*))$. Additionally, (5.7) implies that the sequences $(\underline{\Gamma}_K)$, $(\overline{\Gamma}_K)$ lie in compact subsets of \mathbb{S}^d . Hence, there exist a subsequence, denoted by K again, and $(\underline{\Gamma}, \overline{\Gamma})$ satisfying the inequalities (5.2), such that

$$\lim_{K \to \infty} (\underline{\Gamma}_K, \overline{\Gamma}_K) = (\underline{\Gamma}, \overline{\Gamma})$$

In view of (5.8), (5.9) and Lemma 4.5, we conclude that

$$(\Theta(\underline{x}^*), \Delta, \underline{\Gamma}) \in \overline{\mathscr{P}}^{1,2}_+ u(\underline{\mathfrak{o}}^*, \underline{x}^*), \qquad (\Theta(\overline{x}^*), \Delta, \overline{\Gamma}) \in \overline{\mathscr{P}}^{1,2}_- w(\overline{\mathfrak{o}}^*, \overline{x}^*).$$

Step 5 (Final step). Suppose that $(\mathbf{o}^*, \mathbf{x}^*) = (\underline{\mathbf{o}}^*, \overline{\mathbf{o}}^*, \underline{x}^*, \overline{x}^*)$ is a maximizer of Φ which is not necessarily strict. For $(\underline{\mathbf{o}}, \underline{x}), (\overline{\mathbf{o}}, \overline{x}) \in \mathscr{D}_T$, set

$$\tilde{u}(\underline{\mathfrak{o}},\underline{x}) \mathrel{\mathop:}= u(\underline{\mathfrak{o}},\underline{x}) - \rho(\underline{\mathfrak{o}}-\underline{\mathfrak{o}}^*,\underline{x}-\underline{x}^*)^4, \qquad \tilde{w}(\overline{\mathfrak{o}},\overline{x}) \mathrel{\mathop:}= w(\overline{\mathfrak{o}},\overline{x}) + \rho(\overline{\mathfrak{o}}-\overline{\mathfrak{o}}^*,\overline{x}-\overline{x}^*)^4.$$

Then, $(\mathbf{o}^*, \mathbf{x}^*)$ is a *strict* maximizer of

$$\tilde{\Phi}:\mathscr{D}_T^2\to\mathbb{R},\quad \bar{\Phi}(\mathbf{o},\mathbf{x})=\tilde{u}(\underline{\mathbf{o}},\underline{x})-\tilde{w}(\overline{\mathbf{o}},\overline{x})-\frac{1}{2\varepsilon}\rho(\underline{\mathbf{o}}-\overline{\mathbf{o}},\underline{x}-\overline{x})^2.$$

Moreover,

$$\overline{\mathscr{P}}^{1,2}_+ u(\underline{\mathfrak{o}}^*, \underline{x}^*) = \overline{\mathscr{P}}^{1,2}_+ \tilde{u}(\underline{\mathfrak{o}}^*, \underline{x}^*), \quad \text{and} \quad \overline{\mathscr{P}}^{1,2}_- w(\overline{\mathfrak{o}}^*, \overline{x}^*) = \overline{\mathscr{P}}^{1,2}_- \tilde{w}(\overline{\mathfrak{o}}^*, \overline{x}^*).$$

As $\tilde{u} \in \mathscr{U}_e(\mathscr{D}_T)$, $\tilde{w} \in \mathscr{L}_e(\mathscr{D}_T)$, we can apply the above steps to \tilde{u}, \tilde{w} , constructing elements in their sub and super-differentials with desired properties.

Remark 5.2. We emphasize that, in the proof of the Crandall-Ishii-Lions Lemma we work on the closed set $\mathscr{R}_K \times \mathbb{R}^d$, including its boundary points. In particular, in (5.6) we do not require $(\underline{z}_K, \underline{x}_K)$ and $(\overline{z}_K, \overline{x}_K)$ to be interior points of $\mathscr{R}_K \times \mathbb{R}^d$. In fact, this is also the case in the classical paper [14] and Theorem 3.2 of [13] which is stated on a general subset Ω of a Euclidean space. When we apply this lemma to prove the comparison principle in the next section, however, the viscosity property holds only for interior points of the infinite-dimensional set \mathscr{D}_T .

6 Comparison Principle: Proof of Theorem 3.9

Recall the norms ϱ, ρ defined in subsection 2.2 and q of (4.1). For $(\gamma_1, \gamma_2, \beta) \in \mathbb{R}^3$, $(\mathfrak{o}, x) \in \mathscr{D}_T$, $(\theta, \Delta, \Gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, set

$$\mathscr{H}^{\gamma}_{\beta}(\mathfrak{o}, x, \theta, \Delta, \Gamma) \coloneqq \mathscr{H}(\mathfrak{o}, x, \theta - \beta + \gamma_1 q(x), \Delta + \gamma_2 \nabla q(x), \Gamma + \gamma_2 \nabla^2 q(x)).$$

We next state the main property of the Hamiltonian needed in the comparison result. In the below definition, $(\underline{o}, \underline{x}), (\overline{o}, \overline{x}) \in \mathscr{D}_T, \underline{\theta}, \overline{\theta} \in \mathbb{R}$ are arbitrary points.

Definition 6.1. We say that $\mathscr{H} : \mathscr{D}_T \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \mapsto \mathbb{R}$, has the *Crandall-Ishii-Lions (CIL)* property, if there exists a continuous strictly increasing modulus $\mathfrak{m} : [0, \infty) \mapsto [0, \infty)$ with $\mathfrak{m}(0) = 0$, and constants $C_*, c_0 > 0$, such that for all $\varepsilon, \gamma_1, \gamma_2, \beta \in (0, 1]$ we have,

$$\begin{aligned} \mathscr{H}_{-\beta}^{-\gamma}(\overline{\mathfrak{o}},\overline{x},\overline{\theta},\Delta_{\varepsilon},\overline{\Gamma}_{\varepsilon}) &-\mathscr{H}_{\beta}^{\gamma}(\underline{\mathfrak{o}},\underline{x},\underline{\theta},\Delta_{\varepsilon},\underline{\Gamma}_{\varepsilon}) \\ &\leq -c_{0}\beta + \mathfrak{m}(\zeta_{\varepsilon}) + C_{*}\Big[\frac{1}{\varepsilon}\rho^{2}(\underline{\mathfrak{o}}-\overline{\mathfrak{o}},\underline{x}-\overline{x}) + \gamma_{1}Q^{2}(\underline{x},\overline{x}) + \gamma_{2}Q(\underline{x},\overline{x})\Big], \end{aligned}$$

where $\Delta_{\varepsilon} := (\underline{x} - \overline{x})/\varepsilon$, $Q(\underline{x}, \overline{x}) := q(\underline{x}) + q(\overline{x})$, $\underline{\Gamma}_{\varepsilon}, \overline{\Gamma}_{\varepsilon} \in \mathbb{S}^d$ is any pair satisfying (5.2), and

$$\zeta_{\varepsilon} := Q(\underline{x}, \overline{x})[\varrho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}) + \rho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x})(|\Delta_{\varepsilon}| + |\underline{\theta}| + |\overline{\theta}|) + |\underline{\theta} - \overline{\theta}|].$$

We first establish this property under natural conditions.

Lemma 6.2. Under Assumption 3.4, the Hamiltonian \mathcal{H} defined in (3.11) has the Crandall-Ishii-Lions property.

Proof. First we note that for any $(\mathfrak{o}, x) \in \mathscr{D}_T$,

$$\rho(\mathfrak{o}, x) = \sqrt{\varrho(\mathfrak{o})^2 + |x|^2} \le |\mathfrak{o}| + |x| \le T + |x| \le (T+1)q(x).$$

$$(6.1)$$

Also, as $q \ge 1$, $q(x)\nabla q(x) = x$, and $q(x)\nabla^2 q(x) = I - x \otimes x/q^2(x)$, we have,

$$|\nabla q(\underline{x})| \le 1, \quad \|\nabla^2 q(\underline{x})\|_F \le d.$$

For $\varepsilon > 0$ set

$$\Xi_{\varepsilon} := (\underline{\mathfrak{o}}, \underline{x}, \underline{\theta}, \Delta_{\varepsilon}, \underline{\Gamma}_{\varepsilon}), \qquad \Xi^{\varepsilon} := (\overline{\mathfrak{o}}, \overline{x}, \overline{\theta}, \Delta_{\varepsilon}, \overline{\Gamma}_{\varepsilon}).$$

