

# On Convergence and Rate of Convergence of Policy Improvement Algorithms

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## Abstract

In this paper we provide a simple proof from scratch for the convergence of *Policy Improvement Algorithm* (PIA) for a continuous time entropy-regularized stochastic control problem. Such convergence has been established by Huang-Wang-Zhou [4] by using sophisticated PDE estimates for the iterative PDEs involved in the PIA. Our approach builds on some Feynman-Kac type probabilistic representation formulae for solutions of PDEs and their derivatives. Moreover, in the infinite horizon model with large discount factor and in the finite horizon model, we obtain the exponential rate of convergence with similar arguments. Finally, in the one dimensional setting, we extend the convergence result to the diffusion control case.

**Keywords.** Reinforcement learning, entropy-regularized exploratory control problem, policy improvement algorithm, Feynman-Kac representation formula, rate of convergence.

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# 1 Introduction

The *Policy Improvement Algorithm* (PIA) is a well-known approach in numerical optimal control theory, see, e.g., Jacka-Mijatović [5], Kerimkulov-Siska-Szpruch [6, 7], Puterman [10]. Its main idea is to construct an iteration scheme for the control actions that traces the maximizers/minimizers of the Hamiltonian, so that the corresponding returns are naturally improving. Mathematically this amounts to a type of Picard iteration for the associated HJB equations. Motivated by the above problem but with model uncertainty, the *Reinforcement Learning* (RL) algorithms for the entropy-regularized stochastic control problems have received very strong attention in recent years. By using relaxed control, the control problem is regularized or say penalized by Shannon's entropy, which captures the trade-off between exploitation (to optimize) and exploration (to learn the model). Such entropy-regularized problem in a continuous time model was introduced by Wang-Zariphopoulou-Zhou [14], see also Guo-Xu-Zariphopoulou [3], Reisinger-Zhang [11], and Tang-Zhang-Zhou [12] in this direction, especially on the relation between the entropy-regularized problem and the original control problem.

The convergence of the PIA for the entropy-regularized problem, in terms of both the value and the optimal strategy, is clearly a central issue in the theory. In a linear quadratic model, Wang-Zhou [15] solved the problem explicitly and the convergence is immediate. Our paper is mainly motivated by the work Huang-Wang-Zhou [4], which established the desired convergence in a general infinite horizon diffusion model with drift controls. Note that the values of the iterative sequence in PIA are by nature increasing (and bounded), so the main issue is to identify its limit with the true value function of the entropy-regularized control problem. By using some sophisticated Sobolev estimates, [4] established uniform regularity for the iterative value functions, especially uniform bounds for their derivatives, and then derived the convergence by compactness arguments. We also refer to Bai-Gamage-Ma-Xie [1] and Dong [2] for some related works.

The main purpose of this paper is to provide a simple proof from scratch for the main results in [4]. Our proof builds on Feynman-Kac type representation formulae for derivatives of functions, established in Ma-Zhang [8] and Zhang [16] by using the integration by parts formula for Malliavin derivatives. These formulae enable us to establish the uniform bounds of the derivatives of the iterative value functions rather easily, see §4 Step 1 below, and then the desired convergence under the  $C^2$ -norm follows immediately.

Moreover, in the case that the discount factor is sufficiently large, we obtain an exponen-

tial rate of convergence, which is new in the literature<sup>1</sup>. Instead of applying the compactness arguments, in this case we use the representation formulae to obtain the rate of convergence directly. In particular, while the involved derivatives still have uniform bounds, we do not require them for the proof of the convergence here. We would also like to note that, when the discount factor is small, in general it may not be reasonable to expect a good rate of convergence, see Remark 4.2 and Example 4.3 below.

We also apply our approach to the finite horizon case, which is associated with parabolic PDEs. We shall obtain the exponential rate of convergence under the  $C^{1,2}$ -norm, by first considering small time duration and then extending to arbitrary time duration. In this case, we do not need a constraint corresponding to the large discount factor in the infinite horizon case. This result is also new in the literature, to the best of our knowledge.

Finally, we investigate the infinite horizon model with diffusion controls. In the case that the state process is scalar, we obtain the  $C^2$ -convergence of the value function, which implies the convergence of the optimal strategy. This case is much more subtle, and our current arguments rely heavily on the scalar assumption.

When finalizing the present paper, we learned the very interesting recent paper Tran-Wang-Zhang [13]. In the base case of infinite horizon model with drift control and sufficiently large discount factor, [13] obtained the same exponential rate of convergence. The main ideas are similar, however, they used Schauder estimates from PDE literature while we proved the required estimates from scratch by using the probabilistic representation formulae. It is remarkable that [13] established the convergence for diffusion controlled models when the diffusion control is small in certain sense and the discount factor is sufficiently large, without requiring the state process to be scalar. The key is again the crucial uniform estimates for the associated iterative fully nonlinear PDEs, in the spirit of Evans-Krylov theorem. It will be very interesting to combine our approaches and to explore more general models, especially when there is diffusion control, which we shall leave for future research.

The rest of the paper is organized as follows. In §2 we formulate the problem and state the main results for the infinite horizon case. In §3 and §4 we prove the main result, first in the case when the discounting factor is large, and then for the general case when the discounting factor is small. In §5 we discuss the finite horizon case.

**Notations.** Throughout the paper, we shall use the following notations.

- For  $x, y \in \mathbb{R}^d$ ,  $x \cdot y := \sum_{i=1}^d x_i y_i$  denotes the inner product; and all vectors are viewed as column vectors.

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<sup>1</sup>The same rate is obtained independently by [13], as we will comment in details soon.

- For  $A, B \in \mathbb{R}^{d \times d}$ ,  $A : B := \text{trace}(AB^\top)$ , where  $B^\top$  is the transpose of  $B$ . Moreover,  $I_d$  denote the  $d \times d$  identity matrix.
- For a generic Euclidean space  $E_1, E_2$ , and  $m, k \geq 0$ ,  $C_b^{m,k}([0, T] \times E_1; E_2)$  denotes the set of functions  $\phi : [0, T] \times E_1 \times E_2$  which is  $m$ -th order continuous differentiable in the temporal variable  $t$  and  $k$ -th order continuously differentiable in the spatial variable, and  $\phi$  as well as all its derivatives involved above are bounded. Moreover,  $C_b^k(E_1; E_2)$  denotes the subspace where  $\phi : E_1 \rightarrow E_2$  is independent of the temporal variable  $t$ .
- For  $\phi \in C^2(E_1; E_2)$  and  $\psi \in C^{1,2}([0, T] \times E_1; E_2)$ ,

$$\begin{aligned} \|\phi\|_0 &:= \sup_{x \in E_1} |\phi(x)|, & \|\phi\|_2 &:= \|\phi\|_0 + \|\partial_x \phi\|_0 + \|\partial_{xx} \phi\|_0; \\ \|\psi\|_0 &:= \sup_{(t,x) \in [0,T] \times E_1} |\phi(t,x)|, & \|\psi\|_{1,2} &:= \|\psi\|_0 + \|\partial_t \psi\|_0 + \|\partial_x \psi\|_0 + \|\partial_{xx} \psi\|_0. \end{aligned}$$

- For a domain  $A$  in certain Euclidean space,  $\mathcal{P}_0(A)$  denotes the set of  $\pi : A \rightarrow (0, \infty)$  such that  $\int_A \pi(a) da = 1$ , namely  $\pi$  is a probability measure on  $A$  with density. Moreover, for  $\pi \in \mathcal{P}_0(A)$ ,  $\mathcal{H}(\pi)$  denotes its the Shannon's entropy:

$$\mathcal{H}(\pi) := \int_A \pi(a) \ln \pi(a) da. \quad (1.1)$$

- For any  $\pi \in \mathcal{P}_0(A)$  and  $\phi : E_1 \times A \rightarrow E_2$ , where  $E_1, E_2$  are Euclidean spaces, denote:

$$\tilde{\phi}(x, \pi) := \int_A \phi(x, a) \pi(a) da. \quad (1.2)$$

## 2 The problem and the main results

In this section we focus on stochastic control problems in infinite horizon with drift controls. We shall extend our approach to the finite horizon case in §5 and to the diffusion control case in §6 below.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space,  $W$  a standard  $d$ -dimensional Brownian motion,  $\mathbb{F} := \mathbb{F}^W$ , and the control set  $A$  is a domain in some Euclidean space with finite Lebesgue measure:  $0 < |A| < \infty$ . Our underlying control problem takes the following form:

$$\begin{aligned} X_t^\alpha &= x + \int_0^t b(X_s^\alpha, \alpha_s) ds + \int_0^t \sigma(X_s^\alpha) dW_s; \\ v_0(x) &:= \sup_\alpha \mathbb{E} \left[ \int_0^\infty e^{-\rho t} r(X_t^\alpha, \alpha_t) dt \right]. \end{aligned} \quad (2.1)$$

where  $b : \mathbb{R}^d \times A \rightarrow \mathbb{R}^d$ ,  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ ,  $r : \mathbb{R}^d \times A \rightarrow \mathbb{R}$ ,  $\rho > 0$  is a constant, and  $\alpha$  is an appropriate  $A$ -valued admissible control. It is well known that, under certain technical

conditions,  $v_0$  satisfies an HJB equation and there is a vast literature on numerical methods for  $v_0$ , provided that  $b, \sigma, r$  are known.

Strongly motivated by numerical methods for the above control problem but with model uncertainty, namely  $b, \sigma, r$  are unknown, we consider instead the *entropy-regularized exploratory optimal control problem*. That is, we consider the relaxed control for the purpose of exploration. Recall (1.1), (1.2), and let  $\mathcal{A}$  denote the set of  $\pi : [0, \infty) \times \mathbb{R}^d \rightarrow \mathcal{P}_0(A)$ . Then our entropy-regularized exploratory optimal control problem becomes:

$$\begin{aligned} X_t^\pi &= x + \int_0^t \tilde{b}(X_s^\pi, \pi(s, X_s^\pi)) ds + \int_0^t \sigma(X_s^\pi) dW_s; \\ v(x) &:= \sup_{\pi \in \mathcal{A}} J(x, \pi), \quad J(x, \pi) := \mathbb{E} \left[ \int_0^\infty e^{-\rho t} [\tilde{r}(X_t^\pi, \pi(s, X_s^\pi)) - \lambda \mathcal{H}(\pi(t, X_t^\pi))] dt \right], \end{aligned} \quad (2.2)$$

where  $\lambda > 0$  is the exogenous ‘‘temperature’’ parameter capturing the trade-off between exploitation and exploration. We remark that  $v \rightarrow v_0$  when  $\lambda \downarrow 0$ , see [12].

In this section, we shall assume:

**Assumption 2.1.** (i)  $b, \sigma, r$  are measurable in  $a$  and twice continuously differential in  $x$ , and both the functions and their derivatives are bounded by a constant  $C_0 > 0$ ;

(ii)  $\sigma$  is uniform non-degenerate:  $\sigma \sigma^\top \geq \frac{1}{C_0} I_{d \times d}$ .

Throughout the paper, let  $C > 0$  denote a generic constant depending only on  $d, \lambda, |A|$ , and  $C_0$ , but not on  $\rho$ . When the constant depends further on  $\rho$ , we denote it as  $C_\rho$ .

