

A Dynamic Principal Agent Problem with One-sided Commitment

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Abstract

In this paper we consider a principal agent problem with moral hazard, where the principal is committed to the contract but the agent can quit at a stopping time up to his choice. When the current agent quits the job, both the principal and the agent will incur a cost, and the principal will hire a new agent, possibly with a different type. Being aware of the agent's noncommitment, the principal's problem is to find the optimal contract satisfying the agent's (and all possible future agents') individual rationality. We characterize the principal's dynamic value function, which could be discontinuous on the boundary, as the (unique) minimal solution of an infinite dimensional system of HJB equations, parametrized by the agent's type. In particular, this dynamic problem is time consistent in certain sense.

Some interesting findings are worth mentioning. First, self-enforcing contracts are typically suboptimal. The principal would rather let the agent quit and hire a new one. Next, the standard contract for committed agent may also be suboptimal. This is due to the presence of different types of agents in our model. In some markets, the principal may prefer no commitment from the agent so she can hire a cheaper agent from the market by designing the contract to induce the agent to quit. Moreover, due to the cost incurring to the agent, the principal will see only finitely many quittings.

Keywords. Principal agent problems, contract theory, one-sided commitment, self-enforcing contracts, time inconsistency

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1 Introduction

The contract theory involves two parties: the principal (she) hires an agent (he) by offering a contract, which provides incentives for the agent to work for her. Such theory has had great impact on organizational economics and corporate finance, as well as many related fields. One typical example, which motivates our work, is employer (principal) versus employee (agent) in labor markets. Given a contract, the agent's problem is to choose efforts to maximize his expected utility, depending on both the contract and the efforts. Then the principal's problem is to design an optimal contract to maximize her own expected utility, by anticipating that the agent would always choose his corresponding optimal efforts. There is a constraint on the admissible contracts: the agent's optimal utility should satisfy certain *individual rationality* (IR) constraint, which can be interpreted as the market value of the agent. The continuous time principal agent problems were first studied by the seminal paper Holmström-Milgrom [19], followed by Schättler-Sung [33] and Sung [36, 37]. There have been numerous publications on the subject in the past three decades. We refer to the monograph Cvitanic-Zhang [10] and the references therein. In particular, we refer to Sannikov [32] and Cvitanic-Possamai-Touzi [7, 8] for the dynamic programming approach, which will be used in this paper.

All the above literature consider the principal agent problem on a fixed time horizon $[0, T]$ (or $[0, \infty)$). There is a crucial time inconsistency issue which does not seem to receive much attention in the literature: the optimal contract typically does not remain optimal if the two parties reconsider the contract at a later time t . When both parties are fully committed, this time inconsistency issue is irrelevant. However, in many situations, for example in a typical labor contract the agent is allowed to quit before T , then the principal will have to offer another contract to hire a new agent at the quitting time. The time inconsistency means that the new contract will be different from the current one, which was supposed to be optimal at time 0 as found in the standard literature. Thus, being aware of the agent's possible quitting, the principal would design the contract differently in the beginning. We shall emphasize that, when an agent quits, the new contract should satisfy the new IR constraint determined by the market.

There have been very serious efforts on principal agent problems with one-sided commitment. For example, Grochulski-Zhang [14], Jeon-Koo-Park [21], Niu-Yang-Zou [29], Phelan [30], Ray [31], and Zhang [42] studied models where the agent is not committed. They restricted to the self-enforcing contracts, that is, the agent's continuation utility satisfies the IR constraint at any time. This is in the spirit of the renegotiation-proof equilibria, see

e.g. Farrell-Maskin [12] and Strulovici [35], and under such a contract the agent actually will never quit. We also note that these works mainly consider risk sharing in full information setting, without moral hazard and in particular without the incentive compatibility constraint. The works Gottlieb-Zhang [13] and Karaivanov-Martin [22] require incentive compatibility, but also restricts the contracts to self-enforcing ones¹. The works Ai-Li [2], Miao-Zhang [28], and Thomas-Worrall [38] are also in the realm of self-enforcing contracts, but with two-sided limited commitment, namely both the principal and the agent can terminate the contract. In a different direction, Krueger-Uhlig [23] considered a model with multiple principals and agents where the agents are not committed to the contract and can switch to another principal. Their main focus is to determine endogenously the outside option of the agent, which amounts to the IR constraint, through competition of the principals. However, there is no discussion on the principal's behavior after the agent quits, which is the main focus of our paper. The work Ai-Kiku-Li-Tong [1] also endogenizes the outside options, through assortative matching. However, their stopping times for terminating the contract are exogenously given, and before that the contracts are still required to be self-enforcing. Another related work is Hu-Ren-Yang [20] where one agent can switch among multiple principals. Their main focus is to find the equilibrium (and mean field equilibrium) contract among the principals. We also refer to Capponi-Frei [4], Hajjej-Hillairet-Mnif [15], He-Tan-Zou [17], and Lin-Ren-Touzi-Yang [25] where the principal and/or the agent can choose a stopping time to terminate the contract,² but the principal does not hire a new agent. We emphasize again that, when the contracts are not committed by both parties, their time inconsistency becomes a serious issue.

We remark that, while disincentivizing the agent from quitting, the self-enforcing contracts are typically sub-optimal for the principal. She would rather offer a standard contract, which satisfies the IR constraint only at initial time, and be prepared for a possible quitting of the agent. In this paper we consider a continuous time principal agent problem with moral hazard where the agent is allowed to quit. Our main feature is that the principal would hire a new agent when the current agent quits the job, and we shall investigate the time consistency issue from the principal's perspective. One particular application of our model is the dynamic firm-worker relationships, c.f. Harris-Holmström [16]. There is one risk neutral principal and a family of risk averse agents with different types, parametrized by θ . Unlike the adverse selection problems which involve only one agent but with unknown (to the principal) type, here the principal knows the types of all agents but hires only one at

¹It is called non-lapsing constraint in [13], and the so called front-loading constraint is also studied there.

²In [4] the agent chooses a density function for certain stopping time, not the stopping time itself.

any particular time. The principal would pay the agent continuously in time, without lump sum payments. The current agent with type θ would quit the job at time t if he prefers the new opportunity from the market, quantified by the dynamic individual rationality R_t^θ . If he quits at t , he will incur a cost c_t^θ and the principal will also suffer a loss c_t^P . The principal will hire a new agent, possibly with a different type $\tilde{\theta}$, and the new agent is also allowed to quit before the expiration date T . For simplicity, in this paper we assume $R_t^\theta, c_t^\theta, c_t^P$ are exogenously given and are deterministic.

Given a contract, the agent can choose his efforts and the quitting time, which is a stopping time. We note that we do not keep track of the agent's status once he quits the current job. This agent's problem is a mixed optimal control and stopping problem, and as standard can be solved through a reflected backward SDE, see e.g. Zhang [41]. We follow the idea of [32] to rewrite the agent's continuation (optimal) utility process as a forward process, see also Ma-Yong [27] for similar idea in the contexts of forward backward SDEs. Then the principal's problem becomes a constrained stochastic control problem, where the agent's continuation utility becomes the state process, and the controls are the contract and the target effort, with a constraint that the agent's terminal utility is zero (since there is no lump sum payment). The agent's optimal quitting time is an intrinsic hitting time, when the agent's continuation utility process hits the lower barrier $\underline{L}_t^\theta := R_t^\theta - c_t^\theta$ (for type θ agent). This enables us to use the HJB equation approach to solve the principal's problem.

We now discuss the most crucial component of the problem: the boundary condition of the HJB equation on the lower barrier \underline{L}_t^θ . When the current type θ agent's continuation utility process hits \underline{L}_t^θ at t , the agent would quit, and then the principal would hire a new agent. First, the principal will choose a different type $\tilde{\theta}$, optimal for the principal's utility. Moreover, the new agent is still allowed to quit, and thus the boundary condition at \underline{L}_t^θ should be the principal's optimal utility at $R_t^{\tilde{\theta}}$ (minus the cost c_t^P), which is exactly the solution we are looking for. So we are considering an infinitely dimensional system of HJB equations, parametrized by θ , where the interaction is through the boundary condition. The last feature that the boundary condition involves the solution itself actually reflects the time consistent nature of our formulation. However, this adds to the difficulty of the problem, since the boundary condition is not exogenously given.

We establish the wellposedness of this HJB system by recursive approximation. Let $V_n(\theta; \cdot)$ denote the principal's optimal value function when the current agent is type θ and the principal allows at most n quittings in the remaining time period, that is, after seeing n quittings the principal will only offer self-enforcing contracts for the remaining period. Then the boundary condition for $V_n(\theta; \cdot)$ at \underline{L}_t^θ will be $\sup_{\tilde{\theta}} V_{n-1}(\tilde{\theta}; t, R_t^{\tilde{\theta}}) - c_t^P$, which is

exogenously given in a recursive way. Under some technical conditions, we show that V_n converges uniformly to the unique solution of the HJB system. Another subtle point is that, at the quitting time t , while the agent is indifferent on quitting or staying and thus the agent's optimal utility is continuous at t , the principal's utility may have a jump due to the quitting, and thus the principal's value function is typically discontinuous at the boundary \underline{L}_t^θ . Therefore, the boundary condition is actually an inequality and our value function is characterized as the minimal viscosity solution of the HJB system.

Some findings implied by our results are interesting from practical perspectives. First, self-enforcing contracts are typically only sub-optimal for the principal. That is, given that the agent is not committed to the contract, it is not desirable for the principal to offer self-enforcing contracts to disincentivize the agent from quitting, she would rather let the agent quit and rehire a new one when that happens. This observation is more or less obvious, and is strongly confirmed by our results. Next, the standard optimal contract (satisfying the IR constraint only at the initial time) may also be only sub-optimal. That is, in some markets, the principal may prefer no commitment from the agent so that she can hire a cheaper one from the market at a later time by designing the contract to induce the current agent to quit. The reason is that R^θ (and c^θ, c^P) evolves along the time. While it is optimal for the principal to hire an agent θ at time 0, at some later time t another agent $\tilde{\theta}$ may become much cheaper, and thus the principal would be happy to see agent θ to quit so that she could hire agent $\tilde{\theta}$ to replace him³. Moreover, by assuming uniform lower bound of the cost c^θ , the agents won't quit too frequently and actually the principal will only see finitely many quittings. This is of course consistent with what we observe in practice.

We shall mention another direction in the literature on the time inconsistency issue, see e.g. Balbus-Reffett-Wozny [3], Cetemen-Feng-Urgun [5]⁴, Djehiche-Helgesson [11], Hernández-Possamai [18], Li-Mu-Yang [24], Liu-Mu-Yang [26], Wang-Huang-Liu-Zhang [39], Yilmaz [40], and [13], to mention a few. Here the preference of the agent (and/or the principal) is time inconsistent, for example due to non-exponential discounting. We refer to the survey paper Strotz [34] for time inconsistent stochastic control problems, in particular, one may consider pre-committed, naive, and sophisticated agents. The principal knows what type of agent she is hiring, and would find the optimal contract (or possibly sub-optimal

³Some other principals may still prefer agent θ at time t , and that's why R_t^θ could remain high in the market. But we do not model competitions among principals in this paper. We refer again to [1, 23] for endogenous models for R_t^θ . It will be very interesting to combine our work with those models, and we shall leave that for future research.

⁴In this work both the principal and the agent have time inconsistent preferences, and they consider certain optimal renegotiation-proof contract.

one when the principal's preference is also time inconsistent) by anticipating that the agent would always choose his optimal or sub-optimal efforts in a corresponding way. We emphasize again that the time inconsistency in these works is due to the individual player's preference. This has completely different nature from the time inconsistency we investigate in this paper due to the interaction between the principal and the agents.

Finally we clarify the time consistency we obtain in this paper. First, our agent's problem is always time consistent for the period he stays with the job, and we do not keep track of the agent once he leaves the current principal. Our principal's problem is time consistent at the agents' optimal quitting times, see Theorem 6.5. That is, the principal would never regret when an agent quits and thus she could offer a new contract to another agent. We should note though, if the principal reconsiders the contract within the time period an agent is staying, she may find it not optimal. However, since the principal is required to commit to the contract, she is not allowed to fire the agent and sign a better contract during those time periods, and thus such time inconsistency for the principal is irrelevant from practical considerations.

The rest of the paper is organized as follows. In Section 2 we introduce our model and consider self-enforcing contracts. In Section 3 we consider the model where only one quitting of the agent is allowed, and the principal's dynamic value function in this model is analyzed in Section 4. In Section 5 we extend these results to the case with at most n quittings, and in Section 6 we remove the restriction on the number of quittings. In Section 7 we reinvestigate the problem when the benchmark model is the standard principal agent problem with full commitment. Finally in Section 8 we summarize our findings.

2 Our model and self-enforcing contracts

In this section we introduce our model on the fixed time horizon $[0, T]$. In particular, we shall consider self-enforcing contracts so that the agent has no incentive to quit.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space on $[0, T]$, B a standard one dimensional Brownian motion, and $\mathbb{F} = \mathbb{F}^B$. There are two players: a principal (she) and an agent (he). The agent has a parameter $\theta \in \Theta$, which can be viewed as the type of the agent. In this section, θ is fixed, while in the rest of the paper, we will consider different types of agents.

The principal's control, namely the contract, is a continuous time payment $\eta = \{\eta_t\}_{0 \leq t \leq T}$. The agent's control, namely his action or effort, is a process $\alpha = \{\alpha_t\}_{0 \leq t \leq T}$. In particular, there are no terminal payments, which are not convenient to model when an agent quits. The admissible controls satisfy the following conditions, required for technical convenience.

Definition 2.1. (i) Let \mathcal{A}^P denote the set of \mathbb{F} -progressively measurable processes η on $[0, T]$ such that $-C(\eta) \leq \eta \leq C_0$. That is, η is bounded, with a uniform upper bound $C_0 > 0$, but the lower bound $-C(\eta)$ may depend on η .

(ii) Let \mathcal{A}^A denote the set of \mathbb{F} -progressively measurable processes α on $[0, T]$ such that

$$\mathbb{E}_t^{\mathbb{P}} \left[\int_t^T |\alpha_s|^2 ds \right] \leq C(\alpha), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \quad (2.1)$$

where the constant $C(\alpha) > 0$ may depend on α . That is, $\int_0^t \alpha_s dB_s$ is a BMO martingale.

We consider moral hazard models, and as in the standard literature we use the weak formulation as follows, see e.g. Cvitanović-Wan-Zhang [9]. The state process is:

$$X_t \equiv B_t. \quad (2.2)$$

The agent controls the law of X , denoted as \mathbb{P}^α , through the Girsanov theorem:

$$B_t^\alpha := B_t - \int_0^t \alpha_s ds, \quad M_t^\alpha := \exp \left(\int_0^t \alpha_s dB_s - \frac{1}{2} \int_0^t |\alpha_s|^2 ds \right), \quad d\mathbb{P}^\alpha := M_T^\alpha d\mathbb{P}. \quad (2.3)$$

Note that under (2.1) M^α is a true martingale under \mathbb{P} and hence \mathbb{P}^α is another probability measure. Moreover, there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that (c.f. [41, Section 7.2])

$$\mathbb{E}^{\mathbb{P}} \left[\exp \left(\varepsilon \int_0^T |\alpha_s|^2 ds \right) + |M_T^\alpha|^{1+\varepsilon} \right] < \infty. \quad (2.4)$$

We shall also introduce the following notations:

$$\mathbb{E} := \mathbb{E}^{\mathbb{P}}, \quad \mathbb{E}^\alpha := \mathbb{E}^{\mathbb{P}^\alpha}, \quad \mathbb{E}_t^\alpha := \mathbb{E}^\alpha[\cdot | \mathcal{F}_t]. \quad (2.5)$$

2.1 The agent's problem

Given $\eta \in \mathcal{A}^P$, the agent's optimal expected utility at time 0 is:

$$V_0^A(\theta; 0, \eta) := \sup_{\alpha \in \mathcal{A}^A} \mathbb{E}^\alpha \left[\int_0^T f(\theta; s, \eta_s) ds - \int_0^T h(\theta; s, \alpha_s) ds \right]. \quad (2.6)$$

Here the agent's running utility function f and running cost function h satisfy Assumption 2.2 below. Moreover, the subscript 0 in V_0^A indicates that the agent is allowed to quit 0 time, namely not allowed to quit.

Assumption 2.2. *There exists a constant $\Lambda_0 \geq 1$ satisfying the following.*

(i) f is twice differentiable in η with:

$$f \leq -\frac{1}{\Lambda_0} < 0, \quad \partial_\eta f \geq \frac{1}{\Lambda_0} > 0, \quad \partial_{\eta\eta} f \leq -\frac{1}{\Lambda_0} < 0, \quad \text{for all } \eta \leq C_0; \quad (2.7)$$

$$\text{and } f(\theta; s, C_0) \geq -\Lambda_0.$$

(ii) h is twice differentiable in α with:

$$h \geq 0 \text{ with } h(\theta; s, 0) = 0, \quad \frac{1}{\Lambda_0} \leq \partial_{\alpha\alpha} h \leq \Lambda_0. \quad (2.8)$$

(iii) $f, h, \partial_{\alpha} h$ are uniformly continuous in s , with a common modulus of continuity function ρ , and we assume without loss of generality that $\rho(\delta) \geq \delta$.

Clearly, in this section we may allow Λ_0 and ρ depend on θ . However, for the purpose of later sections, we require them to be common for all θ . Throughout the paper, we shall denote by C a generic constant which may depend on T, C_0 and Λ_0 , but not on ρ or θ . One typical example satisfying Assumption 2.2 is: for some appropriate $\lambda_1(\theta), \lambda_2(\theta) > 0$,

$$f(\theta; s, \eta) = -\lambda_1(\theta)e^{-\lambda_1(\theta)\eta}, \quad h(\theta; s, \alpha) = \frac{1}{2\lambda_2(\theta)}|\alpha|^2. \quad (2.9)$$

Under (2.7) and (2.8), it is clear that $f(\theta; s, \cdot)$ and $\partial_{\alpha} h(\theta; s, \cdot)$ have inverse functions, denoted as $I^f(\theta; s, \cdot)$ and $I^h(\theta; s, \cdot)$, respectively. That is,

$$f(\theta; s, I^f(\theta; s, y)) = y, \quad y \leq f(\theta; s, C_0); \quad \partial_{\alpha} h(\theta; s, I^h(\theta; s, z)) = z, \quad z \in \mathbb{R}. \quad (2.10)$$

Moreover, we introduce the following functions:

$$H(\theta; s, z) := \sup_{\alpha \in \mathbb{R}} [\alpha z - h(\theta; s, \alpha)], \quad \tilde{H}(\theta; s, z) := -h(s, \theta, I^h(\theta; s, z)). \quad (2.11)$$

It is clear that I^h is the maximum argument of H : $H(\theta; s, z) = zI^h(\theta; s, z) - h(s, \theta, I^h(\theta; s, z))$.

Moreover, it is straightforward to verify the following: for some constant $C(\Lambda_0) > 0$,

$$\begin{aligned} 0 < \partial_y I^f \leq \Lambda_0, \quad |\eta|^2 \leq C(\Lambda_0) [1 - f(\theta; s, \eta)], \quad \text{for all } \eta \leq C_0; \\ I^h(\theta; s, 0) = 0, \quad 0 < \frac{1}{\Lambda_0} \leq \partial_z I^h(\theta; s, z) \leq \Lambda_0 < \infty; \\ I^h \text{ is uniformly continuous in } s \text{ with modulus of continuity function } \Lambda_0 \rho; \end{aligned} \quad (2.12)$$

$$\tilde{H}(s, \theta, z) \leq \tilde{H}(s, \theta, 0) = 0 = H(s, \theta, 0) \leq H(s, \theta, z);$$

$$|\partial_z H(\theta; s, z)|, |\partial_z \tilde{H}(\theta; s, z)| \leq \Lambda_0 |z|, \quad |z|^2 \leq -C(\Lambda_0) \tilde{H}(\theta; s, z).$$

The solution to the agent's problem is standard, see e.g. [10].

Lemma 2.3. *Let Assumption 2.2 hold and $\eta \in \mathcal{A}^P$. Then $V_0^A(\theta; 0, \eta) = Y_0^{\theta, \eta}$, where $(Y^{\theta, \eta}, Z^{\theta, \eta})$ is the solution to the following BSDE:*

$$Y_t^{\theta, \eta} = \int_t^T [f(\theta; s, \eta) + H(\theta; s, Z_s^{\theta, \eta})] ds - \int_t^T Z_s^{\theta, \eta} dB_s, \quad t \in [0, T]. \quad (2.13)$$

Moreover, the agent's unique optimal control is $\alpha^* = I^h(\theta; \cdot, Z^{\theta, \eta}) \in \mathcal{A}^A$.

