Instability and Efficiency of Non-cooperative Games

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May 28, 2024

Abstract

It is well known that a non-cooperative game may have multiple equilibria. In this paper we consider the efficiency of games, measured by the ratio between the aggregate payoff over all Nash equilibria and that over all admissible controls. Such efficiency operator is typically unstable with respect to small perturbation of the game. This seemingly bad property can actually be a good news in practice: it is possible that a small change of the game mechanism may improve the efficiency of the game dramatically. We shall introduce a game mediator with limited resources and investigate the mechanism designs aiming to improve the efficiency.

Keywords: Non-cooperative games, Nash equilibrium, set value, price of anarchy, price of stability, efficiency, mechanism design

2020 AMS subject classifications: 91A10, 91B03, 91B32

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1 Introduction

Consider the following two player nonzero-sum game as an illustrative example:

$J(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(0, 102)$	$(0, 102)$
$a_1 = 1$	$(102, 0)$	$(1, 2)$	$(0, 0)$
$a_1 = 2$	$(102, 0)$	$(0, 0)$	$(3, 1)$

Table 1: The illustrative example

where each player i aims to maximize his payoff $J_i(a)$ by choosing his control a_i. Clearly this game has two Nash equilibria $(1, 1)$ and $(2, 2)$, with corresponding payoffs $(1, 2)$ and $(3, 1)$, respectively. It is well known that a game is typically inefficient, in this case both equilibria are much worse in terms of the average payoff (or total payoff) than the socially optimal control $(0, 0)$ with corresponding payoffs $(100, 100)$. In fact, here the two equilibria are Pareto dominated by $(0, 0)$, as in the *prisoners' dilemma*. In the literature, such inefficiency is measured either by the *price of anarchy* corresponding to the worst equilibrium $(1, 1)$, see e.g. Koutsoupias-Papadimitriou [7], or by the *price of stability* corresponding to the best equilibrium $(2, 2)$, see e.g. Anshelevich-Dasgupta-et al [1]. A natural and important question in the game theory is:

$$\textit{Can we improve the efficiency of a game?} \tag{1.1}$$

In this paper we shall focus on the price of stability[1], namely on the best equilibrium. We remark that the best equilibrium is automatically Pareto optimal among all equilibria, so at least some players will be happy to implement it. Moreover, in societies where people tend to trust the authority (the government, or even just the media), once the authority recommends the best equilibrium which is good for the society, each individual player may feel the others (or most others) will follow and then it is also for his own best interest to follow that equilibrium. That is, it is relatively easy to implement the best equilibrium.

One simple answer to the question (1.1) is the bounded rationality, see e.g. Magill-Quinzii [8] and Papadimitriou-Yannakakis [12]. That is, the players are satisfied with a good enough decision which is not necessarily the best one. For example, in Table 1, if the players are willing to sacrifice \$1, then $(0, 1)$ and $(2, 0)$ become acceptable approximate equilibria, which effectively increases the average payoff from $\frac{1}{2}[1+3] = 2$ to $\frac{1}{2}[0+102] = 51$.

[1]Both [7] and [1] consider costs, and thus the corresponding prices are greater than 1. Here we consider payoffs, and hence the price of stability or say the efficiency is less than 1.

Furthermore, if the players are willing to accept 2-equilibrium, then $(0,0)$ becomes an acceptable one and thus the game reaches the socially optimal one with average payoff $\frac{1}{2}[100+100] = 100$. However, in reality quite often this is not the case, either due to selfishness or due to distrust among the players, and people indeed get stuck in bad equilibria (thinking of all the internal fights within a society, or even wars among countries). So our goal is to introduce appropriate mechanism design to improve the efficiency of the game.

One seminal paper in this direction is Monderer-Tennenholtz [10] which proposed the κ-implementation.[2] See also Bachrach-Elkind-ect al [2], Deng-Tang-Zheng [3], Huang-Wang-Wei-Zhang [5], Monderer-Tennenholtz [11], Zhang-Farina-et al [15] for some works along this line. In this approach there is a third party called mediator. According to [10], a mediator cannot design a new game, cannot enforce players' behavior, cannot enforce payments by the players, and cannot prohibit strategies available to the players. However, she enjoys the reliability, and she can make non-negative payments to the players, with the total amount limited to $\$\kappa$. The problem is to design the payments in a way such that they would induce the players to implement a desirable outcome, for example the socially optimal one.

In this paper we study the problem in an abstract framework, with the k-implementation as one of the two main mechanisms we consider. Instead of targeting at a desirable outcome which may require a large κ as in [10], we focus on the effect of a small κ, reflecting the reality that quite often the mediator has only limited resources. It is well known that equilibria are very sensitive to small perturbations of the game parameters, and hence the efficiency of the game is overall unstable. This seemingly bad property turns out to be a good news for our purpose: it is possible that a small investment κ could increase the efficiency of the game dramatically. For example, in Table 1, when $\kappa = 1$, the mediator can pay \$1 to Player 1 when the control is $(0,1)$, this will make $(0,1)$ a new equilibrium and hence improve the efficiency significantly.

By considering the efficiency of the game as a function of κ, our main result is that the efficiency function is non-decreasing and right continuous in κ, but in general it can be discontinuous. The right continuity at $\kappa = 0$ implies that, in order to improve the efficiency of the game, an appropriate amount of investment is needed.[3] However, this

[2] [10] used the notation K, we change to κ since we consider small value here.

[3] [10] emphasized that it is possible that no monetary offer is materialized when the players follow the desired behavior. This does not contradict with our result here. In [10], the targeted outcome could already be the best equilibrium, the promised offers just aim to induce the players to implement that one, which do not improve the efficiency. We should also note that, even if the real payment (along the targeted desired outcome) is 0, the promised offers on the other outcomes may not be small, namely κ could be large. Since the reliability of the mediator is crucial, she has to have $\$\kappa$ available when making the promises, even if she

amount is not necessarily large. It is important to understand this efficiency function in practice, especially the discontinuous points are critical. Indeed, if κ is already close to a discontinuous point, it will be wise to add a little more investment because that small amount of extra investment could increase the efficiency significantly. Moreover, considering the case that the mediator has several projects to take care, each involving a game. Since she may have only limited resources, it will be important to understand for which projects the efficiency can be increased dramatically by a small investment, and it will be wise to set a higher priority on those projects. We would also like to note that, in the case that the mediator is the government and the payoffs of the base game are individuals' incomes, the government can charge tax on the incomes, then the improvement of the efficiency implies the government will receive more tax. When this extra tax exceeds κ (again noting that κ is small), it could be possible that all players as well as the government receive more money, and thus the κ-implementation will create a win-win-win situation.

The second main mechanism we investigate is taxation,[4] again by considering the government as the mediator. There are numerous publications on tax policies, see e.g. the survey paper Mankiw-Weinzierl-Yagan [9] and the references therein. Our focus here is not on optimal tax policy or its impact on the society, but rather on how to redistribute the tax the government receives so as to improve the efficiency of the game. For simplicity, we assume all players are charged with a fixed tax rate $\theta \in [0, 1]$ and again we consider only small θ.[5] This mechanism differs from the κ-implementation in two aspects. First, unlike in the κ-implementation where each player receives non-negative payments from the mediator, here by first paying tax and then receiving certain redistribution payments, the players could end up receiving punishment, see e.g. Ramirez-Kolumbus-Nagel-Wolpert-Jost [13] for mechanism design with both rewarding and punishment. This gives the mediator more power to influence the game. Next, the resource available to the mediator, namely the amount of the tax she receives, depends on the game outcome, while in κ-implementation the total resource κ is a fixed constant. Our main result remains true for this mechanism, that is, the efficiency function is non-decreasing and right continuous in θ, and may typically be discontinuous in θ.

knows she doesn't need to pay it in the end. Then a large κ will restrict the mediator's ability seriously.

[4]In the previous paragraph, the tax is received afterwards and thus is not considered in the mechanism design. Here, we shall consider the tax as the main resource of the mediator.

[5]Alternatively, we may interpret θ as the incremental increase of the tax rate when the government considers to increase the tax rate, which is typically small. In this paper we assume θ is positive. We may also consider to lower the interest rate and then θ becomes negative, which is covered by our abstract framework but is not investigated specifically in this paper.

One consequence of the possible punishment is the difference between closed loop and open loop mechanisms. Motivated by the stochastic control literature, we call a mechanism closed loop if the rewards/punishments depend only on the original payoffs of the game, and open loop if they can depend on the controls as well. For taxation purpose, it could be more appropriate to use closed loop mechanisms since people are required to report their income, but not necessarily their activities. When there are only rewards, we show that the two mechanisms lead to the same effect of efficiency improvement. However, when there is punishment as well, in general open loop mechanisms have larger power to improve the efficiency than the corresponding closed loop mechanisms.