Clearly, under the above conditions, the SDE for  $X^\pi$  has a unique weak solution and  $v(x)$  is well defined and satisfies the following HJB equation:

$$\begin{aligned} \rho v &= \frac{1}{2} \sigma \sigma^\top : \partial_{xx} v + H(x, \partial_x v), \\ \text{where } H(x, z) &:= \sup_{\pi \in \mathcal{P}_0(A)} \left[ \tilde{b}(x, \pi) \cdot z + \tilde{r}(x, \pi) - \lambda \mathcal{H}(\pi) \right]. \end{aligned} \quad (2.3)$$

Moreover, the Hamiltonian  $H$  has optimal relaxed control  $\Gamma$  in Gibbs form:

$$\begin{aligned} \Gamma(x, z, a) &:= \frac{\gamma(x, z, a)}{\int_A \gamma(x, z, a') da'}, \quad \text{where } \gamma(x, z, a) := \exp \left( \frac{1}{\lambda} [b(x, a) \cdot z + r(x, a)] \right), \\ \text{and thus } H(x, z) &= \lambda \ln \left( \int_A \gamma(x, z, a) da \right). \end{aligned} \quad (2.4)$$

Then we have the following simple result.

**Lemma 2.2.** *Let Assumption 2.1 hold.*

(i)  $H$  is twice continuously differentiable in  $(x, z)$ , and there exists a constant  $C > 0$  (again, independent of  $\rho$ ) such that

$$|\partial_z H| \leq C, \quad 0 \leq \partial_{zz} H \leq C I_d, \quad |H(x, z)| + |\partial_x H(x, z)| + |\partial_{xz} H(x, z)| \leq C[1 + |z|]. \quad (2.5)$$

(ii)  $v \in C_b^2(\mathbb{R}^d; \mathbb{R})$  is the unique classical solution of (2.3), with  $\|v\|_2 \leq C_\rho$ .

**Proof** (ii) is rather standard in the PDE literature, given the uniform non-degeneracy of  $\sigma$ . To see (i), since  $b, r$  are twice continuously differentiable in  $x$ , by (2.4) it is clear that  $H$  is also twice continuously differentiable in  $(x, z)$ . Note that

$$\begin{aligned}\partial_z H &= \frac{\int_A b \gamma da}{\int_A \gamma da}, & \partial_{zz} H &= \frac{\int_A b b^\top \gamma da \int_A \gamma da - \int_A b \gamma da \int_A b^\top \gamma da}{(\int_A \gamma da)^2}; \\ \partial_x H &= \frac{\int_A [\partial_x b z + \partial_x r] \gamma da}{\int_A \gamma da}, & \partial_{xz} H &= \frac{\int_A \partial_x b \gamma da}{\int_A \gamma da} - \frac{\int_A [\partial_x b z + \partial_x r] \gamma da \int_A b^\top \gamma da}{(\int_A \gamma da)^2}.\end{aligned}$$

Then it is straightforward to verify (2.5). ■

We now introduce the *Policy Improvement Algorithm* (PIA) for solving (2.3) recursively:

*Step 0.* Set  $v^0 := \int_0^\infty e^{-\rho t} [-C_0 - \lambda \inf_{\pi \in \mathcal{P}_0(A)} \mathcal{H}(\pi)] dt = -\frac{1}{\rho} [C_0 - \lambda(\ln |A|)^+]^2$

*Step n.* For  $n \geq 1$ , define  $\pi^n(x, a) := \Gamma(x, \partial_x v^{n-1}(x), a)$  and  $v^n(x) := J(x, \pi^n)$ .

Then it is clear that  $v^n$  satisfies the following recursive linear PDE:

$$\rho v^n = \frac{1}{2} \sigma \sigma^\top : \partial_{xx} v^n + \partial_z H(x, \partial_x v^{n-1}) \cdot (\partial_x v^n - \partial_x v^{n-1}) + H(x, \partial_x v^{n-1}). \quad (2.6)$$

We remark that here  $\partial_z H(x, \partial_x v^{n-1}) = \tilde{b}(x, \pi^n(x, \cdot)) = \int_A b(x, a) \Gamma(x, \partial_x v^{n-1}, a) da$ . The following result is immediate, see e.g. [4].

**Proposition 2.3.** *Let Assumption 2.1 hold. Then*

- (i) For each  $n \geq 1$ ,  $v^n \in C_b^2(\mathbb{R}^d; \mathbb{R})$  is a classical solution of (2.6);
- (ii)  $v^n$  is increasing in  $n$  and  $v^n \leq \frac{1}{\rho} [C_0 + \lambda(\ln |A|)^+]$ .

By above it is clear that  $v^n \uparrow v^*$  for some function  $v^*$ . Our goal in this paper is to identify  $v^* = v$  and to study the rate of convergence. Our main result is as follows.

**Theorem 2.4.** *Let Assumption 2.1 hold.<sup>3</sup>*

- (i) There exists a constant  $\rho_0$ , depending only on  $d, \lambda, |A|$ , and  $C_0$ , such that,

$$\|v^n - v\|_2 \leq \frac{C}{2^n}, \quad \text{whenever } \rho \geq \rho_0. \quad (2.7)$$

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<sup>2</sup>We set  $v^0$  in this way so that  $v^0(x) \leq J(x, \pi)$  for all  $\pi$ . In particular, why it is not crucial for the remaining analysis, this will imply that  $v^0 \leq v^1$ .

<sup>3</sup>We assume the twice differentiability in Assumption 2.1 in order to get the  $C^2$ -convergence in the theorem. If we content ourselves with the  $C^1$ -convergence, from our proofs one can easily see that the second order differentiability is not required. We also note that the  $C^1$ -convergence is sufficient for the convergence of the optimal strategies, see Remark 2.5.

(ii) In the general case,  $v^n \rightarrow v$  in  $C^2$  uniformly on compacts. That is, for any compact set  $K \subset \subset \mathbb{R}^d$ ,

$$\lim_{n \rightarrow \infty} \sup_{x \in K} \left[ |(v^n - v)(x)| + |\partial_x(v^n - v)(x)| + |\partial_{xx}(v^n - v)(x)| \right] = 0. \quad (2.8)$$

The theorem is proved in the next two sections.

**Remark 2.5.** In the setting of Theorem 2.4, it clear that the iterative optimal strategy  $\pi^n(t, x, a) = \Gamma(x, \partial_x v^{n-1}(x), a)$  also converges to the optimal strategy  $\pi^*(t, x, a) = \Gamma(x, \partial_x v(x), a)$  for the entropy-regularized problem (2.2).

### 3 Proof of Theorem 2.4 (i): the case of large $\rho$

In this section we prove (2.7), provided  $\rho$  is large enough. We emphasize again that in this section the generic constant  $C$  does not depend on  $\rho$ . We proceed in three steps.

**Step 1.** In this step we provide probabilistic representation formulae for  $v, v^n$  and their derivatives, which will be crucial for our estimates. First, fix  $x \in \mathbb{R}^d$  and denote

$$X_t^x = x + \int_0^t \sigma(X_s^x) dW_s. \quad (3.1)$$

Then by standard Feynman-Kac formula we derive from (2.3) and (2.6) that

$$v(x) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(X_t^x) dt \right], \quad v^n(x) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f^n(X_t^x) dt \right], \quad \text{where} \quad (3.2)$$

$$f(x) := H(x, \partial_x v(x)), \quad f^n(x) := \partial_z H(x, \partial_x v^{n-1}) \cdot (\partial_x v^n - \partial_x v^{n-1}) + H(x, \partial_x v^{n-1}).$$

Next, for any  $\phi \in C_b^1(\mathbb{R}^d; \mathbb{R})$ , by [8] and following the arguments in [16] we have the following representation formula for its first and second order derivatives<sup>4</sup>

$$\partial_x \mathbb{E}[\phi(X_t^x)] = \mathbb{E}[\phi(X_t^x) N_t^x], \quad \partial_{xx} \mathbb{E}[\phi(X_t^x)] = \mathbb{E} \left[ N_t^x (\nabla X_t^x \partial_x \phi(X_t^x))^\top + \phi(X_t^x) \nabla N_t^x \right], \quad (3.3)$$

where the kernels are defined by:

$$\begin{aligned} \nabla X_t^x &= I_d + \sum_{i=1}^d \int_0^t \partial_x \sigma^i(X_s^x) \nabla X_s^x dW_s^i, \quad N_t^x := \frac{1}{t} \int_0^t (\sigma^{-1}(X_s^x) \nabla X_s^x)^\top dW_s; \\ \nabla_j^2 X_t^x &= \sum_{i=1}^d \int_0^t \left[ \sum_{k=1}^d \partial_{xx_k} \sigma^i(X_s^x) (\nabla X_s^x)^{kj} \nabla X_s^x + \partial_x \sigma^i(X_s^x) \nabla_j^2 X_s^x \right] dW_s^i, \\ \nabla_i N_t^x &:= \frac{1}{t} \int_0^t \left( \sum_{j=1}^d (\nabla X_s^x)^{ij} \partial_{x_j} \sigma^{-1}(X_s^x) \nabla X_s^x + \sigma^{-1}(X_s^x) \nabla_i^2 X_s^x \right)^\top dW_s. \end{aligned} \quad (3.4)$$

<sup>4</sup>[8] provides only the first one in (3.3). The second one follows the same arguments as in [8, 16] directly by differentiating the first one with respect to the initial value  $x$ . We also note that, the  $N$  in [8, 16] is a row vector, corresponding to the transpose of the  $N$  here.

Here the  $i$ -th column of  $\nabla X^x$  stands for  $\partial_{x_i} X^x$ ,  $\nabla_j^2 X$  stands for  $\partial_{x_j} \nabla X^x$ , and  $\sigma^i$  (resp.  $\nabla_i N_t^x$ ) is the  $i$ -th column of  $\sigma$  (resp.  $\nabla N_t^x$ ). Apply these on (3.2), we obtain

$$\begin{aligned}\partial_x v(x) &= \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f(X_t^x) N_t^x dt \right], \quad \partial_x v^n(x) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} f^n(X_t^x) N_t^x dt \right]; \\ \partial_{xx} v(x) &= \mathbb{E} \left[ \int_0^\infty e^{-\rho t} [N_t^x (\nabla X_t^x \partial_x f(X_t^x))^\top + f(X_t^x) \nabla N_t^x] dt \right]; \\ \partial_{xx} v^n(x) &= \mathbb{E} \left[ \int_0^\infty e^{-\rho t} [N_t^x (\nabla X_t^x \partial_x f^n(X_t^x))^\top + f^n(X_t^x) \nabla N_t^x] dt \right].\end{aligned}\tag{3.5}$$

Moreover, applying standard SDE estimates on (3.4), by Assumption 2.1 we can easily have

$$\mathbb{E} \left[ |\nabla X_t^x|^2 + |\nabla^2 X_t^x|^2 \right] \leq C e^{Ct}, \quad \mathbb{E} \left[ |N_t^x|^2 + |\nabla N_t^x|^2 \right] \leq \frac{C}{t} e^{Ct}.\tag{3.6}$$

**Remark 3.1.** When  $d = 1$ ,  $\sigma \equiv 1$ , we have  $X_t^x = x + W_t$ ,  $\nabla X_t^x \equiv 1$ , and  $N_t^x = \frac{W_t}{t}$ , then the first formula in (3.3) is a direct consequence of the integration by parts formula:

$$\partial_x \mathbb{E}[\phi(X_t^x)] = \int_{\mathbb{R}} \phi'(x+y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy = \int_{\mathbb{R}} \phi(x+y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \frac{y}{t} dy = \mathbb{E}[\phi(X_t^x) N_t^x].$$

The general formula follows from the integration by parts formula for Malliavin derivatives.