Recall (2.3) and (2.5), we denote

$$B^{\theta,Z} := B^{I^h(\theta;\cdot,Z)}, \quad \mathbb{P}^{\theta,Z} := \mathbb{P}^{I^h(\theta;\cdot,Z)}, \quad \mathbb{E}^{\theta,Z} := \mathbb{E}^{I^h(\theta;\cdot,Z)}, \quad \mathbb{E}_t^{\theta,Z} := \mathbb{E}_t^{I^h(\theta;\cdot,Z)}. \quad (2.14)$$

Moreover, introduce

$$\bar{L}_t^\theta := \int_t^T f(\theta; s, C_0) ds, \quad 0 \leq t \leq T. \quad (2.15)$$

We note that, when $\eta \equiv C_0$, obviously $(\bar{L}_t^\theta, 0)$ satisfies BSDE (2.13), so $(Y_t^{\theta,C_0}, Z_t^{\theta,C_0}) = (\bar{L}_t^\theta, 0)$. Since $\eta \leq C_0$, by applying the comparison principle for BSDE (2.13), we have

$$Y_t^{\theta,\eta} \leq \bar{L}_t^\theta, \quad \text{for all } \eta \in \mathcal{A}^P. \quad (2.16)$$

The following estimate will be used frequently in the paper.

Lemma 2.4. *Let Assumption 2.2 hold. For any $t < T$ and $\eta \in \mathcal{A}^P$, we have*

$$\begin{aligned} \mathbb{E}_t^{\theta,Z^{\theta,\eta}} \left[\int_t^T [|Z_s^{\theta,\eta}|^2 + C_0 - \eta_s] ds \right] &\leq C [\bar{L}_t^\theta - Y_t^{\theta,\eta}], \\ \mathbb{E}_t^{\theta,Z^{\theta,\eta}} \left[\int_t^T |\eta_s|^2 ds \right] &\leq C [T - t - Y_t^{\theta,\eta}]. \end{aligned} \quad (2.17)$$

Proof Denote $(Y, Z) := (Y^{\theta,\eta}, Z^{\theta,\eta})$ for notational simplicity. Note that

$$Y_t = \int_t^T [f(\theta; s, \eta) + \tilde{H}(\theta; s, Z_s)] ds - \int_t^T Z_s dB_s^{\theta,Z}.$$

Then, by (2.15),

$$\bar{L}_t^\theta - Y_t = \mathbb{E}_t^{\theta,Z} \left[\int_t^T [f(\theta; s, C_0) - f(\theta; s, \eta_s)] - \tilde{H}(\theta; s, Z_s) ds \right].$$

Since $\eta \leq C_0$ and $\partial_\eta f \geq \frac{1}{\Lambda_0}$, and by (2.12), we have

$$f(\theta; s, C_0) - f(\theta; s, \eta_s) \geq \frac{1}{\Lambda_0} (C_0 - \eta_s) \geq 0; \quad -\tilde{H}(\theta; s, Z_s) \geq \frac{1}{C} |Z_s|^2 \geq 0.$$

Then

$$\begin{aligned} \mathbb{E}_t^{\theta,Z} \left[\int_t^T |Z_s|^2 ds \right] &\leq C \mathbb{E}_t^{\theta,Z} \left[\int_t^T [-\tilde{H}(\theta; s, Z_s)] ds \right] \leq C [\bar{L}_t^\theta - Y_t]; \\ \mathbb{E}_t^{\theta,Z} \left[\int_t^T [C_0 - \eta_s] ds \right] &\leq C \mathbb{E}_t^{\theta,Z} \left[\int_t^T [f(\theta; s, C_0) - f(\theta; s, \eta_s)] ds \right] \leq C [\bar{L}_t^\theta - Y_t]. \end{aligned}$$

Moreover, by (2.12) we have $|\eta_s|^2 \leq C[1 - f(\theta; s, \eta_s)]$. Then,

$$\begin{aligned} \mathbb{E}_t^{\theta,Z} \left[\int_t^T |\eta_s|^2 ds \right] &\leq C \mathbb{E}_t^{\theta,Z} \left[\int_t^T [1 - f(\theta; s, \eta_s)] ds \right] \\ &\leq C \mathbb{E}_t^{\theta,Z} \left[\int_t^T [1 - f(\theta; s, \eta_s) - \tilde{H}(\theta; s, Z_s)] ds \right] = C [T - t - Y_t]. \quad \square \end{aligned}$$

2.2 The principal's problem

We consider a risk neutral principal. Given $\eta \in \mathcal{A}^P$, the principal's expected utility is:

$$J_P(\theta; 0, \eta) := \mathbb{E}^{\alpha^*} \left[X_T - \int_0^T \eta_s ds \right] = \mathbb{E}^{\theta, Z^{\theta, \eta}} \left[\int_0^T [I^h(\theta; s, Z_s^{\theta, \eta}) - \eta_s] ds \right]. \quad (2.18)$$

Remark 2.5. (i) In (2.3) we assume the drift of the state process is α . This is a simplification without loss of generality. Indeed, if we consider a more general drift $b(\theta; s, \alpha_s)$, we may simply set a new control $\tilde{\alpha}_s := b(\theta; s, \alpha_s)$, and one can easily see that the agent's problem (2.6) is equivalent to the problem by replacing h with $\tilde{h}(\theta; s, \tilde{\alpha}_s) := \inf_{\alpha_s: b(\theta; s, \alpha_s) = \tilde{\alpha}_s} h(\theta; s, \alpha_s)$.

(ii) As in the standard literature for principal agent problems with one-sided commitment, in (2.18) we consider a risk neutral principal. We note that the slightly more general case with a discount factor r can be easily transformed into our model. Indeed, consider

$$J_P(\theta; 0, \eta) := \mathbb{E}^{\alpha^*} \left[e^{-rT} X_T - \int_0^T e^{-rs} \eta_s ds \right] = e^{-rT} \mathbb{E}^{\theta, Z^{\theta, \eta}} \left[\int_0^T [I^h(\theta; s, Z_s^{\theta, \eta}) - e^{r(T-s)} \eta_s] ds \right].$$

Denote

$$\tilde{\eta}_s := e^{r(T-s)} \eta_s, \quad \tilde{Z}^{\theta; \tilde{\eta}} := Z^{\theta, \tilde{\eta} e^{r(\cdot - T)}}, \quad \tilde{J}_P(\theta; 0, \tilde{\eta}) := \mathbb{E}^{\theta, \tilde{Z}^{\theta, \tilde{\eta}}} \left[\int_0^T [I^h(\theta; s, \tilde{Z}_s^{\theta, \tilde{\eta}}) - \tilde{\eta}_s] ds \right].$$

Then the principal's problem is equivalent to optimizing \tilde{J}_P , which has no discount factor.

(iii) The case with a risk averse principal is technically harder. Moreover, in (2.6) we assume the agent's utility f and cost h are separable. The general non-separable case in the form $f(\theta; s, \eta, \alpha)$ is also technically harder. We do not pursue such generality in this paper.

Introduce the agent's individual rationality R_t^θ at time t , which can be interpreted as the agent's market value if he works for a principal over period $[t, T]$. In this paper we assume R^θ is deterministic and is given exogenously, and we leave more general cases to future research. In light of (2.16), we assume the following, with the same Λ_0, ρ for simplicity.

Assumption 2.6. (i) $-\Lambda_0 \leq R_s^\theta \leq \bar{L}_s^\theta$, $0 \leq s \leq T$, and $R_T^\theta = 0$.

(ii) R^θ is uniformly Lipschitz continuous in s , with modulus of continuity function ρ .

In the standard literature, the principal's problem is as follows:

$$V_0^{S,P}(\theta; 0) := \sup_{\eta \in \mathcal{A}^P} J_P(\theta; 0, \eta), \quad \text{subject to} \quad V_0^A(\theta; 0, \eta) \geq R_0^\theta. \quad (2.19)$$

Here S stands for standard. We shall provide more discussions on $V_0^{S,P}(\theta; 0)$ in Section 7.1 below. In this paper we mainly focus on self-enforcing contracts: recalling Lemma 2.3,

$$V_0^P(\theta; 0) := \sup_{\eta \in \mathcal{A}^P} J_P(\theta; 0, \eta), \quad \text{subject to} \quad Y_t^{\theta, \eta} \geq R_t^\theta, \quad 0 \leq t \leq T, \text{ a.s.} \quad (2.20)$$

By (2.16) it is clear that $\eta \equiv C_0$ satisfies the constraints and thus

$$V_0^{S,P}(\theta; 0) \geq V_0^P(\theta; 0) \geq J_P(\theta; 0, C_0) = -C_0(T - t). \quad (2.21)$$

2.3 The principal's dynamic value

We first extend the above principal agent problem to arbitrary time interval $[t, T]$. Denote by $B_s^t := B_s - B_t$ the shifted Brownian motion on $[t, T]$, $\mathbb{F}^t := \mathbb{F}^{B^t}$, and \mathcal{T}_t the set of \mathbb{F}^t -stopping times on $[t, T]$. We shall extend all the notations in the previous subsection to the shifted space in the obvious manner. In particular, $\mathcal{A}_t^P, \mathcal{A}_t^A$ will denote the set of \mathbb{F}^t -progressively measurable processes η, α , but we shall still denote by $(Y^{\theta, \eta}, Z^{\theta, \eta})$ the solution to the BSDE (2.13) on $[t, T]$. We then define the principal's dynamic value over self-enforcing contracts:

$$\begin{aligned} V_0^P(\theta; t) &:= \sup_{\eta \in \mathcal{A}_t^P} J_P(\theta; t, \eta), \quad \text{subject to } Y_s^{\theta, \eta} \geq R_s^\theta, \quad t \leq s \leq T, \text{ a.s.} \\ \text{where } J_P(\theta; t, \eta) &:= \mathbb{E}^{\theta, Z^{\theta, \eta}} \left[\int_t^T [I^h(\theta; s, Z_s^{\theta, \eta}) - \eta_s] ds \right]. \end{aligned} \quad (2.22)$$

The following result will be important in the next section.

Proposition 2.7. *Under Assumptions 2.2 and 2.6, $V_0^P(\theta; \cdot)$ is uniformly continuous in t :*

$$|V_0^P(\theta; s) - V_0^P(\theta; t)| \leq C\sqrt{\rho(|s - t|)}, \quad \forall s, t \in [0, T]. \quad (2.23)$$

Proof Without loss of generality we estimate only $|V_0^P(\theta; 0) - V_0^P(\theta; \delta)|$ for $\delta \leq T$.

Step 1. Let $\eta \in \mathcal{A}^P = \mathcal{A}_0^P$ satisfies the constraint in (2.20), and simplify the notation $(Y, Z) := (Y^{\theta, \eta}, Z^{\theta, \eta})$. Clearly $\eta_{[\delta, T]}$ also satisfies the constraint in (2.22) with $t = \delta$, then

$$\mathbb{E}_{\mathcal{F}_\delta}^{\theta, Z} \left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s] ds \right] \leq V_0^P(\theta; \delta), \quad \text{a.s.}$$

Thus, by (2.12) and (2.17),

$$\begin{aligned} J_P(\theta; 0, \eta) - V_0^P(\theta; \delta) &\leq \mathbb{E}^{\theta, Z} \left[\int_0^\delta [I^h(\theta; s, Z_s) - \eta_s] ds \right] \\ &\leq \mathbb{E}^{\theta, Z} \left[\int_0^\delta [C|Z_s| + |\eta_s|] ds \right] \leq C\sqrt{\delta} \left(\mathbb{E}^{\theta, Z} \left[\int_0^T [|Z_s|^2 + |\eta_s|^2] ds \right] \right)^{\frac{1}{2}} \\ &\leq C\sqrt{\delta} \left(\overline{L}_0^\theta - Y_0 + T - Y_0 \right)^{\frac{1}{2}} \leq C\sqrt{\delta} \left(\overline{L}_0^\theta + T - 2R_0^\theta \right)^{\frac{1}{2}} \leq C\sqrt{\delta}. \end{aligned}$$

Since η is arbitrary, we obtain $V_0^P(\theta; 0) - V_0^P(\theta; \delta) \leq C\sqrt{\delta}$.

Step 2. Let $\eta \in \mathcal{A}_\delta^\mathbb{P}$ satisfies the constraint in (2.22). It suffices to prove

$$J_P(\theta; \delta, \eta) - V_0^P(\theta; 0) \leq C\sqrt{\rho(\delta)}. \quad (2.24)$$

Then by the arbitrariness of η we obtain $V_0^P(\theta; \delta) - V_0^P(\theta; 0) \leq C\sqrt{\rho(\delta)}$.

Denote $\delta' := \rho(\delta) + \Lambda_0\delta$ and $(Y, Z) := (Y^{\theta, \eta}, Z^{\theta, \eta})$. We prove (2.24) in two cases.

Case 1. $Y_\delta \geq \bar{L}_\delta^\theta - \delta'$. Then, by (2.12), (2.17), and (2.21),

$$\begin{aligned} J_P(\theta; \delta, \eta) &= \mathbb{E}^{\theta, Z} \left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s] ds \right] \\ &\leq \mathbb{E}^{\theta, Z} \left[\int_\delta^T [C|Z_s| + C_0 - \eta_s] ds \right] - C_0(T - \delta) \\ &\leq C \left[\sqrt{\bar{L}_\delta^\theta - Y_\delta} + \bar{L}_\delta^\theta - Y_\delta \right] + V_0^P(\theta; 0) + C_0\delta \\ &\leq C \left[\sqrt{\delta'} + \delta' \right] + V_0^P(\theta; 0) + C_0\delta \leq V_0^P(\theta; 0) + C\sqrt{\rho(\delta)}. \end{aligned}$$

Case 2. $Y_\delta < \bar{L}_\delta^\theta - \delta'$. Construct

$$\begin{aligned} \tau_\delta &:= \inf \{s \geq \delta : Y_s = \bar{L}_s^\theta - \delta'\} < T, \quad s \in [\delta, T], \\ \eta_s^\delta &:= C_0 \mathbf{1}_{[0, \delta)}(s) + \eta_s \mathbf{1}_{[\delta, \tau_\delta)}(s) + C_0 \mathbf{1}_{[\tau_\delta, T]}(s). \end{aligned}$$

It is clear that $\eta^\delta \in \mathcal{A}^P$. Denote $(Y^\delta, Z^\delta) := (Y^{\theta, \eta^\delta}, Z^{\theta, \eta^\delta})$. One can easily verify that

$$(Y_s^\delta, Z_s^\delta) = (Y_s + \delta', Z_s) \mathbf{1}_{[\delta, \tau_\delta)}(s) + (\bar{L}_s^\theta, 0) \mathbf{1}_{[\tau_\delta, T]}(s), \quad s \in [\delta, T].$$

In particular, $Y_s^\delta \geq Y_s \geq R_s^\theta$ for $s \in [t, T]$. Moreover, for $s \in [0, \delta]$, we have

$$Y_s^\delta = Y_s^\delta + \int_t^\delta [f(\theta; s, \eta_s^\delta) + H(\theta; s, Z_s^\delta)] ds - \int_t^\delta Z_s^\delta dB_s.$$

Since $\eta \in \mathcal{A}_\delta^P$ is \mathbb{F}^{B^δ} -progressively measurable, then Y_δ is deterministic, and hence so is Y_δ^δ .

This implies $Z_s^\delta = 0$ for $s \in [0, \delta]$. Therefore, by (2.7) and the regularity of R^θ ,

$$Y_s^\delta = Y_\delta^\delta + \int_s^\delta f(\theta; r, C_0) dr \geq Y_\delta^\delta - \Lambda_0\delta = Y_\delta + \rho(\delta) \geq R_\delta^\theta + \rho(\delta) \geq R_s^\theta, \quad s \in [0, \delta].$$

That is, η^δ satisfies the constraints in (2.20). Then

$$\begin{aligned} J_P(\theta; \delta, \eta) - V_0^P(\theta; 0) &\leq J_P(\theta; \delta, \eta) - J_P(\theta; 0, \eta^\delta) \\ &= \mathbb{E}^{\theta, Z} \left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s] ds \right] - \mathbb{E}^{\theta, Z^\delta} \left[\int_\delta^{\tau_\delta} [I^h(\theta; s, Z_s) - \eta_s] ds + C_0[T - \tau_\delta - \delta] \right]. \end{aligned}$$

Since $Z^\delta = Z$ on $[\delta, \tau_\delta]$, then $\mathbb{P}^{\theta, Z^\delta} = \mathbb{P}^Z$ on $\mathcal{F}_{\tau_\delta}^\delta$. Thus, by (2.17),

$$\begin{aligned}
& J_P(\theta; \delta, \eta) - V_0^P(\theta; 0) \\
& \leq \mathbb{E}^{\theta, Z} \left[\int_\delta^T [I^h(\theta; s, Z_s) - \eta_s] ds \right] - \mathbb{E}^{\theta, Z} \left[\int_\delta^{\tau_\delta} [I^h(\theta; s, Z_s) - \eta_s] ds + C_0[T - \tau_\delta - \delta] \right] \\
& \leq \mathbb{E}^{\theta, Z} \left[\int_{\tau_\delta}^T [I^h(\theta; s, Z_s) + C_0 - \eta_s] ds \right] \leq \mathbb{E}^{\theta, Z} \left[\int_{\tau_\delta}^T [C|Z_s| + C_0 - \eta_s] ds \right] \\
& \leq C \mathbb{E}^{\theta, Z} \left[\sqrt{\overline{L}_{\tau_\delta}^\theta - Y_{\tau_\delta}} + \overline{L}_{\tau_\delta}^\theta - Y_{\tau_\delta} \right] = C \left[\sqrt{\delta'} + \delta' \right] \leq C \sqrt{\rho(\delta)}. \quad \square
\end{aligned}$$

Remark 2.8. *The dynamic problem (2.22) is time inconsistent in the following sense. Assume the principal's problem (2.20) at time 0 admits an optimal contract $\eta^* \in \mathcal{A}_0^P$. When time evolves to a later time $t > 0$, the remaining part of the contract $\eta_{[t, T]}^*$ may not be optimal for the problem $V^P(\theta; t)$. Indeed, consider an extreme case that $R_0^\theta \in \mathbb{R}$ but $R_t^\theta = -\infty$ for $t > 0$, then $V_0^P(\theta; 0) = V_0^{S, P}(\theta; 0)$. Assume it has an optimal contract η^* . But for $t > 0$, obviously $\eta^* - 1$ still satisfies the required constraints in (2.22) (since $R_s^\theta = -\infty$ for $s \in [t, T]$) and it will provide a larger expected utility than η^* for the principal at time t . That is, if for example the principal can fire the agent, she would do so and hire a new agent at time $t > 0$.*

In this paper, we do not allow the principal to fire the agent. Moreover, given a self-enforcing contract, the agent has no incentive to quit, so the principal has no opportunity to reconsider the contract and thus the above time inconsistency is less relevant. However, as we will discuss in Section 7.2 below, the self-enforcing contract can only be sub-optimal, and it will be better for the principal to offer a standard contract and to hire a new agent if the current agent quits the job. This will be the main subject of the present paper.

We also note that, the dynamic version of the standard problem (2.19), which will be discussed in Section 7.1 below, is also time inconsistent in the same spirit.

We conclude this section by rewriting the BSDE (2.13) in the forward form, following Sannikov [32], which will be crucial in the next section. Recall (2.16). For any $t \in [0, T)$, $x \leq \overline{L}_t^\theta$ and $(\eta, Z) \in \mathcal{A}_t^P \times \mathcal{A}_t^A$, denote

$$\begin{aligned}
\mathcal{X}_s^{\theta; t, x, \eta, Z} & := x - \int_t^s [f(\theta; r, \eta_r) + H(\theta; r, Z_r)] dr + \int_t^s Z_r dB_r \\
& = x - \int_t^s [f(\theta; r, \eta_r) + \tilde{H}(\theta; r, Z_r)] dr + \int_t^s Z_r dB_r^{\theta, Z}, \quad t \leq s \leq T,
\end{aligned} \tag{2.25}$$

and set, recalling $Y_T^{\theta, \eta} = 0$,

$$\mathcal{A}_{t, x}^\theta := \left\{ (\eta, Z) \in \mathcal{A}_t^P \times \mathcal{A}_t^A : \mathcal{X}_T^{\theta; t, x, \eta, Z} = 0 \right\}. \tag{2.26}$$

Remark 2.9. (i) We emphasize that $\mathcal{X}^{\theta;t,x,\eta,Z}$ is the forward form of $Y^{\theta,\eta}$, which stands for the continuation utility of the agent, rather than the state process X in (2.2). In particular, x is the agent's optimal utility at initial time t for the contract η .