By ellipticity (3.9), for any $\beta > 0$,

$$\mathscr{H}_{\beta}^{\gamma}(\Xi_{\varepsilon}) \geq \mathscr{H}_{0}^{\gamma}(\Xi_{\varepsilon}) + \frac{1}{c_{*}}\beta, \qquad \mathscr{H}_{-\beta}^{-\gamma}(\Xi^{\varepsilon}) \leq \mathscr{H}_{0}^{-\gamma}(\Xi^{\varepsilon}) - \frac{1}{c_{*}}\beta.$$

Set $c_{0} := 2/c_{*}, \underline{G}(\gamma, \Xi_{\varepsilon}) := \mathscr{H}(\Xi_{\varepsilon}) - \mathscr{H}_{0}^{\gamma}(\Xi_{\varepsilon}), \text{ and } \overline{G}(\gamma, \Xi^{\varepsilon}) := \mathscr{H}_{0}^{-\gamma}(\Xi^{\varepsilon}) - \mathscr{H}(\Xi^{\varepsilon}), \text{ so that}$
 $\mathscr{H}_{-\beta}^{-\gamma}(\Xi^{\varepsilon}) - \mathscr{H}_{\beta}^{\gamma}(\Xi_{\varepsilon}) \leq -c_{0}\beta + \mathscr{H}_{0}^{-\gamma}(\Xi^{\varepsilon}) - \mathscr{H}_{0}^{\gamma}(\Xi_{\varepsilon})$

We continue by estimating $\underline{G}(\gamma, \Xi_{\varepsilon})$ and $\overline{G}(\gamma, \Xi^{\varepsilon})$ using the uniform Lipschitz continuity assumption on σ, b, ℓ . In view of Assumption 3.4,

 $= -c_0\beta + \mathscr{H}(\Xi^{\varepsilon}) - \mathscr{H}(\Xi_{\varepsilon}) + \overline{G}(\gamma, \Xi^{\varepsilon}) + \underline{G}(\gamma, \Xi_{\varepsilon}).$

$$\begin{aligned} |\underline{G}(\gamma,\Xi_{\varepsilon})| &\leq \sup_{a \in A} \left\{ \gamma_1 \lambda(\underline{\mathfrak{o}},\underline{x},a) \, q(\underline{x}) + \gamma_2 \left[|b(\underline{\mathfrak{o}},\underline{x},a)| \, |\nabla q(\underline{x})| + \frac{1}{2} |\operatorname{tr}((\sigma\sigma^{\mathrm{T}})(\underline{\mathfrak{o}},\underline{x},a)\nabla^2 q(\underline{x}))| \right] \right\} \\ &\leq c_* \rho(\underline{\mathfrak{o}},\underline{x})(\gamma_1 q(\underline{x}) + \gamma_2 [|\nabla q(\underline{x})| + ||\nabla^2 q(\underline{x})||_F]) \\ &\leq \gamma c_* (1+d)(1+T)(\gamma_1 q^2(\underline{x}) + \gamma_2 q(\underline{x})). \end{aligned}$$

We similarly show that

$$|\overline{G}(\gamma, \Xi^{\varepsilon})| \leq \gamma c_* (1+d)(1+T)(\gamma_1 q^2(\overline{x}) + \gamma_2 q(\overline{x})).$$

Set $C_1 := c_*(1+d)(1+T)$, to conclude that

$$\mathscr{H}^{\gamma}_{\beta}(\Xi_{\varepsilon}) - \mathscr{H}^{-\gamma}_{-\beta}(\Xi^{\varepsilon}) \leq -c_0\beta + \mathscr{H}(\Xi_{\varepsilon}) - \mathscr{H}(\Xi^{\varepsilon}) + \gamma C_1(\gamma_1 Q^2(\underline{x}, \overline{x}) + \gamma_2 Q(\underline{x}, \overline{x}))$$

We next estimate $\mathscr{H}(\Xi_\varepsilon)-\mathscr{H}(\Xi^\varepsilon)$ in several steps. Set

$$\begin{aligned} \mathscr{I}_{1} &:= \sup_{a \in A} \{ |\lambda(\underline{\mathfrak{o}}, \underline{x}, a) - \lambda(\overline{\mathfrak{o}}, \overline{x}, a)| \, |\underline{\theta}| \} + \sup_{a \in A} \{ \|\sigma(\overline{\mathfrak{o}}, \overline{x}, a)\|_{F}^{2} \, |\underline{\theta} - \overline{\theta}| \} \\ &\leq c_{*}\rho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x}) |\underline{\theta}| + c_{*}(1 + \rho(\overline{\mathfrak{o}}, \overline{x})) |\underline{\theta} - \overline{\theta}|, \\ \\ \mathscr{I}_{2} &:= \sup_{a \in A} \{ |b(\underline{\mathfrak{o}}, \underline{x}, a) - b(\overline{\mathfrak{o}}, \overline{x}, a)| |\Delta_{\varepsilon}| \} \leq c_{*}\rho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x}) |\Delta_{\varepsilon}|. \end{aligned}$$

In view of (6.1), $\rho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x}) \leq (T+1)(q(\underline{x}) + q(\overline{x})) = (T+1)Q(\underline{x}, \overline{x})$. Hence,

$$\mathscr{I}_1 + \mathscr{I}_2 \le c_*(T+1)Q(\underline{x},\overline{x})|\underline{\theta} - \overline{\theta}| + c_*\rho(\underline{\mathfrak{o}} - \overline{\mathfrak{o}},\underline{x} - \overline{x})(|\Delta_{\varepsilon}| + |\underline{\theta}| + \overline{\theta}|).$$

Finally, set

$$\mathscr{I}_3 := \sup_{a \in A} \{ \operatorname{tr}((\sigma \sigma^{\top})(\underline{\mathfrak{o}}, \underline{x}, a) \underline{\Gamma}_{\varepsilon} - (\sigma \sigma^{\top})(\overline{\mathfrak{o}}, \overline{x}, a) \underline{\Gamma}^{\varepsilon}) \}.$$

We estimate \mathscr{I}_3 as in [13, Example 3.6] using (5.2). Then, for every $a \in A$,

$$\begin{split} \operatorname{tr} \left((\sigma \sigma^{\top})(\underline{\mathfrak{o}}, \underline{x}, a) \underline{\Gamma}_{\varepsilon} - (\sigma \sigma^{\top})(\overline{\mathfrak{o}}, \overline{x}, a) \underline{\Gamma}^{\varepsilon}) \right) \\ &= \operatorname{tr} \left(\operatorname{diag}(\underline{\Gamma}_{\varepsilon}, -\overline{\Gamma}^{\varepsilon}) \left(\begin{matrix} \sigma(\underline{\mathfrak{o}}, \underline{x}, a) \\ \sigma(\overline{\mathfrak{o}}, \overline{x}, a) \end{matrix} \right) \left(\sigma^{\top}(\underline{\mathfrak{o}}, \underline{x}, a), \sigma^{\top}(\overline{\mathfrak{o}}, \overline{x}, a) \right) \right) \\ &\leq \frac{3}{\varepsilon} \operatorname{tr} \left(\mathbf{G} \left(\begin{matrix} \sigma(\underline{\mathfrak{o}}, \underline{x}, a) \\ \sigma(\overline{\mathfrak{o}}, \overline{x}, a) \end{matrix} \right) \left(\sigma^{\top}(\underline{\mathfrak{o}}, \underline{x}, a), \sigma^{\top}(\overline{\mathfrak{o}}, \overline{x}, a) \right) \right) \\ &= \frac{3}{\varepsilon} \| \sigma(\underline{\mathfrak{o}}, \underline{x}, a) - \sigma(\overline{\mathfrak{o}}, \overline{x}, a) \|_{F}^{2} \leq \frac{3c_{*}^{2}}{\varepsilon} \rho^{2}(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x}). \end{split}$$

We now use all above inequalities to arrive at

$$\begin{aligned} |\mathscr{H}(\Xi_{\varepsilon}) - \mathscr{H}(\Xi^{\varepsilon})| &\leq \mathscr{I}_{1} + \mathscr{I}_{2} + \mathscr{I}_{3} \\ &\leq c_{*}(T+1)Q(\underline{x},\overline{x})|\underline{\theta} - \overline{\theta}| + 2c_{*}(|\Delta_{\varepsilon}| + |\underline{\theta}| + \overline{\theta}|) + \frac{3c_{*}^{2}}{\varepsilon}\rho^{2}(\underline{\mathfrak{o}} - \overline{\mathfrak{o}}, \underline{x} - \overline{x}). \end{aligned}$$

We are now ready to prove the main result.

Proof of Theorem 3.9. Towards a counterposition we assume that $\sup_{\mathscr{D}_T}(u-w) > 0$ and proceed in several steps to obtain a contradiction.