**Step 2.** In this step we study the convergence of  $\partial_x v^n$ . Denote

$$\varepsilon_1^n := \|\partial_x v^n - \partial_x v\|_0 < \infty.\tag{3.7}$$

Here the subscript  $_1$  indicates the first order derivative  $\partial_x$ . Note that, by (3.2),

$$|f^n(x) - f(x)| \leq |\partial_z H(x, \partial_x v^{n-1})| (\varepsilon_1^n + \varepsilon_1^{n-1}) + |H(x, \partial_x v) - H(x, \partial_x v^{n-1})|.$$

Recall (2.5) that  $|\partial_z H| \leq C_0$ , we have

$$|f^n(x) - f(x)| \leq C(\varepsilon_1^n + \varepsilon_1^{n-1}).\tag{3.8}$$

Now for any  $n \geq 1$  and  $x \in \mathbb{R}^d$ , by (3.5) and (3.6) we have

$$\begin{aligned}|\partial_x(v^n - v)(x)| &\leq \mathbb{E} \left[ \int_0^\infty e^{-\rho t} |(f^n - f)(X_t^x)| |N_t^x| dt \right] \leq C(\varepsilon_1^n + \varepsilon_1^{n-1}) \int_0^\infty e^{-\rho t} \mathbb{E}[|N_t^x|] dt \\ &\leq C(\varepsilon_1^n + \varepsilon_1^{n-1}) \left[ \int_0^\infty \frac{1}{\sqrt{t}} e^{-\rho t + C_1 t} dt \right] = \frac{C_2}{\sqrt{\rho - C_1}} (\varepsilon_1^n + \varepsilon_1^{n-1}),\end{aligned}$$

where  $C_1, C_2 > 0$  are generic constant independent of  $\rho$  and we assumed  $\rho > C_1$ . Set

$$\rho_0 := C_1 + 9|C_2|^2 \quad \text{so that} \quad \frac{C_2}{\sqrt{\rho_0 - C_1}} = \frac{1}{3}.\tag{3.9}$$



Since  $x$  is arbitrary, then for  $\rho \geq \rho_0$  we obtain

$$\varepsilon_1^n \leq \frac{C_2}{\sqrt{\rho - C_1}}(\varepsilon_1^n + \varepsilon_1^{n-1}) \leq \frac{1}{3}(\varepsilon_1^n + \varepsilon_1^{n-1}), \quad \text{and thus} \quad \varepsilon_1^n \leq \frac{1}{2}\varepsilon_1^{n-1}. \quad (3.10)$$

Moreover, by (3.2) and (2.5) we have

$$|\partial_x v(x)| \leq C(1 + \|\partial_x v\|_0) \int_0^\infty e^{-\rho t} \mathbb{E}[|N_t^x|] dt \leq \frac{C_2}{\sqrt{\rho - C_1}}(1 + \|\partial_x v\|_0) \leq \frac{1}{3}(1 + \|\partial_x v\|_0).$$

This implies  $\|\partial_x v\|_0 \leq \frac{1}{2}$ . Note further that  $\partial_x v^0 \equiv 0$ . Then, by (3.10) we get

$$\varepsilon_1^n \leq \frac{\varepsilon_1^0}{2^n} = \frac{\|\partial_x v\|_0}{2^n} \leq \frac{C}{2^n}, \quad \text{and thus} \quad \|\partial_x v^n\|_0 \leq \|\partial_x v\|_0 + \varepsilon_1^n \leq C. \quad (3.11)$$

Furthermore, it follows from (3.2) that

$$|v^n(x) - v(x)| \leq \mathbb{E} \left[ \int_0^\infty e^{-\rho t} |f^n(X_t^x) - f(X_t^x)| dt \right].$$

Then, by (3.8) we have

$$|v^n(x) - v(x)| \leq C(\varepsilon_1^n + \varepsilon_1^{n-1}) \int_0^\infty e^{-\rho t} dt = \frac{C}{\rho}(\varepsilon_1^n + \varepsilon_1^{n-1}).$$

Plug (3.11) into it and note again that  $\rho \geq \rho_0$ , we obtain

$$\|v^n - v\|_0 \leq \frac{C}{2^n}. \quad (3.12)$$

**Step 3.** We now estimate the difference for the second order derivatives. Denote  $\varepsilon_2^n := \|\partial_{xx} v^n - \partial_{xx} v\|_0 < \infty$ . By (3.5) we have

$$|\partial_{xx}(v^n - v)(x)| \leq \mathbb{E} \left[ \int_0^\infty e^{-\rho t} [|\partial_x(f^n - f)(X_t^x)| |\nabla X_t^x| |N_t^x| + |(f^n - f)(X_t^x)| |\nabla N_t^x|] dt \right].$$

Note that

$$\begin{aligned} \partial_x(f^n - f)(x) &= [\partial_{xz}H(x, \partial_x v^{n-1}) + \partial_{zz}H(x, \partial_x v^{n-1})\partial_{xx}v^{n-1}] [\partial_x v^n - \partial_x v^{n-1}] \\ &\quad + \partial_z H(x, \partial_x v^{n-1}) [\partial_{xx}v^n - \partial_{xx}v^{n-1}] + [\partial_x H(x, \partial_x v^{n-1}) - \partial_x H(x, \partial_x v)] \\ &\quad + [\partial_z H(x, \partial_x v^{n-1})\partial_{xx}v^{n-1} - \partial_z H(x, \partial_x v)\partial_{xx}v]. \end{aligned}$$

Then it follows from (3.11) and Lemma 2.2 that

$$\begin{aligned} \left| \partial_x(f^n - f)(x) \right| &\leq C[1 + \varepsilon_2^{n-1}] [\varepsilon_1^n + \varepsilon_1^{n-1}] + C[\varepsilon_2^n + \varepsilon_2^{n-1}] + C\varepsilon_1^{n-1} + C[\varepsilon_1^{n-1} + \varepsilon_2^{n-1}] \\ &\leq C[1 + \varepsilon_1^n + \varepsilon_1^{n-1}] [\varepsilon_2^n + \varepsilon_2^{n-1}] + C[\varepsilon_1^n + \varepsilon_1^{n-1}]. \end{aligned} \quad (3.13)$$

Thus, for possibly larger  $C_1$  and  $C_2$ , by (3.8), (3.11), and (3.6) we have

$$\begin{aligned} |\partial_{xx}v^n(x) - \partial_{xx}v(x)| &\leq C[\varepsilon_2^n + \varepsilon_2^{n-1} + \frac{1}{2^n}] \int_0^\infty e^{-\rho t} \mathbb{E}[|\nabla X_t^x| |N_t^x| + |\nabla N_t^x|] dt \\ &\leq C[\varepsilon_2^n + \varepsilon_2^{n-1} + \frac{1}{2^n}] \int_0^\infty \frac{1}{\sqrt{t}} e^{-\rho t + C_1 t} \leq \frac{C_2}{\sqrt{\rho - C_1}} [\varepsilon_2^n + \varepsilon_2^{n-1} + \frac{1}{2^n}]. \end{aligned}$$

Since  $x$  is arbitrary, then for  $\rho \geq \rho_0$  we have

$$\varepsilon_2^n \leq \frac{C_2}{\sqrt{\rho - C_1}} [\varepsilon_2^n + \varepsilon_2^{n-1} + \frac{1}{2^n}] \leq \frac{1}{3} [\varepsilon_2^n + \varepsilon_2^{n-1} + \frac{1}{2^n}].$$

This implies

$$\varepsilon_2^n \leq \frac{1}{2} \varepsilon_2^{n-1} + \frac{C}{2^n}, \quad \text{and thus} \quad \varepsilon_2^n \leq \frac{\varepsilon_2^0}{2^n} + \frac{C}{2^n}. \quad (3.14)$$

Moreover, by (2.5) and noting from Step 2 that  $\|\partial_x v\|_0 \leq C$  for  $\rho \geq \rho_0$ , then it holds

$$\begin{aligned} |f(x)| &= |H(x, \partial_x v)| \leq C[1 + |\partial_x v|] \leq C, \\ |\partial_x f(x)| &\leq |\partial_x H(x, \partial_x v)| + |\partial_z H(x, \partial_x v)| |\partial_{xx} v| \leq C[1 + \|\partial_{xx} v\|_0]. \end{aligned}$$

Thus by (3.5) and (3.6) we have, again for  $\rho \geq \rho_0$ ,

$$\begin{aligned} |\partial_{xx}v(x)| &\leq C[1 + \|\partial_{xx}v\|_0] \int_0^\infty e^{-\rho t} \mathbb{E}[|\nabla X_t^x| |N_t^x|] + C \int_0^\infty e^{-\rho t} \mathbb{E}[|\nabla N_t^x|] dt \\ &\leq \frac{C_2}{\sqrt{\rho - C_1}} [1 + \|\partial_{xx}v\|_0] \leq \frac{1}{3} [1 + \|\partial_{xx}v\|_0]. \end{aligned}$$

By the arbitrariness of  $x$ , we have  $\|\partial_{xx}v\|_0 \leq \frac{1}{3} [1 + \|\partial_{xx}v\|_0]$  and thus  $\|\partial_{xx}v\|_0 \leq \frac{1}{2}$ . Note further that  $\partial_{xx}^2 v^0 \equiv 0$ . Then  $\varepsilon_2^0 = \|\partial_{xx}^2 v\|_0 \leq \frac{1}{2}$ , and thus it follows from (3.14) that  $\varepsilon_2^n \leq \frac{C}{2^n}$ . This, together with (3.11) and (3.12), proves (2.7).  $\blacksquare$

## 4 Proof of Theorem 2.4 (ii): the case of small $\rho$

We now prove (2.8) for arbitrary  $\rho$ . Let  $\rho_1 > \rho$  be a large constant which will be specified later. We remark that here we allow  $\rho_1$  to depend on  $\rho$ . We proceed in two steps.