(ii) In [7, 8], the principal has terminal payment ξ , then ξ has one to one correspondence with the corresponding (x, Z) , and thus one may view (x, Z) as the principal's controls. Alternatively (although less natural in this case), we may view (ξ, x, Z) altogether as the principal's control, but then require a constraint $\xi = \mathcal{X}_T^{t,x,Z}$ for some appropriate process $\mathcal{X}^{t,x,Z}$. In our situation, η and (x, Z) also have one to one correspondence, however, it is not easy to express η directly in terms of (x, Z) . So we take the alternative way by viewing (η, x, Z) as the principal's controls, but require the constraint $\mathcal{X}_T^{\theta;t,x,\eta,Z} = 0$ in (2.26).

3 The case with at most one quitting

We now assume the principal offers standard contracts which meets the individual reservation only at initial time. The agent might quit before T , at a stopping time up to his choice, if he finds a better opportunity in the market. Then the principal will hire another agent with a new contract. In this section, we consider the case that, once the agent quits the job, the principal will offer only self-enforcing new contract. That is, the principal will tolerate only one quitting. We assume there is a pool of agents in the market with parameter set Θ . For each agent $\theta \in \Theta$, if he quits at time t , he will incur a cost c_t^θ , and the principal will bear a cost c_t^P , which (for simplicity) does not depend on θ . Denote further that

$$\underline{L}_t^\theta := R_t^\theta - c_t^\theta, \quad D_\theta := \{(t, x) : 0 \leq t < T, \underline{L}_t^\theta < x \leq \overline{L}_t^\theta\}, \quad T_\delta := T - \delta. \quad (3.1)$$

In this paper we assume c^θ, c^P are deterministic and are given exogenously, and we leave more general cases to future research.

Assumption 3.1. R^θ satisfies Assumption 2.6 and, for the same Λ_0, ρ (for simplicity),

- (i) $c_t^\theta \geq \frac{1}{\Lambda_0} > 0$, $c_t^P \geq 0$, and consequently, $\overline{L}_t^\theta - \underline{L}_t^\theta \geq \frac{1}{\Lambda_0}$, $0 \leq t \leq T$;
- (ii) c^θ, c^P are uniformly continuous in t with modulus of continuity function ρ ;
- (iii) c^θ, c^P are bounded by Λ_0 , and $|\underline{L}_t^\theta - \underline{L}_s^\theta| \leq \Lambda_1 |t - s|$ for some $\Lambda_1 > 0$.

From now on, we allow the generic constant $C > 0$ to depend on Λ_1 as well.

3.1 The agent's problem

Consider the problem on time interval $[t, T]$ with an agent θ at initial time t and a contract $\eta \in \mathcal{A}_t^P$. The agent can choose a stopping time $\tau \in \mathcal{T}_t$ to quit the job. In that case, he

would receive the payment η from the principal over $[t, \tau]$, and would get a new job with expected utility R_τ^θ for the remaining period $[\tau, T]$, and in the mean time he would incur the cost c_τ^θ . He may also choose not to quit, and in that case we set $\tau = T$. Therefore, given the contract $\eta \in \mathcal{A}_t^P$, the agent's optimal utility is:

$$V_1^A(\theta; t, \eta) := \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t^A} \mathbb{E}^\alpha \left[\int_t^\tau [f(\theta; s, \eta_s) - h(\theta; s, \alpha_s)] ds + \underline{L}_\tau^\theta \mathbf{1}_{\{\tau < T\}} \right]. \quad (3.2)$$

Here the subscript $_1$ in V_1^A indicates that the agent can quit once. This problem is solved through a reflected BSDE (c.f. [41, Chapter 6]):

$$\begin{aligned} Y_s^{\theta;t,\eta} &= \int_s^T [f(\theta; r, \eta_r) + H(\theta; r, Z_r^{\theta;t,\eta})] dr - \int_s^T Z_r^{\theta;t,\eta} dB_r + K_T^{\theta;t,\eta} - K_s^{\theta;t,\eta}, \\ Y_s^{\theta;t,\eta} &\geq \underline{L}_s, \quad \int_t^T [Y_s^{\theta;t,\eta} - \underline{L}_s] dK_s^{\theta;t,\eta} = 0. \end{aligned} \quad (3.3)$$

Here the solution is a triplet of processes $(Y^{\theta;t,\eta}, Z^{\theta;t,\eta}, K^{\theta;t,\eta})$, and $K^{\theta;t,\eta}$ is a non-decreasing continuous process with $K_t^{\theta;t,\eta} = 0$. In particular, we use the subscript t to denote the solution to RBSDE (3.3), while using $(Y^{\theta,\eta}, Z^{\theta,\eta})$ for BSDE (2.13) (even if it is on $[t, T]$). We introduce the first time that $Y^{\theta;t,\eta}$ hits the barrier:

$$\tau^{\theta;t,\eta} := \inf \{ s \geq t : Y_s^{\theta;t,\eta} = \underline{L}_s \} \wedge T. \quad (3.4)$$

Proposition 3.2. *Let Assumptions 2.2 and 3.1 hold true.*

(i) *For any $\theta \in \Theta$ and $t < T$, we have $\{Y_t^{\theta;t,\eta} : \eta \in \mathcal{A}_t^P\} = [\underline{L}_t^\theta, \overline{L}_t^\theta]$.*

(ii) *$V_1^A(\theta; t, \eta) = Y_t^{\theta;t,\eta}$, the $\tau^{\theta;t,\eta} \in \mathcal{T}_t$ in (3.4) is an optimal quitting time and $\alpha^{\theta;t,\eta} := I^h(\theta; s, Z^{\theta;t,\eta}) \mathbf{1}_{[t, \tau^{\theta;t,\eta}]} \in \mathcal{A}_t^A$ is the agent's unique optimal effort (unique up to $\tau^{\theta;t,\eta}$).*

Remark 3.3. *We remark that the agent's optimal quitting time may not be unique, and among them $\tau^{\theta;t,\eta}$ is the smallest one. Unfortunately, while the agent's optimal utility is invariant on his choices of the optimal quitting time, the principal's optimal utility does depend on this choice. For technical simplicity in this paper we take the convention that the agent would always take this smallest one $\tau^{\theta;t,\eta}$, namely he would choose to quit when he is indifferent to quitting or staying.*

To prove Proposition 3.2, we first establish a simple lemma. Recall (2.25) and (2.26).

Lemma 3.4. *Let Assumption 2.2 hold. Fix $t < T$ and $\delta > 0$.*

(i) *For any $x \leq \overline{L}_t^\theta$, there exists a deterministic $\eta^{\theta;t,x} \in \mathcal{A}_t^P$ such that*

$$(\eta^{\theta;t,x}, 0) \in \mathcal{A}_{t,x}^\theta, \quad \int_t^T [C_0 - \eta_s^{\theta;t,x}] ds \leq \delta. \quad (3.5)$$

(ii) Assume further that Assumption 3.1 holds and $x > \underline{L}_t^\theta$. Then we may and will always construct $\eta^{\theta;t,x}$ in a way such that $(Y, Z, K) := (\mathcal{X}^{\theta;t,x,\eta^{\theta;t,x},0}, 0, 0)$ solve RBSDE (3.3) corresponding to $\eta^{\theta;t,x}$. In particular, $\mathcal{X}_s^{\theta;t,x,\eta^{\theta;t,x},0} > \overline{L}_s^\theta$ for all $s \in [t, T]$ and $\tau^{\theta;t,\eta^{\theta;t,x}} = T$.

Proof (i) Fix $n \geq 1$ sufficiently large, and set

$$\varepsilon_n := \frac{\overline{L}_t^\theta - x}{n} < T - t, \quad \eta_s^n := I^f(\theta; s, f(\theta; s, C_0) - n) \mathbf{1}_{[t, t+\varepsilon_n)} + C_0 \mathbf{1}_{[t+\varepsilon_n, T]}. \quad (3.6)$$

Clearly η^n is bounded from below, and $\partial_y I^f > 0$ implies $\eta_s^n \leq C_0$, thus $\eta^n \in \mathcal{A}_t^P$. Denote

$$\mathcal{X}_s^n := x - \int_t^s f(\theta; r, \eta_r^n) dr = x - \int_t^s [f(\theta; r, C_0) - n] dr = x - \overline{L}_t^\theta + \overline{L}_s^\theta + n(s - t),$$

for $s \in [t, t + \varepsilon_n]$. Then it is clear that $\mathcal{X}_s^n < \overline{L}_s^\theta$ for $s \in [t, t + \varepsilon_n)$ and $\mathcal{X}_{t+\varepsilon_n}^n = \overline{L}_{t+\varepsilon_n}^\theta$. Since $\eta_s^n = C_0$ for $s > t + \varepsilon_n$, this implies further that $\mathcal{X}_s^n = \overline{L}_s^\theta$ for $s \in [t + \varepsilon_n, T]$. In particular, $\mathcal{X}_T^n = \overline{L}_T^\theta = 0$, and thus $(\eta^n, 0) \in \mathcal{A}_{t,x}^\theta$.

Moreover, for $s \in [t, t + \varepsilon_n]$, by (2.12) and (2.7) we have

$$|\eta_s^n|^2 \leq C[1 - f(\theta; s, \eta_s^n)] = C[1 - f(\theta; s, C_0) + n] \leq C[1 + \Lambda_0 + n] \leq Cn.$$

Then

$$\int_t^T [C_0 - \eta_s^n] ds = \int_t^{t+\varepsilon_n} [C_0 - \eta_s^n] ds \leq \int_t^{t+\varepsilon_n} [C_0 + |\eta_s^n|] ds \leq \left[\frac{C_0}{n} + \frac{C}{\sqrt{n}}\right](\overline{L}_t^\theta - x).$$

This verifies (3.5) immediately by choosing n large enough.

(ii) For $s \in [t, t + \varepsilon_n]$, by (2.7) and Assumption 3.1 (iii) we have, whenever $n \geq \Lambda_1$,

$$\mathcal{X}_s^n - \underline{L}_s^\theta > \underline{L}_t^\theta - \int_t^s [f(\theta; r, C_0) - n] dr - \underline{L}_s^\theta \geq n(s - t) - |\underline{L}_s^\theta - \underline{L}_t^\theta| \geq 0.$$

Moreover, for $s \in (t + \varepsilon_n, T]$, we have $\mathcal{X}_s^n = \overline{L}_s^\theta > \underline{L}_s^\theta$. So $(\mathcal{X}^n, 0, 0)$ satisfies RBSDE (3.3) with η^n , and thus $\tau^{\theta;t,\eta^{\theta;t,x}} = T$. \square

Remark 3.5. In the rest of the paper we will often use a random version of Lemma 3.4, under Assumptions 2.2 and 3.1. Fix $\theta \in \Theta$, $(t, x) \in \mathcal{A}_{t,x}^\theta$, $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$, and $\tau \in \mathcal{T}_t$ such that $\tau \leq \tau^{\theta;t,\eta} \wedge T_\delta$. Then we may construct $\eta^{\theta;\tau, \mathcal{X}_\tau^{\theta;t,x,\eta,Z}}$ in a way such that:

$$(\tilde{\eta}, \tilde{Z}) := (\eta, Z) \mathbf{1}_{[t,\tau)} + (\eta^{\theta;\tau, \mathcal{X}_\tau^{\theta;t,x,\eta,Z}}, 0) \mathbf{1}_{[\tau,T]} \in \mathcal{A}_{t,x}^\theta.$$

Indeed, from the construction in Lemma 3.4 we can easily see that the above $\tilde{\eta}, \tilde{Z}$ are \mathbb{F}^t -progressively measurable, $\tilde{Z} \in \mathcal{A}_t^A$, and $\mathcal{X}_\tau^{\theta;t,x,\tilde{\eta},\tilde{Z}} = 0$. Moreover, since $\tau \leq \tau^{\theta;t,\eta}$, then $\mathcal{X}_\tau^{\theta;t,x,\tilde{\eta},\tilde{Z}} \in [\underline{L}_\tau^\theta, \overline{L}_\tau^\theta]$ is bounded. Note further that $T - \tau \geq T - T_\delta = \delta$, then from the construction in Lemma 3.4 we can easily see that $\eta^{\theta;\tau, \mathcal{X}_\tau^{\theta;t,x,\eta,Z}}$ is uniformly bounded, which implies that $\tilde{\eta}$ has a uniform lower bound, and hence $\tilde{\eta} \in \mathcal{A}_t^P$.

Proof of Proposition 3.2. (i) First, by definition $Y_t^{\theta;t,\eta} \geq \underline{L}_t^\theta$. Next, since $\bar{L}^\theta > \underline{L}^\theta$, then $(\bar{L}^\theta, 0, 0)$ is the unique solution of the RBSDE (3.3) with $\eta \equiv C_0$, and thus it follows from the comparison principle of RBSDEs that $Y_t^{\theta;t,\eta} \leq \bar{L}_t^\theta$. On the other hand, for any $x \in (\underline{L}_t^\theta, \bar{L}_t^\theta)$, by Lemma 3.4 we see that $\eta^{\theta;t,x} \in \mathcal{A}_t^P$ and $x = Y_t^{\theta;t,\eta^{\theta;t,x}}$. Moreover, $x = \underline{L}_t^\theta$, from the arguments in Lemma 3.4 we can easily see that we still have $x = Y_t^{\theta;t,\eta^{\theta;t,x}}$, however, in this case $Y_s^{\theta;t,\eta^{\theta;t,x}} > \underline{L}_s^\theta$ holds only for $s \in (t, T]$. Then we obtain the claimed equality.

(ii) First, for fixed $\tau \in \mathcal{T}_t$, the agent's optimal effort can be solved through a BSDE:

$$Y_s^{\theta;t,\eta,\tau} = \underline{L}_\tau^\theta \mathbf{1}_{\{\tau < T\}} + \int_s^\tau [f(\theta; r, \eta_r) + H(\theta; r, Z_r^{\theta;t,\eta,\tau})] dr - \int_s^\tau Z_r^{\theta;t,\eta,\tau} dB_r. \quad (3.7)$$

Similarly to Lemma 2.3, we see that $V_1^A(\theta; t, \eta) = \sup_{\tau \in \mathcal{T}_t} Y_t^{\theta;t,\eta,\tau}$ and, once τ is given, the agent's optimal effort is $\alpha^* = I^h(\theta; s, Z^{\theta;t,\eta,\tau})$ on $[t, \tau]$. Then it follows from the standard RBSDE theory that $V_1^A(\theta; t, \eta) = Y_t^{\theta;t,\eta}$ and $\tau^{\theta;t,\eta}$ is the smallest optimal stopping time. Notice further that $K_s^{\theta;t,\eta} = 0$ for $t \leq s \leq \tau^{\theta;t,\eta}$, then (3.3) coincide with (3.7) on $[t, \tau^{\theta;t,\eta}]$. Thus $Z_s^{\theta;t,\eta} = Z_s^{\theta;t,\eta,\tau^{\theta;t,\eta}}$, $s \in [t, \tau^{\theta;t,\eta}]$. Since $\underline{L}_\tau^\theta \mathbf{1}_{\{\tau < T\}}$ is bounded, by (3.7) we see that $Z^{\theta;t,\eta,\tau} \in \mathcal{A}_t^A$ (see e.g. [41, Chapter 7]), and hence $\alpha^{\theta;t,\eta} = I^h(\theta; s, Z^{\theta;t,\eta}) \mathbf{1}_{[t, \tau^{\theta;t,\eta}]} \in \mathcal{A}_t^A$. \square

3.2 The principal's problem

Assume the principal hires an agent θ at time t with a contract $\eta \in \mathcal{A}_t^P$. At time $\tau^{\theta;t,\eta}$, the agent would quit the job, then the principal will hire a new agent from the market, with possibly a different type $\tilde{\theta}$. In this section, we assume the principal does not want the new agent to quit, by offering only self-enforcing contracts. Then, by the previous section the principal's optimal value for the new contract will be $V_0^P(\theta; \tau^{\theta;t,\eta})$. Clearly the principal will find $\tilde{\theta}$ to maximize this value, and recall that she will incur a cost c^P , so we introduce

$$\bar{u}_0(t) := \sup_{\theta \in \Theta} V_0^P(\theta; t) - c_t^P. \quad (3.8)$$

Thus the principal's problem is:

$$V_1^P(\theta; t) := \sup_{\eta \in \mathcal{A}_t^P} \mathbb{E}^{\theta, Z^{\theta;t,\eta}} \left[\int_t^{\tau^{\theta;t,\eta}} [I^h(\theta; s, Z^{\theta;t,\eta}) - \eta_s] ds + \bar{u}_0(\tau^{\theta;t,\eta}) \mathbf{1}_{\{\tau^{\theta;t,\eta} < T\}} \right], \quad (3.9)$$

subject to $V_1^A(\theta; t, \eta) \geq R_t^\theta$.

We remark that, here we consider the principal's utility for a given θ . In practice, of course, the principal would choose an optimal θ in the beginning, which amounts to computing

$$V_1^P(t) := \sup_{\theta \in \Theta} V_1^P(\theta; t).$$

We emphasize that in our model there are a family of agents whose types are indexed by θ . The principal knows the value θ of each agent and she can choose the optimal θ . This is completely different from the adverse selection problems. In the latter case there is only one agent, the principal does not know the agent's type θ and cannot choose θ , c.f. [10].

We first need the following simple lemma, which ensures the integrability in (3.9).