In the general setting, both for one period static games and for continuous time stochastic differential games, we consider weighted average of the payoffs, reflecting the fact that some players (or say some sectors) can be more important for the society than some others. We remark that these weights only affect the mediator's decision on choosing a best mechanism design. The available mechanisms to the mediator remain the same, and for a given mechanism, the mediated game also has the same equilibria. In particular, the efficiency is stable with respect to the weights.

To study the efficiency rigorously, we invoke the set value of the game proposed by Feinstein-Rudloff-Zhang [4], which roughly speaking is the set of values over all possible equilibria. It is showed in [4] that these set values enjoy stability/regularity in certain sense, which is consistent with our result that the efficiency function is right continuous at $k = 0$ and $\theta = 0$. The point of this paper is that, such stability/regularity is not uniform in the sense that discontinuity may appear for small k and θ, and thus it becomes possible and crucial to design mechanisms so as to improve the efficiency of the game.

We shall also discuss briefly two related issues. First, by nature the mechanism design is a leader follower problem, with the mediator as the leader and the players as multiple followers. So the problem is also intrinsically connected to the principal agent problems with multiple agents, see e.g. Segal [14]. Next, by reinterpreting the θ in the taxation mechanism as a portion the government will control the economic outcome, then the θ is typically large in a central planned economy. In the extreme case that $\theta = 1$, the game problem reduces to a centralized control problem. In this case the payoffs of the base game may already depend on θ. We shall investigate the optimization of θ, combined with the mechanism design for every given θ.

Finally we remark that in this paper we only investigate the efficiency of the game in a theoretical way. There are extensive studies on algorithms and their complexity analysis for desired equilibria, which we do not consider in this paper.[6]

The rest of the paper is organized as follows. In Section 2 we illustrate our ideas through the example in Table 1. In Section 3 we study the general static games. Section 4 is devoted to stochastic differential games. Finally we provide further discussions in Section 5 and concluding remarks in Section 6.

2 The illustrative example

Consider the game specified in Table 1. This game has two Nash equilibria $\mathcal{E} = \big\{(1,1),(2,2)\big\}$. As defined in [4], the raw set value of the game is:

$$\mathbb{V}_0 = \big\{J(a^*) : a^* \in \mathcal{E}\big\} = \big\{(1,2),(3,1)\big\}. \tag{2.1}$$

Introduce the average payoff:

$$\overline{J}(a) := \frac{1}{2}[J_1(a) + J_2(a)]. \tag{2.2}$$

Denote by $A = A_1 \times A_2 = \{0,1,2\}^2$ the admissible control set. We define the optimal value of the game problem[7] and optimal value of the control problem as follows:

$$\begin{aligned} V &:= \sup_{a^*\in\mathcal{E}} \overline{J}(a^*) = \sup_{y\in\mathbb{V}_0} \frac{1}{2}[y_1+y_2] = \max\Big\{\frac{1+2}{2}, \frac{3+1}{2}\Big\} = 2;\\ \widehat{V} &:= \sup_{a\in A} \overline{J}(a) = \overline{J}(0,0) = \frac{1}{2}[100+100] = 100. \end{aligned} \tag{2.3}$$

We then define the efficiency of the game as:

$$E := \frac{V}{\widehat{V}} = \frac{2}{100} = 2\%. \tag{2.4}$$

That is, by restricting to equilibria, the players can achieve at most 2% of the optimal value the system could provide them. This is of course a huge waste of the resources, both for the individual players and for the society. The main goal of this paper is to improve the efficiency by modifying the game slightly. In particular, we shall introduce two possible mechanisms: the κ-implementation and the taxation mechanism.

Remark 2.1 *The efficiency E corresponds to the price of stability in [1]. However, [1] considers costs, and thus the price of stability is greater than 1. Here we consider payoffs, and thus our E is less than 1.*

[6] In particular, our efficiency of the game is completely different from the efficiency of numerical algorithms.

[7] In general we shall use the set value $\mathbb{V}$, rather than the raw set value $\mathbb{V}_0$, to define the optimal game value, see the next section. However, in this special case, one can easily verify that $\mathbb{V} = \mathbb{V}_0$. Since it is easier to explain the intuition by using the raw set value, we shall use $\mathbb{V}_0$ in this section.

Remark 2.2 *We remark that V is a lot easier to achieve than $\widehat{V}$ in practice. To achieve $\widehat{V}$, one needs to implement $(0,0)$. Since it is not an equilibrium, both players have the incentive to move away from it. To achieve V, one needs to implement $(2,2)$, which is an equilibrium. The mediator can simply announce that this is her preferred equilibrium. Note that by nature this equilibrium is Pareto optimal among all equilibria and thus some players (in this case, Player 1) are willing to follow it. Then, as long as the players think the others will follow that (especially when there are a large number of players, instead of two players), this equilibrium will be adopted.*

2.1 The κ-implementation

As in [10], we assume the mediator has an extra resource \$$\kappa$ to add into the system. She may introduce a distribution function $\pi : \mathbb{R}^2_+ \to \mathbb{R}^2$ such that

$$\pi_i(y) \geq 0,\ \ i = 1,2; \quad \pi_1(y) + \pi_2(y) \leq \kappa. \tag{2.5}$$

Let Π_κ denote the set of all these functions π. Introduce

$$J_i^\pi(a) := J_i(a) + \pi_i(J(a)). \tag{2.6}$$

Note that, alternatively we may consider π as a function of a instead of y. We shall investigate this in Subsection 3.2 below, in particular we refer to Remark 3.9.

We may define $E(\pi)$ as the efficiency of the game J^π, which would count the amount $\pi_1(J(a))+\pi_2(J(a))$ in the calculation. However, it is slightly better to exclude that, so that π will only affect the equilibria $\mathcal{E}(\pi)$, not the average payoff. That is, for the $\widehat{V}$ in (2.3),

$$E(\pi) := \frac{V(\pi)}{\widehat{V}}, \quad \text{where} \quad V(\pi) := \sup\big\{\overline{J}(a^\pi) :\ a^\pi \in \mathcal{E}(\pi)\big\}. \tag{2.7}$$

We remark that, since κ is supposed to be small, so this modification of the definition does not impact our analysis much. See also Remark 2.5 below. We now define

$$E(\kappa) := \sup_{\pi\in\Pi_\kappa} E(\pi). \tag{2.8}$$

Clearly $\Pi_{\kappa_1} \subset \Pi_{\kappa_2}$ and thus $E(\kappa_1) \leq E(\kappa_2)$ when $\kappa_1 \leq \kappa_2$.

We next analyze $E(\kappa)$. When $\kappa < 1$, we are not able to change the equilibria, and the equilibria remain to be $\{(1,1),(2,2)\}$, then

$$E(\kappa) = E(0) = 2\%.$$

When $1 \le \kappa < 4$, we can make $(0,1),(2,0)$ equilibria by setting:

$$\pi(0,102) = (1,0), \quad \pi(102,0) = (0,1), \quad \pi(y) = (0,0) \text{ for all other } y.$$

Then

$$V(\pi) = \overline{J}(0,1) = \frac{1}{2}[0+102] = 51, \quad E(\kappa) = \frac{51}{100} = 51\%.$$

When $\kappa \ge 4$, we can make $(0,0)$ an equilibrium by setting:

$$\pi(100,100) = (2,2), \quad \pi(y) = (0,0) \text{ for all other } y.$$

Then

$$V(\pi) = \overline{J}(0,0) = \frac{1}{2}[100+100] = 100, \quad E(\kappa) = \frac{100}{100} = 100\%.$$

In summary,

$$E(\kappa) = \begin{cases} 2\%, & 0 \le \kappa < 1; \\ 51\%, & 1 \le \kappa < 4; \\ 100\%, & \kappa \ge 4. \end{cases} \tag{2.9}$$

That is, by investing \$1 into the system, the mediator may improve the efficiency of the game from 2% to 51%. If she can invest \$4, the efficiency can increase further to 100%.

Remark 2.3 *It is important to understand the efficiency function like* (2.9) *in practice.*

(i) As we can see, $E(\kappa)$ is discontinuous in κ. It is crucial to figure out these discontinuous points. For example, if the mediator has already invested $\kappa = 0.99$, it is better to add a little more investment to increase it to $\kappa = 1$ so that the efficiency of the game can increase from 2% *to* 51%*. Similarly, it is not wise to invest $\kappa = 3.99$, one would rather increase to $\kappa = 4$ in that case.*

(ii) Consider the situation that the mediator, say a government, takes care of two projects, each involving a game. Assume the efficiency of one game can be improved significantly by small investment while the other cannot. Since the total resource of the mediator is typically limited, then it makes sense to set a higher priority of its investment on the former one.