Step 1 (Set up). We first note that the occupation derivative of $\psi(\mathfrak{o}, x) := |\mathfrak{o}|$ is equal to $\partial_{\mathfrak{o}}\psi(\mathfrak{o}, x) = 1$ for any $(\mathfrak{o}, x) \in \mathscr{D}$. Next, for $(\gamma_1, \gamma_2, \beta) \in (0, 1], (\underline{\mathfrak{o}}, \underline{x}), (\overline{\mathfrak{o}}, \overline{x}) \in \mathscr{D}_T^2$ we set,

$$\begin{split} u_{\beta}^{\gamma}(\underline{\mathbf{o}},\underline{x}) &:= u(\underline{\mathbf{o}},\underline{x}) - \gamma_1 \underline{\mathbf{o}}(q) - \gamma_2 q(\underline{x}) + \beta(|\underline{\mathbf{o}}| - T), \\ w_{\gamma}^{\beta}(\overline{\mathbf{o}},\overline{x}) &:= w(\overline{\mathbf{o}},\overline{x}) + \gamma_1 \overline{\mathbf{o}}(q) + \gamma_2 q(\overline{x}) - \beta(|\overline{\mathbf{o}}| - T). \end{split}$$

A direct argument using the fact that $\partial_{\mathfrak{o}}\psi(\mathfrak{o},x) = 1$ shows that u_{β}^{γ} is a viscosity subsolution of $\mathscr{H}_{\beta}^{\gamma} \leq 0$ in $\mathring{\mathscr{D}}_{T}$, and w_{γ}^{β} is a viscosity supersolution of $\mathscr{H}_{-\beta}^{-\gamma} \leq 0$. Moreover by Lemma 4.2, $u_{\beta}^{\gamma} \in \mathscr{U}_{e}(\mathscr{D}_{T}), w_{\gamma}^{\beta} \in \mathscr{L}_{e}(\mathscr{D}_{T})$. Hence, there is $\beta_{0} > 0$ such that

$$\sup_{\mathscr{D}_{T}}(u_{\beta}^{\gamma}-w_{\gamma}^{\beta})>0\geq \sup_{\partial\mathscr{D}_{T}}(u_{\beta}^{\gamma}-w_{\gamma}^{\beta}),\qquad\forall 0\leq\beta,\gamma_{1},\gamma_{2}\leq\beta_{0}.$$
(6.2)

In the remainder of the proof, we fix $\beta = \beta_0 > 0$ and assume that the parameters always satisfy $0 < \gamma_1, \gamma_2 \leq \beta_0$. Additionally, to simplify the presentation we write

$$u^{\gamma} := u^{\gamma}_{\beta_0}, \quad \text{and} \quad w_{\gamma} := w^{\beta_0}_{\gamma}$$

Step 2 (Doubling the variables). For $(\mathbf{o}, \mathbf{x}) = ((\underline{\mathbf{o}}, \underline{x}), (\overline{\mathbf{o}}, \overline{x})) \in \mathscr{D}_T^2, \, \gamma_1, \gamma_2, \varepsilon \in (0, \beta_0], \, \text{set}$

$$\Phi(\mathbf{o}, \mathbf{x}) := \Phi_{\varepsilon}^{\gamma}(\mathbf{o}, \mathbf{x}) = u^{\gamma}(\underline{\mathbf{o}}, \underline{x}) - w_{\gamma}(\overline{\mathbf{o}}, \overline{x}) - \frac{1}{2\varepsilon}\rho^{2}(\underline{\mathbf{o}} - \overline{\mathbf{o}}, \underline{x} - \overline{x})$$

By Lemma 4.2, there exists a maximizer $(\mathbf{o}_{\varepsilon}^{\gamma}, \mathbf{x}_{\varepsilon}^{\gamma}) = ((\underline{\mathbf{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma}), (\overline{\mathbf{o}}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})) \in \mathscr{D}_{T}^{2}$ of $\Phi_{\varepsilon,\beta}^{\gamma}$. Then, in view of Lemma 5.1, there exists $(\underline{\Gamma}_{\varepsilon}, \overline{\Gamma}_{\varepsilon}) = (\underline{\Gamma}_{\varepsilon}^{\gamma}, \overline{\Gamma}_{\varepsilon}^{\gamma}) \in (\mathbb{S}^{d})^{2}$ satisfying (5.2) such that

$$(\underline{\theta}_{\varepsilon}^{\gamma}, \Delta_{\varepsilon}, \underline{\Gamma}_{\varepsilon}) \in \overline{\mathscr{P}}_{+}^{1,2} u^{\gamma}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma}), \qquad (\overline{\theta}_{\varepsilon}^{\gamma}, \Delta_{\varepsilon}, \overline{\Gamma}_{\varepsilon}) \in \overline{\mathscr{P}}_{-}^{1,2} w_{\gamma}(\overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}), \tag{6.3}$$

$$\underline{\theta}_{\varepsilon}^{\gamma} := \Theta_{\varepsilon}^{\gamma}(\underline{x}_{\varepsilon}^{\gamma}), \qquad \overline{\theta}_{\varepsilon}^{\gamma} := \Theta_{\varepsilon}^{\gamma}(\overline{x}_{\varepsilon}^{\gamma}), \qquad \Theta_{\varepsilon}^{\gamma} = \frac{1}{\varepsilon} \sum_{k} (\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma})(f_{k}) f_{k}, \qquad \Delta_{\varepsilon} := \Delta_{\varepsilon}^{\gamma} = \frac{1}{\varepsilon} (\underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}).$$

Step 3 (Viscosity property). By (6.2), for all sufficiently small γ, ε , maximizers of $\Phi_{\varepsilon}^{\gamma}$ satisfy $((\underline{o}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma}), (\overline{o}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})) \in \mathscr{D}_{T}^{2}$. Then, in view of (6.3) and the viscosity properties of u^{γ}, w_{γ} ,

$$\mathscr{H}^{\gamma}_{\beta_{0}}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma},\underline{x}_{\varepsilon}^{\gamma},\underline{\theta}_{\varepsilon}^{\gamma},\Delta_{\varepsilon},\underline{\Gamma}_{\varepsilon}) \leq 0 \leq \mathscr{H}^{-\gamma}_{-\beta_{0}}(\overline{\mathfrak{o}}_{\varepsilon}^{\gamma},\overline{x}_{\varepsilon}^{\gamma},\overline{\theta}_{\varepsilon}^{\gamma},\Delta_{\varepsilon},\overline{\Gamma}_{\varepsilon})$$

We now use the above inequalities and the CIL property from Lemma 6.2 to arrive at,

$$0 \leq \mathscr{H}_{-\beta_{0}}^{-\gamma}(\overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}, \overline{\theta}_{\varepsilon}^{\gamma}, \Delta_{\varepsilon}, \overline{\Gamma}_{\varepsilon}) - \mathscr{H}_{\beta_{0}}^{\gamma}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma}, \underline{\theta}_{\varepsilon}^{\gamma}, \Delta_{\varepsilon}, \underline{\Gamma}_{\varepsilon})$$

$$\leq -c_{0}\beta_{0} + \mathfrak{m}(\zeta_{\varepsilon}^{\gamma}) + C_{*}[\rho^{2}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma})/\varepsilon + \gamma_{1}Q^{2}(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) + \gamma_{2}Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})],$$

$$(6.4)$$

where as before $Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) = q(\underline{x}_{\varepsilon}^{\gamma}) + q(\overline{x}_{\varepsilon}^{\gamma})$ and

$$\zeta_{\varepsilon}^{\gamma} = Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) \left[\varrho(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}) + \rho(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}) \left(|\Delta_{\varepsilon}| + |\underline{\theta}_{\varepsilon}^{\gamma}| + |\overline{\theta}_{\varepsilon}^{\gamma}| \right) + |\underline{\theta}_{\varepsilon}^{\gamma} - \overline{\theta}_{\varepsilon}^{\gamma}| \right].$$

Step 4 (Passage to limit). We let the parameters tend to zero in the inequality (6.4) to obtain a contradiction resulting from (6.2), hence completing the proof. In order, we first let $\varepsilon \downarrow 0$ then $\gamma_1 \downarrow 0$, and finally $\gamma_2 \downarrow 0$.