Lemma 3.6. *Under Assumptions 2.2 and 3.1, \bar{u}_0 is bounded with $\bar{u}_0(T) = -c_T^P$. Moreover, \bar{u}_0 is uniformly continuous on $[0, T]$ with a modulus of continuity function $\rho_0 := C\sqrt{\rho}$.*

Proof First, for any $t < T$ and $(\eta, Z) \in \mathcal{A}_{t, R_t^\theta}^\theta$, by (2.17) we have

$$\begin{aligned} |J_P(\theta; t, \eta)| &\leq \mathbb{E}^{\theta, Z} \left[\int_t^T [C|Z_s| + |\eta_s|] ds \right] \leq C\sqrt{T-t} \left(\mathbb{E}^{\theta, Z} \left[\int_t^T [|Z_s|^2 + |\eta_s|^2] ds \right] \right)^{\frac{1}{2}} \\ &\leq C\sqrt{T-t} \left(\bar{L}_t^\theta - R_t^\theta + T - t - R_t^\theta \right)^{\frac{1}{2}}. \end{aligned}$$

Then

$$|V_0^P(\theta; t)| \leq C\sqrt{T-t} \left(\bar{L}_t^\theta - R_t^\theta + T - t - R_t^\theta \right)^{\frac{1}{2}}$$

This implies \bar{u}_0 is bounded and $\lim_{t \uparrow T} |\bar{u}_0(t)| = 0$. Moreover, by Proposition 2.7 and Assumption 3.1, clearly $\bar{u}_0(t)$ is uniformly continuous in $[0, T]$ and hence on $[0, T]$. \square

Similarly to (2.25), we can rewrite RBSDE (3.3) in a forward form, which leads to the following dynamic utility function:

$$u_1(\theta; t, x) := \sup_{(\eta, Z) \in \mathcal{A}_{t, x}^\theta} J_1(\theta; t, x, \eta, Z), \quad (t, x) \in D_\theta,$$

$$\text{where } \tau^{\theta; t, x, \eta, Z} := \inf\{s \geq t : \mathcal{X}_s^{\theta; t, x, \eta, Z} = \underline{L}_s^\theta\} \wedge T, \quad (3.10)$$

$$J_1(\theta; t, x, \eta, Z) := \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta; t, x, \eta, Z}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} < T\}} \right].$$

Proposition 3.7. *Under Assumptions 2.2 and 3.1, for any $\theta \in \Theta$ and $t < T$, we have*

$$V_1^P(\theta; t) = \sup_{R_t^\theta \leq x \leq \bar{L}_t^\theta} u_1(\theta; t, x). \quad (3.11)$$

Proof Let $\tilde{J}_1(\theta; t, \eta)$ denote the expectation in the right side of (3.9) for a fixed η . For any $\eta \in \mathcal{A}_t^P$ satisfying the constraint in (3.9), by Proposition 3.2 we have $x := Y_t^{\theta; t, \eta} = V_1^A(\theta; t, \eta) \in [R_t^\theta, \bar{L}_t^\theta]$. Fix $\delta > 0$ small and set

$$\tau_\delta := \tau^{\theta; t, \eta} \wedge T_\delta, \quad (\eta^\delta, Z^\delta)_s := (\eta, Z^{\theta; t, \eta})_s \mathbf{1}_{[t, \tau_\delta)}(s) + (\eta_s^{\theta; \tau_\delta, Y_{\tau_\delta}^{\theta; t, \eta}}, 0) \mathbf{1}_{[\tau_\delta, T]}(s), \quad (3.12)$$

where $\eta^{\theta; \tau_\delta, Y_{\tau_\delta}^{\theta; t, \eta}}$ is constructed in Lemma 3.4. By Remark 3.5 we have $(\eta^\delta, Z^\delta) \in \mathcal{A}_{t, x}^\theta$. By (3.4) we have $Y_s^{\theta; t, \eta} > \underline{L}_s^\theta$ and hence $K_s^{\theta; t, \eta} = 0$ for $s \in [t, \tau^{\theta; t, \eta})$. Then

$$\mathcal{X}_s^{\theta; t, x, \eta^\delta, Z^\delta} = Y_s^{\theta; t, \eta} > \underline{L}_s^\theta, \quad s \in [t, \tau_\delta), \quad \text{and} \quad \tau^{\theta; t, x, \eta^\delta, Z^\delta} = \tau^{\theta; t, \eta} \text{ on } \{\tau^{\theta; t, \eta} \leq T_\delta\}.$$

Moreover, by Lemma 3.4 (ii) we have $\tau^{\theta; t, x, \eta^\delta, Z^\delta} = T$ on $\{\tau^{\theta; t, \eta} > T_\delta\}$. Then, since $\eta_s^{\theta; T_\delta, Y_{T_\delta}^{\theta; t, \eta}}$ is \mathcal{F}_{T_δ} -measurable, we have

$$J_1(\theta; t, x, \eta^\delta, Z^\delta) = \mathbb{E}^{\theta, Z^\delta}[\Xi] = \mathbb{E}^{\theta, Z^{\theta; t, \eta}}[\Xi], \quad \text{where}$$

$$\Xi := \int_t^{\tau_\delta} [I^h(\theta; s, Z_s^{\theta; t, \eta}) - \eta_s] ds + \mathbf{1}_{\{\tau^{\theta; t, \eta} \leq T_\delta\}} \bar{u}_0(\tau^{\theta; t, \eta}) - \mathbf{1}_{\{\tau^{\theta; t, \eta} > T_\delta\}} \int_{T_\delta}^T \eta_s^{\theta; T_\delta, Y_{T_\delta}^{\theta; t, \eta}} ds.$$

Thus, by omitting the superscripts $\theta; t, \eta$, namely denoting $(\tau, Y, Z) := (\tau^{\theta; t, \eta}, Y^{\theta; t, \eta}, Z^{\theta; t, \eta})$,

$$\begin{aligned} & \tilde{J}_1(\theta; t, \eta) - u_1(\theta; t, x) \leq \tilde{J}_1(\theta; t, \eta) - J_1(\theta; t, x, \eta^\delta, Z^\delta) \\ & = \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau > T_\delta\}} \left[\int_{T_\delta}^\tau [I^h(\theta; s, Z_s) - \eta_s] ds + \int_{T_\delta}^T \eta_s^{\theta; T_\delta, Y_{T_\delta}} ds + \bar{u}_0(\tau) \mathbf{1}_{\{\tau < T\}} \right] \right] \\ & \leq \mathbb{E}^{\theta, Z} \left[\int_{T_\delta}^T [|I^h(\theta; s, Z_s)| + |\eta_s|] ds + C \mathbf{1}_{\{T_\delta < \tau < T\}} \right] + C_0 \delta, \end{aligned}$$

where the last inequality is due to Lemma 3.6 and the fact $\eta_s^{\theta; T_\delta, Y_{T_\delta}} \leq C_0$. Send $\delta \rightarrow 0$, we obtain $\tilde{J}_1(\theta; t, \eta) \leq u_1(\theta; t, x)$. Since η is arbitrary, we prove the "≤" part of the proposition.

To see the opposite direction, for any $x \in [R_t^\theta, \bar{L}_t^\theta]$, $(\eta, Z) \in \mathcal{A}_{t, x}^\theta$, and $\delta > 0$ small, set

$$\tilde{\tau}_\delta := \tau^{\theta; t, x, \eta, Z} \wedge T_\delta, \quad (\tilde{\eta}^\delta, \tilde{Z}^\delta)_s := (\eta, Z)_s \mathbf{1}_{[t, \tilde{\tau}_\delta)}(s) + (\eta_s^{\theta; \tilde{\tau}_\delta, \mathcal{X}_{\tilde{\tau}_\delta}^{\theta; t, x, \eta, Z}}, 0) \mathbf{1}_{[\tilde{\tau}_\delta, T]}(s).$$

Following similar arguments as above, by sending $\delta \rightarrow 0$ one can show that $J_1(\theta; t, x, \eta, Z) \leq V_1^P(\theta; t)$. Then, since x and (η, Z) are arbitrary, we obtain the desired inequality. \square

Remark 3.8. *In this section we assume the principal will offer self-enforcing contracts to hire a new agent when the current one quits. There can be other alternatives, for example, the principal may close the business, or run the firm by herself. In those cases, we should model the principal's outside option, denoted as $\bar{u}'_0(t)$, and then study the principal's problem by replacing the \bar{u}_0 in (3.9) with \bar{u}'_0 . Provided \bar{u}'_0 satisfies the properties in Lemma 3.6, the main results in this paper will remain true.*

4 The dynamic value function u_1

In this section we shall characterize u_1 as the minimal solution of an HJB equation. For that purpose, we shall establish its monotonicity, regularity, and boundary conditions.

We first note that, for any $t < T$, $x \leq \bar{L}_t^\theta$, and $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$, under Assumption 2.2 it follows from Lemma 2.4 that

$$\mathbb{E}^{\theta,Z} \left[\int_t^T [|Z_s|^2 + C_0 - \eta_s] ds \right] \leq C[\bar{L}_t^\theta - x], \quad \mathbb{E}^{\theta,Z} \left[\int_t^T |\eta_s|^2 ds \right] \leq C[T - t - x]. \quad (4.1)$$

Moreover, we have the following technical lemma.

Lemma 4.1. *Let Assumptions 2.2 and 3.1 hold. Let $\delta > 0$ be small, $t \leq T_{2\delta}$, and $\underline{L}_t^\theta < x \leq \underline{L}_t^\theta + \delta$. Set $\underline{\tau}_\delta := \tau^{\theta;t,x,C_0,1} \wedge T_\delta$ and*

$$(\underline{\eta}_s^{\theta;t,x}, \underline{Z}_s^{\theta;t,x}) := \mathbf{1}_{[t, \underline{\tau}_\delta)}(s)(C_0, 1) + \mathbf{1}_{[\underline{\tau}_\delta, T)}(s)(\eta_s^{\theta; \underline{\tau}_\delta, \mathcal{X}_{\underline{\tau}_\delta}^{\theta;t,x,C_0,1}}, 0). \quad (4.2)$$

Then $(\underline{\eta}^{\theta;t,x}, \underline{Z}^{\theta;t,x}) \in \mathcal{A}_{t,x}^\theta$, and

$$\bar{u}_0(t) - J_1(\theta; t, x, \underline{\eta}^{\theta;t,x}, \underline{Z}^{\theta;t,x}) \leq C\rho_0(\delta). \quad (4.3)$$

Proof Denote

$$(\underline{\eta}, \underline{Z}) := (\underline{\eta}^{\theta;t,x}, \underline{Z}^{\theta;t,x}), \quad \mathcal{X} := \mathcal{X}^{\theta;t,x,C_0,1}, \quad \tau := \tau^{\theta;t,x,C_0,1}, \quad \underline{\tau} := \tau^{\theta;t,x, \underline{\eta}^{\theta;t,x}, \underline{Z}^{\theta;t,x}}.$$

By Lemma 3.4 one can easily verify that $(\underline{\eta}, \underline{Z}) \in \mathcal{A}_{t,x}^\theta$ and $\underline{\tau} = \tau \mathbf{1}_{\{\tau \leq T_\delta\}} + T \mathbf{1}_{\{\tau > T_\delta\}}$. Then

$$\begin{aligned} \bar{u}_0(t) - J_1(\theta; t, x, \underline{\eta}, \underline{Z}) &= \bar{u}_0(t) - \mathbb{E}^{\theta,1} \left[\underline{u}_0(\tau) \mathbf{1}_{\{\tau \leq T_\delta\}} \right. \\ &\quad \left. + \mathbf{1}_{\{\tau \leq T_\delta\}} \int_t^\tau [I^h(\theta; s, 1) - C_0] ds + \mathbf{1}_{\{\tau > T_\delta\}} \left[\int_t^{T_\delta} [I^h(\theta; s, 1) - C_0] ds - \int_{T_\delta}^T \eta_s^{\theta; T_\delta, \mathcal{X}_{T_\delta}} ds \right] \right]. \end{aligned}$$

Note that $t \leq T_{2\delta}$, then $\{\tau > T_\delta\} \subset \{\tau > t + \delta\}$. Then, by considering the two cases $\{\tau \leq t + \delta\}$ and $\{\tau > t + \delta\}$, it follows from (3.5) and Lemma 3.6 that

$$\begin{aligned} \bar{u}_0(t) - J_1(\theta; t, x, \underline{\eta}, \underline{Z}) &\leq \rho_0(\delta) + \delta + C\mathbb{P}^{\theta,1}(\tau > t + \delta) \\ &\leq C\rho_0(\delta) + \mathbb{P}^{\theta,1} \left(\inf_{t \leq s \leq t+\delta} [\mathcal{X}_s - \underline{L}_s^\theta] > 0 \right) \\ &= C\rho_0(\delta) + \mathbb{P}^{\theta,1} \left(\inf_{t \leq s \leq t+\delta} \left[x - \int_t^s [f(\theta; r, C_0) + \tilde{H}(\theta; r, 1)] dr + [B_s^{\theta,1} - B_t^{\theta,1}] - \underline{L}_s^\theta \right] > 0 \right). \end{aligned}$$

Note that

$$- \int_t^s [f(\theta; r, C_0) + \tilde{H}(\theta; r, 1)] dr \leq C\delta, \quad x - \underline{L}_s^\theta = x - \underline{L}_t^\theta + \underline{L}_t^\theta - \underline{L}_s^\theta \leq \delta + \Lambda_1 \delta.$$

Thus

$$\begin{aligned} \bar{u}_0(t) - J_1(\theta; t, x, \underline{\eta}, \underline{Z}) &\leq C\rho_0(\delta) + \mathbb{P}^{\theta,1} \left(\inf_{t \leq s \leq t+\delta} [B_s^{\theta,1} - B_t^{\theta,1}] > -C\delta \right) \\ &\leq C\rho_0(\delta) + C\sqrt{\delta} \leq C\rho_0(\delta). \quad \square \end{aligned}$$

We next establish the monotonicity and regularity of u_1 with respect to x .

Proposition 4.2. *Let Assumptions 2.2 and 3.1 hold. For any $\theta \in \Theta$ and $t < T$, the value function $u_1(\theta; t, x)$ is decreasing and uniformly continuous in $x \in (\underline{L}_t^\theta, \overline{L}_t^\theta]$. That is, for any $\underline{L}_t^\theta < x < x + \delta \leq \overline{L}_t^\theta$, we have*

$$0 \leq u_1(\theta; t, x) - u_1(\theta; t, x + \delta) \leq C\rho_0(\delta). \quad (4.4)$$

where ρ_0 is the modulus of continuity function of \overline{u}_0 specified in Lemma 3.6. Consequently,

$$V_1^P(\theta; t) = u_1(\theta; t, R_t^\theta). \quad (4.5)$$

Remark 4.3. *In general the monotonicity property and the continuity of u_1 may fail at the boundary point $x = \underline{L}_t^\theta$. For example, for any $x \in (\underline{L}_t^\theta, \overline{L}_t^\theta]$, construct η^n as in (3.6) and send $n \rightarrow \infty$, one can easily see that under these contracts the agent never quits and $u_1(\theta; t, x) \geq -C_0(T - t)$. However, for $x = \underline{L}_t^\theta$, the agent quits immediately at t , and thus $u_1(t, \underline{L}_t^\theta) = \overline{u}_0(t) = \sup_{\tilde{\theta}} V_0^P(\tilde{\theta}; t) - c_t^P$. Clearly it is possible that $-C_0(T - t) > \overline{u}_0(t)$ when c_t^P is sufficiently large.*

Proof of Proposition 4.2. First, (4.5) follows directly from Proposition 3.7 and the monotonicity of u_1 . Without loss of generality we prove (4.4) only at $t = 0$ and for $\delta > 0$ sufficiently small. Denote $x_\delta := x + \delta$.

Step 1. We first show that u_1 is decreasing in x . For any $(\eta, Z) \in A_{0, x_\delta}^\theta$ and $n \geq 1$ such that $\frac{\delta}{n} < T$, similarly to (3.6) and (3.12) we construct:

$$\begin{aligned} t_n &:= \frac{\delta}{n}, & \eta_s^n &:= I^f(\theta; s, f(\theta; s, \eta_s) - n), & Z_s^n &:= Z_s, & 0 \leq s \leq t_n; \\ \tau_n &:= \inf\{s \geq 0 : \mathcal{X}_s^{\theta; 0, x, \eta^n, Z^n} = \underline{L}_s^\theta\} \wedge t_n, & \tau^\delta &:= \tau^{\theta; 0, x_\delta, \eta, Z}, \\ (\eta_s^n, Z_s^n) &:= (\eta_s^{\theta; \tau_n, \underline{L}_{\tau_n}^\theta}, 0) \mathbf{1}_{\{\tau_n < t_n\}} + (\eta_s, Z_s) \mathbf{1}_{\{\tau_n = t_n\}}, & \tau_n &< s \leq T, \end{aligned} \quad (4.6)$$

where $\eta^{\theta; \tau_n, \underline{L}_{\tau_n}^\theta}$ is as in Lemma 3.4 (with an arbitrary δ in (3.5)) and Remark 3.5. Denote $\mathcal{X}^n := \mathcal{X}^{\theta; 0, x, \eta^n, Z^n}$. It is clear that $(\eta^n, Z^n) \in \mathcal{A}_0^P \times \mathcal{A}_0^A$, and by Lemma 3.4 we see that $\mathcal{X}_T^n = 0$ on $\{\tau_n < t_n\}$. Note further that $f(\theta; s, \eta_s^n) = f(\theta; s, \eta_s) - n$, then

$$\mathcal{X}_s^n - \mathcal{X}_s^{\theta; 0, x_\delta, \eta, Z} = ns - \delta < 0 \text{ for } s \in [0, t_n]; \quad \text{and} \quad \mathcal{X}_{t_n}^n - \mathcal{X}_{t_n}^{\theta; 0, x_\delta, \eta, Z} = 0. \quad (4.7)$$

Then, since $(\eta^n, Z^n) = (\eta, Z)$ on $[t_n, T]$, we have $\mathcal{X}_T^n = \mathcal{X}_T^{\theta; 0, x_\delta, \eta, Z} = 0$ on $\{\tau_n = t_n\}$. Therefore, $\mathcal{X}_T^n = 0$ in all the cases, and thus $(\eta^n, Z^n) \in \mathcal{A}_{0, x}^\theta$.

Moreover, note that (4.7) implies that

$$\tau^{\theta; 0, x, \eta^n, Z^n} = \tau_n \mathbf{1}_{\{\tau_n < t_n\}} + \tau^\delta \mathbf{1}_{\{\tau_n = t_n\}} \leq \tau^\delta, \quad Z_s^n = Z_s, \quad 0 \leq s \leq \tau^{\theta; 0, x, \eta^n, Z^n}.$$

Then

$$\begin{aligned} J_1(\theta; 0, x, \eta^n, Z^n) &= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau_n < t_n\}} \left[\int_t^{\tau_n} [I^h(\theta; s, Z_s) - \eta_s^n] ds + \bar{u}_0(\tau_n) \right] \right. \\ &\quad \left. + \mathbf{1}_{\{\tau_n = t_n\}} \left[\int_0^{\tau^\delta} [I^h(\theta; s, Z_s) - \eta_s^n] ds + \bar{u}_0(\tau^\delta) \mathbf{1}_{\{\tau^\delta < T\}} \right] \right], \end{aligned}$$

and therefore,

$$\begin{aligned} J_1(\theta; 0, x_\delta, \eta, Z) - u_1(\theta; 0, x) &\leq J_1(\theta; 0, x_\delta, \eta, Z) - J_1(\theta; 0, x, \eta^n, Z^n) \\ &= \mathbb{E}^{\theta, Z} \left[\int_0^{\tau_n} [\eta_s^n - \eta_s] ds + \mathbf{1}_{\{\tau_n < t_n\}} \left[\int_{\tau_n}^{\tau^\delta} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\tau^\delta) \mathbf{1}_{\{\tau^\delta < T\}} - \bar{u}_0(\tau_n) \right] \right] \\ &\leq \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau_n < t_n\}} \left[\int_{\tau_n}^T [|Z_s| + C_0 - \eta_s] ds + C \right] \right], \end{aligned}$$

where the last inequality thanks to (2.12), Lemma 3.6, and the fact $\eta^n \leq \eta$. Moreover, noting that D_θ is bounded, by (4.1) we have

$$\mathbb{E}_{\mathcal{F}_{\tau_n}}^{\theta, Z} \left[\int_{\tau_n}^T [|Z_s| + C_0 - \eta_s] ds \right] \leq C \sqrt{\bar{L}_{\tau_n}^\theta - \mathcal{X}_{\tau_n}^\delta} + C(\bar{L}_{\tau_n}^\theta - \mathcal{X}_{\tau_n}^\delta) \leq C.$$

Then

$$J_1(\theta; 0, x_\delta, \eta, Z) - u_1(\theta; 0, x) \leq C \mathbb{P}^{\theta, Z}(\tau_n < t_n).$$

Recall (2.25) and note that $f < 0$, $\tilde{H} \leq 0$. Then, by Assumption 3.1 (iii) we have

$$\mathcal{X}_s^{\theta; 0, x, \eta^n, Z^n} - \underline{L}_s^\theta \geq x + \int_0^s Z_r dB_r^{\theta, Z} - \underline{L}_0^\theta - \Lambda_1 t_n, \quad 0 \leq s \leq t_n.$$

Thus, since $x - \underline{L}_0^\theta > 0$, for n large enough such that $\Lambda_1 t_n < \frac{x - \underline{L}_0^\theta}{2}$,

$$\begin{aligned} \mathbb{P}^{\theta, Z}(\tau_n < t_n) &\leq \mathbb{P}^{\theta, Z} \left(\inf_{0 \leq s \leq t_n} [\mathcal{X}_s^{\theta; 0, x, \eta^n, Z^n} - \underline{L}_s^\theta] \leq 0 \right) \\ &\leq \mathbb{P}^{\theta, Z} \left(\inf_{0 \leq s \leq t_n} \int_0^s Z_r dB_r^{\theta, Z} \leq -\frac{x - \underline{L}_0^\theta}{2} \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This implies that $J_1(\theta; 0, x_\delta, \eta, Z) \leq u_1(\theta; 0, x)$, and hence $u_1(\theta; 0, x_\delta) \leq u_1(\theta; 0, x)$.

Step 2. We next prove the right inequality of (4.4). Fix $(\eta, Z) \in \mathcal{A}_{0, x}^\theta$ and construct

$$\begin{aligned} (\eta_s^\delta, Z_s^\delta) &:= (\eta_s, Z_s) \mathbf{1}_{[0, \tau)}(s) + \mathbf{1}_{[\tau, T)}(s) \times \\ &\left[\mathbf{1}_{\{\tau = \bar{\tau}\}}(C_0, 0) + \mathbf{1}_{\{\tau = T_{2\delta} < \bar{\tau} \wedge \underline{\tau}\}} \left(\eta_t^{\theta; T_{2\delta}, \mathcal{X}_{T_{2\delta}}^{\theta; 0, x, \eta, Z} + \delta}, 0 \right) + \mathbf{1}_{\{\tau = \underline{\tau}\}} \left(\eta_s^{\theta; \tau, \underline{L}_\tau^\theta + \delta}, \underline{Z}_s^{\theta; \tau, \underline{L}_\tau^\theta + \delta} \right) \right]; \end{aligned} \quad (4.8)$$

where $\bar{\tau} := \inf\{s \geq 0 : \mathcal{X}_s^{\theta; 0, x, \eta, Z} = \bar{L}_s^\theta - \delta\}$; $\underline{\tau} := \tau^{\theta; 0, x, \eta, Z}$; $\tau := \bar{\tau} \wedge \underline{\tau} \wedge T_{2\delta}$.