**Step 1.** Note that we may rewrite (2.6) as

$$\rho_1 v^n = \frac{1}{2} \sigma \sigma^\top : \partial_{xx} v^n + \partial_z H(x, \partial_x v^{n-1}) \cdot (\partial_x v^n - \partial_x v^{n-1}) + H(x, \partial_x v^{n-1}) + (\rho_1 - \rho) v^n.$$

Denote

$$L_1^n := \|\partial_x v^n\|_0, \quad L_2^n := \|\partial_{xx} v^n\|_0, \quad \tilde{f}^n := f^n + (\rho_1 - \rho) v^n.$$

First, similarly to (3.5) we have

$$\partial_x v^n(x) = \mathbb{E} \left[ \int_0^\infty e^{-\rho_1 t} \tilde{f}^n(X_t^x) N_t^x dt \right].$$

By Proposition 2.3 (ii) we have  $\|v^n\|_0 \leq \frac{C}{\rho}$ . Then, by (3.2) and Lemma 2.2,

$$|\tilde{f}^n(x)| \leq C(L_1^n + L_1^{n-1}) + C_\rho(\rho_1 - \rho). \quad (4.1)$$

Thus, by (3.6),

$$\begin{aligned} |\partial_x v^n(x)| &\leq C[L_1^n + L_1^{n-1} + C_\rho(\rho_1 - \rho)] \int_0^\infty e^{-\rho_1 t} \mathbb{E}[|N_t^x|] dt \\ &\leq C[L_1^n + L_1^{n-1} + C_\rho(\rho_1 - \rho)] \int_0^\infty \frac{1}{\sqrt{t}} e^{-\rho_1 t + Ct} dt \\ &\leq \frac{C_2}{\sqrt{\rho_1 - C_1}} (L_1^n + L_1^{n-1}) + \frac{C_\rho(\rho_1 - \rho)}{\sqrt{\rho_1 - C_1}}. \end{aligned}$$

Here  $C_1, C_2$  are generic constants independent of  $\rho$ . Now set  $\rho_1 > \rho$  be large enough as in (3.9) such that  $\frac{C_2}{\sqrt{\rho_1 - C_1}} \leq \frac{1}{3}$ . Then, by the arbitrariness of  $x$ , we obtain

$$L_1^n \leq \frac{1}{3}(L_1^n + L_1^{n-1}) + C_\rho.$$

Note further that  $\partial_x v^0 = 0$  and thus  $L_1^0 = 0$ . Then it follows from standard arguments that,

$$L_1^n \leq \frac{L_1^0}{2^n} + C_\rho = C_\rho. \quad (4.2)$$

Next, similarly to (3.5) we have

$$\partial_{xx} v^n(x) = \mathbb{E} \left[ \int_0^\infty e^{-\rho_1 t} [N_t^x (\nabla X_t^x \partial_x \tilde{f}^n(X_t^x))^\top + \tilde{f}^n(X_t^x) \nabla N_t^x] dt \right]. \quad (4.3)$$

By (4.1) and (4.2) it is clear that  $|\tilde{f}^n(x)| \leq C_\rho(\rho_1 - \rho + 1)$ . Moreover, following similar arguments as in (3.13) and by using (4.2) again, we have

$$|\partial_x \tilde{f}^n(x)| \leq |\partial_x f^n(x)| + (\rho_1 - \rho) |\partial_x v^n| \leq C_\rho [L_2^n + L_2^{n-1} + \rho_1 - \rho + 1].$$

Then, by (3.6),

$$\begin{aligned} |\partial_{xx} v^n(x)| &\leq C_\rho [L_2^n + L_2^{n-1} + \rho_1 - \rho + 1] \mathbb{E} \left[ \int_0^\infty \frac{1}{\sqrt{t}} e^{-\rho_1 t + Ct} dt \right] \\ &\leq \frac{C_\rho}{\sqrt{\rho_1 - C_1}} [L_2^n + L_2^{n-1} + \rho_1 - \rho + 1]. \end{aligned}$$

Set  $\rho_1 := C_1 + 9|C_\rho|^2$  so that  $\frac{C_\rho}{\sqrt{\rho_1 - C_1}} = \frac{1}{3}$ . Then, by the arbitrariness of  $x$ ,

$$L_2^n \leq \frac{1}{3} [L_2^n + L_2^{n-1} + \rho_1 - \rho + 1].$$

Note that  $L_2^0 = \|\partial_{xx}v^0\|_0 = 0$ . By standard arguments this implies that

$$L_2^n \leq \frac{1}{2} \left[ L_2^{n-1} + \rho_1 - \rho + 1 \right] \quad \text{and thus} \quad L_2^n \leq C(\rho_1 - \rho + 1) \leq C\rho. \quad (4.4)$$

**Step 2.** We now prove the desired convergence. First, by the monotonicity and boundedness of  $v^n$ , there exists bounded  $v^*$  such that  $v^n \uparrow v^*$ . By (4.2)  $\{v^n\}_{n \geq 1}$  are equicontinuous, then the above convergence is uniform on compacts. Next, by (4.2) and (4.4) we see that  $\{\partial_x v^n\}_{n \geq 1}$  are bounded and equicontinuous. Then by applying Arzella-Ascoli Theorem there exist a subsequence  $\{n_k\}_{k \geq 1}$  such that  $\partial_x v^{n_k}$  converge uniformly on compacts. Note that the derivative operator is a closed operator, and since  $v^n \rightarrow v^*$ , then we must have  $\partial_x v^{n_k} \rightarrow \partial_x v^*$ . This implies that the limit of the subsequence  $\{\partial_x v^{n_k}\}_{k \geq 1}$  is unique, then we must have the convergence of the whole sequence  $\partial_x v^n$ , namely  $\partial_x v^n \rightarrow \partial_x v^*$  uniformly on compacts. In particular, this implies that

$$f^n \rightarrow f^* \text{ uniformly on compacts, where } f^*(x) := H(x, \partial_x v^*).$$

Moreover, by (4.4) it is clear that  $\partial_x v^*$  and hence  $f^*$  are uniformly Lipschitz continuous.

Note that  $v^n$  is the classical solution of the following PDE:

$$\rho v^n = \frac{1}{2} \sigma \sigma^\top : \partial_{xx} v^n + f^n.$$

Let  $\tilde{v}$  denote the unique viscosity solution of the PDE:

$$\rho \tilde{v} = \frac{1}{2} \sigma \sigma^\top : \partial_{xx} \tilde{v} + f^*. \quad (4.5)$$

By the stability of the viscosity solution, we see that  $v^* = \lim_{n \rightarrow \infty} v^n$  is a viscosity solution of (4.5). Moreover, since  $\partial_x v^*$  is (Lipschitz) continuous, for any smooth test function  $\phi$  of  $v^*$  at  $x$ , we must have  $\partial_x \phi(x) = \partial_x v^*(x)$ , then it is clear that  $v^*$  is also a viscosity solution of the PDE:

$$\rho \tilde{v} = \frac{1}{2} \sigma \sigma^\top : \partial_{xx} \tilde{v} + H(x, \partial_x \tilde{v}).$$

This PDE identifies with (2.3), then by the uniqueness of its viscosity solution, we obtain  $v^* = v$ . That is,  $(v^n, \partial_x v^n) \rightarrow (v, \partial_x v)$  uniformly on compacts.

It remains to prove the desired convergence of  $\partial_{xx} v^n$ .<sup>5</sup> For this purpose, we shall introduce another representation formula for  $\partial_{xx} v^n$ . Recall (3.4) and denote

$$R_t^x := N_t^x (N_t^x)^\top - \frac{1}{t} \int_0^t D_s N_t^x \sigma^{-1}(X_s^x) \nabla X_s ds + \nabla N_t^x, \quad (4.6)$$

---

<sup>5</sup>This can be done by PDE arguments. In particular, once we have a uniform Hölder continuity of  $\partial_{xx} v^n$ , then it follows from the same compactness argument at above to derive the convergence of  $\partial_{xx} v^n$ . Nevertheless, we provide a probabilistic proof for the convergence directly here.

where  $D_s N_t^x$  is the Malliavin derivative, see [9]. Note that, denoting by  $D_s^i N^x$  (resp.  $\nabla_i X^x$ ) the  $i$ -th column of  $D_s N^x$  (resp.  $\nabla X^x$ ),

$$\begin{aligned} D_s^i N_t^x &= \frac{1}{t} \sigma^{-1}(X_s^x) \nabla_i X_s^x + \frac{1}{t} \int_s^t \left( \sum_{j=1}^d \partial_{x_j} \sigma^{-1}(X_l^x) D_s^i X_l^{x,j} \nabla X_s + \sigma^{-1}(X_s^x) D_s^i \nabla X_l \right)^\top dW_l, \\ D_s^i X_t^x &= \sigma^i(X_s^x) + \int_s^t \sum_{j=1}^d \partial_{x_j} \sigma(X_l^x) D_s^i X_l^{x,j} dW_l, \\ D_s^i \nabla X_t^x &= \partial_x \sigma^i(X_s^x) \nabla X_s^x + \sum_{j=1}^d \int_s^t \left[ \sum_{k=1}^d \partial_{x_k x} \sigma^j(X_l^x) D_s^i X_l^{x,k} \nabla X_l^x + \partial_x \sigma^j(X_l^x) D_s^i \nabla X_l^x \right] dW_l^i. \end{aligned}$$

Fix  $s$ , and consider  $D_s^i X^x, D_s^i \nabla X^x$  as the solution to the above linear SDE systems for  $t \in [s, \infty)$ . One can easily check that

$$\mathbb{E}[|D_s N_t^x|^4] \leq \frac{C}{t^4} e^{Ct}, \quad \text{and thus} \quad \mathbb{E}[|R_t^x|^2] \leq \frac{C}{t^2} e^{Ct}. \quad (4.7)$$

Then, for any  $\phi \in C_b^0(\mathbb{R}^d; \mathbb{R})$ , by [16, Chapter 2] we have<sup>6</sup>

$$\partial_{xx} \mathbb{E}[\phi(X_t^x)] = \mathbb{E}[\phi(X_t^x) R_t^x], \quad (4.8)$$

Thus, for any  $\delta > 0$  small, we may rewrite (4.3) as

$$\partial_{xx} v^n(x) = \mathbb{E} \left[ \int_0^\delta e^{-\rho_1 t} [N_t^x (\nabla X_t^x \partial_x \tilde{f}^n(X_t^x))^\top + \tilde{f}^n(X_t^x) \nabla N_t^x] dt + \int_\delta^\infty e^{-\rho_1 t} \tilde{f}^n(X_t^x) R_t^x dt \right].$$

We remark that  $\mathbb{E}[|\tilde{f}^n(X_t^x) R_t^x|] \leq \frac{C}{t} e^{Ct}$  which is not integrable around  $t = 0$ , so at above we use different representations for small  $t$  and large  $t$ . Similarly we have

$$\begin{aligned} \partial_{xx} v(x) &= \mathbb{E} \left[ \int_0^\delta e^{-\rho_1 t} [N_t^x (\nabla X_t^x \partial_x \tilde{f}(X_t^x))^\top + \tilde{f}(X_t^x) \nabla N_t^x] dt + \int_\delta^\infty e^{-\rho_1 t} \tilde{f}(X_t^x) R_t^x dt \right]; \\ &\quad \text{where} \quad \tilde{f}(x) = H(x, \partial_x v) + (\rho_1 - \rho)v. \end{aligned}$$

Now fix a compact set  $K \subset \subset \mathbb{R}^d$ , and let  $M_0 > 0$  be such that  $|x| \leq M_0$  for all  $x \in K$ . Denote, for all  $M > M_0$ ,

$$\rho_M^n := \sup_{|x| \leq M} \left[ |(v^n - v)(x)| + |\partial_x (v^n - v)(x)| \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

---

<sup>6</sup>When  $d = 1$  and  $\sigma \equiv 1$ , we have  $\nabla X_t^x = 1$ ,  $N_t^x = \frac{W_t}{t}$ ,  $D_s N_t^x = \frac{1}{t}$ ,  $\nabla N_t^x = 0$ , then  $R_t^x = \frac{W_t^2 - t}{t^2}$ , and thus

$$\partial_{xx} \phi(X_t^x) = \partial_{xx} \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy = \int_{\mathbb{R}} \phi(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \frac{(y-x)^2 - t}{t^2} dy = \mathbb{E}[\phi(X_t^x) R_t^x].$$