Remark 2.4 *Now assume the mediator is the government (and nevertheless still consider two players for illustrative purpose), and it can charge income tax afterwards with rate θ. For simplicity assume the amount π from the government is tax free. So for the original* *base game, the optimal equilibrium is* $(2,2)$*, then Player 1, Player 2, and the government will receive, respectively:*

$$P1 : 3(1-\theta), \quad P2 : 1-\theta, \quad G : 4\theta.$$

When $\kappa = 4$*, the optimal equilibrium is* $(0,0)$*, the payoffs they will receive become:*

$$P1 : 2 + 100(1-\theta), \quad P2 : 2 + 100(1-\theta), \quad G : 200\theta - 4.$$

Let's say $\theta = 5\%$*, then we have the results in Table 2.*

	P1	P2	G	E
$\kappa = 0$	2.85	0.95	0.2	2%
$\kappa = 4$	97	97	6	100%

Table 2: win-win-win example

So, by improving the efficiency, this is a win-win-win situation.

2.2 A taxation mechanism

Inspired by Remark 2.4, actually the government doesn't have to invest extra resources. It can just invest the tax it's going to receive (after all the government does not make money by itself).[8] That is, assume the tax rate is $\theta \in [0,1]$, the government can introduce a distribution function $\pi : \mathbb{R}^2_+ \to \mathbb{R}^2$ such that

$$\pi_i(y) \geq 0, \ i = 1,2; \quad \pi_1(y) + \pi_2(y) \leq \theta[y_1 + y_2]. \tag{2.10}$$

Let Π_θ denote the set of all these functions π. By abusing the notation with (2.6), introduce

$$J_i^\pi(a) := (1-\theta)J_i(a) + \pi_i(J(a)). \tag{2.11}$$

Then, for the $E(\pi)$ defined in (2.7) corresponding to this J^π, we define

$$E(\theta) := \sup_{\pi \in \Pi_\theta} E(\pi). \tag{2.12}$$

Remark 2.5 *(i) In* (2.7) *we use* $\overline{J}(a^\pi)$*, instead of* $\overline{J}^\pi(a^\pi)$*. Note that* $\overline{J}^\pi(a^\pi)$ *is what the two players actually receive, while* $\overline{J}(a^\pi)$ *is what they "produce" or contribute to the society.*

[8] Instead of the actual tax it receives, we may also view the right of making tax policy as the resource of the government. Then a small θ amounts to a limited resource.

Since our discussion of efficiency focuses more on the interest of the society, and what's really crucial is how the redistribution affect the equilibria, so it makes sense to use $\overline{J}(a^\pi)$*, which technically is also easier to study. In any case, since we are talking about small* θ*, the difference between the two are minor.*

(ii) We should note that in the previous section $\overline{J}^\pi(a^\pi) \geq \overline{J}(a^\pi)$*, while here* $\overline{J}^\pi(a^\pi) \leq \overline{J}(a^\pi)$*. In particular, this means that here the two players can receive less than what they actually produce. So this setting includes the mechanism of punishment. The government can discourage the players to choose some controls, e.g.* $(1,1)$ *and* $(2,2)$*, through punishment by setting* $\pi(1,2) = \pi(3,1) = (0,0)$*.*

(iii) This setting also includes the mechanism of rewarding. For example, we will set $\pi(0,102) = (102\theta, 0)$*, and thus* $J^\pi(0,1) = (102\theta, 102(1-\theta)$*. So we are rewarding Player 1 in this case. Moreover, since the tax is charged proportionally, so the rich players are contributing more.*

Remark 2.6 *(i) A special case of the taxation mechanism is the uniform redistribution:* $\pi_i(y) := \frac{\theta}{2}[y_1 + y_2]$*. Then* (2.11) *becomes:*

$$J_i^\pi(a) := (1-\theta)J_i(a) + \theta\overline{J}(a). \tag{2.13}$$

One may justify the model (2.13) *by the bounded rationality (cf. [8]). That is, the individual players are willing to sacrifice his/her own utility for the benefit of the whole society to certain degree, and* θ *is a measure for this degree.*

(ii) The model (2.13) *provides a natural bridge between the game problem (when* $\theta = 0$*) and the control problem (when* $\theta = 1$*).*

(iii) One may also use the bounded rationality to answer our core question (1.1)*. Indeed, if the players are willing to sacrificing \$2, then* $(0,0)$ *is already an acceptable approximate equilibrium and thus the game is already efficient. However, as we see often that in practice people may not be willing to sacrifice much and they do get stuck in bad equilibria. So in this paper we focus on the mechanism design and see how small external incentives/penalties can help pull the players with unbounded rationality out of the bad equilibria.*

In the spirit of Remark 2.5 (ii) and (iii), given θ, we would like to choose π to encourage "good" outcomes and discourage "bad" outcomes. By direct analysis, the best π^* and then the corresponding J^{π^*} should be as in Table 3. Consequently, the efficiency function is as in Table 4. In summary,

$$E(\theta) = E(\pi^*) = \begin{cases} 2\%, & 0 \le \theta < \frac{1}{103}; \\ 51\%, & \frac{1}{103} \le \theta < \frac{1}{51}; \\ 100\%, & \frac{1}{51} \le \theta \le 1. \end{cases} \tag{2.14}$$

π^*	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100\theta, 100\theta)$	$(102\theta, 0)$	$(102\theta, 0)$
$a_1 = 1$	$(0, 102\theta)$	$(0, 0)$	$(0, 0)$
$a_1 = 2$	$(0, 102\theta)$	$(0, 0)$	$(0, 0)$

$J^{\pi^*}(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(102\theta, 102(1-\theta))$	$(102\theta, 102(1-\theta))$
$a_1 = 1$	$(102(1-\theta), 102\theta)$	$(1-\theta, 2(1-\theta))$	$(0, 0)$
$a_1 = 2$	$(102(1-\theta), 102\theta)$	$(0, 0)$	$(3(1-\theta), 1-\theta)$

Table 3: The taxation mechanism

θ	$\mathcal{E}(\pi^*)$	best NE	$E(\theta) = E(\pi^*)$
$\theta < \frac{1}{103}$	$(1, 1), (2, 2)$	$(2, 2)$	2%
$\theta = \frac{1}{103}$	$(0, 1), (1, 0), (1, 1), (2, 2)$	$(0, 1), (1, 0)$	51%
$\frac{1}{103} < \theta < \frac{1}{51}$	$(0, 1), (1, 0)$	$(0, 1), (1, 0)$	51%
$\theta = \frac{1}{51}$	$(0, 0), (0, 1), (1, 0)$	$(0, 0)$	100%
$\frac{1}{51} < \theta \le 1$	$(0, 0)$	$(0, 0)$	100%

Table 4: The efficiency in the taxation mechanism

That is, by charging $\frac{1}{103} \approx 1\%$ of tax and by designing the redistribution mechanism π optimally, one may improve the efficiency of the game from 2% to 51%. If one can charge $\frac{1}{51} \approx 2\%$ of tax, the efficiency can increase to 100%. Moreover, as in Remark 2.3, we see that $E(\theta)$ is discontinuous in θ, and it is crucial to figure out these discontinuous points, which are $\frac{1}{103}$ and $\frac{1}{51}$ in this example. It's not wise to set a tax rate right below these discontinuous points.

3 A static game

In this section we consider general static N-player game. Player i has control set A_i, and denote $A := A_1 \times \cdots \times A_N$. Given a control $a \in A$, player i will receive payoff $J_i(a) > 0$. We first consider the N-player control problem, which is a weighted sum of the payoffs:

$$\widehat{V}_\lambda := \sup_{a\in A} \sum_{i=1}^N \lambda_i J_i(a). \tag{3.1}$$

Here $\lambda_i > 0$ with $\sum_{i=1}^N \lambda_i = 1$ are appropriate weights given exogenously. The typical example is $\lambda_i \equiv \frac{1}{N}$, however, in general some players can be more important for the society than some others,[9] and then the mediator may set a larger λ_i for the former ones.