One can verify straightforwardly that $(\eta^\delta, Z^\delta) \in \mathcal{A}_{0, x_\delta}^\theta$, where the lower boundedness of η^δ is due to (3.6). Denote

$$\mathcal{X}^x := \mathcal{X}^{\theta; 0, x, \eta, Z}; \quad \mathcal{X}^\delta := \mathcal{X}^{\theta; 0, x_\delta, \eta^\delta, Z^\delta}, \quad \tau^\delta := \tau^{\theta; 0, x_\delta, \eta^\delta, Z^\delta}.$$

Then one can easily check that

$$\begin{aligned} \mathcal{X}_t^\delta &= \mathcal{X}_t + \delta, \quad 0 \leq t \leq \tau; \quad \mathcal{X}_\tau^\delta = \overline{L}_\tau^\theta \text{ on } \{\tau = \overline{\tau}\}; \quad \mathcal{X}_\tau^\delta = \underline{L}_\tau^\theta + \delta \text{ on } \{\tau = \underline{\tau}\}; \\ \tau^\delta &> \tau \text{ on } \{\tau = \underline{\tau}\}, \quad \text{and} \quad \tau^\delta = T \text{ on } \{\tau \neq \underline{\tau}\}. \end{aligned}$$

Thus

$$\begin{aligned} &J_1(\theta; 0, x, \eta, Z) - u_1(\theta; 0, x_\delta) \leq J_1(\theta; 0, x, \eta, Z) - J_1(\theta; 0, x_\delta, \eta^\delta, Z^\delta) \\ &= \mathbb{E}^{\theta, Z} \left[\int_\tau^{\overline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \overline{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} \right] \\ &\quad - \mathbb{E}^{\theta, Z^\delta} \left[\int_\tau^{\tau^\delta} [I^h(\theta; s, Z_s^\delta) - \eta_s^\delta] ds + \overline{u}_0(\tau^\delta) \mathbf{1}_{\{\tau^\delta < T\}} \right] \\ &= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = \overline{\tau}\}} I_1 + \mathbf{1}_{\{\tau = T_{2\delta} < \overline{\tau} \wedge \underline{\tau}\}} I_2 + \mathbf{1}_{\{\tau = \underline{\tau}\}} I_3 \right], \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} I_1 &:= \mathbb{E}_\tau^{\theta, Z} \left[\int_\tau^{\overline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \overline{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} + C_0(T - \overline{\tau}) \right]; \\ I_2 &:= \mathbb{E}_{T_{2\delta}}^{\theta, Z} \left[\int_{T_{2\delta}}^{\underline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \overline{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} + \int_{T_{2\delta}}^T \eta_s^{\theta; T_{2\delta}, \mathcal{X}_{T_{2\delta}}^\delta} ds \right]; \\ I_3 &:= \overline{u}_0(\underline{\tau}) - J_1(\theta; \underline{\tau}, \underline{L}_\tau^\theta + \delta, \underline{\eta}^{\theta; \underline{\tau}, \underline{L}_\tau^\theta + \delta}, \underline{Z}^{\theta; \underline{\tau}, \underline{L}_\tau^\theta + \delta}). \end{aligned}$$

We now estimate I_1, I_2, I_3 separately. First, on $\{\tau = \overline{\tau}\}$, since \overline{u}_0 is bounded, we have

$$\begin{aligned} I_1 &= \mathbb{E}_\tau^{\theta, Z} \left[\int_\tau^{\overline{\tau}} [I^h(\theta; s, Z_s) + C_0 - \eta_s] ds \right] + C \mathbb{P}_\tau^{\theta, Z}(\underline{\tau} < T) \\ &\leq \mathbb{E}_\tau^{\theta, Z} \left[\int_\tau^{\overline{\tau}} [\Lambda_0 |Z_s| + C_0 - \eta_s] ds \right] + C \mathbb{P}_\tau^{\theta, Z}(\underline{\tau} < T) \\ &\leq C[\sqrt{\delta} + \delta] + C \mathbb{P}_\tau^{\theta, Z}(\underline{\tau} < T) \leq C\sqrt{\delta} + C \mathbb{P}_\tau^{\theta, Z}(\underline{\tau} < T), \end{aligned}$$

where the second inequality is due to (4.1) and the fact $\overline{L}_\tau^\theta - \mathcal{X}_\tau^x = \delta$. Moreover, note that $\mathcal{X}_\tau^x = \underline{L}_\tau^\theta \mathbf{1}_{\{\underline{\tau} < T\}}$ and $\mathcal{X}_T^x = \overline{L}_T^\theta = 0$, then by Assumption 3.1 (i) we have

$$\begin{aligned} 0 &\leq \mathbb{E}_\tau^{\theta, Z} \left[\int_\tau^{\overline{\tau}} [f(\theta; r, C_0) - f(\theta; r, \eta_r) - \tilde{H}(\theta; r, Z_r)] dr \right] \\ &= \mathbb{E}_\tau^{\theta, Z} \left[[\mathcal{X}_\tau^x - \mathcal{X}_\tau^x] - [\overline{L}_\tau^\theta - \overline{L}_\tau^\theta] \right] \\ &= \mathbb{E}_\tau^{\theta, Z} \left[\mathbf{1}_{\{\underline{\tau} < T\}} [\underline{L}_\tau^\theta - \overline{L}_\tau^\theta] + \delta \right] \leq -\frac{1}{\Lambda_0} \mathbb{P}_\tau^{\theta, Z}(\underline{\tau} < T) + \delta. \end{aligned}$$

Thus $\mathbb{P}_{\bar{\tau}}^{\theta, Z}(\underline{\tau} < T) \leq \Lambda_0 \delta$, and therefore,

$$I_1 \leq C\sqrt{\delta} + \Lambda_0 \delta \leq C\sqrt{\delta}, \quad \text{on } \{\tau = \bar{\tau}\}. \quad (4.10)$$

Next, on $\{\tau = T_{2\delta} < \bar{\tau} \wedge \underline{\tau}\}$, by (4.1) and noting that $\eta_s^{\theta; T_{2\delta}, \mathcal{X}_{T_{2\delta}}^\delta} \leq C_0$, we have

$$I_2 \leq \mathbb{E}_{T_{2\delta}}^{\theta, Z} \left[C \int_{T_{2\delta}}^T [|Z_s| + |\eta_s| + C_0] ds + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} \right] \leq C\sqrt{\delta} + C \mathbb{E}_{T_{2\delta}}^{\theta, Z} [\bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}}].$$

Since $\bar{u}_0(t)$ is uniformly continuous and $\bar{u}_0(T) = -c_T^P < 0$, we have

$$\bar{u}_0(\underline{\tau}) \leq \underline{u}_0(\underline{\tau}) - \bar{u}_0(T) \leq \sup_{T_{2\delta} \leq s \leq T} |\bar{u}_0(s) - \bar{u}_0(T)| \leq \rho_0(2\delta) \leq 2\rho_0(\delta).$$

Here $\rho_0(2\delta) \leq 2\rho_0(\delta)$ is without loss of generality. Thus

$$I_2 \leq C\sqrt{\delta} + 2\rho_0(\delta) \leq C\rho_0(\delta), \quad \text{on } \{\tau = T_{2\delta} < \bar{\tau} \wedge \underline{\tau}\}. \quad (4.11)$$

Finally, on $\{\tau = \underline{\tau}\}$, by Lemma 4.1 we have $I_3 \leq C\rho_0(\delta)$. Plug this and (4.10), (4.11) into (4.9), we get

$$J_1(\theta; 0, x, \eta, Z) - u_1(\theta; 0, x_\delta) \leq C\rho_0(\delta).$$

Since $(\eta, Z) \in \mathcal{A}_{0, x}^\theta$ is arbitrary, we prove the right inequality of (4.4) at $t = 0$. \square

We now prove the regularity of u_1 with respect to (t, x) .

Proposition 4.4. *Let Assumptions 2.2 and 3.1 hold. For any $(t_i, x_i) \in D_\theta$, we have*

$$|u_1(\theta; t_1, x_1) - u_1(\theta; t_2, x_2)| \leq C\rho_0(\rho(|t_1 - t_2|) + |x_1 - x_2|). \quad (4.12)$$

Proof For simplicity we assume $t_1 = 0$, $x_1 = x$, $t_2 = \delta$, and denote $\Delta x := x_2 - x_1$. We assume without loss of generality that $\delta > 0$ is sufficiently small.

Step 1. First, recall Assumption 3.1 (iii) and denote

$$\begin{aligned} \bar{t} &:= \inf\{s \geq 0 : x + \Lambda_1 s = \bar{L}_s^\theta\} \wedge \delta; \quad \eta_s^0 := I^f(\theta; s, -\Lambda_1) \mathbf{1}_{[0, \bar{t}]}(s) + C_0 \mathbf{1}_{[(\bar{t}, \delta)]}(s), \quad 0 \leq s \leq \delta; \\ x_\delta &:= \mathcal{X}_\delta^{\theta; 0, x, \eta^0, 0} = x + \Lambda_1 \bar{t} - \int_{\bar{t}}^\delta f(\theta; s, C_0) ds. \end{aligned}$$

Then it is clear that

$$\underline{L}_s^\theta < \mathcal{X}_s^{\theta; 0, x, \eta^0, 0} \leq \bar{L}_s^\theta, \quad s \in [0, \delta]; \quad |x_\delta - x| \leq C\delta.$$

Now for any $(\eta, Z) \in \mathcal{A}_{\delta, x_\delta}^\theta$, denote

$$(\eta_s^\delta, Z_s^\delta) := (\eta_s^0, 0) \mathbf{1}_{[0, \delta)}(s) + (\eta_s, Z_s) \mathbf{1}_{[\delta, T)}(s).$$

One can easily see that $\mathcal{X}_s^{\theta;0,x,\eta^\delta,Z^\delta} = \mathcal{X}_s^{\theta;\delta,x_\delta,\eta,Z}$ for $s \in [\delta, T]$, and $\tau^{\theta;0,x,\eta^\delta,Z^\delta} = \tau^{\theta;\delta,x_\delta,\eta,Z}$. Then $(\eta^\delta, Z^\delta) \in \mathcal{A}_{0,x}^\theta$ and thus

$$J_1(\theta; \delta, x_\delta, \eta, Z) - u_1(\theta; 0, x) \leq J_1(\theta; \delta, x_\delta, \eta, Z) - J_1(\theta; 0, x, \eta^\delta, Z^\delta) = \int_0^\delta \eta_s^0 ds \leq C_0 \delta.$$

By the arbitrariness of $(\eta, Z) \in \mathcal{A}_{\delta,x_\delta}^\theta$, we obtain $u_1(\theta; \delta, x_\delta) - u_1(\theta; 0, x) \leq C_0 \delta$. Then, by (4.4) and noting that $|x_2 - x_\delta| \leq |\Delta x| + C\delta$ we have

$$\begin{aligned} u_1(\theta; \delta, x_2) - u_1(\theta; 0, x) &= u_1(\theta; \delta, x_2) - u_1(\theta; \delta, x_\delta) + u_1(\theta; \delta, x_\delta) - u_1(\theta; 0, x) \\ &\leq C\rho_0(|\Delta x| + C\delta) + C_0\delta \leq C\rho_0(|\Delta x| + \delta). \end{aligned}$$

This implies immediately one direction of (4.12) at $t = 0$.

Step 2. To see the opposite direction of (4.12), we denote

$$\delta' := \Lambda_1 \delta + 2T\rho(\delta). \quad (4.13)$$

We first consider the case that $\overline{L}_0^\theta - x \leq \delta'$. Then

$$0 \leq \overline{L}_\delta^\theta - x_2 = \overline{L}_\delta^\theta - \overline{L}_0^\theta + \overline{L}_0^\theta - x + x - x_2 \leq C\rho(\delta) + |\Delta x|.$$

Thus, by (4.4) we have

$$u_1(\theta; 0, x) - u_1(\theta; \delta, x_2) \leq u_1(\theta; 0, \overline{L}_0^\theta) - u_1(\theta; \delta, \overline{L}_\delta^\theta) + C\rho_0(\rho(\delta) + |\Delta x|).$$

Moreover, one can easily see that $u_1(\theta; t, \overline{L}_t^\theta) = -C_0(T - t)$. Then

$$u_1(\theta; 0, x) - u_1(\theta; \delta, x_2) \leq -C_0\delta + C\rho_0(\rho(\delta) + |\Delta x|) \leq C\rho_0(\rho(\delta) + |\Delta x|). \quad (4.14)$$

We now consider the case that $x < \overline{L}_0^\theta - \delta'$. Since $x > \underline{L}_0^\theta$, we can easily see that

$$x'_\delta := x + \delta' \in (\underline{L}_\delta^\theta, \overline{L}_\delta^\theta). \quad (4.15)$$

For any $(\eta, Z) \in \mathcal{A}_{0,x}^\theta$, we shall construct in Step 3 below $(\eta^\delta, Z^\delta) \in \mathcal{A}_{\delta,x'_\delta}^\theta$ such that

$$J_1(\theta; 0, x, \eta, Z) - J_1(\theta; \delta, x'_\delta, \eta^\delta, Z^\delta) \leq C\rho_0(\rho(\delta)). \quad (4.16)$$

Then $J_1(\theta; 0, x, \eta, Z) - u_1(\theta; \delta, x'_\delta) \leq C\rho_0(\rho(\delta))$. Thus, by the arbitrariness of $(\eta, Z) \in \mathcal{A}_{0,x}^\theta$ we obtain $u_1(\theta; 0, x) - u_1(\theta; \delta, x'_\delta) \leq C\rho_0(\rho(\delta))$. Note further that $|x'_\delta - x_2| \leq |\Delta x| + \delta'$, then

$$\begin{aligned} u_1(\theta; 0, x) - u_1(\theta; \delta, x_2) &= u_1(\theta; 0, x) - u_1(\theta; \delta, x'_\delta) + u_1(\theta; \delta, x'_\delta) - u_1(\theta; \delta, x_2) \\ &\leq C\rho_0(\rho(\delta)) + C\rho_0(|\Delta x| + \delta') \leq C\rho_0(|\Delta x| + \rho(\delta)). \end{aligned}$$

This, together with (4.14), proves the other direction of (4.12) in both cases.

Step 3. We now construct $(\eta^\delta, Z^\delta) \in \mathcal{A}_{\delta, x'_\delta}^\theta$ satisfying (4.16). Denote

$$\hat{\mathcal{X}}_s^{\theta; \delta, x, \eta, Z} := x - \int_0^s [f(\theta; r + \delta, \eta_r) + H(\theta; r + \delta, Z_r)] dr + \int_0^s Z_r dB_r, \quad 0 \leq s \leq T_\delta, \quad (4.17)$$

and introduce the following processes by shifting the space:

$$B'_s := B_{s+\delta}^\delta, \quad \eta'_s(B_{[\delta, s]}^\delta) := \eta_{s-\delta}(B'_{[0, s-\delta]}), \quad Z'_s(B_{[\delta, s]}^\delta) := Z_{s-\delta}(B'_{[0, s-\delta]}). \quad (4.18)$$

Denote

$$\begin{aligned} \mathcal{X}^x &:= \mathcal{X}^{\theta; 0, x, \eta, Z}, \quad \hat{\mathcal{X}}^x := \hat{\mathcal{X}}^{\theta; \delta, x, \eta, Z}, \quad \mathcal{X}'_t := \hat{\mathcal{X}}_{t-\delta}^x(B'); \\ \bar{\tau} &:= \inf \{t \geq 0 : \hat{\mathcal{X}}_t^x = \bar{L}_{t+\delta}^\theta - \delta'\}; \quad \underline{\tau} := \inf \{t \geq 0 : \mathcal{X}_t^x = \underline{L}_t^\theta\}; \quad \tau := \bar{\tau} \wedge \underline{\tau} \wedge T_{3\delta}; \\ \bar{\tau}' &:= \bar{\tau}(B') + \delta; \quad \underline{\tau}' := \underline{\tau}(B') + \delta; \quad \tau' := \tau(B') + \delta = \bar{\tau}' \wedge \underline{\tau}' \wedge T_{2\delta}. \end{aligned} \quad (4.19)$$

We see that $(B_{[\delta, T]}^\delta, \eta'_{[\delta, T]}, Z'_{[\delta, T]}, \mathcal{X}'_{[\delta, T]}, \bar{\tau}', \underline{\tau}', \tau')$ and $(B_{[0, T_\delta]}, \eta_{[0, T_\delta]}, Z_{[0, T_\delta]}, \hat{\mathcal{X}}_{[0, T_\delta]}^x, \bar{\tau} + \delta, \underline{\tau} + \delta, \tau + \delta)$ have the same joint distribution. We then construct, similar to (4.8),

$$\begin{aligned} (\eta_s^\delta, Z_s^\delta) &:= (\eta'_s, Z'_s) \mathbf{1}_{[\delta, \tau']}(s) + \mathbf{1}_{[\tau', T]}(s) \times \\ &\left[\mathbf{1}_{\{\tau' = \bar{\tau}'\}}(C_0, 0) + \mathbf{1}_{\{\tau' = T_{2\delta} < \bar{\tau}' \wedge \underline{\tau}'\}} \left(\eta_s^{\theta; T_{2\delta}, \mathcal{X}'_{T_{2\delta}} + \delta'}, 0 \right) + \mathbf{1}_{\{\tau' = \underline{\tau}'\}} \left(\eta_s^{\theta; \underline{\tau}', L_{\underline{\tau}'}^\theta + \delta'}, \underline{Z}_s^{\theta; \underline{\tau}', L_{\underline{\tau}'}^\theta + \delta'} \right) \right]. \end{aligned}$$

Denote $\mathcal{X}^\delta := \mathcal{X}^{\theta; \delta, x'_\delta, \eta^\delta, Z^\delta}$ and $\tau^\delta := \tau^{\theta; \delta, x'_\delta, \eta^\delta, Z^\delta}$. It is clear that

$$\mathcal{X}_s^\delta = \mathcal{X}'_s + \delta', \quad \delta \leq s \leq \tau'. \quad (4.20)$$

Moreover, by Assumption 2.2 (iii) and (2.12), one can easily see that

$$|\hat{\mathcal{X}}_s^x - \mathcal{X}_s^x| \leq 2T\rho(\delta), \quad 0 \leq s \leq T_\delta. \quad (4.21)$$

Then, by Assumption 3.1 (iii),

$$\hat{\mathcal{X}}_s^x + \delta' \geq \mathcal{X}_s^x + \delta' - 2T\rho(\delta) > \underline{L}_s^\theta + \Lambda_1\delta \geq \underline{L}_{s+\delta}^\theta, \quad \text{for } 0 \leq s < \underline{\tau}.$$

By (4.20) and recalling that $\hat{\mathcal{X}}^x$ and \mathcal{X}' have the same distribution, this implies $\mathcal{X}_s^\delta > \underline{L}_s^\theta$ for $\delta \leq s < \underline{\tau}'$, a.s. Then one can easily see that $(\eta^\delta, Z^\delta) \in \mathcal{A}_{\delta, x'_\delta}^\theta$, and

$$\begin{aligned} \mathcal{X}_{\tau'}^\delta &= \bar{L}_{\tau'}^\theta \text{ on } \{\tau' = \bar{\tau}'\}; \quad 0 \leq \mathcal{X}_{\tau'}^\delta - \underline{L}_{\tau'-\delta}^\theta \leq 2\delta' \text{ on } \{\tau' = \underline{\tau}'\}; \\ \tau^\delta &> \tau' \text{ on } \{\tau' = \underline{\tau}'\}, \quad \text{and} \quad \tau^\delta = T \text{ on } \{\tau' \neq \underline{\tau}'\}. \end{aligned}$$

Note further that

$$\begin{aligned}\mathbb{E}^{\theta, Z^\delta} \left[\int_\delta^{\tau'} [I^h(\theta; s, Z_s^\delta) - \eta_s^\delta] ds \right] &= \mathbb{E}^{\theta, Z'} \left[\int_\delta^{\tau(B') + \delta} [I^h(\theta; s, Z'_s) - \eta'_s] ds \right] \\ &= \mathbb{E}^{\theta, Z} \left[\int_0^{\tau'} [I^h(\theta; s + \delta, Z_s) - \eta_s] ds \right].\end{aligned}$$