Then one can easily see that, for any  $x \in K$ ,

$$\begin{aligned}
& |\partial_{xx}(v^n - v)(x)| \\
& \leq \mathbb{E} \left[ \int_0^\delta e^{-\rho_1 t} \left[ (|\partial_x \tilde{f}^n(X_t^x)| + |\partial_x \tilde{f}^n(X_t^x)|) |\nabla X_t^x| |N_t^x| + (|\tilde{f}^n(X_t^x)| + |\tilde{f}(X_t^x)|) |\nabla N_t^x| \right] dt \right. \\
& \quad \left. + \int_\delta^\infty e^{-\rho_1 t} |(\tilde{f}^n - \tilde{f})(X_t^x)| |R_t^x| dt \right] \\
& \leq C_\rho \int_0^\delta e^{-\rho_1 t} \mathbb{E} [|\nabla X_t^x| |N_t^x| + |\nabla N_t^x|] dt + \int_\delta^\infty e^{-\rho_1 t} \mathbb{E} \left[ (\rho_M^n + C_\rho \mathbf{1}_{\{|X_t^x| \geq M\}}) |R_t^x| \right] dt \\
& \leq C_\rho \sqrt{\delta} + C_\rho \rho_M^n \int_\delta^\infty \frac{1}{t} e^{-\rho_1 t + Ct} dt + \frac{C_\rho}{M} \mathbb{E} \left[ \int_\delta^\infty e^{-\rho_1 t} |X_t^x| |R_t^x| dt \right] \\
& \leq C_\rho \sqrt{\delta} + \rho_M^n C_\rho \ln \frac{1}{\delta} + \frac{C_\rho}{M} (1 + |x|) \mathbb{E} \left[ \int_\delta^\infty \frac{1}{t} e^{-\rho_1 t + Ct} dt \right] \\
& \leq C_\rho \sqrt{\delta} + \rho_M^n C_\rho \ln \frac{1}{\delta} + \frac{C_\rho}{M} (1 + |M_0|) \ln \frac{1}{\delta}.
\end{aligned}$$

Thus

$$\sup_{x \in K} |\partial_{xx}(v^n - v)(x)| \leq C_\rho \sqrt{\delta} + \rho_M^n C_\rho \ln \frac{1}{\delta} + \frac{C_\rho}{M} (1 + |M_0|) \ln \frac{1}{\delta}.$$

Fix  $M$ ,  $\delta$  and send  $n \rightarrow \infty$ , we obtain

$$\overline{\lim}_{n \rightarrow \infty} \sup_{x \in K} |\partial_{xx}(v^n - v)(x)| \leq C_\rho \sqrt{\delta} + \frac{C_\rho}{M} (1 + |M_0|) \ln \frac{1}{\delta}.$$

By first sending  $M \rightarrow \infty$  and then  $\delta \rightarrow 0$ , we obtain the desired estimate:

$$\overline{\lim}_{n \rightarrow \infty} \sup_{x \in K} |\partial_{xx}(v^n - v)(x)| = 0. \quad \blacksquare$$

**Remark 4.1.** When  $d = 1$ , the uniform estimate for  $\|\partial_{xx} v^n\|_0$  in the end of Step 1 and the convergence of  $\partial_{xx} v^n$  in the end of Step 2 become trivial. Indeed, in this case we have

$$\partial_{xx} v^n = \frac{2}{\sigma^2(x)} \left[ \rho v^n - \partial_x H(x, \partial_x v^{n-1}) (\partial_x v^n - \partial_x v^{n-1}) - H(x, \partial_x v^{n-1}) \right].$$

Then the desired boundedness and convergence of  $\partial_{xx} v^n$  follow directly from those of  $\partial_x v^n$ .

**Remark 4.2.** The convergence in this case relies heavily on the fact that  $v^n$  is monotone and hence converging. When  $\rho$  is small, in general Picard iteration may not converge, not to mention with certain rate of convergence, as we see in the following example.

**Example 4.3.** Let  $d = 1$ . Consider the following (linear) PDE with unique bounded classical solution  $v \equiv 0$ :

$$\rho v = \frac{1}{2} \partial_{xx} v + \partial_x v.$$

Set  $v^0(x) := -\cos x$ , and define  $v^n$  recursively by Picard iteration:

$$\rho v^n = \frac{1}{2} \partial_{xx} v^n + \partial_x v^{n-1}.$$

Then

$$\partial_x v^n(0) = \begin{cases} 0, & n \text{ is even;} \\ \frac{(-1)^{\frac{n-1}{2}}}{(\rho + \frac{1}{2})^n}, & n \text{ is odd.} \end{cases} \quad (4.9)$$

In particular,  $|\partial_x v^{2m+1}(0)| \rightarrow \infty$  as  $m \rightarrow \infty$ , whenever  $\rho < \frac{1}{2}$ .

**Proof** By (3.5), we have

$$\begin{aligned} \partial_x v^n(x) &= \mathbb{E} \left[ \int_0^\infty e^{-\rho t_1} \partial_x v^{n-1}(x + W_{t_1}) \frac{W_{t_1}}{t_1} dt_1 \right], \\ \partial_x v^{n-1}(x + W_{t_1}) &= \mathbb{E} \left[ \int_0^\infty e^{-\rho t_2} \partial_x v^{n-2}(x + W_{t_2}) \frac{W_{t_1+t_2} - W_{t_1}}{t} dt_2 \middle| \mathcal{F}_{W_{t_1}} \right]. \end{aligned}$$

Plug the second into the first, we get

$$\partial_x v^n(x) = \mathbb{E} \left[ \int_{\mathbb{R}_+^2} e^{-\rho(t_1+t_2)} \partial_x v^{n-2}(x + W_{t_1+t_2}) \frac{W_{t_1}}{t_1} \frac{W_{t_1+t_2} - W_{t_1}}{t} dt_2 dt_1 \right].$$

Repeat the arguments and note that  $\partial_x v^0(x) = \sin x$ , we obtain

$$\partial_x v^n(x) = \mathbb{E} \left[ \int_{\mathbb{R}_+^n} e^{-\rho T_n} \sin(x + W_{T_n}) \frac{W_{t_1}}{t_1} \dots \frac{W_{T_n} - W_{T_{n-1}}}{t_n} dt_n \dots dt_1 \right],$$

where  $T_i := t_1 + \dots + t_i$ . Note that  $E[\cos W_t \frac{W_t}{t}] = 0$ , and

$$\mathbb{E}[\sin W_t \frac{W_t}{t}] = \sum_{k=0}^{\infty} (-1)^k \mathbb{E} \left[ \frac{W_t^{2k+1}}{(2k+1)!} \frac{W_t}{t} \right] = \sum_{k=0}^{\infty} (-t)^k \frac{(2k+1)!!}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-t)^k}{2^k k!} = e^{-\frac{t}{2}}.$$

Let  $\text{Im}$  denote the imaginary part of a complex number. Then

$$\begin{aligned} \partial_x v^n(0) &= \text{Im} \mathbb{E} \left[ \int_{\mathbb{R}_+^n} e^{-\rho T_n} e^{\sqrt{-1} W_{T_n}} \frac{W_{t_1}}{t_1} \dots \frac{W_{T_n} - W_{T_{n-1}}}{t_n} dt_n \dots dt_1 \right] \\ &= \text{Im} \int_{\mathbb{R}_+^n} \prod_{i=1}^n e^{-\rho t_i} \mathbb{E} \left[ e^{\sqrt{-1}(W_{T_i} - W_{T_{i-1}})} \frac{W_{T_i} - W_{T_{i-1}}}{t_i} \right] dt_n \dots dt_1 \\ &= \text{Im} \int_{\mathbb{R}_+^n} \prod_{i=1}^n e^{-\rho t_i} \mathbb{E} \left[ e^{\sqrt{-1} W_{t_i}} \frac{W_{t_i}}{t_i} \right] dt_n \dots dt_1 = \text{Im} \left( \int_0^\infty e^{-\rho t} \mathbb{E} \left[ e^{\sqrt{-1} W_t} \frac{W_t}{t} \right] dt \right)^n \\ &= \text{Im} \left( \int_0^\infty e^{-\rho t} \sqrt{-1} e^{-\frac{t}{2}} dt \right)^n = \text{Im} \left( \frac{\sqrt{-1}}{\rho + \frac{1}{2}} \right)^n. \end{aligned}$$

This implies (4.9) immediately. ■

## 5 The finite horizon case

In this section we consider the problem on a finite time horizon  $[0, T]$ . In this case we may allow  $b, \sigma, r$  to depend on  $t$ , and all the results remain valid with the same arguments. However, in order not to complicate and confuse the notations, we still consider time homogeneous coefficients. Moreover, in this case we allow a terminal condition  $g : \mathbb{R}^d \rightarrow \mathbb{R}$ .

**Assumption 5.1.**  $b, \sigma, r$  satisfy Assumption 2.1,<sup>7</sup> and  $g \in C_b^2(\mathbb{R}^d; \mathbb{R})$  with  $\|g\|_2 \leq C_0$ .

Throughout this section, we let  $C > 0$  denote a generic constant depending only on  $d, \lambda, |A|$ , and  $C_0$ , but not on  $T$ . When the constant depends further on  $T$ , we denote it as  $C_T$ .

Recall (1.1), (1.2), and let  $\mathcal{A}_T$  denote the set of  $\pi : [0, T] \times \mathbb{R}^d \rightarrow \mathcal{P}_0(A)$ . Fix  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Our entropy-regularized exploratory optimal control problem in this case is:

$$\begin{aligned} X_s^{t,x,\pi} &= x + \int_t^s \tilde{b}(X_l^{t,x,\pi}, \pi(l, X_l^{t,x,\pi})) dl + \int_0^t \sigma(X_l^{t,x,\pi}) dW_s; \\ J(t, x, \pi) &:= \mathbb{E} \left[ g(X_T^{t,x,\alpha}) + \int_t^T [\tilde{r}(X_s^{t,x,\pi}, \pi(s, X_s^{t,x,\pi})) - \lambda \mathcal{H}(\pi(s, X_s^\pi))] ds \right]; \\ u(t, x) &:= \sup_{\pi \in \mathcal{A}_T} J(t, x, \pi). \end{aligned} \quad (5.1)$$

Clearly, for the same  $H, \Gamma$  as before,  $u$  satisfies the following parabolic HJB equation:

$$\partial_t u + \frac{1}{2} \sigma \sigma^\top : \partial_{xx} u + H(x, \partial_x u) = 0, \quad u(T, x) = g(x). \quad (5.2)$$

The following simple result is rather standard.

**Lemma 5.2.** *Under Assumption 5.1,  $u$  is the unique bounded classical solution of (5.2), with  $\|u\|_{1,2} \leq C e^{CT}$ .*

We now introduce the *Policy Improvement Algorithm* (PIA) for solving (5.2) recursively:

*Step 0.* Set  $u^0(t, x) := -C_0 + \int_t^T [-C_0 - \lambda \inf_{\pi \in \mathcal{P}_0(A)} \mathcal{H}(\pi)] dt = -C_0 - [C_0 - \lambda (\ln |A|)^+] (T - t)$ ;

*Step n.* For  $n \geq 1$ , define  $\pi^n(t, x, a) := \Gamma(x, \partial_x u^{n-1}(t, x), a)$  and  $u^n(t, x) := J(t, x, \pi^n)$ .

Then it is clear that  $u^n$  satisfies the following recursive linear PDE:

$$\begin{aligned} \partial_t u^n + \frac{1}{2} \sigma \sigma^\top : \partial_{xx} u^n + \partial_z H(x, \partial_x u^{n-1}) \cdot (\partial_x u^n - \partial_x u^{n-1}) + H(x, \partial_x u^{n-1}) &= 0, \\ u^n(T, x) &= g(x). \end{aligned} \quad (5.3)$$

The following result is immediate, as in Proposition 2.3.

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<sup>7</sup>When  $b, \sigma, r$  depend on  $t$ , we require only their continuity in  $t$ .