We start by using the standard direct arguments as in [13, Lemma 3.1] to obtain

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \rho^2 (\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}) = 0, \qquad \lim_{\gamma_2 \downarrow 0} \lim_{\gamma_1 \downarrow 0} \lim_{\varepsilon \downarrow 0} \gamma_2 Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) = 0.$$
(6.5)

In particular,

$$c(\gamma_1) := \sup_{\varepsilon, \gamma_2 \in (0, \beta_0]} Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) < \infty.$$
(6.6)

Step 4.a ($\varepsilon \downarrow 0$). By their definitions, Cauch-Schwarz inequality, and (2.4),

$$\begin{split} |\underline{\theta}_{\varepsilon}^{\gamma} - \overline{\theta}_{\varepsilon}^{\gamma}| &= |\Theta_{\varepsilon}^{\gamma}(\underline{x}_{\varepsilon}^{\gamma}) - \Theta_{\varepsilon}^{\gamma}(\overline{x}_{\varepsilon}^{\gamma})| \leq \frac{1}{\varepsilon} \sum_{k} |(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma})(f_{k})| \, |f_{k}(\underline{x}_{\varepsilon}^{\gamma}) - f_{k}(\overline{x}_{\varepsilon}^{\gamma})| \\ &\leq \frac{1}{\varepsilon} \sum_{k} |(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma})(f_{k})| \, ||f_{k}||_{\mathscr{C}^{1}} |\underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}| \\ &\leq \frac{1}{2\varepsilon} \sum_{k} |(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma})(f_{k})|^{2} + \frac{1}{2\varepsilon} \, |\underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}|^{2} \sum_{k} ||f_{k}||_{\mathscr{C}^{1}}^{2} \\ &\leq \frac{1}{2\varepsilon} \left(\varrho^{2}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}) + |\underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}|^{2} \right) = \frac{1}{2\varepsilon} \, \rho^{2}(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}). \end{split}$$

Additionally, writing $\rho_{\varepsilon,\gamma} = \rho(\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}, \underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma})$, we have

$$\begin{split} \rho_{\varepsilon,\gamma} |\underline{\theta}_{\varepsilon}^{\gamma}| &= \frac{1}{2\varepsilon} \rho_{\varepsilon,\gamma}^{2} + \frac{\varepsilon}{2} |\underline{\theta}_{\varepsilon}^{\gamma}|^{2} = \frac{1}{2\varepsilon} \rho_{\varepsilon,\gamma}^{2} + \frac{1}{2\varepsilon} \Big(\sum_{k} (\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \overline{\mathfrak{o}}_{\varepsilon}^{\gamma}) (f_{k}) f_{k}(\underline{x}_{\varepsilon}^{\gamma}) \Big)^{2} \\ &\leq \frac{1}{2\varepsilon} \rho_{\varepsilon,\gamma}^{2} + \frac{1}{2\varepsilon} \varrho^{2} (\underline{\mathfrak{o}}_{\varepsilon}^{\gamma} - \underline{\mathfrak{o}}_{\varepsilon}^{\gamma}). \end{split}$$

Same estimate also holds for $\rho_{\varepsilon,\gamma}|\overline{\theta}_{\varepsilon}^{\gamma}|$. Proceeding similarly, we obtain

$$\rho_{\varepsilon,\gamma}|\Delta_{\varepsilon}| \leq \frac{1}{2\varepsilon}\rho_{\varepsilon,\gamma}^2 + \frac{1}{2\varepsilon} |\underline{x}_{\varepsilon}^{\gamma} - \overline{x}_{\varepsilon}^{\gamma}|^2.$$

In view of (6.5), (6.6), and the above inequalities, $\lim_{\varepsilon \downarrow 0} \zeta_{\varepsilon}^{\gamma} = 0$, for every $\gamma_1, \gamma_2 > 0$. Then, we take the limit as $\varepsilon \downarrow 0$ in the inequality (6.4). Since *m* in (6.4) is continuous with m(0) = 0, the limit of (6.4) is the following,

$$0 \leq -c_0\beta_0 + C_* \lim_{\varepsilon \downarrow 0} [\gamma_1 Q^2(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) + \gamma_2 Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})].$$

Step 4.b ($\gamma_1 \downarrow 0$, then $\gamma_2 \downarrow 0$). In view of (6.6), $\lim_{\gamma_1 \downarrow 0} (\gamma_1 Q^2(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})) = 0$. Hence,

$$0 \leq -c_*\beta_0 + C_* \lim_{\gamma_1 \downarrow 0} \lim_{\varepsilon \downarrow 0} [\gamma_1 Q^2(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) + \gamma_2 Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma})] = -c_*\beta_0 + C_* \lim_{\gamma_1 \downarrow 0} \lim_{\varepsilon \downarrow 0} \gamma_2 Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}).$$

We now let γ_2 tend to zero and use (6.5) to conclude that

$$0 \leq -c_*\beta_0 + C_* \lim_{\gamma_2 \downarrow 0} \lim_{\gamma_1 \downarrow 0} \lim_{\varepsilon \downarrow 0} \gamma_2 Q(\underline{x}_{\varepsilon}^{\gamma}, \overline{x}_{\varepsilon}^{\gamma}) = -c_*\beta_0.$$

As $\beta_0, c_* > 0$ this clear contradiction implies that (6.2) cannot hold completing the proof.

7 Examples

7.1 Heat Equation

Let X be a d-dimensional, uncontrolled Brownian motion, that is $b \equiv 0$ and $\sigma = I \in \mathbb{S}^d$ in (3.3). We consider the standard time $\Lambda_t = t$ in (3.1) (or equivalently $\Lambda_t = \frac{1}{d} \operatorname{tr}(\langle X \rangle_t)$ since X is Brownian motion) and set $\ell \equiv 0$. Given $g \in \mathscr{C}_b(\partial \mathscr{D}_T)$, the value function (3.8) is given by

$$v(\mathbf{o}, x) = \mathbb{E}^{\mathbb{Q}}_{\mathbf{o}, x}[g(\mathcal{O}_{T-|\mathbf{o}|}, X_{T-|\mathbf{o}|})] = \mathbb{E}^{\mathbb{Q}}[g(\mathbf{o} + \mathcal{O}^{x}_{T-|\mathbf{o}|}, x + X_{T-|\mathbf{o}|})],$$
(7.1)

with $(\mathfrak{o}, x) \in \mathscr{D}_T$ and the shifted measure $\mathcal{O}_s^x(B) = \mathcal{O}_s(B - x)$, $B \in \mathscr{B}(\mathbb{R}^d)$. The associated Hamiltonian is $\mathscr{H}(\mathfrak{o}, x, \theta, \Delta, \Gamma) = -\theta - \frac{1}{2} \operatorname{tr}(\Gamma)$, and the dynamic programming equation coincides with the (backward) occupied heat equation

$$\begin{cases} -\partial_{\mathbf{o}}u - \frac{1}{2} \Delta u = 0 & \text{on } \mathring{\mathscr{D}}_{T}, \\ u = g, & \text{on } \partial \mathscr{D}_{T}, \end{cases}$$
(7.2a)
(7.2b)

and the Laplacian $\triangle = \sum_{i=1}^{d} \partial_{x_i x_i}$.

Remark 7.1. Recall the space of continuous paths $\Omega = \mathcal{C}([0, T]; \mathbb{R}^d)$ seen in the Introduction. It is well-known that the path-dependent heat equation

$$-\partial_t u(t,\omega) - \frac{1}{2} \mathrm{tr}(\partial_\omega^2 u(t,\omega)) = 0, \quad u(T,\cdot) = G \in \mathcal{C}(\Omega),$$

does not always admit a classical solution. One example is $G(\omega) = \omega_{t_0}$ for some fixed time $t_0 < T$; see [37, Chapter 11]. We note that functionals depending on specific values of the path before the final time T cannot be expressed in terms of the occupation measure $\mathcal{O}_T(\omega)$ since the latter erases the chronology of ω . Hence, it is not possible to translate the above counterexample to occupied PDEs. In fact, we conjecture that the occupied heat equation (7.2a)-(7.2b) always admits a classical solution whenever the terminal functional $g : \partial \mathscr{D}_T \to \mathbb{R}$ is continuous with respect to the weak \times Euclidean topology.

We now discuss some examples.

Example 7.2. Let $g(\mathfrak{o}, x) = \mathfrak{o}(B)$ with $B = \{|x| \leq 1\}$. Then $g(\mathcal{O}_t, X_t) = \int_0^t \mathbb{1}_B(X_s) ds$ is the occupation time of Brownian motion in the unit ball. While g is continuous in \mathfrak{o} with respect to the stronger total variation distance, it is not continuous with respect to the weak topology and the corresponding path functional is not continuous either. The value function is given by

$$v(\mathfrak{o}, x) = \mathfrak{o}(B) + \mathbb{E}^{\mathbb{Q}}[\mathcal{O}_{T-|\mathfrak{o}|}^{x}(B)] = \mathfrak{o}(B) + \int_{0}^{T-|\mathfrak{o}|} \mathbb{Q}(|x+X_{s}| \leq 1) ds.$$

Observe that the occupation derivative $\partial_{\mathfrak{o}} v(\mathfrak{o}, x) = \mathbb{1}_B(x) - \mathbb{Q}(|x+X_{T-|\mathfrak{o}|}| \leq 1)$ is discontinuous in x. Hence the occupied heat equation (7.2a)–(7.2b) does not admit a classical solution, motivating the viscosity theory developed throughout.