Then

$$\begin{aligned}J_1(\theta; 0, x, \eta, Z) - J_1(\theta; \delta, x'_\delta, \eta^\delta, Z^\delta) &= \mathbb{E}^{\theta, Z} \left[\int_0^{\underline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} \right] \\ &\quad - \mathbb{E}^{\theta, Z^\delta} \left[\int_\delta^{\tau^\delta} [I^h(\theta; s, Z_s^\delta) - \eta_s^\delta] ds + \bar{u}_0(\tau^\delta) \mathbf{1}_{\{\tau^\delta < T\}} \right] \\ &= I_0 + I_1 + I_2 + I_3,\end{aligned}\tag{4.22}$$

where, by abusing the notations slightly with (4.9),

$$\begin{aligned}I_0 &:= \mathbb{E}^{\theta, Z} \left[\int_0^{\tau'} [I^h(\theta; s, Z_s) - I^h(\theta; s + \delta, Z_s)] ds \right]; \\ I_1 &:= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = \bar{\tau}\}} \left[\int_{\bar{\tau}}^{\underline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} \right] \right] + C_0 \mathbb{E}^{\theta, Z'} \left[\mathbf{1}_{\{\tau' = \bar{\tau}'\}} (T - \bar{\tau}') \right]; \\ I_2 &:= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = T_{3\delta} < \bar{\tau} \wedge \underline{\tau}\}} \left[\int_{T_{3\delta}}^{\underline{\tau}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} < T\}} \right] \right] \\ &\quad + \mathbb{E}^{\theta, Z'} \left[\mathbf{1}_{\{\tau' = T_{2\delta} < \bar{\tau}' \wedge \underline{\tau}'\}} \int_{T_{2\delta}}^T \eta_s^{\theta; T_{2\delta}, \mathcal{X}_{T_{2\delta}}^\delta} ds \right]; \\ I_3 &:= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = \underline{\tau}\}} \bar{u}_0(\underline{\tau}) \right] - \mathbb{E}^{\theta, Z'} \left[\mathbf{1}_{\{\tau' = \underline{\tau}'\}} J_1(\theta; \underline{\tau}', \mathcal{X}_{\underline{\tau}'}^\delta, \underline{\eta}^{\theta; \underline{\tau}', \mathcal{X}_{\underline{\tau}'}^\delta}, \underline{Z}^{\theta; \underline{\tau}', \mathcal{X}_{\underline{\tau}'}^\delta}) \right].\end{aligned}$$

We now estimate I_i . First, by the regularity of h we have $|I_0| \leq C\rho(\delta)$. Next, note that

$$\begin{aligned}\mathbb{E}^{\theta, Z'} \left[\mathbf{1}_{\{\tau' = \bar{\tau}'\}} (T - \bar{\tau}') \right] &= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = \bar{\tau}\}} [T - \tau - \delta] \right] \leq \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau = \bar{\tau}\}} [T - \tau] \right]; \\ \bar{L}_{\bar{\tau}}^\theta - \mathcal{X}_{\bar{\tau}}^x &\leq \bar{L}_{\bar{\tau}}^\theta - \hat{\mathcal{X}}_{\bar{\tau}}^x + \delta' = \bar{L}_{\bar{\tau}}^\theta - \bar{L}_{\bar{\tau} + \delta}^\theta + 2\delta' \leq 2\delta'.\end{aligned}$$

Then, similarly to (4.10) and (4.11), we have

$$I_1 \leq C\sqrt{\delta'} \leq C\sqrt{\rho(\delta)} \quad \text{and} \quad I_2 \leq C\rho_0(\delta') \leq C\rho_0(\rho(\delta)).$$

Finally, when $\underline{\tau} = \tau \leq T_{3\delta}$, by (4.19) and (4.21) we have

$$\hat{\mathcal{X}}_{\underline{\tau}}^x - \underline{L}_{\underline{\tau}}^\theta = \hat{\mathcal{X}}_{\underline{\tau}}^x - \mathcal{X}_{\underline{\tau}}^x \leq C\rho(\delta).$$

Again due the the related identical distribution, we have $\mathcal{X}'_{\underline{\tau}'} - \underline{L}'_{\underline{\tau}' - \delta}^\theta \leq C\rho(\delta)$ a.s. on $\{\underline{\tau} = \underline{\tau}'\}$. Then, by (4.20) and the regularity of \underline{L}^θ ,

$$\mathcal{X}'_{\underline{\tau}'} - \underline{L}'_{\underline{\tau}'}^\theta \leq \mathcal{X}'_{\underline{\tau}'} + \delta' - \underline{L}'_{\underline{\tau}' - \delta}^\theta + \Lambda_1 \delta \leq C\rho(\delta), \quad \text{a.s.}$$

Now it follows from Lemma 4.1 (iii) that

$$\begin{aligned} I_3 &\leq \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau=\underline{t}\}} \bar{u}_0(\underline{t}) \right] - \mathbb{E}^{\theta, Z'} \left[\mathbf{1}_{\{\tau'=\underline{t}'\}} \bar{u}_0(\underline{t}') \right] + C\rho_0(\rho(\delta)) \\ &= \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau=\underline{t}\}} \bar{u}_0(\underline{t}) \right] - \mathbb{E}^{\theta, Z} \left[\mathbf{1}_{\{\tau=\underline{t}\}} \bar{u}_0(\underline{t} + \delta) \right] + C\rho_0(\rho(\delta)) \leq C\rho_0(\rho(\delta)). \end{aligned}$$

Plug all these estimates into (4.22), we obtain (4.16). \square

We next turn to the boundary conditions.

Proposition 4.5. *Under Assumptions 2.2 and 3.1, $u_1(\theta; \cdot, \cdot)$ has boundary conditions:*

$$\begin{aligned} u_1(\theta; t, \bar{L}_t^\theta) &= -C_0(T-t), \quad \forall t \in [0, T]; \quad u_1(\theta; T-, x) = 0, \quad \forall x \in (\underline{L}_T^\theta, 0); \\ u_1(\theta; t, \underline{L}_t^\theta) &\geq \bar{u}_0(t), \quad \forall t \in [0, T]. \end{aligned} \quad (4.23)$$

Proof Since $\mathcal{A}_{t, \bar{L}_t^\theta}^\theta = \{(C_0, 0)\}$, the equality $u_1(\theta; t, \bar{L}_t^\theta) = -C_0(T-t)$ is obvious, and actually has already been used in the proof of Proposition 4.4. Next, by using Lemma 3.4,

$$u_1(\theta; t, x) \geq J_1(\theta; t, x, \eta^{\theta; t, x}, 0) = - \int_t^T \eta_s^{\theta; t, x} ds \geq -C_0(T-t).$$

Then $\lim_{t \uparrow T} u_1(\theta; t, x) \geq 0$. On the other hand, for any $\underline{L}_T^\theta < x < 0$, any $t < T$ large enough such that $\underline{L}_t^\theta < x$, and any $(\eta, Z) \in \mathcal{A}_{t, x}^\theta$, by (4.1) and Lemma 3.6 we have

$$\begin{aligned} J_1(\theta; t, x, \eta, Z) &= \mathbb{E}^{\theta, Z} \left[\int_t^T [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_0(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} < T\}} \right] \\ &\leq \mathbb{E}^{\theta, Z} \left[\int_t^T [C|Z_s| + |\eta_s|] ds + \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} < T\}} [\bar{u}_0(T) + \sup_{t \leq s \leq T} |\bar{u}_0(s) - \bar{u}_0(T)|] \right] \\ &\leq C\sqrt{T-t} + [\rho_0(T-t) - c_T^P] \mathbb{P}(\tau^{\theta; t, x, \eta, Z} < T) \leq C\sqrt{T-t} + \rho_0(T-t). \end{aligned}$$

By the arbitrariness of (η, Z) , we have $u_1(\theta; t, x) \leq C\sqrt{T-t} + \rho_0(T-t)$. Send $t \uparrow T$, together with $\lim_{t \uparrow T} u_1(\theta; t, x) \geq 0$, we obtain $u_1(\theta; T-, x) = 0$.

Finally, fix $t < T$. For any $\delta > 0$ small, by (4.3) we have $\bar{u}_0(t) \leq u_1(\theta; t, x) + C\rho_0(\delta)$ for all $x \in (\underline{L}_t^\theta, \underline{L}_t^\theta + \delta]$. Send $\delta \downarrow 0$, since u_1 is decreasing, we obtain $u_1(\theta; t, \underline{L}_t^\theta) \geq \bar{u}_0(t)$. \square

Finally, we characterize u_1 through an HJB equation. Introduce

$$\mathcal{L}^\theta u(t, x) := \partial_t u + \sup_{\eta \leq C_0, z \in \mathbb{R}} \left[\frac{1}{2} |z|^2 \partial_{xx} u - [f(\theta; t, \eta) + \tilde{H}(\theta; t, z)] \partial_x u + I^h(\theta; t, z) - \eta \right]. \quad (4.24)$$

Theorem 4.6. *Under Assumptions 2.2 and 3.1, the value function $u_1(\theta; \cdot, \cdot)$ is the minimal continuous viscosity solution of the following HJB equation with boundary conditions (4.23):*

$$\mathcal{L}^\theta u(t, x) = 0, \quad (t, x) \in D_\theta. \quad (4.25)$$

Proof First, by the regularity in Proposition 4.2, one may follow rather standard arguments to obtain the dynamic programming principle: for any $(t, x) \in D_\theta$ and $\tau \in \mathcal{T}_t$,

$$u_1(\theta; t, x) := \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau \wedge \tau^{\theta; t, x, \eta, Z}} [I^h(\theta; r, Z_r) - \eta_r] dr \right. \\ \left. + u_1(\theta; \tau, \mathcal{X}_\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau < \tau^{\theta; t, x, \eta, Z}\}} + \bar{u}_0(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} \leq \tau, \tau^{\theta; t, x, \eta, Z} < T\}} \right]. \quad (4.26)$$

Then, by (2.25), it is clear that u_1 is a viscosity solution of (4.25). Moreover, by Proposition 4.5 it satisfies the boundary conditions (4.23).

We next show that u_1 is minimal. Assume u is an arbitrary continuous viscosity solution of (4.25) satisfying (4.23). In particular, $u(t, \underline{L}_t^\theta) \geq \bar{u}_0(t)$. For each $n \geq 1$, introduce

$$u^{(n)}(t, x) := \sup_{(\eta, Z) \in \mathcal{A}_t^P \times \mathcal{A}_t^A, -n \leq \eta \leq C_0} \mathbb{E}^{\theta, Z} \left[\int_t^\tau [I^h(\theta; r, Z_r) - \eta_r] dr + u(\tau, \mathcal{X}_\tau^{\theta; t, x, \eta, Z}) \right], \quad (4.27)$$

where $\tau := \underline{\tau} \wedge \bar{\tau}$, $\underline{\tau} := \tau^{\theta; t, x, \eta, Z}$, $\bar{\tau} := \inf\{s \geq t : \mathcal{X}_s^{\theta; t, x, \eta, Z} = \bar{L}_s^\theta\} \wedge T$.

Since η is uniformly bounded here, one can easily see that $u^{(n)}$ is continuous on \bar{D}_θ , the closure of D_θ . Then $u^{(n)}$ is the unique viscosity solution of the truncated HJB equation:

$$\partial_t u^{(n)} + \sup_{-n \leq \eta \leq C_0, z \in \mathbb{R}} \left[\frac{1}{2} |z|^2 \partial_{xx} u^{(n)} - [f(\theta; t, \eta) + \tilde{H}(\theta; t, z)] \partial_x u^{(n)} + I^h(\theta; t, z) - \eta \right] = 0; \\ u^{(n)}(t, \bar{L}_t^\theta) = -C_0(T - t), \quad u^{(n)}(T, x) = 0, \quad u^{(n)}(t, \underline{L}_t^\theta) = u(t, \underline{L}_t^\theta). \quad (4.28)$$

Note that u is a viscosity supersolution of the above HJB equation with the same boundary conditions, then by the comparison principle for (4.28) we have $u \geq u^{(n)}$.

Now for any $(t, x) \in D_\theta$ and $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$, recall Definition 2.1 and let $n \geq C(\eta)$, then (η, Z) satisfy the requirements in (4.27). Let $\tau, \underline{\tau}, \bar{\tau}$ be defined in (4.27). Note that

$$u_1(\theta; \bar{\tau}, \bar{L}_{\bar{\tau}}^\theta) = -C_0(T - \bar{\tau}) = u(\bar{\tau}, \bar{L}_{\bar{\tau}}^\theta), \quad \bar{u}_0(\underline{\tau}) \leq u(\underline{\tau}, \underline{L}_{\underline{\tau}}^\theta) \text{ on } \{\underline{\tau} < T\}.$$

Similarly to the DPP (4.26), we have

$$J_1(\theta; t, x, \eta, Z) \\ \leq \mathbb{E}^{\theta, Z} \left[\int_t^{\bar{\tau} \wedge \underline{\tau}} [I^h(\theta; r, Z_r) - \eta_r] dr + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} \leq \bar{\tau}\}} + u_1(\theta; \bar{\tau}, \mathcal{X}_{\bar{\tau}}^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\underline{\tau} > \bar{\tau}\}} \right] \\ = \mathbb{E}^{\theta, Z} \left[\int_t^\tau [I^h(\theta; r, Z_r) - \eta_r] dr + \bar{u}_0(\underline{\tau}) \mathbf{1}_{\{\underline{\tau} \leq \bar{\tau}\}} + u_1(\theta; \bar{\tau}, \bar{L}_{\bar{\tau}}^\theta) \mathbf{1}_{\{\underline{\tau} > \bar{\tau}\}} \right] \\ \leq \mathbb{E}^{\theta, Z} \left[\int_t^\tau [I^h(\theta; r, Z_r) - \eta_r] dr + u(\underline{\tau}, \underline{L}_{\underline{\tau}}^\theta) \mathbf{1}_{\{\underline{\tau} \leq \bar{\tau}\}} + u(\bar{\tau}, \bar{L}_{\bar{\tau}}^\theta) \mathbf{1}_{\{\underline{\tau} > \bar{\tau}\}} \right] \\ = \mathbb{E}^{\theta, Z} \left[\int_t^\tau [I^h(\theta; r, Z_r) - \eta_r] dr + u(\tau, \mathcal{X}_\tau^{\theta; t, x, \eta, Z}) \right] \leq u^{(n)}(t, x) \leq u(t, x).$$

Since $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$ is arbitrary, we obtain $u_1(\theta; t, x) \leq u(t, x)$. \square

Remark 4.7. (i) As explained in Remark 4.3, in general it is possible that $u_1(\theta; t, \underline{L}_t^\theta) > \bar{u}_0(t)$. One can show that $u_1(\theta; \cdot, \cdot)$ is the unique viscosity solution to (4.25) if we replace the boundary condition at $x = \underline{L}_t^\theta$ with $u_1(\theta; t, \underline{L}_t^\theta)$. However, we are not able to compute $u_1(\theta; t, \underline{L}_t^\theta)$ explicitly, so we characterize u_1 as the minimal solution in Theorem 4.6.

(ii) The differential operator \mathcal{L}^θ in (4.24) is the same as the case without constraint on (η, Z) . The constraint $(\eta, Z) \in \mathcal{A}_{t,x}^\theta$ in (3.10) is reflected in the boundary conditions (4.23).

5 The case with at most n quitings

In this section we extend the results in the previous section to the case that the principal allows for at most n quitings. That is, if there have been n agents quitting the job, the principal will only offer a self-enforcing contract so that the $(n+1)$ -th agent will not quit anymore. We remark that, for each agent, the agent's problem remains the same as in the previous section, in particular, we do not keep track of the agent after he quits the job. Then, given the setting in the beginning of Section 3, in particular under (3.1) and Assumption 3.1, we introduce the principal's dynamic utility function recursively as in (3.10):

$$u_n(\theta; t, x) := \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta; t, x, \eta, Z}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_{n-1}(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} < T\}} \right]; \quad (5.1)$$

$$\bar{u}_n(t) := \sup_{\theta \in \Theta} u_n(\theta; t, R_t^\theta) - c_t^P,$$

for $n \geq 1$ and $(t, x) \in D_\theta$, and as in (3.9) the principal's problem is:

$$V_n^P(\theta; t) := \sup_{\eta \in \mathcal{A}_t^P} \mathbb{E}^{\theta, Z^{\theta; t, \eta}} \left[\int_t^{\tau^{\theta; t, \eta}} [I^h(\theta; s, Z_s^{\theta; t, \eta}) - \eta_s] ds + \bar{u}_{n-1}(\tau^{\theta; t, \eta}) \mathbf{1}_{\{\tau^{\theta; t, \eta} < T\}} \right], \quad (5.2)$$

subject to $V_1^A(\theta; t, \eta) \geq R_t^\theta$.

By the same arguments as in the previous section, the following results are obvious.

Theorem 5.1. (i) \bar{u}_n is bounded and uniformly continuous on $[0, T]$ with certain modulus of continuity function ρ_n , and $\bar{u}_n(T) = -c_T^P$.

(ii) u_n is decreasing in x . In particular, $V_n^P(\theta; t) = u_n(\theta; t, R_t^\theta)$.

(iii) For any $(t_i, x_i) \in D_\theta$, $i = 1, 2$, with $\Delta t := t_2 - t_1$, $\Delta x := x_2 - x_1$, we have

$$|u_n(\theta; t_1, x_1) - u_n(\theta; t_2, x_2)| \leq C \rho_{n-1}(\rho(|\Delta t|) + |\Delta x|). \quad (5.3)$$

(iv) The value function $u_n(\theta; \cdot, \cdot)$ is the minimal continuous viscosity solution of the HJB

equation (4.25) with the following boundary conditions:

$$\begin{aligned} u_n(\theta; t, \bar{L}_t^\theta) &= -C_0(T-t), \quad \forall t \in [0, T]; & u_n(\theta; T-, x) &= 0, \quad \forall x \in (\underline{L}_T^\theta, 0); \\ u_n(\theta; t, \underline{L}_t^\theta) &\geq \bar{u}_{n-1}(t), \quad \forall t \in [0, T]. \end{aligned} \quad (5.4)$$

Moreover, we have the following monotonicity result.

Proposition 5.2. *Under Assumptions 2.2 and 3.1, u_n and V_n^P are increasing in n .*

Proof For any $\theta \in \Theta$, $t < T$, and $\eta \in \mathcal{A}_t^P$ satisfying the constraint in (2.22), by setting $\tau = T$ in (3.2) we have $V_1^A(\theta; t, \eta) \geq Y_t^{\theta; \eta} \geq R_t^\theta$, where $Y^{\theta; \eta}$ solves the BSDE (2.13). Moreover, by (3.3) and (3.4) clearly $(Y^{\theta; t, \eta}, Z^{\theta; t, \eta}, K^{\theta; t, \eta}) = (Y^{\theta; \eta}, Z^{\theta; \eta}, 0)$ and $\tau^{\theta; t, \eta} = T$. Then by (2.22) and (3.9) we see that $V_1^P(\theta; t) \geq J_P(\theta; t, \eta)$, and thus $V_1^P(\theta; t) \geq V_0^P(\theta; t)$. Now by (4.5), (3.9), and (5.2) we see that $u_2 \geq u_1$. Then by (5.2) again we may prove recursively that u_n is increasing in n , which implies further that V_n^P is also increasing. \square

Remark 5.3. *The above monotonicity holds true even when there is only one type of agent, namely $\Theta = \{\theta_0\}$ is a singleton, as in the standard literature. This proposition implies that, it is indeed not optimal for the principal to restrict to self-enforcing contracts. She would rather allow the agents to quit and then hire a new one if the current agent quits the job.*

5.1 An alternative representation formula for u_n

In this subsection we provide an alternative representation formula for u_n , which will be used in the next section. To illustrate the idea, consider $u_2(\theta_0; t, x)$ for some $\theta_0 \in \Theta$ and $(t, x) \in D_{\theta_0}$. First, there exists an (approximately) optimal $(\eta^0, Z^0) \in \mathcal{A}_{t,x}^{\theta_0}$. Denote $\tau_1 := \tau^{\theta_0; t, x, \eta^0, Z^0}$. When $\{\tau_1 < T\}$, by (5.1) we may choose an (approximately) optimal θ_1 , which is an \mathcal{F}_{τ_1} -measurable Θ -valued random variable, such that

$$\bar{u}_1(\tau_1) \approx u_1(\theta_1; \tau_1, R_{\tau_1}^{\theta_1}) - c_{\tau_1}^P.$$

We may continue to find (approximately) optimal $(\eta^1, Z^1) \in \mathcal{A}_{\tau_1, R_{\tau_1}^{\theta_1}}^{\theta_1}$ for $u_1(\theta_1; \tau_1, R_{\tau_1}^{\theta_1})$.