We next study the game value. We say $a^* \in A$ is an equilibrium, denoted as $a^* \in \mathcal{E}$, if

$$J_i(a^*) \geq J_i(a^{*,-i}, a_i), \quad \forall a_i \in A_i, \forall i. \tag{3.2}$$

Similarly, for $\varepsilon > 0$, we say $a^\varepsilon \in A$ is an ε-equilibrium, denoted as $a^\varepsilon \in \mathcal{E}_\varepsilon$, if

$$J_i(a^\varepsilon) \geq J_i(a^{\varepsilon,-i}, a_i) - \varepsilon, \quad \forall a_i \in A_i, \forall i. \tag{3.3}$$

As in [4], we define raw set value $\mathbb{V}_0$ and set value $\mathbb{V}$ of the game as follows:

$$\begin{gathered}\mathbb{V}_0 := \Big\{J(a^*) : a^* \in \mathcal{E}\Big\} \subset \mathbb{R}^N;\\ \mathbb{V} := \cap_{\varepsilon>0}\mathbb{V}_\varepsilon, \quad \mathbb{V}_\varepsilon := \Big\{y \in \mathbb{R}^N : |y - J(a^\varepsilon)| \leq \varepsilon, \text{ for some } a^\varepsilon \in \mathcal{E}_\varepsilon\Big\} \subset \mathbb{R}^N.\end{gathered} \tag{3.4}$$

Remark 3.1 *(i) In stochastic control problem, the value $v := \sup_a J(a) = \lim_{\varepsilon\to 0} J(a^\varepsilon)$ is defined through ε-optimal controls a^ε, rather than through optimal controls a^* which may not exist. So the set value $\mathbb{V}$, not the raw set value $\mathbb{V}_0$, is the natural counterpart for games.*

(ii) For fixed $\varepsilon > 0$, the ε-equilibrium can be interpreted as bounded rationality, while the true equilibrium corresponds to unbounded rationality. Therefore, the values in $\mathbb{V}$ can be viewed as the values with asymptotically unbounded rationality.

(iii) The set value $\mathbb{V}$ also enjoys many other properties, such as regularity/stability, and it is possible that $\mathbb{V} \neq \emptyset = \mathbb{V}_0$. We refer to [4] for more details.

Throughout the paper, we assume $\mathbb{V} \neq \emptyset$.[10] We then define the optimal game value as:

$$V_\lambda := \sup_{y\in\mathbb{V}} \sum_{i=1}^N \lambda_i y_i = \lim_{\varepsilon\to 0} \sup_{a^\varepsilon\in\mathcal{E}_\varepsilon} \sum_{i=1}^N \lambda_i J_i(a^\varepsilon). \tag{3.5}$$

[9] Instead of individual players, we may think of some sectors which are critical for the society.

[10] When $\mathbb{V} = \emptyset$, we may define $V_\lambda = 0$ in (3.5). The interpretation is that, in this case the system could be in chaos, and in many practical situations, a bad equilibrium is still better than chaos.

Our objective is again the efficiency of the game:

$$E_\lambda := \frac{V_\lambda}{\widehat{V}_\lambda} \in (0, 1]. \tag{3.6}$$

Remark 3.2 *(i) If we replace* $\mathbb{V}$ *with raw set value* $\mathbb{V}_0$ *in* (3.5)*, then*

$$V_{0,\lambda} := \sum_{y\in\mathbb{V}_0} \lambda_i y_i = \sup_{a^*\in\mathcal{E}} \sum_{i=1}^N \lambda_i J_i(a^*), \tag{3.7}$$

as in (2.3)*. It is clear that* $V_\lambda \geq V_{0,\lambda}$ *and in general the inequality can be strict.*

(ii) Assume (3.7) *has an optimal equilibrium* $\widehat{a}^*$*. Since* $\lambda_i > 0$*, then* $\widehat{a}^*$ *is Pareto optimal among all equilibria, namely there does not exist another equilibrium* a^* *such that* $J_i(a^*) \geq J_i(\widehat{a}^*)$ *for all* i *with strict inequality for at least one* i*. However,* $\widehat{a}^*$ *still may not be Pareto optimal among all admissible controls, namely there may exist a control* a *(not an equilibrium) such that* $J_i(a) \geq J_i(\widehat{a}^*)$ *for all* i *with strict inequality for at least one* i*.*

Remark 3.3 *(i) The weights* λ *are only under the consideration of the mediator. The individual players do not care about* λ*, in particular, the set value* $\mathbb{V}$ *does not depend on* λ*. Since the instability is solely due to the game structure, or say the set value* $\mathbb{V}$*, one can easily verify that* $\widehat{V}_\lambda, V_\lambda, E_\lambda$ *are stable in terms of small perturbation of* λ*.*

(ii) $\widehat{V}_\lambda$ *is also stable in terms of small perturbation of the model parameters of the game, that's why in the previous section we can fix the same* $\widehat{V}$ *in* (2.7)*. As we already saw in the previous section, however,* V_λ *and hence* E_λ *are typically unstable*[11] *in terms of small perturbation of the model parameters of the game. This is the main focus of the present paper, in particular, we are interested in the mechanism design such that a small perturbation could possibly increase the efficiency of the game significantly.*

Example 3.4 *For the example in Table 1, noting that* $\lambda_2 = 1 - \lambda_1$*, we have*

$$\widehat{V}_\lambda = \max\Big(100,\ 102\lambda_1,\ 102\lambda_2\Big) = \max\Big(100,\ 102\lambda_1,\ 102(1-\lambda_1)\Big),$$
$$\mathbb{V} = \mathbb{V}_0 = \big\{(1,2),(3,1)\big\}, \quad V_\lambda = \max\Big(\lambda_1 + 2\lambda_2,\ 3\lambda_1 + \lambda_2\Big) = \max\Big(2-\lambda_1,\ 1+2\lambda_1\Big),$$

and thus

$$E_\lambda = \begin{cases} \frac{2-\lambda_1}{102(1-\lambda_1)}, & 0 < \lambda_1 < \frac{1}{51}; \\ \frac{2-\lambda_1}{100}, & \frac{1}{51} \leq \lambda_1 < \frac{1}{3}; \\ \frac{1+2\lambda_1}{100}, & \frac{1}{3} \leq \lambda_1 < \frac{50}{51}; \\ \frac{1+2\lambda_1}{102\lambda_1}, & \frac{50}{51} \leq \lambda_1 < 1. \end{cases}$$

[11] More precisely, they are not uniformly stable. See Remark 3.7 below.

We note that, the optimal control for $\widehat{V}_\lambda$ and the optimal equilibrium for V_λ may depend on λ and have jumps at $\lambda_1 = \frac{1}{51}, \frac{1}{3}, \frac{50}{51}$, but the values $\widehat{V}_\lambda, V_\lambda, E_\lambda$ are continuous in λ.

We now turn to the mechanisms which could improve the efficiency of the game.

3.1 The mechanism schemes

We call Π a mechanism scheme if it is a set of functions $\pi : \mathcal{R} \subset \mathbb{R}^N_+ \to \mathbb{R}^N$, where $\mathcal{R} := \{J(a) : a \in A\}$ is the range of J. Note that we allow π to be negative. For each $\pi \in \Pi$, define J^π as in (2.6):

$$J^\pi_i(a) := J_i(a) + \pi_i(J(a)), \tag{3.8}$$

and ε-equilibria $\mathcal{E}^\pi_\varepsilon$ for J^π in the spirit of (3.3). Recall (3.5), we then define:

$$V^\pi_\lambda := \lim_{\varepsilon\to 0} \sup_{a^\varepsilon \in \mathcal{E}^\pi_\varepsilon} \sum_{i=1}^N \lambda_i J_i(a^\varepsilon), \quad E^\pi_\lambda := \frac{V^\pi_\lambda}{\widehat{V}_\lambda}, \quad E_\lambda(\Pi) := \sup_{\pi\in\Pi} E^\pi_\lambda. \tag{3.9}$$

We consider the Hausdorff distance for the sets Π, that is, given Π_1, Π_2,

$$d(\Pi_1,\Pi_2) := \max\Big(\sup_{\pi_1\in\Pi_1} d(\pi_1,\Pi_2), \ \sup_{\pi_2\in\Pi_2} d(\pi_2,\Pi_1)\Big), \ d(\pi_i,\Pi_j) := \inf_{\pi_j\in\Pi_j} \|\pi_i - \pi_j\|_\infty. \tag{3.10}$$

Our main theorem is as follows.

Theorem 3.5 *(i) E_λ is increasing in Π in the sense that:*

$$E_\lambda(\Pi_1) \le E_\lambda(\Pi_2), \quad \textit{whenever} \quad \Pi_1 \subset \Pi_2.$$

(ii) E_λ is upper semi-continuous on compact Π in the sense that: when Π is compact,

$$\overline{\lim_{n\to\infty}} E_\lambda(\Pi_n) \le E_\lambda(\Pi), \quad \textit{whenever} \quad \lim_{n\to\infty} d(\Pi_n,\Pi) = 0.$$

In particular, if Π is compact, $\Pi_n \downarrow \Pi$ and $d(\Pi_n,\Pi)\to 0$, then $E_\lambda(\Pi_n) \downarrow E_\lambda(\Pi)$.