Example 7.3. Let $g(\mathfrak{o}, x) = \psi(\mathfrak{o}(\phi))$ for some functions $\phi \in \mathscr{C}(\mathbb{R}^d)$, $\psi \in \mathscr{C}(\mathbb{R})$. X is again a Brownian motion. Then, $v(\mathfrak{o}, x) = u(|\mathfrak{o}|, x, \mathfrak{o}(\phi))$, where

$$u(t,x,y) := \mathbb{E}^{\mathbb{Q}} \left[\psi \left(y + \int_0^{T-t} \phi(x+X_s) ds \right) \right].$$

It is clear that u is the unique viscosity solution of the following degenerate parabolic linear PDE

$$-\partial_t u - \frac{1}{2} \Delta_x u - \phi(x) \partial_y u = 0, \quad u(T, x, y) = \psi(y).$$

When ψ is smooth, u is also smooth and

$$\partial_{\mathfrak{o}} v(\mathfrak{o}, x) = \mathbb{E}^{\mathbb{Q}} \left[\psi' \Big(o(\phi) + \int_{0}^{T-|o|} \phi(x+X_s) ds \Big) \Big(\phi(x) - \phi(x+X_{T-|\mathfrak{o}|}) \Big) \right].$$

We additionally have that

$$\partial_{\mathfrak{o}} v(\mathfrak{o}, x) = \partial_t u(|\mathfrak{o}|, x, \mathfrak{o}(\phi)) + \partial_y u(|\mathfrak{o}|, x, \mathfrak{o}(\phi))\phi(x) = -\frac{1}{2} \triangle_x u(|\mathfrak{o}|, x, \mathfrak{o}(\phi)).$$

However, when ψ is not smooth, in general u is not smooth either. This on the other hand does not immediately imply that v is also not smooth. To illustrate this point, suppose that $\phi \equiv 1$. Then, $u(t, x, y) = \psi(y + T - t)$, is not smooth. But as $\mathfrak{o}(\phi) = |\mathfrak{o}|, v(\mathfrak{o}, x) = u(|\mathfrak{o}|, x, |\mathfrak{o}|) = \psi(T)$, and hence, v is obviously smooth. This is due to the fact that the existence of $\partial_{\mathfrak{o}} v$ depends on the smoothness of the combined operator $\partial_t u + \phi(x)\partial_y u$, and not each one separately. A related discussion can be found in [6] in the case $\phi(x) = x$.

7.2 Timer options

Timer options [3, 22], are contingent claims where the maturity is floating and depends on the realized variance of the underlying asset. More precisely, the buyer of the option receives the payoff as soon as the realized variance exceeds a given threshold which we denote by T > 0. We now derive the pricing PDE of timer options written on a single asset, i.e. d = 1. See also Section 5.2 in [35]. Suppose that the log price of the asset and its occupation flow evolve according to the OSDE (assuming zero interest rate)

$$d\mathcal{O}_t = \sigma_t^2 \,\delta_{X_t} \,dt, \quad dX_t = -\sigma_t^2/2dt + \sigma_t dW_t, \quad \sigma_t = \sigma(\mathcal{O}_t, X_t)$$

We therefore choose $\lambda(\mathfrak{o}, x, a) = \sigma(\mathfrak{o}, x)^2$ in (3.2). Given $g: \partial \mathscr{D}_T \to \mathbb{R}$, consider a timer option that pays $g(\mathcal{O}_\tau, X_\tau)$ at $\tau = \inf\{t \ge 0 : |\mathcal{O}_t| \ge T\}$. Since $|\mathcal{O}_t| = \langle X \rangle_t = \int_0^t \sigma_s^2 ds$ is the realized variance of the asset up to time t, the maturity τ indeed corresponds to the first time the realized variance exceeds the budget T. Note also that the option is *exotic* in the sense that the payoff $g(\mathcal{O}_\tau, X_\tau)$ depends on the path $(X_t)_{t \le \tau}$ (through \mathcal{O}_τ). For instance, a fixed strike Asian put option is obtained by setting

$$g(\mathbf{o}, x) = \left(K - \frac{1}{|\mathbf{o}|} \int_{\mathbb{R}} y \mathbf{o}(dy)\right)^+ \implies g(\mathcal{O}_{\tau}, X_{\tau}) = \left(K - \frac{1}{\tau} \int_0^{\tau} X_t dt\right)^+.$$

See [35, Section 5] for a list of exotic payoffs expressible in terms of the occupied process. In light of the control problem (3.7), we set $\ell \equiv 0$, so the value function corresponds to the price of the option, namely

$$v(\mathfrak{o}, x) = \mathbb{E}^{\mathbb{Q}}_{\mathfrak{o}, x}[g(\mathcal{O}_{\tau}, X_{\tau})].$$

The associated Hamiltonian reads

$$\mathscr{H}(\mathfrak{o}, x, \theta, \Delta, \Gamma) = -\left(\sigma^2(\mathfrak{o}, x)\theta - \frac{1}{2}\sigma^2(\mathfrak{o}, x)\Delta + \frac{1}{2}\sigma^2(\mathfrak{o}, x)\Gamma\right),\tag{7.3}$$

with the Greeks $\theta = \partial_{\mathfrak{o}} v(\mathfrak{o}, x)$, $\Delta = \partial_x v(\mathfrak{o}, x)$, and $\Gamma = \partial_{xx} v(\mathfrak{o}, x)$. We therefore conclude from Theorem 3.10 that the price function v is the unique viscosity solution of (7.4a)-(7.4b). Moreover, observe that the square volatility in the Hamiltonian (7.3) can be factored out and due to the ellipticity condition (3.9), the pricing PDE can be simplified to

$$\begin{cases} -\partial_{\mathfrak{o}}v + \frac{1}{2}\partial_{x}v - \frac{1}{2}\partial_{xx}v = 0 \quad \text{on} \quad \mathring{\mathscr{D}}_{T}, \tag{7.4a} \end{cases}$$

$$v = g,$$
 on $\partial \mathscr{D}_T.$ (7.4b)

In other words, the volatility used to price the option is here irrelevant. This exhibits an important virtue of timer options, namely is that they do not entail any model risk with respect to volatility. We note that this is no longer true when interest rates are nonzero; see [3].

7.3 Exotic Options in the Uncertain Volatility Model

Following the uncertain volatility model introduced by Avellaneda et al. [1], we consider an asset price X^{α} with dynamics $dX_t^{\alpha} = \alpha_t X_t^{\alpha} dW_t$. The control process α represents the volatility of the asset. The control set is $A = [\underline{a}, \overline{a}] \subset (0, \infty)$ meaning that volatility, albeit uncertain, is kept in a compact interval. In the occupied SDE (3.3), we thus choose $b \equiv 0$ and $\sigma(\mathfrak{o}, x, a) = ax$, so that σ satisfies the growth and Lipschitz conditions with constant $c_* = \overline{a}$.

Consider an exotic option with payoff $\varphi = \varphi(\mathcal{O}_T^{\alpha}, X_T^{\alpha})$ and assume calendar time for \mathcal{O}^{α} , i.e. $\lambda \equiv 1$. Observe that $J(\mathfrak{o}, x, \alpha) = \mathbb{E}_{\mathfrak{o}, x}^{\mathbb{Q}}[\varphi(\mathcal{O}_{T-|\mathfrak{o}|}^{\alpha}, X_{T-|\mathfrak{o}|}^{\alpha})]$ is the price of the option under the volatility process α . Suppose we are interested in the seller's price $p : \mathscr{D}_T \to \mathbb{R}$ of the option, which is obtained from the volatility α that maximizes $J(\mathfrak{o}, x, \alpha)$. We therefore set $\ell \equiv 0, g = -\varphi$ in (3.7), so the price of the option coincides with the negative of the value function. That is,

$$p(\mathbf{o}, x) = -v(\mathbf{o}, x) = \sup_{\alpha \in \mathscr{A}} \mathbb{E}^{\mathbb{Q}}_{\mathbf{o}, x}[\varphi(\mathcal{O}^{\alpha}_{T-|\mathbf{o}|}, X^{\alpha}_{T-|\mathbf{o}|})].$$

The associated dynamic programming equation is

p

$$\begin{cases} \partial_{\mathfrak{o}} p + \sup_{a \in A} \frac{1}{2} a^2 x^2 \partial_{xx} p = 0 \quad \text{on} \quad \mathring{\mathscr{D}}_T, \end{cases}$$
(7.5a)

$$=\varphi,$$
 on $\partial \mathscr{D}_T,$ (7.5b)

where (7.5a) can be rewritten as the (occupied) Black-Scholes-Barenblatt equation [1, 22],

$$\partial_{\mathfrak{o}}p + \frac{1}{2}x^2 V(\partial_{xx}p) = 0, \qquad (7.6)$$

with $V(\Gamma) = (\underline{a}^2 \mathbb{1}_{\{\Gamma < 0\}} + \overline{a}^2 \mathbb{1}_{\{\Gamma \ge 0\}})\Gamma$. Applying Theorem 3.10 to v = -p, it is then immediate to see that p is the unique viscosity solution of (7.5a)–(7.5b).