Denote $\tau_2 := \tau^{\theta_1; \tau_1, R_{\tau_1}^{\theta_1}, \eta^1, Z^1}$, then formally we may rewrite (5.1) as

$$\begin{aligned} u_1(\theta_0; t, x) &= \sup_{(\eta^0, Z^0) \in \mathcal{A}_{t,x}^{\theta_0}} \sup_{\theta_1} \sup_{(\eta^1, Z^1) \in \mathcal{A}_{\tau_1, R_{\tau_1}^{\theta_1}}^{\theta_1}} \mathbb{E}^{\theta_0, Z^0} \left[\int_t^{\tau_1} [I^h(\theta_0; s, Z_s^0) - \eta_s^0] ds \right. \\ &\quad \left. + \mathbb{E}_{\mathcal{F}_{\tau_1}}^{\theta_1, Z^1} \left[\int_{\tau_1}^{\tau_2} [I^h(\theta_1; s, Z_s^1) - \eta_s^1] ds + \bar{u}_0(\tau_2) \mathbf{1}_{\{\tau_2 < T\}} \right] - c_{\tau_1}^P \mathbf{1}_{\{\tau_1 < T\}} \right]. \end{aligned} \quad (5.5)$$

Moreover, denote

$$\zeta_s := \theta_0 \mathbf{1}_{[t, \tau_1)}(s) + \theta_1 \mathbf{1}_{[\tau_1, T)}(s), \quad (\eta_s, Z_s) := (\eta_s^0, Z_s^0) \mathbf{1}_{[t, \tau_1)}(s) + (\eta_s^1, Z_s^1) \mathbf{1}_{[\tau_1, T)}(s), \quad (5.6)$$

and let $\mathbb{L}^0(\mathbb{F}^t; \Theta)$ denote the set of \mathbb{F}^t -progressively measurable Θ -valued processes. Then the expectation in (5.5) can be rewritten as:

$$\mathbb{E}^{\zeta, Z} \left[\int_t^{\tau_2} [I^h(\zeta_s; s, Z_s) - \eta_s] ds - c_{\tau_1}^P \mathbf{1}_{\{\tau_1 < T\}} + \bar{u}_0(\tau_2) \mathbf{1}_{\{\tau_2 < T\}} \right]. \quad (5.7)$$

Recall (2.1) it is clear that $Z \in \mathcal{A}_t^A$. However, we remark that the lower bound of η^1 may not be uniform, and thus in general the above $\eta \notin \mathcal{A}_t^P$.

We now present rigorously the alternative representation formula for u_n . Fix $\theta \in \Theta$ and $(t, x) \in D_\theta$, let $\mathcal{A}_{t,x}^{\infty, \theta}$ denote the set of $(\eta, Z, \zeta) \in \mathcal{A}_t^P \times \mathcal{A}_t^A \times \mathbb{L}^0(\mathbb{F}^t; \Theta)$ such that there exist associated $(\{\tau_i\}_{i \geq 0}, \mathcal{X})$ satisfying:

$$\begin{aligned} \tau_0 &:= t, \quad \mathcal{X}_{\tau_0} := x, \quad \zeta_{\tau_0} = \theta, \quad \text{and for } i \geq 0, \\ \tau_{i+1} &:= \inf\{s \geq \tau_i : \mathcal{X}_s^{\zeta_{\tau_i}; \tau_i, \mathcal{X}_{\tau_i}, \eta, Z} = \underline{L}_s^{\zeta_{\tau_i}}\} \wedge T; \\ \zeta_s &= \zeta_{\tau_i}, \quad \mathcal{X}_s := \mathcal{X}_s^{\zeta_{\tau_i}; \tau_i, \mathcal{X}_{\tau_i}, \eta, Z}, \quad \tau_i \leq s < \tau_{i+1}; \\ \mathcal{X}_{\tau_{i+1}} &:= R_{\tau_{i+1}}^{\zeta_{\tau_{i+1}}} \mathbf{1}_{\{\tau_{i+1} < T\}} + \mathcal{X}_T^{\zeta_{\tau_i}; \tau_i, \mathcal{X}_{\tau_i}, \eta, Z} \mathbf{1}_{\{\tau_{i+1} = T\}}. \end{aligned} \quad (5.8)$$

Here τ_i stands for the quitting time of the i -th agent, and when he quits, the principal would hire a new agent with type ζ_{τ_i} . We note that \mathcal{X} have jumps at τ_i .

Proposition 5.4. *Let Assumptions 2.2 and 3.1 hold. Then*

$$\begin{aligned} u_n(\theta; t, x) &= \sup_{(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}} J_n(t, x, \eta, Z, \zeta), \quad \text{subject to } \mathcal{X}_T = 0 \text{ on } \{\tau_n = T\}, \quad \text{where} \\ J_n(t, x, \eta, Z, \zeta) &:= \mathbb{E}^{\zeta, Z} \left[\int_t^{\tau_n} [I^h(\zeta_s; s, Z_s) - \eta_s] ds - \sum_{i=1}^{n-1} c_{\tau_i}^P \mathbf{1}_{\{\tau_i < T\}} + \bar{u}_0(\tau_n) \mathbf{1}_{\{\tau_n < T\}} \right]. \end{aligned} \quad (5.9)$$

Proof The result is trivial when $n = 1$. We shall prove (5.9) only for $n = 2$. The general case $n > 2$ can be proved following similar but slightly more involved arguments. Let $\tilde{u}_2(\theta; t, x)$ denote the right side of the first line in (5.9) with $n = 2$.

Step 1. We first prove $\tilde{u}_2(\theta; t, x) \leq u_2(\theta; t, x)$. For any (η, Z, ζ) satisfying the required properties in (5.9), and for any $\delta > 0$ small, denote

$$\begin{aligned} (\eta_s^0, Z_s^0) &:= (\eta_s, Z_s) \mathbf{1}_{[t, \tau_1 \wedge T_\delta)}(s) + (\eta_s^{\theta; \tau_1 \wedge T_\delta, \mathcal{X}_{\tau_1 \wedge T_\delta}^{\theta; t, x, \eta, Z}}, 0) \mathbf{1}_{[\tau_1 \wedge T_\delta, T)}(s); \\ (\eta_s^1, Z_s^1) &:= (\eta_s, Z_s) \mathbf{1}_{[\tau_1, \tau_2 \wedge T_\delta)}(s) + (\eta_s^{\zeta_{\tau_1}; \tau_2 \wedge T_\delta, \mathcal{X}_{\tau_2 \wedge T_\delta}^{\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}, \eta, Z}}, 0) \mathbf{1}_{[\tau_2 \wedge T_\delta, T)}(s), \quad \text{on } \{\tau_1 \leq T_\delta\}. \end{aligned}$$

Then

$$\begin{aligned} (\eta^0, Z^0) &\in \mathcal{A}_{t,x}^\theta, \quad \tau_1^\delta := \tau^{\theta; t, x, \eta^0, Z^0} = \tau_1 \mathbf{1}_{\{\tau_1 \leq T_\delta\}} + T \mathbf{1}_{\{\tau_1 > T_\delta\}}; \\ (\eta^1, Z^1) &\in \mathcal{A}_{\tau_1, R_{\tau_1}^{\zeta_{\tau_1}}}^{\zeta_{\tau_1}}, \quad \tau_2^\delta := \tau^{\zeta_{\tau_1}; \tau_2, R_{\tau_1}^{\zeta_{\tau_1}}, \eta^1, Z^1} = \tau_2 \mathbf{1}_{\{\tau_2 \leq T_\delta\}} + T \mathbf{1}_{\{\tau_2 > T_\delta\}}, \quad \text{on } \{\tau_1 \leq T_\delta\}. \end{aligned}$$

Thus, by (5.1),

$$\begin{aligned}
u_2(\theta; t, x) &\geq \mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau_1^\delta} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds + \bar{u}_1(\tau_1) \mathbf{1}_{\{\tau_1 \leq T_\delta\}} \right] \\
&\geq \mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau_1^\delta} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds + [u_1(\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}) - c_{\tau_1}^P] \mathbf{1}_{\{\tau_1 \leq T_\delta\}} \right] \\
&\geq \mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau_1^\delta} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds \right. \\
&\quad \left. + \left(\mathbb{E}_{\tau_1}^{\zeta_{\tau_1}, Z^1} \left[\int_{\tau_1}^{\tau_2^\delta} [I^h(\theta; s, Z_s^1) - \eta_s^1] ds + \bar{u}_0(\tau_2) \mathbf{1}_{\{\tau_2 \leq T_\delta\}} \right] - c_{\tau_1}^P \right) \mathbf{1}_{\{\tau_1 \leq T_\delta\}} \right] \\
&= \mathbb{E}^{\zeta, Z} \left[\mathbf{1}_{\{\tau_2 \leq T_\delta\}} \left[\int_t^{\tau_2} [I^h(\zeta; s, Z_s) - \eta_s] ds - c_{\tau_1}^P + \bar{u}_0(\tau_2) \right] \right. \\
&\quad \left. + \mathbf{1}_{\{\tau_1 \leq T_\delta < \tau_2\}} \left[\int_t^{T_\delta} [I^h(\zeta; s, Z_s) - \eta_s] ds - \int_{T_\delta}^T \eta_s^{\zeta_{\tau_1}; T_\delta, \mathcal{X}_{T_\delta}^{\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}, \eta, Z}} ds - c_{\tau_1}^P \right] \right. \\
&\quad \left. + \mathbf{1}_{\{\tau_1 > T_\delta\}} \left[\int_t^{T_\delta} [I^h(\zeta; s, Z_s) - \eta_s] ds - \int_{T_\delta}^T \eta_s^{\theta; T_\delta, \mathcal{X}_{T_\delta}^{\theta; t, x, \eta, Z}} ds \right] \right].
\end{aligned}$$

This implies that

$$\begin{aligned}
&J_2(t, x, \eta, Z, \zeta) - u_2(\theta; t, x) \\
&\leq \mathbb{E}^{\zeta, Z} \left[\mathbf{1}_{\{\tau_2 > T_\delta\}} \int_{T_\delta}^{\tau_2} [I^h(\zeta; s, Z_s) - \eta_s] ds + \bar{u}_0(\tau_2) \mathbf{1}_{\{T_\delta < \tau_2 < T\}} - c_{\tau_1}^P \mathbf{1}_{\{T_\delta < \tau_1 < T\}} \right. \\
&\quad \left. + \mathbf{1}_{\{\tau_1 \leq T_\delta < \tau_2\}} \int_{T_\delta}^T \eta_s^{\zeta_{\tau_1}; T_\delta, \mathcal{X}_{T_\delta}^{\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}, \eta, Z}} ds + \mathbf{1}_{\{\tau_1 > T_\delta\}} \int_{T_\delta}^T \eta_s^{\theta; T_\delta, \mathcal{X}_{T_\delta}^{\theta; t, x, \eta, Z}} ds \right] \\
&\leq \mathbb{E}^{\zeta, Z} \left[\mathbf{1}_{\{\tau_2 > T_\delta\}} \int_{T_\delta}^{\tau_2} [I^h(\zeta; s, Z_s) - \eta_s] ds + \bar{u}_0(\tau_2) \mathbf{1}_{\{T_\delta < \tau_2 < T\}} - c_{\tau_1}^P \mathbf{1}_{\{T_\delta < \tau_1 < T\}} \right. \\
&\quad \left. + C_0 \delta [\mathbf{1}_{\{\tau_1 \leq T_\delta < \tau_2\}} + \mathbf{1}_{\{\tau_1 > T_\delta\}}] \right].
\end{aligned}$$

Send $\delta \rightarrow 0$, we obtain $J_2(t, x, \eta, Z, \zeta) \leq u_2(\theta; t, x)$. Since (η, Z, ζ) are arbitrary, we have $\tilde{u}_2(\theta; t, x) \leq u_2(\theta; t, x)$.

Step 2. We next show that $u_2(\theta; t, x) \leq \tilde{u}_2(\theta; t, x)$. Let $\tilde{\rho}$ be a generic modulus of continuity function. Fix a small $\delta > 0$. First, by (5.1) there exists $(\eta^0, Z^0) \in \mathcal{A}_{t,x}^\theta$ such that

$$u_2(\theta; t, x) \leq \mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau_0} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds + \bar{u}_1(\tau_0) \mathbf{1}_{\{\tau_0 < T\}} \right] + \delta.$$

where $\tau_0 := \tau^{\theta; t, x, \eta^0, Z^0}$. Let $t = t_0 < \dots < t_n = T$ be a partition of $[t, T]$ such that $t_{n-1} = T_\delta$ and $t_i - t_{i-1} \leq \delta$, $i = 1, \dots, n-1$. Now fix an $i = 1, \dots, n-1$. Choose θ_i such that

$$\bar{u}_1(t_i) \leq u_1(\theta_i; t_i, R_{t_i}^\theta) - c_{t_i}^P + \delta.$$

Denote $x_i := \inf_{t_{i-1} \leq t \leq t_i} R_t^{\theta_i}$. Since $R_t^\theta - \underline{L}_t^\theta = c_t^\theta \geq \frac{1}{\Lambda_0}$, by the uniform regularity of R_t^θ we have $x_i > \underline{L}_t^\theta + \Lambda_0 \delta$ for $t_{i-1} \leq t \leq t_i$, provided δ is small enough. Let $(\eta^i, Z^i) \in \mathcal{A}_{t_i, x_i}^{\theta_i}$ be such that, denoting $\tau_i := \tau^{\theta_i; t_i, x_i, \eta^i, Z^i}$,

$$u_1(\theta_i; t_i, x_i) \leq \mathbb{E}^{\theta_i, Z^i} \left[\int_{t_i}^{\tau_i} [I^h(\theta_i; s, Z_s^i) - \eta_s^i] ds + \bar{u}_0(\tau_i) \mathbf{1}_{\{\tau_i < T\}} \right] + \delta.$$

Denote, on $\{t_{i-1} < \tau_0 \leq t_i\}$,

$$x'_i := \mathcal{X}_{t_i}^{\theta_i; \tau_0, R_{\tau_0}^{\theta_i}, C_0, 0} = R_{\tau_0}^{\theta_i} - \int_{\tau_0}^{t_i} f(\theta_i; s, C_0) ds.$$

We shall note that x'_i is random here, and it is obvious that $0 \leq x'_i - x_i \leq \tilde{\rho}(\delta)$. Following the arguments in Proposition 4.2 Step 1, we can construct $(\tilde{\eta}^i, \tilde{Z}^i) \in \mathcal{A}_{t_i, x'_i}^{\theta_i}$ such that

$$\begin{aligned} & \mathbb{E}^{\theta_i, Z^i} \left[\int_{t_i}^{\tau_i} [I^h(\theta_i; s, Z_s^i) - \eta_s^i] ds + \bar{u}_0(\tau_i) \mathbf{1}_{\{\tau_i < T\}} \right] \\ & \leq \mathbb{E}^{\theta_i, \tilde{Z}^i} \left[\int_{t_i}^{\tilde{\tau}_i} [I^h(\theta_i; s, \tilde{Z}_s^i) - \tilde{\eta}_s^i] ds + \bar{u}_0(\tilde{\tau}_i) \mathbf{1}_{\{\tilde{\tau}_i < T\}} \right] + \tilde{\rho}(\delta), \end{aligned}$$

where $\tilde{\tau}_i := \tau^{\theta_i; t_i, x'_i, \tilde{\eta}^i, \tilde{Z}^i}$, and $\tilde{\eta}^i$ has a lower bound. Now extend $(\tilde{\eta}^i, \tilde{Z}^i)$ to $[\tau_0, T]$ with $(\tilde{\eta}^i, \tilde{Z}^i) = (C_0, 0)$ on $[\tau_0, t_i]$. Note that, for $\delta > 0$ small enough, $\mathcal{X}_s^{\theta_i; \tau_0, R_{\tau_0}^{\theta_i}, C_0, 0} > \underline{L}_s^\theta$, $\tau_0 \leq s \leq t_i$. Then we have $(\tilde{\eta}^i, \tilde{Z}^i) \in \mathcal{A}_{\tau_0, R_{\tau_0}^{\theta_i}}^{\theta_i}$, and $\tau^{\theta_i; \tau_0, R_{\tau_0}^{\theta_i}, \tilde{\eta}^i, \tilde{Z}^i} = \tilde{\tau}_i$, and thus we can easily check that, again on $\{t_{i-1} < \tau_0 \leq t_i\}$,

$$\begin{aligned} & \mathbb{E}^{\theta_i, Z^i} \left[\int_{t_i}^{\tau_i} [I^h(\theta_i; s, Z_s^i) - \eta_s^i] ds + \bar{u}_0(\tau_i) \mathbf{1}_{\{\tau_i < T\}} \right] \\ & \leq \mathbb{E}^{\theta_i, \tilde{Z}^i} \left[\int_{\tau_0}^{\tilde{\tau}_i} [I^h(\theta_i; s, \tilde{Z}_s^i) - \tilde{\eta}_s^i] ds + \bar{u}_0(\tilde{\tau}_i) \mathbf{1}_{\{\tilde{\tau}_i < T\}} \right] + \tilde{\rho}(\delta), \end{aligned}$$

We now define

$$\begin{aligned} & (\eta_s, Z_s, \zeta_s) := (\eta_s^0, Z_s^0, \theta) \mathbf{1}_{[t, \tau_0 \wedge T_\delta]}(s) + \mathbf{1}_{[\tau_0 \wedge T_\delta, T]}(s) \times \\ & \left[\sum_{i=1}^{n-1} (\tilde{\eta}^i, \tilde{Z}^i, \theta_i) \mathbf{1}_{\{t_{i-1} < \tau_0 \leq t_i\}} + (\eta_s^{\theta; T_\delta, \mathcal{X}_{T_\delta}^{\theta; t, x, \eta^0, Z^0}}, 0, \theta) \mathbf{1}_{\{\tau_0 > t_{n-1}\}} \right]. \end{aligned}$$

Recall Remark 3.5, and since the above expression involves only finitely many η^i , the above η has a uniform lower bound, then one can verify that $(\eta, Z, \zeta) \in \mathcal{A}_{t, x}^{\infty, \theta}$ and

$$u_2(\theta; t, x) \leq J_2(t, x, \eta, Z, \zeta) + \tilde{\rho}(\delta) \leq \tilde{u}_2(\theta; t, x) + \tilde{\rho}(\delta).$$

Send $\delta \rightarrow 0$, we obtain $u_2(\theta; t, x) \leq \tilde{u}_2(\theta; t, x)$. □

6 The general case with arbitrary number of quitings

In light of (5.9), we first introduce the principal's dynamic value function when there is no constraint on the number of agent's quitings: for any $\theta \in \Theta$ and $(t, x) \in D_\theta$,

$$u_\infty(\theta; t, x) = \sup_{(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}} J_\infty(t, x, \eta, Z, \zeta), \quad \text{subject to} \quad \lim_{i \rightarrow \infty} \tau_i = T, \mathcal{X}_{T-} = 0, \quad (6.1)$$

where $J_\infty(t, x, \eta, Z, \zeta) := \mathbb{E}^{\zeta, Z} \left[\int_t^T [I^h(\zeta_s; s, Z_s) - \eta_s] ds - \sum_{i=1}^{\infty} c_{\tau_i}^P \mathbf{1}_{\{\tau_i < T\}} \right].$

Then, if the principal hires an agent θ at time t and allows the agent as well as all future agents to quit as they want, the principal's value at time t will be:

$$V_\infty^P(\theta, t) := \sup_{R_i^\theta \leq x \leq \bar{L}_i^\theta} u_\infty(\theta; t, x), \quad V_\infty^P(t) := \sup_{\theta \in \Theta} V_\infty^P(\theta, t). \quad (6.2)$$

Here $V_\infty^P(t)$ will be principal's real optimal utility if he hires an agent at time t , by hiring the best agent for her.

We remark that, in general it may look possible that agents quit too frequently and there are already infinitely many agents quitting at some time before T , namely $\tau_\infty := \lim_{i \rightarrow \infty} \tau_i < T$. However, mainly due to our assumption $c_t^\theta \geq \frac{1}{\Lambda_0} > 0$, the following result shows that this can never be the case. In fact, we shall have $\tau_n = T$, for n large enough, a.s. for all $(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}$, and thus the constraint $\lim_{i \rightarrow \infty} \tau_i = T$ in (6.1) is actually redundant. In particular, there is no self-enforcing contract involved anymore in (6.1) or (6.2).