Proof By (3.9), (i) is obvious. To see (ii), let $\Pi_n \to \Pi$ and Π be compact. Recall (3.9) and fix an arbitrary $\varepsilon > 0$. For each $n \ge 1$, there exist $\pi_n \in \Pi_n$ and $a^\varepsilon_n \in \mathcal{E}^{\pi_n}_\varepsilon$ such that

$$\widehat{V}_\lambda E_\lambda(\Pi_n) \le \widehat{V}_\lambda E^{\Pi_n}_\lambda + \varepsilon = V^{\Pi_n}_\lambda + \varepsilon \le \sup_{a^\varepsilon\in\mathcal{E}^{\pi_n}_\varepsilon} \sum_{i=1}^N \lambda_i J_i(a^\varepsilon) + \varepsilon \le \sum_{i=1}^N \lambda_i J_i(a^\varepsilon_n) + 2\varepsilon.$$

By (3.10), there exists $\pi'_n \in \Pi$ such that $\|\pi'_n - \pi_n\|_\infty \le d(\Pi_n,\Pi) + \varepsilon$. Moreover, since Π is compact, there exists a subsequence, stilled denoted as π'_n, such that $\pi'_n \to \pi^* \in \Pi$. Now for n large enough such that $d(\Pi_n,\Pi) \le \varepsilon$ and $\|\pi'_n - \pi^*\|_\infty \le \varepsilon$, we have

$$\|\pi_n - \pi^*\|_\infty \le \|\pi_n - \pi'_n\|_\infty + \|\pi'_n - \pi^*\|_\infty \le d(\Pi_n,\Pi) + \varepsilon + \|\pi'_n - \pi^*\|_\infty \le 3\varepsilon.$$

Recall (3.3), for any i and a_i we have

$$\begin{aligned} J_i^{\pi^*}((a_n^\varepsilon)^{-i}, a_i) &\le J_i^{\pi_n}((a_n^\varepsilon)^{-i}, a_i) + \|\pi^* - \pi_n\|_\infty \le J_i^{\pi_n}(a_n^\varepsilon) + \varepsilon + \|\pi'_n - \pi_n\|_\infty \\ &\le J_i^{\pi^*}(a_n^\varepsilon) + 2\|\pi'_n - \pi_n\|_\infty + \varepsilon \le J_i^{\pi^*}(a_n^\varepsilon) + 7\varepsilon. \end{aligned}$$

That is, $a_n^\varepsilon \in \mathcal{E}_{7\varepsilon}^{\pi^*}$. Thus, for n large enough,

$$\widehat{V}_\lambda E_\lambda(\Pi_n) \le \sum_{i=1}^N \lambda_i J_i(a_n^\varepsilon) + 2\varepsilon \le \sup_{a^\varepsilon \in \mathcal{E}_{7\varepsilon}^{\pi^*}} \sum_{i=1}^N \lambda_i J_i(a^\varepsilon) + 2\varepsilon.$$

Then, send $n \to \infty$ and $\varepsilon \to 0$,

$$\overline{\lim_{n\to\infty}} E_\lambda(\Pi_n) \le \lim_{\varepsilon \to 0} \frac{1}{\widehat{V}_\lambda} \sup_{a^\varepsilon \in \mathcal{E}_{7\varepsilon}^{\pi^*}} \sum_{i=1}^N \lambda_i J_i(a^\varepsilon) = \frac{V_\lambda^{\pi^*}}{\widehat{V}_\lambda} \le E_\lambda(\Pi).$$

Finally, when $\Pi_n \downarrow \Pi$, since $E_\lambda(\Pi_n) \ge E_\lambda(\Pi)$ and $\overline{\lim}_{n\to\infty} E_\lambda(\Pi_n) \le E_\lambda(\Pi)$, we obtain $E_\lambda(\Pi_n) \downarrow E_\lambda(\Pi)$ immediately. ■

Inspired by Subsections 2.1 and 2.2, we have the following two examples.

Example 3.6 *(i) For $\kappa \ge 0$, let Π_κ denote the set of functions $\pi : \mathcal{R} \to \mathbb{R}^N$ such that*

$$\pi_i(y) \ge 0, \ i = 1, \cdots, N; \quad \sum_{i=1}^N \pi_i(y) \le \kappa. \tag{3.11}$$

Then $\kappa \mapsto E_\lambda(\Pi_\kappa)$ is increasing. Moreover, when $\mathcal{R}$ is discrete, then clearly Π_κ is compact and thus $\kappa \mapsto E_\lambda(\Pi_\kappa)$ is right continuous in κ.

(ii) For $\theta \in [0,1]$, let Π_θ denote the set of functions $\pi : \mathcal{R} \to \mathbb{R}^N$ such that[12]

$$\pi_i(y) = \psi_i(y) - \theta y_i, \quad \text{where} \quad \psi_i(y) \ge 0, \ i = 1, \cdots, N; \quad \sum_{i=1}^N \psi_i(y) \le \theta \sum_{i=1}^N y_i. \tag{3.12}$$

Then $\theta \mapsto E_\lambda(\Pi_\theta)$ is increasing. Moreover, when $\mathcal{R}$ is discrete, then clearly Π_θ is compact and thus $\theta \mapsto E_\lambda(\Pi_\theta)$ is also right continuous. We remark that π can be negative here.

Remark 3.7 *(i) By [4] the mapping $\Pi \mapsto \mathbb{V}(\Pi)$ is stable in the sense:*

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \mathbb{V}_\varepsilon(\Pi_n) = \mathbb{V}(\Pi) = \lim_{\varepsilon \to 0} \underline{\lim}_{n\to\infty} \mathbb{V}_\varepsilon(\Pi_n), \quad \text{if} \quad \Pi_n \to \Pi.$$

This is consistent with the right continuity of E_λ in Theorem 3.5 (ii). In particular, it is consistent with the right continuity of E_λ at $\kappa = 0$ or $\theta = 0$ in Example 3.6. We note that the raw set value $\mathbb{V}_0$ does not enjoy such stability.

(ii) However, such stability is not uniform, in particular, in Example 3.6 E_λ may have discontinuity at small κ or small θ. These discontinuous points κ or θ are crucial for the our mechanism design, as pointed out in Remark 2.3 (i).

[12] We are abusing the notation here with those in Subsection 2.2. The π there corresponds to the ψ here.

3.2 The open loop mechanisms

We call Π^o an open loop mechanism scheme if it is a set of functions $\pi^o : A \to \mathbb{R}^N$, namely π^o is a function on the control a directly. Correspondingly, we call the functions π on the payoff y as closed loop mechanisms. For each $\pi^o \in \Pi^o$, we modify (3.8) as:

$$J_i^{\pi^o}(a) := J_i(a) + \pi_i^o(a). \tag{3.13}$$

We then define $\mathcal{E}_\varepsilon^{\pi^o}$, and $V_\lambda^{\pi^o}$, $E_\lambda^{\pi^o}$, and $E_\lambda(\Pi^o)$ in the obvious manner, as in (3.3) and (3.9). Then, similarly to Theorem 3.5, the following result is obvious.

Proposition 3.8 *(i) E_λ is increasing in Π^o.*

(ii) Assume J is continuous in a, then E_λ is upper semi-continuous on compact Π^o. In particular, if Π^o is compact, $\Pi_n^o \downarrow \Pi^o$ and $d(\Pi_n^o, \Pi^o) \to 0$, then $E_\lambda(\Pi_n^o) \downarrow E_\lambda(\Pi^o)$.

Remark 3.9 *(i) In many practical situations, it is easier for the mediator to observe $J(a)$ than to observe a. For example, for tax purpose people are required to report their income, but not necessarily their activities. Then it's feasible to define π on y, but not on a.*

(ii) It is possible that there exist an equilibrium a^* and a non-equilibrium a such that $J(a) = J(a^*)$. For example, in Table 5, $J(0,0) = J(0,1) = (1,1)$, while $(0,0)$ is an equilibrium, but $(0,1)$ is not. By Proposition 3.10 below, overall speaking the open loop mechanism has larger power on improving the efficiency. Moreover, as we will see Example 3.11 below, in general the two are not equal. However, the consideration in (i) outperforms and in the rest of the paper we will consider closed loop mechanisms only.

$J(a)$	$a_2 = 0$	$a_2 = 1$
$a_1 = 0$	$(1,1)$	$(1,1)$
$a_1 = 1$	$(0,0)$	$(2,2)$

Table 5: open loop controls with same value

Proposition 3.10 *Let Π (resp. Π^o) be a closed loop (resp. open loop) mechanism scheme. Assume, for each $\pi \in \Pi$, the following $\pi^o \in \Pi^o$: $\pi^o(a) := \pi(J(a))$. Then $E_\lambda(\Pi) \le E_\lambda(\Pi^o)$.*

Proof For any $\pi \in \Pi$ and the corresponding π^o as above, we have $J^\pi = J^{\pi^o}$. Then it is clear that $\mathcal{E}_\varepsilon^\pi = \mathcal{E}_\varepsilon^{\pi^o}$ and $E_\lambda^\pi = E_\lambda^{\pi^o} \le E_\lambda(\Pi^o)$. Now by the arbitrariness of $\pi \in \Pi$ we obtain $E_\lambda(\Pi) \le E_\lambda(\Pi^o)$. ■

In general the open loop mechanisms may have a strictly larger power on improving the efficiency than closed loop mechanisms.