**Proposition 5.3.** *Let Assumption 5.1 hold. Then*

- (i) For each  $n \geq 1$ ,  $u^n \in C_b^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R})$  is the unique classical solution of (5.3);
- (ii)  $u^n$  is increasing in  $n$  and  $u^n(t, x) \leq C_0 + [C_0 + \lambda(\ln |A|)^+](T - t)$ .

Our main result of this section is as follows. Recall Theorem 2.4 and Remark 2.5.

**Theorem 5.4.** *Under Assumption 5.1, we have*

$$\|u^n - u\|_{1,2} \leq \frac{C_T}{2^n}. \quad (5.4)$$

Consequently, the iterative optimal strategy  $\pi^n(t, x, a) = \Gamma(x, \partial_x u^{n-1}(t, x), a)$  converges to the optimal strategy  $\pi^*(t, x, a) = \Gamma(x, \partial_x u(t, x), a)$  for the entropy-regularized problem (5.1).

**Proof** We proceed in several steps.

**Step 1.** Fix  $(t, x) \in [0, T] \times \mathbb{R}^d$  and denote

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s \sigma(X_l^{t,x}) dW_l; \\ \nabla X_s^{t,x} &= I_d + \sum_{i=1}^d \int_t^s \partial_x \sigma^i(X_l^{t,x}) \nabla X_l^{t,x} dW_l^i, \\ N_s^{t,x} &:= \frac{1}{s-t} \int_t^s (\sigma^{-1}(X_l^{t,x}) \nabla X_l^{t,x})^\top dW_l; \\ \nabla_j^2 X_s^{t,x} &= \sum_{i=1}^d \int_t^s \left[ \sum_{k=1}^d \partial_{xx_k} \sigma^i(X_l^{t,x}) (\nabla X_l^{t,x})^{kj} \nabla X_s^{t,x} + \partial_x \sigma^i(X_l^{t,x}) \nabla_j^2 X_s^{t,x} \right] dW_l^i, \\ \nabla_i N_s^{t,x} &:= \frac{1}{s-t} \int_t^s \left( \sum_{j=1}^d (\nabla X_l^{t,x})^{ij} \partial_{x_j} \sigma^{-1}(X_l^{t,x}) \nabla X_l^{t,x} + \sigma^{-1}(X_l^{t,x}) \nabla_i^2 X_l^{t,x} \right)^\top dW_l. \end{aligned} \quad (5.5)$$

One can easily see that

$$\mathbb{E} \left[ |\nabla X_s^{t,x}|^2 + |\nabla^2 X_s^{t,x}|^2 \right] \leq C e^{C(s-t)}, \quad \mathbb{E} \left[ |N_s^{t,x}|^2 + |\nabla N_s^{t,x}|^2 \right] \leq \frac{C}{s-t} e^{C(s-t)}. \quad (5.6)$$

Denote

$$\begin{aligned} f^n(t, x) &:= \partial_z H(x, \partial_x u^{n-1}(t, x)) \cdot (\partial_x u^n - \partial_x u^{n-1})(t, x) + H(x, \partial_x u^{n-1}(t, x)), \\ f(t, x) &:= H(x, \partial_x u(t, x)). \end{aligned} \quad (5.7)$$

First, by standard Feynman-Kac formula we have

$$u^n(t, x) = \mathbb{E} \left[ g(X_T^{t,x}) + \int_t^T f^n(s, X_s^{t,x}) ds \right], \quad u(t, x) = \mathbb{E} \left[ g(X_T^{t,x}) + \int_t^T f(s, X_s^{t,x}) ds \right]. \quad (5.8)$$

Next, recall the representation formulae for derivatives, see [8],

$$\begin{aligned} \partial_x \mathbb{E}[\phi(X_s^{t,x})] &= \mathbb{E} \left[ \partial_x \phi(X_s^{t,x}) \nabla X_s^{t,x} \right] = \mathbb{E} \left[ \phi(X_s^{t,x}) N_s^{t,x} \right], \\ \partial_{xx}^2 \mathbb{E}[\phi(X_s^{t,x})] &= \mathbb{E} \left[ \partial_{xx} \phi(X_s^{t,x}) (\nabla X_s^{t,x})^2 + \partial_x \phi(X_s^{t,x}) \nabla^2 X_s^{t,x} \right] \\ &= \mathbb{E} \left[ (\partial_x \phi(X_s^{t,x}))^\top \nabla X_s^{t,x} N_s^{t,x} + \phi(X_s^{t,x}) \nabla N_s^{t,x} \right]. \end{aligned} \quad (5.9)$$

Then we have the representation formulae for the first order derivatives

$$\begin{aligned}\partial_x u^n(t, x) &= \mathbb{E} \left[ \partial_x g(X_T^{t,x}) \nabla X_T^{t,x} + \int_t^T f^n(s, X_s^{t,x}) N_s^{t,x} ds \right]; \\ \partial_x u(t, x) &= \mathbb{E} \left[ \partial_x g(X_T^{t,x}) \nabla X_T^{t,x} + \int_t^T f(s, X_s^{t,x}) N_s^{t,x} ds \right];\end{aligned}\tag{5.10}$$

and that for the second order derivatives

$$\begin{aligned}\partial_{xx} u^n(t, x) &= \mathbb{E} \left[ \partial_{xx} g(X_T^{t,x}) (\nabla X_T^{t,x})^2 + \partial_x g(X_T^{t,x}) \nabla^2 X_T^{t,x} \right. \\ &\quad \left. + \int_t^T [N_s^{t,x} (\nabla X_s^{t,x} \partial_x f^n(s, X_s^{t,x}))^\top + f^n(s, X_s^{t,x}) \nabla N_s^{t,x}] ds \right], \\ \partial_{xx} u(t, x) &= \mathbb{E} \left[ \partial_{xx} g(X_T^{t,x}) (\nabla X_T^{t,x})^2 + \partial_x g(X_T^{t,x}) \nabla^2 X_T^{t,x} \right. \\ &\quad \left. + \int_t^T [N_s^{t,x} (\nabla X_s^{t,x} \partial_x f(s, X_s^{t,x}))^\top + f(s, X_s^{t,x}) \nabla N_s^{t,x}] ds \right].\end{aligned}\tag{5.11}$$

**Step 2.** In this step we assume  $T \leq \delta$  for  $\delta > 0$  which will be specified later, and estimate  $\varepsilon_1^n := \|\partial_x(u^n - u)\|_0 < \infty$ . By the same arguments as in (3.8) we have

$$|(f^n - f)(t, x)| \leq C(\varepsilon_1^n + \varepsilon_1^{n-1}).\tag{5.12}$$

Now for any  $n \geq 1$  and  $(t, x) \in [0, T] \times \mathbb{R}^d$ , by (5.10) we have

$$\begin{aligned}|\partial_x(u^n - u)(t, x)| &\leq \mathbb{E} \left[ |\partial_x(u^n - u)(T, X_T^{t,x})| |\nabla X_T^{t,x}| + \int_t^T |(f^n - f)(s, X_s^{t,x})| |N_s^{t,x}| ds \right] \\ &\leq C \|\partial_x(u^n - u)(T, \cdot)\|_0 \mathbb{E}[|\nabla X_T^{t,x}|] + C(\varepsilon_1^n + \varepsilon_1^{n-1}) \int_t^T \mathbb{E}[|N_s^{t,x}|] ds.\end{aligned}$$

We remark that here obviously  $\partial_x(u^n - u)(T, \cdot) = 0$ , however, for later purpose we would like to keep this term. Then, for a constant  $C_1 > 0$  independent of  $T$ , by (5.6) we have

$$|\partial_x(u^n - u)(t, x)| \leq C_1 e^{C_1(T-t)} \left[ \|\partial_x(u^n - u)(T, \cdot)\|_0 + (\varepsilon_1^n + \varepsilon_1^{n-1}) \sqrt{T-t} \right].$$

Since  $x$  is arbitrary, we obtain

$$\varepsilon_1^n \leq C_1 e^{C_1(T-t)} \left[ \|\partial_x(u^n - u)(T, \cdot)\|_0 + (\varepsilon_1^n + \varepsilon_1^{n-1}) \sqrt{T-t} \right].$$

We now set  $\delta > 0$  small such that

$$C_1 e^{C_1 \delta} \sqrt{\delta} \leq \frac{1}{3}.\tag{5.13}$$

Then, for  $T \leq \delta$ ,

$$\varepsilon_1^n \leq \frac{1}{3}(\varepsilon_1^n + \varepsilon_1^{n-1}) + C \|\partial_x(u^n - u)(T, \cdot)\|_0, \text{ and thus } \varepsilon_1^n \leq \frac{1}{2} \varepsilon_1^{n-1} + C \|\partial_x(u^n - u)(T, \cdot)\|_0.$$

Note that  $\partial_x u^0 \equiv 0$ , then it follows from Lemma 5.2 that  $\varepsilon_1^0 = \|\partial_x u\|_0 \leq C e^{C\delta} \leq C$ . Note further that  $\partial_x(u^n - u)(T, \cdot) = 0$ . Then

$$\varepsilon_1^n \leq \frac{\varepsilon_1^0}{2^n} + C \|\partial_x(u^n - u)(T, \cdot)\|_0 \leq \frac{C}{2^n} + C \|\partial_x(u^n - u)(T, \cdot)\|_0 = \frac{C}{2^n}. \quad (5.14)$$

**Step 3.** We next estimate  $\varepsilon_1^n$  for general  $T$ . First, let  $0 = t_0 < \dots < t_m = T$  be such that  $t_i - t_{i-1} \leq \delta$ , where  $\delta$  satisfies (5.13) and is independent of  $T$ . For each  $i$ , apply the arguments in Step 2 on  $[t_{i-1}, t_i]$ , then the first inequality in (5.14) leads to

$$\sup_{(t,x) \in [t_{i-1}, t_i] \times \mathbb{R}^d} |\partial_x(u^n - u)(t, x)| \leq \frac{C_T}{2^n} + C \sup_{x \in \mathbb{R}^d} |\partial_x(u^n - u)(t_i, x)|.$$

Note that  $\sup_{x \in \mathbb{R}^d} |\partial_x(u^n - u)(t_m, x)| = 0$ . Then by backward induction on  $i = m, \dots, 1$ , we obtain immediately that

$$\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} |\partial_x(u^n - u)(t, x)| \leq \frac{C_T}{2^n}. \quad (5.15)$$

Moreover, by (5.8), (5.12), and (5.14), the above leads to

$$|(u^n - u)(t, x)| \leq \mathbb{E} \left[ \int_t^T |(f^n - f)(s, X_s^{t,x})| ds \right] \leq (T - t) \|f^n - f\|_0 \leq \frac{C_T}{2^n}.$$

This implies

$$\|u^n - u\|_0 \leq \frac{C_T}{2^n}. \quad (5.16)$$

**Step 4.** We now assume  $T \leq \delta'$  for  $\delta' > 0$  which will be specified later, and estimate  $\varepsilon_2^n := \|\partial_{xx}(u^n - u)\|_0 < \infty$ . By (5.11) we have

$$\begin{aligned} |\partial_{xx}(u^n - u)(t, x)| &\leq \mathbb{E} \left[ \|\partial_{xx}(u^n - u)(T, \cdot)\|_0 |\nabla X_T^{t,x}|^2 + \|\partial_x(u^n - u)(T, \cdot)\|_0 |\nabla^2 X_T^{t,x}| \right. \\ &\quad \left. + \int_t^T [N_s^{t,x} (\nabla X_s^{t,x} \partial_x(f^n - f)(s, X_s^{t,x}))^\top + (f^n - f)(s, X_s^{t,x}) \nabla N_s^{t,x}] ds \right]. \end{aligned} \quad (5.17)$$