A Proof of Proposition 3.3

Proof. Given $t \ge 0$, let $0 = t_0 < \cdots < t_N = t$ be a sequence of partitions of [0, t] that satisfies $\max_{n \le N} |t_n - t_{n-1}| \to 0$ as $N \to \infty$. Consider the piecewise constant process

$$X_s^N = X_0 \mathbb{1}_{\{s=0\}} + \sum_n X_{t_n} \mathbb{1}_{(t_{n-1}, t_n]}(s), \quad s \le t.$$
(A.1)

Write also $\Lambda_t = \int_0^t \lambda_s ds$ and introduce the discretized occupation flow $\mathcal{O}_s^N = \int_0^s \delta_{X_u^N} d\Lambda_u$. We note that here X^N and \mathcal{O}^N are not \mathbb{F} -adapted, however, $X_{t_n}^N$, $\mathcal{O}_{t_n}^N$ are \mathcal{F}_{t_n} -measurable, which we will actually use. We first establish that

$$\lim_{N \to \infty} \sup_{0 \le s \le t} \varrho(\mathcal{O}_s - \mathcal{O}_s^N) = 0, \quad \mathbb{Q}\text{-almost surely.}$$
(A.2)

For each $k \in \mathbb{N}$ and f_k introduced in Section 2.2, we compute

$$\begin{aligned} |(\mathcal{O}_{s} - \mathcal{O}_{s}^{N})(f_{k})|^{2} &= \Big| \int_{0}^{s} (f_{k}(X_{u}) - f_{k}(X_{u}^{N})) d\Lambda_{u} \Big|^{2} \leq \Lambda_{s} \int_{0}^{s} |f_{k}(X_{u}) - f_{k}(X_{u}^{N})|^{2} d\Lambda_{u} \\ &\leq \Lambda_{s} \int_{0}^{s} \|f_{k}\|_{\mathscr{C}^{1}}^{2} |X_{u} - X_{u}^{N}|^{2} d\Lambda_{u}. \end{aligned}$$

As the last term is nondecreasing in s for all k, we can set s = t to obtain

$$\sup_{s \le t} \varrho (\mathcal{O}_s - \mathcal{O}_s^N)^2 \le \sum_k \Lambda_t \int_0^t \|f_k\|_{\mathscr{C}^1}^2 |X_u - X_u^N|^2 d\Lambda_u \le \Lambda_t \int_0^t |X_u - X_u^N|^2 d\Lambda_u$$

For \mathbb{Q} -almost all $\omega \in \Omega$, $s \mapsto X_s(\omega)$ is uniformly continuous on the compact [0, t], hence it admits a modulus of continuity $\mathfrak{m} : \mathbb{R}_+ \to \mathbb{R}_+$ (which depends on ω). Consequently,

$$\int_{0}^{t} |X_{u} - X_{u}^{N}|^{2} d\Lambda_{u} = \sum_{n \leq N} \int_{t_{n-1}}^{t_{n}} |X_{u} - X_{t_{n-1}}|^{2} d\Lambda_{u} \leq \mathfrak{m} \Big(\max_{n \leq N} |t_{n} - t_{n-1}| \Big)^{2} \Lambda_{t} \to 0, \quad N \to \infty,$$

which proves (A.2). Together with the continuity of v, we also have that

$$\lim_{N \to \infty} \sup_{0 \le s \le t} \left| v(\mathcal{O}_s^N, X_s) - v(\mathcal{O}_s, X_s) \right| = 0.$$
(A.3)

Next, we write

$$v(\mathcal{O}_{t}^{N}, X_{t}) - v(\mathcal{O}_{0}^{N}, X_{0}) = \sum_{n \leq N} (v(\mathcal{O}_{t_{n}}^{N}, X_{t_{n}}) - v(\mathcal{O}_{t_{n-1}}^{N}, X_{t_{n-1}})).$$
(A.4)

Denote $\Delta_n^{\Lambda} := \Lambda_{t_n} - \Lambda_{t_{n-1}}$. Then

$$\begin{aligned} v(\mathcal{O}_{t_{n}}^{N}, X_{t_{n}}) &- v(\mathcal{O}_{t_{n-1}}^{N}, X_{t_{n-1}}) \\ &= \left[v(\mathcal{O}_{t_{n-1}}^{N} + \Delta_{n}^{\Lambda} \delta_{X_{t_{n}}}, X_{t_{n}}) - v(\mathcal{O}_{t_{n-1}}^{N}, X_{t_{n}}) \right] + \left[v(\mathcal{O}_{t_{n-1}}^{N}, X_{t_{n}}) - v(\mathcal{O}_{t_{n-1}}^{N}, X_{t_{n-1}}) \right] \\ &= \int_{0}^{1} \partial_{\mathfrak{o}} v(\mathcal{O}_{t_{n-1}}^{N} + h\Delta_{n}^{\Lambda} \delta_{X_{t_{n}}}, X_{t_{n}}) dh \Delta_{n}^{\Lambda} \\ &+ \int_{t_{n-1}}^{t_{n}} \left[\nabla v(\mathcal{O}_{t_{n-1}}^{N}, X_{s}) \cdot dX_{s} + \frac{1}{2} \mathrm{tr}(\nabla^{2} v(\mathcal{O}_{t_{n-1}}^{N}, X_{s}) \sigma_{s} \sigma_{s}^{\top}) ds \right]. \end{aligned}$$

Here in the last equality, the first term is due to the definition and continuity of $\partial_o v$, and the second term is due to the classical Itô formula. Then we can rewrite (A.4) as

$$v(\mathcal{O}_{t}^{N}, X_{t}) - v(\mathcal{O}_{0}^{N}, X_{0}) = \sum_{n \leq N} \int_{0}^{1} \partial_{\sigma} v(\mathcal{O}_{t_{n-1}}^{N} + h\Delta_{n}^{\Lambda}\delta X_{t_{n}}, X_{t_{n}})dh \ \Delta_{n}^{\Lambda}$$
$$+ \sum_{n \leq N} \int_{t_{n-1}}^{t_{n}} \left[\nabla v(\mathcal{O}_{t_{n-1}}^{N}, X_{s}) \cdot dX_{s} + \frac{1}{2} \mathrm{tr}(\nabla^{2} v(\mathcal{O}_{t_{n-1}}^{N}, X_{s})\sigma_{s}\sigma_{s}^{\top})ds \right].$$
(A.5)

Similarly to (A.2), one can easily show that

$$\lim_{N \to \infty} \max_{n} \sup_{h \in [0,1]} \sup_{t_{n-1} \leq s \leq t_n} \rho \Big(\mathcal{O}_{t_{n-1}}^N + h \Delta_n^\Lambda \delta_{X_{t_n}} - \mathcal{O}_s, X_{t_n} - X_s \Big) = 0.$$

Then, by (A.3), (A.5), and the regularity of v, we obtain (3.6) immediately.

B Proof of Proposition 3.5

We show that $J(\mathfrak{o}, x, \alpha)$ is locally 1/2-Hölder continuous with constant \hat{c} (to be determined) independent of $\alpha \in \mathscr{A}$. It is then immediate to see that $v(\mathfrak{o}, x)$ is locally 1/2-Hölder continuous with the same constant. Fix $(\mathfrak{o}, x), (\mathfrak{o}', x') \in \mathscr{D}_T, \alpha \in \mathscr{A}$, and assume without loss of generality that $\rho(\mathfrak{o} - \mathfrak{o}', x - x') \leq 1$, i.e., $\delta = 1$ in the statement. Write $(\mathcal{O}, X), (\mathcal{O}', X')$ for the solution of (3.2)-(3.3) controlled by α with initial value $(\mathfrak{o}, x), (\mathfrak{o}', x')$, respectively. The exit times of $(\mathcal{O}, X), (\mathcal{O}', X')$ from \mathscr{D}_T are denoted by τ, τ' . We also set $\varphi_t = \varphi(\mathcal{O}_t, X_t, \alpha_t), \varphi'_t = \varphi(\mathcal{O}'_t, X'_t, \alpha_t), \varphi \in \{\lambda, b, \sigma, \ell\}$ and $\Lambda = \int_0^{-1} \lambda_s ds, \Lambda' = \int_0^{-1} \lambda'_s ds$. Due to the triangle inequality

$$|J(\mathfrak{o}, x, \alpha) - J(\mathfrak{o}, x', \alpha)| \leq \mathbb{E}^{\mathbb{Q}}[|\int_{0}^{\tau} \ell_{s} ds - \int_{0}^{\tau'} \ell'_{s} ds|] + \mathbb{E}^{\mathbb{Q}}[|g(\mathcal{O}_{\tau}, X_{\tau}) - g(\mathcal{O}'_{\tau'}, X'_{\tau'})|],$$

we can estimate the terminal and running cost separately.