Example 3.11 *Let Π denote the set of functions $\pi : \mathcal{R} \to \mathbb{R}^N$ and Π^o the set of functions $\pi^o : A \to \mathbb{R}^N$ such that*

$$\pi_i, \pi_i^o \le 0, \quad \sum_{i=1}^N \pi_i(y) \ge -1, \quad \sum_{i=1}^N \pi_i^o(a) \ge -1. \tag{3.14}$$

Then $E_\lambda(\Pi) \le E_\lambda(\Pi^o)$, and in general the inequality can be strict.

Proof First it follows from Proposition 3.10 that $E_\lambda(\Pi) \le E_\lambda(\Pi^o)$. To see the strict inequality, let's consider the following example in Table 6 with $\lambda_1 = \lambda_2 = \frac{1}{2}$. Since A is discrete, it is clear that $\mathbb{V} = \mathbb{V}_0$ and thus we may focus on true equilibria. Moreover, it is clear that $\widehat{V}_\lambda = \overline{J}(0,1) = 101$.

$J(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(101, 101)$	$(1, 1)$
$a_1 = 1$	$(101, 101)$	$(1, 1)$	$(2, 103)$
$a_1 = 2$	$(1, 1)$	$(103, 2)$	$(1, 1)$

$\pi^o(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(0, 0)$	$(0, -1)$	$(0, 0)$
$a_1 = 1$	$(-1, 0)$	$(0, 0)$	$(0, 0)$
$a_1 = 2$	$(0, 0)$	$(0, 0)$	$(0, 0)$

$J^{\pi^o}(a)$	$a_2 = 0$	$a_2 = 1$	$a_2 = 2$
$a_1 = 0$	$(100, 100)$	$(101, 100)$	$(1, 1)$
$a_1 = 1$	$(100, 101)$	$(1, 1)$	$(1, 103)$
$a_1 = 2$	$(1, 1)$	$(103, 1)$	$(1, 1)$

Table 6: an open loop example

For open loop mechanism, we set π^o as in Table 6, and then obtain J^{π^o}, also reported in Table 6. One can verify straightforwardly that $(0,0) \in \mathcal{E}_0^{\pi^o}$. Moreover, for any $\tilde{\pi}^o \in \Pi^o$, by (3.14) we have $-1 \le \tilde{\pi}_i^o(a) \le 0$, then $J_1^{\tilde{\pi}^o}(0,1) \le 101 < 102 \le J_1^{\tilde{\pi}^o}(2,1)$ and $J_2^{\tilde{\pi}^o}(1,0) \le 101 < 102 \le J_2^{\tilde{\pi}^o}(1,2)$. This implies that $(0,1),(1,0) \notin \mathcal{E}_1^{\tilde{\pi}^o}$. Then one can easily see that

$$E_\lambda(\Pi^o) = E_\lambda^{\pi^o} = \frac{\overline{J}(0,0)}{\widehat{V}_\lambda} = \frac{100}{101}.$$

For any closed loop mechanism $\pi \in \Pi$, however, since $J_1^\pi(0,0) \le 100, J_2^\pi(0,0) \le 100$, and in this case $J^\pi(0,1) = J^\pi(1,0)$ with $J_1^\pi(0,1) + J_2^\pi(0,1) = 202 + \pi_1(101,101) + \pi_2(101,101) \ge 201$, we must have $J_2^\pi(0,1) \ge 100.5 > 100 \ge J_2^\pi(0,0)$ or $J_1^\pi(1,0) = J_1^\pi(0,1) \ge 100.5 > 100 \ge J_1^\pi(0,0)$, so $(0,0) \notin \mathcal{E}_{0.5}^\pi$. Similarly as in the open loop case, we may check that $(0,1),(1,0) \notin \mathcal{E}_1^\pi$ and $(1,2),(2,1) \in \mathcal{E}^\pi$. Then we have $E_\lambda(\Pi) = \frac{\frac{1}{2}[2+103]}{101} = \frac{52.5}{101} < \frac{100}{101} = E_\lambda(\Pi^o)$. ■

The next result shows that, for the k-implementation and the taxation mechanism, the two schemes have the same effect. However, due to technical reasons, we restrict to the raw set values, namely the true equilibria. Recall (3.7).

Proposition 3.12 *(i) For $\kappa \geq 0$, let Π^o_κ denote the set of functions $\pi^o : A \to \mathbb{R}^N$ such that*

$$\pi^o_i(a) \geq 0,\ \ i = 1, \cdots, N; \quad \sum_{i=1}^N \pi^o_i(a) \leq \kappa. \tag{3.15}$$

Then $\sup_{\pi^o \in \Pi^o_k} \sup_{a^* \in \mathcal{E}^{\pi^o}} \sum_{i=1}^N \lambda_i J_i(a^*) = \sup_{\pi \in \Pi_k} \sup_{a^* \in \mathcal{E}^{\pi}} \sum_{i=1}^N \lambda_i J_i(a^*)$.

(ii) For $\theta \in [0,1]$, let Π^o_θ denote the set of functions $\pi^o : A \to \mathbb{R}^N$ such that

$$\pi^o_i(a) = \psi_i(a) - \theta J_i(a),\ \ \textit{where}\ \psi_i(a) \geq 0,\ \ i = 1, \cdots, N; \ \sum_{i=1}^N \psi_i(a) \leq \theta \sum_{i=1}^N J_i(a). \tag{3.16}$$

Then $\sup_{\pi^o \in \Pi^o_\theta} \sup_{a^* \in \mathcal{E}^{\pi^o}} \sum_{i=1}^N \lambda_i J_i(a^*) = \sup_{\pi \in \Pi_\theta} \sup_{a^* \in \mathcal{E}^{\pi}} \sum_{i=1}^N \lambda_i J_i(a^*)$.

Proof (i) The inequality "$\geq$" follows from the same arguments as in Proposition 3.10. To see the opposite inequality "$\leq$", we fix $\pi^o \in \Pi^o_\kappa$ and $a^* \in \mathcal{E}^{\pi^o}$. Introduce $\pi \in \Pi_\kappa$ by:

$$\pi(y) := \pi^o(a^*) \ \text{ for } y = J(a^*), \quad \text{ and } \quad \pi(y) := 0 \text{ for } y \in \mathcal{R} \backslash \{J(a^*)\}. \tag{3.17}$$

Now fix arbitrary $i = 1, \cdots, N$ and $a_i \in A_i$. If $J(a^{*,-i}, a_i) = J(a^*)$, then $\pi(J(a^{*,-i}, a_i)) = \pi(J(a^*)) = \pi^o(a^*)$, and thus

$$J^\pi_i(a^{*,-i}, a_i) - J^\pi_i(a^*) = J(a^{*,-i}, a_i) - J(a^*) = 0.$$

If $J(a^{*,-i}, a_i) \neq J(a^*)$, then $\pi(J(a^{*,-i}, a_i)) = 0 \leq \pi^o(a^{*,-i}, a_i)$ and $\pi(J(a^*)) = \pi^o(a^*)$, thus

$$J^\pi_i(a^{*,-i}, a_i) - J^\pi_i(a^*) \leq J^{\pi^o}_i(a^{*,-i}, a_i) - J^{\pi^o}_i(a^*) \leq 0.$$

This implies that $a^* \in \mathcal{E}^\pi$. Since $\pi^o \in \Pi^o_\kappa$ and $a^* \in \mathcal{E}^{\pi^o}$ are arbitrary, we obtain the desired inequality "$\leq$".

(ii) Again the inequality "$\geq$" is obvious. To see the inequality "$\leq$", fix $\pi^o \in \Pi^o_\theta$ with corresponding ψ and $a^* \in \mathcal{E}^{\pi^o}$. We modify (3.17) and introduce $\pi \in \Pi_\theta$ by:

$$\pi(y) := \pi^o(a^*) = \psi(a^*) - \theta y \ \text{ for } y = J(a^*), \quad \text{ and } \quad \pi(y) := -\theta y \text{ for } y \in \mathcal{R} \backslash \{J(a^*)\}.$$

Similarly we can show that $a^* \in \mathcal{E}^\pi$, which implies the desired inequality "$\leq$". ■

Remark 3.13 *(i) The condition $\pi^o_i \geq 0$ in* (3.15) *is crucial in the proof of the above Proposition 3.12 (i). For mechanisms with rewarding only, following similar arguments the closed loop scheme and the corresponding open loop scheme would have the same efficiency improving effect.*

(ii) In Proposition 3.12 (ii), the condition $\psi_i \geq 0$ in (3.16) plays the same role. However, this relies on our convention that the tax rate θ is a fixed constant and thus the punishment doesn't really provide deterrent to the players. When we allow the tax rate to depend on the payoffs and/or the controls, this essentially falls into the setting of Example 3.11 (where $\psi \equiv 0$), as we saw for mechanisms with punishment in general the open loop scheme may have larger power on improving the efficiency.