Here again we keep the terminal difference term for later purpose. Recall (5.12), and following essentially the same arguments as in (3.13) we have

$$\left| \partial_x(f^n - f)(t, x) \right| \leq C[1 + \varepsilon_1^n + \varepsilon_1^{n-1}][\varepsilon_2^n + \varepsilon_2^{n-1}] + C[\varepsilon_1^n + \varepsilon_1^{n-1}].$$

Thus, by (5.14), it follows from (5.17) that

$$\begin{aligned} |\partial_{xx}(u^n - u)(t, x)| &\leq C e^{C\delta'} \left[ \|\partial_{xx}(u^n - u)(T, \cdot)\|_0 + \|\partial_x(u^n - u)(T, \cdot)\|_0 \right] \\ &\quad + C_1 e^{C_1 \delta'} \sqrt{\delta'} [1 + \varepsilon_1^n + \varepsilon_1^{n-1}][\varepsilon_2^n + \varepsilon_2^{n-1}] + C[\varepsilon_1^n + \varepsilon_1^{n-1}]. \end{aligned}$$

Since  $x$  is arbitrary, we obtain

$$\begin{aligned} \varepsilon_2^n &\leq C e^{C\delta'} \left[ \|\partial_{xx}(u^n - u)(T, \cdot)\|_0 + \|\partial_x(u^n - u)(T, \cdot)\|_0 \right] \\ &\quad + C_1 e^{C_1\delta'} \sqrt{\delta'} [1 + \varepsilon_1^n + \varepsilon_1^{n-1}] [\varepsilon_2^n + \varepsilon_2^{n-1}] + C [\varepsilon_1^n + \varepsilon_1^{n-1}]. \end{aligned} \quad (5.18)$$

**Step 5.** Finally we estimate  $\varepsilon_2^n$  for arbitrary  $T$ . Let  $0 = t'_0 < \dots < t'_{m'} = T$  be another partition such that  $t'_i - t'_{i-1} \leq \delta'$ , where, for the  $C_1$  in (5.18) and  $C_T$  in (5.15),

$$C_1 e^{C_1\delta'} \sqrt{\delta'} \left[ 1 + \frac{C_T}{2^n} + \frac{C_T}{2^{n-1}} \right] \leq \frac{1}{3}.$$

We remark that here we allow  $\delta'$  to depend on  $T$ . For each  $i$ , apply the arguments in Step 4 on  $[t'_{i-1}, t'_i]$ , then by (5.18) and (5.15) we have

$$\begin{aligned} \sup_{(t,x) \in [t'_{i-1}, t'_i] \times \mathbb{R}^d} |\partial_{xx}(u^n - u)(t, x)| &\leq C e^{C\delta'} \sup_{x \in \mathbb{R}^d} |\partial_{xx}(u^n - u)(t'_i, x)| \\ &\quad + \frac{1}{3} \left[ \sup_{(t,x) \in [t'_{i-1}, t'_i] \times \mathbb{R}^d} |\partial_{xx}(u^n - u)(t, x)| + \sup_{(t,x) \in [t'_{i-1}, t'_i] \times \mathbb{R}^d} |\partial_{xx}(u^{n-1} - u)(t, x)| \right] + \frac{C_T}{2^n}. \end{aligned}$$

By standard arguments, this leads to

$$\sup_{(t,x) \in [t'_{i-1}, t'_i] \times \mathbb{R}^d} |\partial_{xx}(u^n - u)(t, x)| \leq C e^{C\delta'} \sup_{x \in \mathbb{R}^d} |\partial_{xx}(u^n - u)(t'_i, x)| + \frac{C_T}{2^n}.$$

Note that  $\sup_{x \in \mathbb{R}^d} |\partial_{xx}(u^n - u)(t'_{m'}, x)| = 0$ . Then by backward induction on  $i = m', \dots, 1$ , we obtain immediately that

$$\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} |\partial_{xx}(u^n - u)(t, x)| \leq \frac{C_T}{2^n}. \quad (5.19)$$

Finally, by the PDEs (5.2) and (5.3), we obtain from (5.16), (5.15), and (5.19) the desired estimate for  $\|\partial_t(u^n - u)\|_0$ , and thus prove (5.4).  $\blacksquare$

## 6 The scalar case with diffusion control

We now investigate the infinite horizon model with diffusion control, corresponding to fully nonlinear PDEs. The general case is much more challenging than the drift control case and we shall leave to future research. In this section we consider only a special case:

$$d = 1. \quad (6.1)$$

Consider the setting in §2, our entropy-regularized problem is:

$$\begin{aligned} X_t^\pi &= x + \int_0^t \tilde{b}(X_s^\pi, \pi(s, X_s^\pi)) ds + \int_0^t \sqrt{\tilde{\sigma}^2(X_s^\pi, \pi(s, X_s^\pi))} dW_s; \\ v(x) &:= \sup_{\pi \in \mathcal{A}_{Lip}} J(x, \pi), \quad J(x, \pi) := \mathbb{E} \left[ \int_0^\infty e^{-\rho t} [\tilde{r}(X_t^\pi, \pi(s, X_s^\pi)) - \lambda \mathcal{H}(\pi(t, X_t^\pi))] dt \right], \end{aligned} \quad (6.2)$$

For simplicity, we restrict to Lipschitz (in  $x$ ) continuous  $\pi$  so that the  $X^\pi$  above has a unique strong solution. In this section, we shall assume:

**Assumption 6.1.**  $d = 1$ ;  $b, \sigma, r$  satisfy Assumption 2.1, with  $\sigma$  depending on  $a$ ; and  $b, \sigma$  are uniformly continuous in  $a$ , uniformly in  $x$ .

In this case,  $v$  satisfies the following fully nonlinear HJB equation:

$$\begin{aligned} \rho v &= H(x, \partial_x v, \partial_{xx} v), \\ \text{where } H(x, z, q) &:= \sup_{\pi \in \mathcal{P}_0(A)} \left[ \frac{1}{2} \widetilde{\sigma}^2(x, \pi) q + \tilde{b}(x, \pi) z + \tilde{r}(x, \pi) - \lambda \mathcal{H}(\pi) \right]. \end{aligned} \quad (6.3)$$

Moreover, the Hamiltonian  $H$  has optimal relaxed control  $\Gamma$  in Gibbs form:

$$\begin{aligned} \Gamma(x, z, q, a) &:= \frac{\gamma(x, z, q, a)}{\int_A \gamma(x, z, q, a') da'}, \\ \text{where } \gamma(x, z, q, a) &:= \exp \left( \frac{1}{\lambda} \left[ \frac{1}{2} \sigma^2(x, a) q + b(x, a) z + r(x, a) \right] \right), \\ \text{and thus } H(x, z, q) &= \lambda \ln \left( \int_A \gamma(x, z, q, a) da \right). \end{aligned} \quad (6.4)$$

Then we have the following simple result.

**Lemma 6.2.** *Let Assumption 6.1 hold. Then  $H$  is twice continuously differentiable in  $(x, z, q)$ ; jointly convex in  $(z, q)$ ; and there exists a constant  $C > 0$  such that*

$$|\partial_z H|, |\partial_q H|, |\partial_{zz} H|, |\partial_{zq} H|, |\partial_{qq} H| \leq C, \quad \partial_q H \geq \frac{1}{C}, \quad |H(x, z, q)| \leq C[1 + |z| + |q|]. \quad (6.5)$$

The following observation is crucial for our analysis.

**Proposition 6.3.** *Let Assumption 6.1 hold. For any  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that<sup>8</sup>*

$$|H(x, z, q) - \partial_q H(x, z, q)q| \leq \varepsilon |q| + C|z| + C_\varepsilon. \quad (6.7)$$

**Proof** We prove the result only for  $q > 0$ . The case  $q < 0$  can be proved similarly<sup>9</sup>.

First, note that

$$H(x, z, q) - \partial_q H(x, z, q)q = H(x, z, 0) + q \int_0^1 [\partial_q H(x, z, \theta q) - \partial_q H(x, z, q)] d\theta.$$

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<sup>8</sup>We conjecture the following stronger result:

$$|H(x, z, q) - \partial_q H(x, z, q)q| \leq C[1 + |z| + \ln(1 + |q|)]. \quad (6.6)$$

<sup>9</sup>Actually this case is not needed, because later on we can easily show that  $\partial_{xx} v^n \geq -C$  for all  $n$

Since  $H$  is convex in  $q$ , we have  $\partial_q H(x, z, \theta q) \leq \partial_q H(x, z, q)$ , and thus

$$H(x, z, q) - \partial_q H(x, z, q)q \leq H(x, z, 0) \leq C[1 + |z|]. \quad (6.8)$$

To see the opposite inequality, denote  $\theta(x, z, q, a) := \frac{1}{2}\sigma^2(x, a) + b(x, a)\frac{z}{q}$ . When there is no confusion, we omit the variables  $(x, z, q)$  in  $\theta$  and  $\gamma$ . Then

$$\partial_z H(x, z, q)z + \partial_q H(x, z, q)q - H(x, z, q) = \frac{q \int_A \theta(a)\gamma(a)da}{\int_A \gamma(a)da} - \lambda \ln \left( \int_A \gamma(a)da \right). \quad (6.9)$$

Denote  $C_1 := \sup_{a \in A, x \in \mathbb{R}} \frac{2|b(x, a)|}{|\sigma^2(x, a)|} < \infty$ . When  $q \leq 2C_1|z|$ , the result is obviously true. We now assume  $q \geq 2C_1|z|$ , which implies  $\theta(a) \geq \frac{1}{C_2} > 0$  for all  $(x, a)$ . Fix  $(x, q, z)$  and denote

$$\bar{\theta} := \sup_{a \in A} \theta(a) > \frac{1}{C_2}, \quad A_\varepsilon := \{a \in A : \theta(a) \geq \bar{\theta} - \varepsilon\}.$$

Since  $b, \sigma$  are uniformly continuous in  $a$ , there exists  $\mu_\varepsilon > 0$ , depending only on the model parameters, such that  $|A_\varepsilon| \geq \mu_\varepsilon$  for all  $(x, z, q)$  satisfying  $q \geq 2C_1|z|$ . Note that

$$\begin{aligned} \int_A \gamma(a)da &\geq \int_{A_\varepsilon} \gamma(a)da \geq e^{\frac{(\bar{\theta}-\varepsilon)q}{\lambda}-C} \mu_\varepsilon; \\ \frac{\int_{A \setminus A_\varepsilon} \gamma(a)da}{\int_{A_\varepsilon} \gamma(a)da} &\leq \frac{\int_{A \setminus A_\varepsilon} \gamma(a)da}{e^{\frac{(\bar{\theta}-\varepsilon)q}{\lambda}-C} |A \setminus A_\varepsilon|} \leq e^{-\frac{\varepsilon}{2\lambda}q+C} \frac{|A|}{\mu_\varepsilon}; \\ \int_A \theta(a)\gamma(a)da &\leq \bar{\theta} \int_{A_\varepsilon} \gamma(a)da + (\bar{\theta} - \varepsilon) \int_{A \setminus A_\varepsilon} \gamma(a)da \\ &\leq \left[ \bar{\theta} + (\bar{\theta} - \varepsilon)e^{-\frac{\varepsilon}{2\lambda}q+C} \frac{|A|}{\mu_\varepsilon} \right] \int_{A_\varepsilon} \gamma(a)da \end{aligned}$$