B.1 Terminal Cost

Using the Lipschitz condition satisfied by g, then

$$\mathbb{E}^{\mathbb{Q}}[|g(\mathcal{O}_{\tau}, X_{\tau}) - g(\mathcal{O}_{\tau'}', X_{\tau'}')|] \le c_* \mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}', X_{\tau} - X_{\tau'}')],$$

with the parabolic norm $\rho(\mathfrak{o}, x) = \sqrt{\varrho(\mathfrak{o})^2 + |x|^2}$. Next, consider the decomposition,

$$\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}', X_{\tau} - X_{\tau'}') \le \rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}, X_{\tau} - X_{\tau'}) + \rho(\mathcal{O}_{\tau'} - \mathcal{O}_{\tau'}', X_{\tau'} - X_{\tau'}').$$
(B.1)

First, we estimate the rightmost term in (B.1).

Lemma B.1. There exists a positive constant C_1 such that

$$\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau'} - \mathcal{O}_{\tau'}', X_{\tau'} - X_{\tau'}')^2] \le C_1 \rho(\mathfrak{o} - \mathfrak{o}', x - x')^2.$$
(B.2)

Proof. Recalling that condition (3.9) implies $\tau' \leq c_*T =: T_*$, then

$$\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau'} - \mathcal{O}_{\tau'}', X_{\tau'} - X_{\tau'}')^2] \le R(T_*), \quad R(t) := \mathbb{E}^{\mathbb{Q}}[\sup_{s \le t \land \tau'} \rho(\mathcal{O}_s - \mathcal{O}_s', X_s - X_s')^2].$$
(B.3)

Fix $t \leq T_*$. From Cauchy-Schwarz and Doob inequalities, it is classical that

$$\frac{1}{3} \mathbb{E}^{\mathbb{Q}} [\sup_{s \le t \land \tau'} |X_s - X'_s|^2] \le |x - x'|^2 + t \mathbb{E}^{\mathbb{Q}} [\int_0^{t \land \tau'} |b_s - b'_s|^2 ds] + 4 \mathbb{E}^{\mathbb{Q}} [\int_0^{t \land \tau'} |\sigma_s - \sigma'_s|^2 ds] \\\le |x - x'|^2 + c_*^2 (t+4) \int_0^t \mathbb{E}^{\mathbb{Q}} [\sup_{u \le s \land \tau'} \rho(\mathcal{O}_u - \mathcal{O}'_u, X_u - X'_u)^2] ds \\\le |x - x'|^2 + c_*^2 (T_* + 4) \int_0^t R(s) ds.$$

Moreover, for each $k \in \mathbb{N}$, the triangle inequality and occupation time formula yields

$$\sup_{s \le t \land \tau'} |(\mathcal{O}_s - \mathcal{O}'_s)(f_k)| \le |(\mathfrak{o} - \mathfrak{o}')(f_k)| + \int_0^{t \land \tau'} |f_k(X_s)\lambda_s - f_k(X'_s)\lambda'_s| ds.$$
(B.4)

Next, use $|f\lambda - f'\lambda'| \le |f||\lambda - \lambda'| + |f - f'||\lambda'|$ to obtain

$$|f_k(X_s)\lambda_s - f_k(X'_s)\lambda'_s| \leq |f_k(X_s)||\lambda_s - \lambda'_s| + |f_k(X_s) - f_k(X'_s)|\lambda'_s$$

$$\leq ||f_k||_{\infty}|\lambda_s - \lambda'_s| + ||\nabla f_k||_{\infty}|X_s - X'_s|\lambda'_s$$

$$\leq c_* ||f_k||_{\mathscr{C}^1}(1 + \lambda'_s)\rho(\mathcal{O}_s - \mathcal{O}'_s, X_s - X'_s), \qquad (B.5)$$

using the Lipschitz property of λ in the last inequality. Using $\int_0^{t \wedge \tau'} \lambda'_s ds \leq \Lambda'_{\tau'} = |\mathcal{O}'_{\tau'}| = T$, $\tau' \leq T_*$, and (B.4), we obtain by integrating (B.5) that

$$\frac{1}{2} \sup_{s \le t \land \tau'} |(\mathcal{O}_s - \mathcal{O}'_s)(f_k)|^2 \le |(\mathfrak{o} - \mathfrak{o}')(f_k)|^2 + c_*^2 ||f_k||_{\mathscr{C}^1}^2 (T_* + T)^2 \int_0^t \sup_{u \le s \land \tau'} \rho(\mathcal{O}_u - \mathcal{O}'_u, X_u - X'_u)^2 ds.$$

summing over k and using $\sum_{k} \|f_k\|_{\mathscr{C}^1}^2 \leq 1$ leads to

$$\frac{1}{2} \mathbb{E}^{\mathbb{Q}}[\sup_{s \le t \land \tau'} \varrho(\mathcal{O}_s - \mathcal{O}'_s)^2] \le \varrho(\mathfrak{o} - \mathfrak{o}')^2 + c_*^2 (T + T_*)^2 \int_0^t R(s) ds.$$

Altogether, we have established that

$$R(t) \leq \mathbb{E}^{\mathbb{Q}}[\sup_{s \leq t \wedge \tau'} \varrho(\mathcal{O}_s - \mathcal{O}'_s)^2] + \mathbb{E}^{\mathbb{Q}}[\sup_{s \leq t \wedge \tau'} |X_s - X'_s|^2] \leq 3\rho(\mathfrak{o} - \mathfrak{o}', x - x')^2 + C \int_0^t R(s) ds$$

where $C = c_*^2 [3(T_* + 4) + 2(T + T_*)^2]$. We therefore conclude from Grönwall's Lemma that

$$\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau'} - \mathcal{O}_{\tau'}', X_{\tau'} - X_{\tau'}')^2] \le R(T_*) \le C_1 \rho(\mathfrak{o} - \mathfrak{o}', x - x')^2, \qquad C_1 = 3e^{CT_*}.$$

We now treat the remaining term in (B.1), namely $\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}, X_{\tau} - X_{\tau'})$. First observe that

$$\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}, X_{\tau} - X_{\tau'})] \leq \mathbb{E}^{\mathbb{Q}}[\varrho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'})] + \mathbb{E}^{\mathbb{Q}}[|X_{\tau} - X_{\tau'}|].$$
(B.6)

Let $\underline{\tau} := \tau \wedge \tau'$ and $\overline{\tau} := \tau \vee \tau'$. Using the linear growth of λ , then for all $k \ge 1$,

$$\frac{1}{2}[|(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'})(f_k)|^2 - |(\mathfrak{o} - \mathfrak{o}')(f_k)|^2] \le \left(\int_{\underline{\tau}}^{\overline{\tau}} |f_k(X_s)|\lambda_s ds\right)^2 \le \|f_k\|_{\mathscr{C}^1}^2 Z_*^2 |\tau - \tau'|^2,$$

where $Z_* = c_* \sup_{t \leq T_*} (1 + \rho(\mathcal{O}_t, X_t))$. Summing over k and rearranging yields

$$\mathbb{E}^{\mathbb{Q}}[\varrho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'})^2] \le 2\varrho(\mathfrak{o} - \mathfrak{o}')^2 + 2C^2 \|\tau - \tau'\|_{L^2(\mathbb{Q})}^2, \tag{B.7}$$

with the finite constant $C = ||Z_*||_{L^2(\mathbb{Q})}$. For the rightmost term in (B.6), we first note that

$$\mathbb{E}^{\mathbb{Q}}[|X_{\tau} - X_{\tau'}|] \leq \mathbb{E}^{\mathbb{Q}}\left[|\int_{\underline{\tau}}^{\overline{\tau}} b_s ds|\right] + \mathbb{E}^{\mathbb{Q}}\left[|\int_{\underline{\tau}}^{\overline{\tau}} \sigma_s dW_s|\right]$$

Using the linear growth of b,σ and Burkholder-Davis-Gundy (BDG) inequality [31, Chapter IV], it follows that

$$\mathbb{E}^{\mathbb{Q}}\left[\left|\int_{\underline{\tau}}^{\overline{\tau}} b_s ds\right|\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\int_{\underline{\tau}}^{\overline{\tau}} |b_s| ds\right] \leq \mathbb{E}^{\mathbb{Q}}[Z_*|\tau - \tau'|] \leq C \|\tau - \tau'\|_{L^2(\mathbb{Q})},\tag{B.8}$$
$$\mathbb{E}^{\mathbb{Q}}\left[\left|\int_{\underline{\tau}}^{\overline{\tau}} \sigma_s dW_s\right|\right] \leq C_1^{\text{BDG}} \mathbb{E}^{\mathbb{Q}}\left[\|\sigma\|_{L^2([\underline{\tau},\overline{\tau}])}\right] \leq C_1^{\text{BDG}} \mathbb{E}^{\mathbb{Q}}[Z_*|\tau - \tau'|^{1/2}] \leq \tilde{C} \|\tau - \tau'\|_{L^1(\mathbb{Q})}^{1/2},\tag{B.9}$$

with $\tilde{C} = CC_1^{\text{BDG}}$. We now prove the following lemma.