4 A dynamic game

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, B a d-dimensional Brownian motion. We shall consider N-player games with drift controls only and thus use weak formulation. Set

$$X \equiv B.$$

Let $A = A_1 \times \cdots \times A_N$ be the control set, and $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$ be the admissible controls which are $\mathbb{F}^X$-progressively measurable and A-valued process $\alpha = (\alpha^1, \cdots, \alpha^N)$. Define

$$J_i(\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\Big[g_i(X_T) + \int_0^T f_i(t, X_t, \alpha_t^i)dt\Big], \quad i = 1, \cdots, N, \tag{4.1}$$

where

$$dP^\alpha = M_T^\alpha d\mathbb{P}, \quad M_T^\alpha = \exp\Big(\int_0^T b(t, X_t, \alpha_t)dB_t - \frac{1}{2}\int_0^T |b(t, X_t, \alpha_t)|^2 dt\Big). \tag{4.2}$$

Fix $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{R}_+^N$ with $\sum_{i=1}^N \lambda_i = 1$, and let $\mathcal{E}_\varepsilon$ denote the set of ε-equilibria of the game. The optimal control value $\hat{V}_\lambda$ and the optimal game value V_λ are:

$$\widehat{V}_\lambda := \sup_{\alpha \in \mathcal{A}} \sum_{i=1}^N \lambda_i J_i(\alpha), \quad V_\lambda := \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon} \sum_{i=1}^N \lambda_i J_i(\alpha^\varepsilon). \tag{4.3}$$

In this section the following assumption will always be in force, and thus the above problems are well defined.

Assumption 4.1 $(b, f) : [0, T] \times \mathbb{R}^d \times A \to (\mathbb{R}^d, \mathbb{R}^N)$ *is progressively measurable, bounded, and continuous in a; and $g : \mathbb{R}^d \to \mathbb{R}^N$ is measurable and bounded.*

We emphasize that the boundedness of f and g are just for simplicity and can be replaced with appropriate integrability conditions.

In this setting, a mechanism scheme Π consists of $\pi = (\tilde{\pi}, \pi')$, where $\tilde{\pi} : C([0,T]) \times \mathbb{R}^N \to \mathbb{R}^N$ and $\pi' : [0,T] \times C([0,T]) \times \mathbb{R}^N \to \mathbb{R}^N$ are progressively measurable and π' is adpated. Given $\pi \in \Pi$, we consider the game:

$$J_i^\pi(\alpha) = \mathbb{E}^{\mathbb{P}^\alpha}\Big[g_i(X_T) + \tilde{\pi}_i(X_\cdot, g(X_T)) + \int_0^T \big[f_i(t, X_t, \alpha_t^i) + \pi_i'(t, X_\cdot, f(t, X_t, \alpha_t^i))\big]dt\Big]. \tag{4.4}$$

Let $\mathcal{E}_\varepsilon^\pi$ denote the set of ε-equilibria $\alpha^{\pi,\varepsilon}$ of the game J^π, we then define

$$V_\lambda^\pi := \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^\pi} \sum_{i=1}^N \lambda_i J_i(\alpha^\varepsilon), \quad V_\lambda(\Pi) := \sup_{\pi \in \Pi} V_\lambda^\pi. \tag{4.5}$$

Remark 4.2 *(i) We are using closed loop mechanisms, so π does not depend on α, which can be viewed as moral hazard in the literature of contract theory. We may consider open loop mechanisms as well, then π depend on X and α, provided the mediator can observe α.*

(ii) We allow π to depend on the paths of X. We may restrict to state dependent π, and we may also restrict π to those depending on X only or depending on f and g only.

(iii) Again we use $J_i(\alpha^\varepsilon)$ instead of $J_i^\pi(\alpha^\varepsilon)$ in this definition. However, when technically more convenient, we may reformulate $\tilde{V}_\lambda^\pi := \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^\pi} \sum_{i=1}^N \lambda_i J_i^\pi(\alpha^\varepsilon)$. Since we shall consider only small perturbation, the difference $|\tilde{V}_\lambda^\pi - V_\lambda^\pi|$ is small, so we can choose whichever is mathematically more convenient.

Remark 4.3 *In this section we allow f, g, and hence J to be negative. It is clear that $V_\lambda(\Pi) \le \widehat{V}_\lambda$, and when $f, g \ge 0$, we may define naturally the efficiency of the game $E_\lambda(\Pi) := \frac{V_\lambda(\Pi)}{\widehat{V}_\lambda}$. Since $\widehat{V}_\lambda$ is fixed, when it is positive, clearly the optimization of $E_\lambda(\Pi)$ is equivalent to that of $V_\lambda(\Pi)$. So from now on we consider $V_\lambda(\Pi)$, which does not require the positiveness.*

Theorem 3.5 obviously remain true, and we shall state it in terms of V_λ.

Theorem 4.4 *(i) V_λ is increasing in Π in the sense that:*

$$V_\lambda(\Pi_1) \le V_\lambda(\Pi_2), \quad \textit{whenever} \quad \Pi_1 \subset \Pi_2.$$

(ii) V_λ is upper semi-continuous on compact Π in the sense that:

$$\overline{\lim_{n\to\infty}} V_\lambda(\Pi_n) \le V_\lambda(\Pi), \quad \textit{whenever} \quad \lim_{n\to\infty} d(\Pi_n, \Pi) = 0.$$

In particular, if Π is compact, $\Pi_n \downarrow \Pi$ and $d(\Pi_n, \Pi) \to 0$, then $V_\lambda(\Pi_n) \downarrow E_\lambda(\Pi)$.

We now introduce the κ-implementation and the taxation mechanism in this setting.

Example 4.5 *(i) Given* $\kappa = (\tilde{\kappa}, \kappa') \in [0, \infty)^2$*,* Π_κ *denotes the set of functions* $\pi = (\tilde{\pi}, \pi')$ *such that*

$$\tilde{\pi}_i \geq 0, \quad \sum_{i=1}^N \tilde{\pi}_i \leq \tilde{\kappa}; \qquad \pi'_i \geq 0, \quad \sum_{i=1}^N \pi'_i \leq \kappa'.$$

(ii) For $\theta = (\tilde{\theta}, \theta') \in [0, 1]^2$*,* Π_θ *denotes the set of functions* $\pi = (\tilde{\pi}, \pi')$ *such that, there exist functions* $\psi = (\tilde{\psi}, \psi')$ *satisfying*

$$\tilde{\pi}(\mathbf{x}, y) = \tilde{\psi}(\mathbf{x}, y) - \tilde{\theta} y, \quad \pi'(t, \mathbf{x}, y) = \psi'(t, \mathbf{x}, y) - \theta' y;$$
$$\tilde{\psi}_i \geq 0, \quad \sum_{i=1}^N \tilde{\psi}_i(\mathbf{x}, y) \leq \tilde{\theta} \sum_{i=1}^N y_i; \quad \psi'_i \geq 0, \quad \sum_{i=1}^N \tilde{\psi}'_i(t, \mathbf{x}, y) \leq \theta' \sum_{i=1}^N y_i.$$

It is interesting and challenging to characterize $V_\lambda(\Pi_\kappa)$ and $V_\lambda(\Pi_\theta)$ in this setting. We shall provide a brief discussion in Subsection 5.1 below. The following example shows that again V_λ could be discontinuous.

Example 4.6 *Let* $N = 2$*,* $\lambda_1 = \lambda_2 = \frac{1}{2}$*,* A *as in Section 2,* $g \equiv 0$*,* $f(t, x, a) = J(a)$ *for the* J *in Table 1. For* $\kappa = (0, \kappa')$ *with* $\kappa' \geq 0$*, it is straightforward to show that* $E_\lambda(\Pi_\kappa) = \frac{V_\lambda(\Pi_\kappa)}{\widehat{V}_\lambda}$ *in Example 4.5 (i) is the same as the* $E(\Pi_{\kappa'})$ *in* (2.9)*. Then* $E_\lambda(\Pi_\kappa)$ *and hence* $V_\lambda(\Pi_\kappa)$ *is discontinuous at* $\kappa' = 1$ *and* $\kappa' = 4$*.*

Remark 4.7 *All the results in this section can be extended to mean field games, which consist of infinitely many players. In particular, this is appropriate for social problems where the system is by nature large. However, to avoid the heavy notations, we do not provide details here. We refer to Iseri-Zhang [6] for set values of mean field games.*

5 Further discussions

In this section we provide two possible extensions of our problem.

5.1 A leader follower problem

By nature the problems (3.9) and (4.5) are leader follower problems with multiple followers, also called Stackelberg games, where the mediator is the leader and the players are followers. Hence the problem is also intrinsically connected to the principal agent problems with one principal and multiple agents.