Then, by (6.9) and assuming without loss of generality that  $\varepsilon < \frac{1}{C_2}$  so that  $\bar{\theta} - \varepsilon > 0$ ,

$$\begin{aligned} &\partial_z H(x, z, q)z + \partial_q H(x, z, q)q - H(x, z, q) \\ &\leq q \left[ \bar{\theta} + (\bar{\theta} - \varepsilon)e^{-\frac{\varepsilon}{2\lambda}q+C} \frac{|A|}{\mu_\varepsilon} \right] - \lambda \ln \left( e^{\frac{(\bar{\theta}-\varepsilon)q}{\lambda}-C} \mu_\varepsilon \right) \\ &= q \left[ \bar{\theta} + (\bar{\theta} - \varepsilon)e^{-\frac{\varepsilon}{2\lambda}q+C} \frac{|A|}{\mu_\varepsilon} \right] - \left[ (\bar{\theta} - \varepsilon)q - \lambda C + \lambda \ln \mu_\varepsilon \right] \\ &\leq \varepsilon q + C_\varepsilon. \end{aligned}$$

Thus

$$\partial_q H(x, z, q)q - H(x, z, q) \leq \varepsilon q + C_\varepsilon - \partial_z H(x, z, q)z \leq \varepsilon q + C_\varepsilon + C|z|.$$

This, together with (6.8), proves (6.7). ■

For the PIA for solving (6.3), we set  $v^0$  the same as in §2, and for  $n \geq 1$ , define  $\pi^n(x, a) := \Gamma(x, \partial_{xx}v^{n-1}(x), a)$  and  $v^n(x) := J(x, \pi^n)$ . Then it is clear that  $v^n$  satisfies the following recursive linear PDE:

$$\begin{aligned} \rho v^n &= \partial_q H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_{xx} v^n - \partial_{xx} v^{n-1}) \\ &+ \partial_z H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_x v^n - \partial_x v^{n-1}) + H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1}). \end{aligned} \quad (6.10)$$

The following result is similar to Proposition 2.3.

**Proposition 6.4.** *Let Assumption 6.1 (i) hold. Then*

- (i) *For each  $n \geq 1$ ,  $v^n \in C_b^2(\mathbb{R}^d; \mathbb{R})$  is a classical solution of (6.10);*
- (ii)  *$v^n$  is increasing in  $n$  and  $v^n \leq \frac{1}{\rho}[C_0 + \lambda(\ln|A|)^+]$ .*

Our main result is as follows.

**Theorem 6.5.** *Let Assumption 6.1 hold. Then  $v^n \rightarrow v$  in  $C^2$  uniformly on compacts in the sense of (2.8). Consequently,  $\pi^n \rightarrow \Gamma$  as well.*

**Proof** We proceed in several steps. Denote

$$L_1^n := \|\partial_x v^n\|_0, \quad L_2^n := \|\partial_{xx} v^n\|_0, \quad \bar{L}_1^n := \sum_{k=1}^n \frac{L_1^k}{3^{n-k+1}}.$$

**Step 1.** First, since  $d = 1$ , by (6.10) we main isolate  $\partial_{xx} v^n$ :

$$\begin{aligned} \partial_{xx} v^n &= \frac{1}{\partial_q H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})} \left[ \rho v^n - \partial_z H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_x v^n - \partial_x v^{n-1}) \right. \\ &\quad \left. - (H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1}) - \partial_q H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1}) \partial_{xx} v^{n-1}) \right]. \end{aligned} \quad (6.11)$$

Then, for  $\varepsilon > 0$ , by (6.5) we have

$$|\partial_{xx} v^n(x)| \leq \frac{\varepsilon}{C_1} L_2^{n-1} + C(L_1^n + L_1^{n-1}) + C_{\rho, \varepsilon}.$$

Set  $\varepsilon := \frac{C_1}{3}$ , by the arbitrariness of  $x$  we have

$$L_2^n \leq \frac{1}{3} L_2^{n-1} + C(L_1^n + L_1^{n-1}) + C_\rho.$$

Then it follows from standard arguments that

$$L_2^n \leq C \bar{L}_1^n + C_\rho. \quad (6.12)$$

**Step 2.** Let  $\rho_1$  be a large constant which will be specified later. Rewrite (6.10) as:

$$\begin{aligned} \rho_1 v^n &= \frac{1}{2} \partial_{xx} v^n + f_n(x), \quad \text{where} \\ f_n(x) &:= \partial_q H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_{xx} v^n - \partial_{xx} v^{n-1}) - \frac{1}{2} \partial_{xx} v^n \\ &+ \partial_z H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_x v^n - \partial_x v^{n-1}) + H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1}) + (\rho_1 - \rho)v^n. \end{aligned}$$

Then, denoting  $X_t^x := x + W_t$ , by Remark 3.1 we have

$$\partial_x v^n(x) := \mathbb{E} \left[ \int_0^\infty e^{-\rho_1 t} f_n(X_t^x) \frac{W_t}{t} dt \right].$$

By (6.12) we get

$$\begin{aligned} |\partial_x v^n(x)| &\leq C \left[ L_2^n + L_2^{n-1} + L_1^n + L_1^{n-1} + 1 + C_\rho |\rho_1 - \rho| \right] \int_0^\infty e^{-\rho_1 t} \frac{1}{\sqrt{t}} dt \\ &\leq \frac{C}{\sqrt{\rho_1}} \left[ \bar{L}_1^n + \bar{L}_1^{n-1} + L_1^n + L_1^{n-1} + C_\rho + C_\rho |\rho_1 - \rho| \right] \\ &\leq \frac{C_1}{\sqrt{\rho_1}} \left[ L_1^n + \bar{L}_1^{n-1} \right] + \frac{C_\rho}{\sqrt{\rho_1}} [|\rho_1 - \rho| + 1]. \end{aligned}$$

Setting  $\rho_1 = 16C_1^2$  and by the arbitrariness of  $x$ , we get

$$L_1^n \leq \frac{1}{4}(L_1^n + \bar{L}_1^{n-1}) + C_\rho, \quad \text{and thus} \quad L_1^n \leq \frac{1}{3}\bar{L}_1^{n-1} + C_\rho$$

Note that

$$\bar{L}_1^n = \frac{1}{3}L_1^n + \frac{1}{3}\bar{L}_1^{n-1} \leq \frac{1}{9}\bar{L}_1^{n-1} + \frac{1}{3}\bar{L}_1^{n-1} + C_\rho \leq \frac{1}{2}\bar{L}_1^{n-1} + C_\rho.$$

This, together with (6.12), implies immediately that

$$\bar{L}_1^n \leq C_\rho, \quad \text{and thus} \quad L_1^n \leq C_\rho, \quad L_2^n \leq C_\rho. \quad (6.13)$$

**Step 3.** Follow the arguments in the beginning of Step 2 in §4, we have  $(v^n, \partial_x v^n) \rightarrow (v^*, \partial_x v^*)$  uniformly on compacts for some function  $v^* \in C_b^1(\mathbb{R}; \mathbb{R})$  such that  $\partial_x v^*$  is Lipschitz continuous. Fix an arbitrary  $x_0$ , and denote  $q_* := \underline{\lim}_{n \rightarrow \infty} \partial_{xx} v^n(x_0) = \lim_{k \rightarrow \infty} \partial_{xx} v^{n_k}(x_0)$  for some subsequence  $\{n_k\}_{k \geq 1}$ , which may depend on  $x_0$ . By (6.10) we have, at  $x_0$ ,

$$\begin{aligned} \partial_{xx} v^n &= \partial_{xx} v^{n-1} + \frac{1}{\partial_q H(x_0, \partial_x v^{n-1}, \partial_{xx} v^{n-1})} \times \\ &\quad \left[ \rho v^n - \partial_z H(x_0, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_x v^n - \partial_x v^{n-1}) - H(x_0, \partial_x v^{n-1}, \partial_{xx} v^{n-1}) \right] \\ &\leq \partial_{xx} v^{n-1} + \frac{\rho v^* - H(x_0, \partial_x v^*, \partial_{xx} v^{n-1})}{\partial_q H(x_0, \partial_x v^*, \partial_{xx} v^{n-1})} + C\varepsilon_n, \end{aligned} \quad (6.14)$$

where

$$\begin{aligned} \varepsilon_n &:= \varepsilon'_n + \varepsilon'_{n-1} + \tilde{\varepsilon}_n \rightarrow 0, \quad \text{as } n \rightarrow \infty, \\ \varepsilon'_n &:= |v^n(x_0) - v^*(x_0)| + |\partial_x v^n(x_0) - \partial_x v^*(x_0)|, \quad \tilde{\varepsilon}_n := q_* - \inf_{m \geq n} \partial_{xx} v^{m-1}(x_0) \geq 0. \end{aligned}$$



Since  $H$  is convex in  $(z, q)$ , by (6.10) again we have

$$\begin{aligned} \rho v^n &= H(x_0, \partial_x v^n, \partial_{xx} v^n) - \left[ H(x_0, \partial_x v^n, \partial_{xx} v^n) - H(x_0, \partial_x v^{n-1}, \partial_{xx} v^{n-1}) \right. \\ &\quad \left. - \partial_q H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_{xx} v^n - \partial_{xx} v^{n-1}) - \partial_z H(x, \partial_x v^{n-1}, \partial_{xx} v^{n-1})(\partial_x v^n - \partial_x v^{n-1}) \right] \\ &\leq H(x_0, \partial_x v^n, \partial_{xx} v^n). \end{aligned} \tag{6.15}$$

Set  $n = n_k + 1$  and send  $k \rightarrow \infty$ , we have  $\rho v^* \leq H(x_0, \partial_x v^*, q_*)$ . Then by (6.14) we have

$$\begin{aligned} \partial_{xx} v^n &\leq \partial_{xx} v^{n-1} + \frac{H(x_0, \partial_x v^*, q_*) - H(x_0, \partial_x v^*, \partial_{xx} v^{n-1})}{\partial_q H(x_0, \partial_x v^*, \partial_{xx} v^{n-1})} + C\varepsilon_n \\ &\leq \partial_{xx} v^{n-1} - \frac{1}{C_1}(\partial_{xx} v^{n-1} - \tilde{q}_*) + C_\rho \varepsilon_n, \end{aligned}$$

where the second inequality thanks to the fact that  $q_* \leq \partial_{xx} v^{n-1}(x_0) + \varepsilon_n$ . This implies

$$\partial_{xx} v^n - q_* \leq \left(1 - \frac{1}{C}\right)(\partial_{xx} v^{n-1} - q_*) + C_\rho \varepsilon_n.$$

Then by standard arguments we have  $\overline{\lim}_{n \rightarrow \infty} (\partial_{xx} v^n - q_*) \leq 0$ . This, together with the definition of  $q_*$ , implies the limit  $\lim_{n \rightarrow \infty} \partial_{xx} v^n(x_0) = q_*$  exists. Then it follows from the closeness of the differentiation operator that  $\lim_{n \rightarrow \infty} \partial_{xx} v^n(x) = \partial_{xx} v^*(x)$ . Now send  $n \rightarrow \infty$  in (6.10), we see that  $v^*$  satisfies (6.3). Note further that  $q \mapsto H(x, z, q)$  has an inverse function, then from (6.3) we conclude that  $\partial_{xx} v^*$  is uniformly Lipschitz continuous. Finally, it follows from the uniqueness of classical solutions to (6.3) that  $v^* = v$ .  $\blacksquare$

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