Lemma B.2. There exists $C_2 > 0$ such that $\|\tau - \tau'\|_{L^2(\mathbb{Q})} \leq C_2 \rho(\mathfrak{o} - \mathfrak{o}', x - x')$.

Proof. Let $\Delta^{\Lambda}_* := \sup_{t \leq T_*} |\Lambda_t - \Lambda'_t|$, where we recall that Λ (respectively Λ') coincides with the total mass process $|\mathcal{O}|$ (resp. $|\mathcal{O}'|$). Since $\Lambda'_{\tau'} = T$, then $\Lambda_{\tau'} = \Lambda'_{\tau'} + (\Lambda_{\tau'} - \Lambda'_{\tau'}) \geq T - \Delta^{\Lambda}_*$. Moreover, the nondegeneracy condition (3.9) implies that

$$\Lambda_{\tau'+s} = \Lambda_{\tau'} + \int_{\tau'}^{\tau'+s} \lambda_u du \ge T - \Delta_*^{\Lambda} + s/c_*, \quad \forall s \ge 0.$$

Choosing $s = c_* \Delta^{\Lambda}_*$ therefore gives $\tau \leq \tau' + c_* \Delta^{\Lambda}_*$. Similarly, $\tau' \leq \tau + c_* \Delta^{\Lambda}_*$, which shows that

$$|\tau - \tau'| \le c_* \sup_{t \le T^*} |\Lambda_t - \Lambda_t'|, \qquad \mathbb{Q} - a.s.$$
(B.10)

Next, noting that $\Lambda_0 = |\mathfrak{o}|, \Lambda'_0 = |\mathfrak{o}'|$, then for all $t \leq T_*, |\Lambda_t - \Lambda'_t| \leq ||\mathfrak{o}| - |\mathfrak{o}'|| + \int_0^t |\lambda_s - \lambda'_s| ds$. Recalling that the first element of the separating family $(f_k)_{k \in \mathbb{N}}$ is constant, say $f_0 \equiv c_0$ with $c_0 \in (0, 1)$, we have

$$||\mathfrak{o}| - |\mathfrak{o}'|| = |\int_{\mathbb{R}^d} (\mathfrak{o} - \mathfrak{o}')(dx)| = c_0^{-1} |(\mathfrak{o} - \mathfrak{o}')(f_0)| \le c_0^{-1} \varrho(\mathfrak{o} - \mathfrak{o}').$$

Moreover, we have from the Lipschitz continuity of λ that

$$\int_0^t |\lambda_s - \lambda'_s| ds \le c_* \int_0^t \rho(\mathcal{O}_s - \mathcal{O}'_s, X_s - X'_s) ds \le c_* T_* \sup_{s \le T_*} \rho(\mathcal{O}_s - \mathcal{O}'_s, X_s - X'_s).$$

Using similar arguments as in the proof of Lemma B.1, we obtain

$$\frac{1}{2} \mathbb{E}^{\mathbb{Q}}[\sup_{t \leq T_*} |\Lambda_t - \Lambda'_t|^2] \leq c_0^{-2} \varrho(\mathfrak{o} - \mathfrak{o}')^2 + c_*^2 T_*^2 C_1 \rho(\mathfrak{o} - \mathfrak{o}', x - x')^2.$$

Together with (B.10), this yields

$$\|\tau - \tau'\|_{L^2(\mathbb{Q})} \le c_* \|\sup_{t \le T^*} |\Lambda_t - \Lambda'_t|\|_{L^2(\mathbb{Q})} \le C_2 \rho(\mathfrak{o} - \mathfrak{o}', x - x'), \quad C_2 = c_* \sqrt{2(c_0^{-2} + c_*^2 T_*^2 C_1)}.$$

Combining the above Lemma with equations (B.6), (B.7) yields

 $\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'})] \leq (2\rho(\mathfrak{o} - \mathfrak{o}')^2 + 2C^2C_2\rho(\mathfrak{o} - \mathfrak{o}', x - x')^2)^{1/2} \leq (2 + 2C^2C_2)^{1/2}\rho(\mathfrak{o} - \mathfrak{o}', x - x').$ Similarly, using (B.8), (B.9), and $\|\tau - \tau'\|_{L^1(\mathbb{Q})} \leq \|\tau - \tau'\|_{L^2(\mathbb{Q})}$, we have

$$\mathbb{E}^{\mathbb{Q}}[|X_{\tau} - X_{\tau'}|] \leq C_2 \varrho(\mathfrak{o} - \mathfrak{o}', x - x') + (C_2 \varrho(\mathfrak{o} - \mathfrak{o}', x - x'))^{1/2}.$$

From $\rho(\mathfrak{o} - \mathfrak{o}', x - x') \leq 1$ and the inequality $\rho \leq \rho^{1/2}$, $\rho \in [0, 1]$, we have thus shown the existence of a constant \hat{c}_g that depends on C_1, C_2 , and $C = ||Z_*||_{L^2(\mathbb{Q})}$ such that

$$\mathbb{E}^{\mathbb{Q}}[|g(\mathcal{O}_{\tau}, X_{\tau}) - g(\mathcal{O}_{\tau'}', X_{\tau'}')|] \le c_* \left(\mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau} - \mathcal{O}_{\tau'}, X_{\tau} - X_{\tau'})] + \mathbb{E}^{\mathbb{Q}}[\rho(\mathcal{O}_{\tau'} - \mathcal{O}_{\tau'}', X_{\tau'} - X_{\tau'}')] \right)$$
$$\le \hat{c}_g \ \rho(\mathfrak{o} - \mathfrak{o}', x - x')^{1/2}.$$

B.2 Running Cost

Suppose that $\tau' \leq \tau$. Then, the growth and Lipschitz property of ℓ, ℓ' implies that

$$\begin{split} |\int_0^\tau \ell_s ds - \int_0^{\tau'} \ell'_s ds| &\leq \int_0^{\tau'} |\ell_s - \ell'_s| ds + \int_{\tau'}^\tau |\ell_s| ds \\ &\leq c_* T_* \sup_{s \leq \tau \wedge \tau'} \rho(\mathcal{O}_s - \mathcal{O}'_s, X_s - X'_s) + Z'_* |\tau' - \tau|, \end{split}$$

where $Z'_* = c_* \sup_{t \leq T_*} (1 + \rho(\mathcal{O}_t, X_t))$. The case $\tau \leq \tau'$ follows analogously, with Z_* in lieu of Z'_* . Taking expectation, we obtain

$$\mathbb{E}^{\mathbb{Q}}[|\int_0^\tau \ell_s ds - \int_0^{\tau'} \ell'_s ds|] \le c_* T_* \mathbb{E}^{\mathbb{Q}}[\sup_{s \le \tau \wedge \tau'} \rho(\mathcal{O}_s - \mathcal{O}'_s, X_s - X'_s)] + C \|\tau - \tau'\|_{L^2(\mathbb{Q})},$$

with $C = ||Z_* \vee Z'_*||_{L^2(\mathbb{Q})}$. Defining $\hat{c}_\ell = c_*T_*C_1 + CC_2$, we have from Lemmas B.1 and B.2 that

$$\mathbb{E}^{\mathbb{Q}}[|\int_{0}^{\tau}\ell_{s}ds - \int_{0}^{\tau'}\ell'_{s}ds|] \leq \hat{c}_{\ell}\rho(\mathfrak{o}-\mathfrak{o}',x-x') \leq \hat{c}_{\ell}\rho(\mathfrak{o}-\mathfrak{o}',x-x')^{1/2},$$

which completes the proof of Proposition 3.5 with the constant $\hat{c} = \hat{c}_g + \hat{c}_\ell$.

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