We note that in (4.5) the leader does not have her own interest, she represents the followers' aggregate interests. In particular, as pointed out in Remark 2.2, given the leader's control $\pi \in \Pi$, V_λ^π corresponds to the best equilibrium which is Pareto optimal among the followers' equilibria. It will be very interesting to solve this leader follower problem (4.5), especially the two examples in Example 4.5. However, since $\mathbb{V}_\lambda^\pi$ has bad stability with respect to π, in general this is a challenging problem and we leave it for future research.

We may alternatively study the worst equilibrium, corresponding to the price of anarchy:

$$\underline{V}_\lambda^\pi := \lim_{\varepsilon \to 0} \inf_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^\pi} \sum_{i=1}^N \lambda_i J_i(\alpha^\varepsilon), \quad \underline{V}_\lambda(\Pi) := \sup_{\pi \in \Pi} \underline{V}_\lambda^\pi. \tag{5.6}$$

This is the robust approach from the mediator's point of view. The above problem is a max-min problem, however, the min problem is over all (approximate) equilibria, and thus the problem is much more challenging than the standard max-min problems.

Moreover, we can consider more general leader follower problems where the leader has her own interest, or say utility $J_P(\pi, a)$. Here for simplicity let's assume the leader's control is still the mechanisms $\pi \in \Pi$. Then correspondingly we can have the following problems corresponding to the worst equilibrium and the best equilibrium, respectively:

$$\begin{aligned} \underline{V}^\pi &:= \lim_{\varepsilon \to 0} \inf_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^\pi} J_P(\pi, \alpha^\varepsilon), \quad \underline{V}(\Pi) := \sup_{\pi \in \Pi} \underline{V}^\pi; \\ \overline{V}^\pi &:= \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^\pi} J_P(\pi, \alpha^\varepsilon), \quad \overline{V}(\Pi) := \sup_{\pi \in \Pi} \overline{V}^\pi. \end{aligned} \tag{5.7}$$

Here $\underline{V}(\Pi)$ measures the leader's optimal utility in the worst scenario, provided the agents would implement an (approximate) equilibrium. The problem $\overline{V}(\Pi)$ corresponding to the best scenario is problematic in practice, however. Unlike in (3.9) and (4.5), here the (approximate) optimal equilibrium for the leader, denoted as α^*, may not be Pareto optimal for the followers among all (approximate) equilibria, hence it is hard for the leader to induce the followers to implement α^*.[13] The problem is closely related to the selection problem, namely given π, which equilibrium (or even non-equilibrium control) the leader expects the followers will implement. While there are other alternatives, one possibility which sounds reasonable from practical point of view is to consider the best equilibrium, best for the leader, among all Pareto optimal equilibria, Pareto optimal for the followers. That is, let $\mathcal{E}_\varepsilon^{Pareto,\pi}$ denote the set of $a^\varepsilon \in \mathcal{E}_\varepsilon^\pi$ which is ε-Pareto optimal in the following sense: there is *no* $\tilde{a}^\varepsilon \in \mathcal{E}_\varepsilon^\pi$ such that $J_i^\pi(\tilde{\alpha}^\varepsilon) \geq J_i^\pi(\alpha^\varepsilon) + \varepsilon$ for all i. We then consider

$$V^\pi := \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^{Pareto,\pi}} J_P(\pi, \alpha^\varepsilon), \quad V(\Pi) := \sup_{\pi \in \Pi} V^\pi. \tag{5.8}$$

[13] We do not have the same concern for $\underline{V}(\pi)$, because here the leader is just considering the worst scenario, and she has no intention to induce the followers to implement the worst equilibrium.

However, it is in general hard to characterize $\mathcal{E}_\varepsilon^{Pareto,\pi}$, so (5.8) could be even more challenging than (5.7) and (4.5).

5.2 A central planned economy with large θ

The θ in the taxation mechanism can also be interpreted as the portion the mediator, say the government, controls the economy. For example, in a central planned economy, θ is typically large. In particular, when $\theta = 1$, then in Example 3.6 (ii) we have $E_\lambda(\Pi_\theta) = 1$, or say the game problem reduces to the control problem. However, for large θ, typically the value function J of the base game may depend on θ: $J = J(\theta, a)$. Consequently the value of the optimal control problem (3.1) will also depend on θ:

$$\widehat{V}_\lambda(\theta) := \sup_{a \in A} \sum_{i=1}^N \lambda_i J_i(\theta, a). \tag{5.9}$$

In this case, especially when J is decreasing in θ, it is not desirable to maximize $E_\lambda(\Pi_\theta)$ for the Π_θ in Example 3.6 (ii). The more reasonable goal is the following problem:

$$\sup_{\theta \in [0,1]} V_\lambda(\theta), \quad V_\lambda(\theta) := E_\lambda(\Pi_\theta)\widehat{V}_\lambda(\theta) = \sup_{\pi \in \Pi_\theta} \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^{\theta,\pi}} \sum_{i=1}^N \lambda_i J_i(\theta, \alpha^\varepsilon), \tag{5.10}$$

where $\mathcal{E}_\varepsilon^{\theta,\pi}$ is the set of ε-equilibria of the mediated game $J^\pi(\theta, a)$.

Example 5.1 *In the setting of Subsection 2.2, assume $J(\theta, a) = (2-\theta)J(a)$ is proportional to the $J(a)$ there. Then one can easily see that $E(\Pi_\theta) = E(\theta)$ is the same as in (2.14), and by (2.3) we have $\hat{V}(\theta) = 100(2-\theta)$. Thus, for $\lambda_1 = \lambda_2 = \frac{1}{2}$,*

$$V_\lambda(\theta) = \begin{cases} 2(2-\theta), & 0 \le \theta < \frac{1}{103}; \\ 51(2-\theta), & \frac{1}{103} \le \theta < \frac{1}{51}; \\ 100(2-\theta), & \frac{1}{51} \le \theta \le 1. \end{cases} \tag{5.11}$$

Then, one can easily verify that the optimal $\theta^* = \frac{1}{51}$.

The following problem in continuous time model is again very challenging. Let $\theta = (\tilde{\theta}, \theta') \in [0,1]^2$ and $\alpha \in \mathcal{A}$ be as in the setting of Section 4, define

$$\begin{aligned} &J_i(\theta, \alpha) = \mathbb{E}^{\mathbb{P}^{\theta,\alpha}}\Big[g_i(\theta, X_T) + \int_0^T f_i(t, \theta; X_t, \alpha_t^i)dt\Big], \quad i = 1, \cdots, N, \quad \text{where} \\ &dP^{\theta,\alpha} = M_T^{\theta,\alpha} d\mathbb{P}, \quad M_T^{\theta,\alpha} = \exp\Big(\int_0^T b(t, \theta, X_t, \alpha_t) dB_t - \frac{1}{2}\int_0^T |b(t, \theta, X_t, \alpha_t)|^2 dt\Big). \end{aligned} \tag{5.12}$$

Let Π_θ be as in Example 4.5 (ii), for for each $\pi \in \Pi_\theta$, define $J^\pi(\theta, \alpha)$ and $\mathcal{E}_\varepsilon^{\theta,\pi}$ in the obvious manner. We then have the following optimization problem corresponding to (5.10):

$$\sup_{\theta \in [0,1]^2} \sup_{\pi \in \Pi_\theta} \lim_{\varepsilon \to 0} \sup_{\alpha^\varepsilon \in \mathcal{E}_\varepsilon^{\theta,\pi}} \sum_{i=1}^N \lambda_i J_i(\theta, \alpha^\varepsilon). \tag{5.13}$$

Again we shall leave it for future research.

6 Conclusion

It is well known that a non-cooperative game is typically inefficient, in the sense that an equilibrium may have less aggregate payoff than the socially optimal control. In this paper we study mechanism design by a mediator aiming to improve the efficiency of a game, equivalent to the price of stability concerning the best equilibrium. In particular, we introduce two mechanisms, the κ-implementation and the taxation scheme. The former one contains rewarding only, while the latter one provides both rewarding and punishment.

We focus on the mechanism design with small perturbations of the game. This is possible because the efficiency operator is typically unstable, and thus a small perturbation could improve the efficiency dramatically. The restriction to small perturbation is important in practice, because quite often the mediator has only limited resources (including the right to make certain policy). In particular, when the mediator has several games to mediate but has only limited resources, it will be wise to set a higher priority for her resources on the game(s) whose efficiency can be improved more easily. Moreover, the efficiency operator could be discontinuous at a small parameter, and at those discontinuous points a very small change of the investment can make a big difference on the efficiency. Then it is important to figure out those points, so that the mediator can use her resources in full effect.

However, it is mathematically very challenging to find the optimal mechanism for the mediator, especially in continuous time models. We shall leave them for future research.

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