Lemma 6.1. *Let Assumptions 2.2 and 3.1 hold. For any $\theta \in \Theta$, $(t, x) \in D_\theta$, and $(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}$ with corresponding $(\{\tau_i\}_{i \geq 0}, \mathcal{X})$ defined in (5.8), it holds that, for any $n \geq 1$,*

$$\mathbb{P}^{\zeta, Z}(\tau_n < T) \leq \frac{C}{n}, \quad \text{and consequently,} \quad \mathbb{P}^{\zeta, Z}(\tau_n < T, \text{ for all } n) = 0. \quad (6.3)$$

Proof Note that, for each $i \geq 1$, on $\{\tau_i < T\}$ we have

$$\begin{aligned} & \underline{L}_{\tau_{i+1}}^{\zeta_{\tau_i}} \mathbf{1}_{\{\tau_{i+1} < T\}} + \mathcal{X}_{T-} \mathbf{1}_{\{\tau_{i+1} = T\}} = \mathcal{X}_{\tau_{i+1}-} \\ & = R_{\tau_i}^{\zeta_{\tau_i}} - \int_{\tau_i}^{\tau_{i+1}} [f(\zeta_{\tau_i}; s, \eta_s) + \tilde{H}(\zeta_{\tau_i}; s, Z_s)] ds + \int_{\tau_i}^{\tau_{i+1}} Z_s dB_s^{\zeta_{\tau_i}, Z} \\ & \geq \underline{L}_{\tau_i}^{\zeta_{\tau_i}} + \frac{1}{\Lambda_0} + \int_{\tau_i}^{\tau_{i+1}} Z_r dB_r^{\zeta_{\tau_i}, Z}. \end{aligned}$$

Then

$$\mathbb{E}_{\tau_i}^{\zeta, Z} \left[\underline{L}_{\tau_{i+1}}^{\zeta_{\tau_i}} \mathbf{1}_{\{\tau_{i+1} < T\}} + \mathcal{X}_{T-} \mathbf{1}_{\{\tau_{i+1} = T > \tau_i\}} \right] \geq \mathbb{E}_{\tau_i}^{\zeta, Z} \left[\left[\underline{L}_{\tau_i}^{\zeta_{\tau_i}} + \frac{1}{\Lambda_0} \right] \mathbf{1}_{\{\tau_i < T\}} \right],$$

and thus, noting that \underline{L} and \mathcal{X} are uniformly bounded,

$$\begin{aligned} \frac{1}{\Lambda_0} \mathbb{P}^{\zeta, Z}(\tau_i < T) &\leq \mathbb{E}^{\zeta, Z} \left[\underline{L}_{\tau_{i+1}}^{\zeta_{\tau_i}} \mathbf{1}_{\{\tau_{i+1} < T\}} - \underline{L}_{\tau_i}^{\zeta_{\tau_i}} \mathbf{1}_{\{\tau_i < T\}} + \mathcal{X}_{T-} \mathbf{1}_{\{\tau_{i+1} = T > \tau_i\}} \right] \\ &\leq \mathbb{E}^{\zeta, Z} \left[\Lambda_1 [\tau_{i+1} - \tau_i] \mathbf{1}_{\{\tau_i < T\}} + [\mathcal{X}_{T-} - \underline{L}_T^{\zeta_{\tau_i}}] \mathbf{1}_{\{\tau_{i+1} = T > \tau_i\}} \right] \\ &\leq C \mathbb{E}^{\zeta, Z} \left[\tau_{i+1} - \tau_i + \mathbf{1}_{\{\tau_{i+1} = T > \tau_i\}} \right]. \end{aligned}$$

Take summation over i , we have

$$\frac{n}{\Lambda_0} \mathbb{P}^{\zeta, Z}(\tau_n < T) \leq \frac{1}{\Lambda_0} \sum_{i=1}^n \mathbb{P}^{\zeta, Z}(\tau_i < T) \leq C \sum_{i=1}^n \mathbb{E}^{\zeta, Z} \left[\tau_{i+1} - \tau_i + \mathbf{1}_{\{\tau_{i+1} = T > \tau_i\}} \right] \leq C.$$

This implies (6.3) immediately. \square

As a direct consequence, we have the following uniform convergence result. Define

$$\bar{u}_\infty(t) := \sup_{\theta \in \Theta} u_\infty(\theta; t, R_t^\theta) - c_t^P. \quad (6.4)$$

Proposition 6.2. *Let Assumptions 2.2 and 3.1 hold. Then u_n converges to u_∞ uniformly:*

$$|u_n(\theta; t, x) - u_\infty(\theta; t, x)| \leq \frac{C}{n}. \quad (6.5)$$

Consequently,

(i) u_∞ is decreasing in x . In particular,

$$V_\infty^P(\theta; t) = u_\infty(\theta; t, R_t^\theta), \quad \text{and thus} \quad \bar{u}_\infty(t) = V_\infty^P(t) - c_t^P.$$

(ii) u_∞ is uniformly continuous in D_θ , uniformly in θ .

(iii) \bar{u}_∞ is bounded and uniformly continuous on $[0, T]$ with $\bar{u}_\infty(T) = -c_T^P$.

(iv) $u_\infty(\theta; t, x)$ has boundary conditions:

$$\begin{aligned} u_\infty(\theta; t, \bar{L}_t^\theta) &= -C_0(T - t), \quad \forall t \in [0, T]; \quad u_\infty(\theta; T-, x) = 0, \quad \forall x \in (\underline{L}_T^\theta, 0); \\ u_\infty(\theta; t, \underline{L}_t^\theta) &\geq \bar{u}_\infty(t), \quad \forall t \in [0, T]. \end{aligned} \quad (6.6)$$

Proof It suffices to prove the uniform convergence, then the rest properties are a direct consequence of Theorem 5.1. Fix $\theta \in \Theta$, $(t, x) \in D_\theta$, and $n \geq 1$.

Step 1. First, for any $(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}$ as in (5.9) and $\delta > 0$ small, construct

$$\tau_n^\delta := \tau_n \wedge T_\delta, \quad (\tilde{\eta}, \tilde{Z}, \tilde{\zeta})_s := (\eta, Z, \zeta)_s \mathbf{1}_{[0, \tau_n^\delta)}(s) + (\eta_s^{\zeta_{\tau_n^\delta}; \tau_n^\delta, R_{\tau_n^\delta}^{\zeta_{\tau_n^\delta}}}, 0, \zeta_{\tau_n^\delta}) \mathbf{1}_{[\tau_n^\delta, T)}(s). \quad (6.7)$$

Then clearly $(\tilde{\eta}, \tilde{Z}, \tilde{\zeta}) \in \mathcal{A}_{t,x}^{\infty,\theta}$ satisfies the constraints in (6.1), with corresponding $\tilde{\tau}_i = \tau_i$ on $\{\tau_i \leq T_\delta\}$ for $i \leq n$, and $\tilde{\tau}_i = T$ otherwise. Then

$$\begin{aligned} J_n(t, x, \eta, Z, \zeta) - u_\infty(\theta; t, z) &\leq J_n(t, x, \eta, Z, \zeta) - J_\infty(t, x, \tilde{\eta}, \tilde{Z}, \tilde{\zeta}) \\ &= \mathbb{E}^{\zeta, Z} \left[\int_{\tau_n^\delta}^{\tau_n} [I^h(\zeta_s; s, Z_s) - \eta_s] ds + \int_{\tau_n^\delta}^T \eta_s^{\zeta_{\tau_n^\delta}; \tau_n^\delta, R_{\tau_n^\delta}^{\zeta_{\tau_n^\delta}}} ds \right. \\ &\quad \left. - \sum_{i=1}^n c_{\tau_i}^P \mathbf{1}_{\{T_\delta < \tau_i < T\}} + \sup_{\theta} V_0^P(\theta; \tau_n) \mathbf{1}_{\{\tau_n < T\}} \right] \\ &\leq \mathbb{E}^{\zeta, Z} \left[\int_{\tau_n^\delta}^{\tau_n} [I^h(\zeta_s; s, Z_s) - \eta_s] ds + C_0(T - \tau_n^\delta) + C \mathbf{1}_{\{\tau_n < T\}} \right]. \end{aligned}$$

Send $\delta \rightarrow 0$, one can easily see that

$$J_n(t, x, \eta, Z, \zeta) - u_\infty(\theta; t, z) \leq \mathbb{E}^{\zeta, Z} \left[C_0(T - \tau_n) + C \mathbf{1}_{\{\tau_n < T\}} \right] \leq C \mathbb{P}^{\zeta, Z}(\tau_n < T) \leq \frac{C}{n},$$

and thus

$$u_n(\theta; t, x) - u_\infty(\theta; t, z) \leq \frac{C}{n}.$$

Step 2. On the other hand, for any $(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty,\theta}$ satisfying the constraints in (6.1), again let $(\tilde{\eta}, \tilde{Z}, \tilde{\zeta})$ be constructed by (6.7). Then $(\tilde{\eta}, \tilde{Z}, \tilde{\zeta}) \in \mathcal{A}_{t,x}^{\infty,\theta}$ satisfies the constraints in (5.9) with $\tilde{\tau}_i = \tau_i$ on $\{\tau_i \leq T_\delta\}$, $i = 0, \dots, n$. Thus, by (3.5),

$$\begin{aligned} J_\infty(t, x, \eta, Z, \zeta) - u_n(\theta; t, x) &\leq J_\infty(t, x, \eta, Z, \zeta) - J_n(t, x, \tilde{\eta}, \tilde{Z}, \tilde{\zeta}) \\ &= \mathbb{E}^{\zeta, Z} \left[\int_{\tau_n^\delta}^T [I^h(\zeta_s; s, Z_s) - \eta_s + \eta_s^{\zeta_{\tau_n^\delta}; \tau_n^\delta, R_{\tau_n^\delta}^{\zeta_{\tau_n^\delta}}}] ds \right. \\ &\quad \left. - \sum_{i=1}^n c_{\tau_i}^P \mathbf{1}_{\{T_\delta < \tau_i < T\}} - \sum_{i=n+1}^{\infty} c_{\tau_i}^P \mathbf{1}_{\{\tau_i < T\}} - \sup_{\theta} V_0^P(\theta; \tau_n) \mathbf{1}_{\{\tau_n < T\}} \right] \\ &\leq \mathbb{E}^{\zeta, Z} \left[\int_{\tau_n^\delta}^T [C|Z_s| + C_0 - \eta_s] ds + C_0 \mathbf{1}_{\{\tau_n < T\}} \right] \\ &\leq C \mathbb{E}^{\zeta, Z} \left[\sqrt{(T - \tau_n^\delta)(\bar{L}_{\tau_n^\delta}^{\zeta_{\tau_n^\delta}} - \mathcal{X}_{\tau_n^\delta})} + (\bar{L}_{\tau_n^\delta}^{\zeta_{\tau_n^\delta}} - \mathcal{X}_{\tau_n^\delta}) + \mathbf{1}_{\{\tau_n < T\}} \right] \\ &\leq C \mathbb{P}^{\zeta, Z}(\tau_n < T) + C \mathbb{E}^{\zeta, Z} \left[\mathbf{1}_{\{\tau_n = T\}} \sqrt{\delta(\bar{L}_{T_\delta}^{\zeta_{T_\delta}} - \mathcal{X}_{T_\delta})} + (\bar{L}_{T_\delta}^{\zeta_{T_\delta}} - \mathcal{X}_{T_\delta}) \right] \end{aligned}$$

where the thrid inequality thanks to (4.1). Send $\delta \rightarrow 0$ and note that $\mathcal{X}_{T-} = 0$, we have

$$J_\infty(t, x, \eta, Z, \zeta) - u_n(\theta; t, x) \leq C \mathbb{P}^{\zeta, Z}(\tau_n < T) \leq \frac{C}{n}.$$

This implies $u_\infty(\theta; t, x) - u_n(\theta; t, z) \leq \frac{C}{n}$, and hence $|u_n(\theta; t, x) - u_\infty(\theta; t, z)| \leq \frac{C}{n}$. \square

We now extend Theorem 5.1 (iv).

Theorem 6.3. *The value function $\{u_\infty(\theta; \cdot, \cdot)\}_{\theta \in \Theta}$ is the minimal continuous viscosity solution of the HJB system (4.25) for all $\theta \in \Theta$ with boundary conditions (6.6).*

Remark 6.4. *In Theorem 5.1, the boundary condition \bar{u}_{n-1} is given (recursively), and then $u_n(\theta; \cdot, \cdot)$ can be solved through (4.25) for fixed θ . The situation here is quite different. By (6.4), the boundary condition \bar{u}_∞ relies on the solution u_∞ itself, and it involves all θ . So here the equations for different θ are interacting through the boundary condition and thus we are considering an infinitely dimensional system of HJB equations.*

Proof of Theorem 6.3. By extending (4.26) to u_n and sending $n \rightarrow \infty$, it follows from the uniform convergence of u_n that u_∞ satisfies the dynamic programming principle:

$$u_\infty(\theta; t, x) := \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau \wedge \tau^{\theta; t, x, \eta, Z}} [I^h(\theta; r, Z_r) - \eta_r] dr \right. \\ \left. + u_\infty(\theta; \tau, \mathcal{X}_\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau < \tau^{\theta; t, x, \eta, Z}\}} + \bar{u}_\infty(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} \leq \tau, \tau^{\theta; t, x, \eta, Z} < T\}} \right], \quad (6.8)$$

for any $\theta \in \Theta$, $(t, x) \in D_\theta$, and $\tau \in \mathcal{T}_t$. Then it is straightforward to show that $\{u_\infty(\theta; \cdot, \cdot)\}_{\theta \in \Theta}$ is a continuous viscosity solution of the HJB system (4.25) for all $\theta \in \Theta$ with boundary conditions (6.6). Thus it remains to prove the minimal property.

Let $\{u(\theta; \cdot, \cdot)\}_{\theta \in \Theta}$ be an arbitrary continuous viscosity solution of the HJB system (4.25) for all $\theta \in \Theta$ with boundary conditions (6.6). We emphasize that here the boundary condition at \underline{L}_t^θ is not \bar{u}_∞ , but rather

$$\bar{u}(t) := \sup_{\theta \in \Theta} u(\theta; t, R_t^\theta) - c_t^P.$$

Fix (θ, t, x) and $(\eta, Z, \zeta) \in \mathcal{A}_{t,x}^{\infty, \theta}$ with corresponding $(\{\tau_i\}_{i \geq 0}, \mathcal{X})$ satisfying $\mathcal{X}_{T-} = 0$. Given \bar{u} , recall (3.10) and introduce

$$\tilde{u}_1(\theta; t, x) := \sup_{(\eta^0, Z^0) \in \mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau^{\theta; t, x, \eta^0, Z^0}} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds + \bar{u}(\tau^{\theta; t, x, \eta^0, Z^0}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta^0, Z^0} < T\}} \right].$$

Note that $u(\theta; \cdot, \cdot)$ is a viscosity solution to (4.25) with boundary condition \bar{u} . Then by Theorem 4.6 we see that $\tilde{u}_1(\theta; t, x) \leq u(\theta; t, x)$. Thus for any $(\eta^0, Z^0) \in \mathcal{A}_{t,x}^\theta$ we have

$$\mathbb{E}^{\theta, Z^0} \left[\int_t^{\tau^{\theta; t, x, \eta^0, Z^0}} [I^h(\theta; s, Z_s^0) - \eta_s^0] ds + \bar{u}(\tau^{\theta; t, x, \eta^0, Z^0}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta^0, Z^0} < T\}} \right] \leq u(\theta; t, x).$$

Note that in general $(\eta, Z) \notin \mathcal{A}_{t,x}^\theta$, and thus at above one cannot set $(\eta^0, Z^0) = (\eta, Z)$. However, following the arguments in Proposition 5.4 one can still derive from above that

$$\mathbb{E}^{\theta, Z} \left[\int_t^{\tau_1} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}(\tau_1) \mathbf{1}_{\{\tau_1 < T\}} \right] \leq u(\theta; t, x).$$

Moreover, by the definition of \bar{u} , we have $u(\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}) - c_{\tau_1}^P \leq \bar{u}(\tau_1)$ on $\{\tau_1 < T\}$, then

$$\mathbb{E}^{\theta, Z} \left[\int_t^{\tau_1} [I^h(\theta; s, Z_s) - \eta_s] ds + [u(\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}) - c_{\tau_1}^P] \mathbf{1}_{\{\tau_1 < T\}} \right] \leq u(\theta; t, x).$$

Similarly, on $\{\tau_1 < T\}$ we have

$$\mathbb{E}_{\mathcal{F}_{\tau_1}}^{\zeta_{\tau_1}, Z} \left[\int_{\tau_1}^{\tau_2} [I^h(\zeta_{\tau_1}; s, Z_s) - \eta_s] ds + \bar{u}(\tau_2) \mathbf{1}_{\{\tau_2 < T\}} \right] \leq u(\zeta_{\tau_1}; \tau_1, R_{\tau_1}^{\zeta_{\tau_1}}).$$

Then

$$\mathbb{E}^{\zeta; \cdot, Z} \left[\int_t^{\tau_2} [I^h(\zeta_s; s, Z_s) - \eta_s] ds + \bar{u}(\tau_2) \mathbf{1}_{\{\tau_2 < T\}} - c_{\tau_1}^P \mathbf{1}_{\{\tau_1 < T\}} \right] \leq u(\theta; t, x).$$

Repeating the arguments we have, for any $n \geq 1$,

$$\mathbb{E}^{\zeta, Z} \left[\int_t^{\tau_n} [I^h(\zeta_s; s, Z_s) - \eta_s] ds + \bar{u}(\tau_n) \mathbf{1}_{\{\tau_n < T\}} - \sum_{i=1}^{n-1} c_{\tau_i}^P \mathbf{1}_{\{\tau_i < T\}} \right] \leq u(\theta; t, x).$$

Send $n \rightarrow \infty$, by (6.3) we have

$$\mathbb{E}^{\zeta, Z} \left[\int_t^T [I^h(\zeta_s; s, Z_s) - \eta_s] ds - \sum_{i=1}^{\infty} c_{\tau_i}^P \mathbf{1}_{\{\tau_i < T\}} \right] \leq u(\theta; t, x).$$

Now by the arbitrariness of (η, Z, ζ) , we prove $u_{\infty}(\theta; t, x) \leq u(\theta; t, x)$. \square

We conclude this section with the following result which shows that the principal agent problem (6.2) is time consistent in the following sense.

Theorem 6.5. *The principal agent problem (6.2) satisfies the following dynamic programming principle: for any $\theta \in \Theta$ and $t < T$, and abbreviating the notation $\tau^{\theta} := \tau^{\theta; t, R_t^{\theta}, \eta, Z}$,*

$$\begin{aligned} V_{\infty}^P(\theta; t) &= \sup_{(\eta, Z) \in \mathcal{A}_{t, R_t^{\theta}}^{\theta}} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta}} [I^h(\theta; r, Z_r) - \eta_r] dr + \sup_{\tilde{\theta} \in \Theta} V_{\infty}^P(\tilde{\theta}; \tau^{\theta}) - c_{\tau^{\theta}}^P \mathbf{1}_{\{\tau^{\theta} < T\}} \right]; \\ V_{\infty}^P(t) &= \sup_{\theta \in \Theta} \sup_{(\eta, Z) \in \mathcal{A}_{t, R_t^{\theta}}^{\theta}} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta}} [I^h(\theta; r, Z_r) - \eta_r] dr + V_{\infty}^P(\tau^{\theta}) - c_{\tau^{\theta}}^P \mathbf{1}_{\{\tau^{\theta} < T\}} \right]. \end{aligned} \quad (6.9)$$

Moreover, let $(\eta^*, Z^*, \zeta^*) \in \mathcal{A}_{t, R_t^{\theta}}^{\infty, \theta}$ with corresponding $(\{\tau_i^*\}_{i \geq 0}, \mathcal{X}^*)$ be an optimal control for $V_{\infty}^P(\theta; t) = u_{\infty}(\theta; t, R_t^{\theta})$, then for any $i \geq 1$, $(\eta^*, Z^*, \zeta^*)|_{[\tau_i^*, T]}$ remains optimal for the problem $V_{\infty}^P(\zeta_{\tau_i^*}^*; \tau_i^*)$.

Proof Recall Proposition 6.2 (i) and (6.4). The first line of (6.9) follows directly from (6.8) by setting $\tau \equiv T$, which implies further the second line of (6.9). We shall emphasize that, (6.9) involves the term $V_{\infty}^P(\tilde{\theta}; \tau^{\theta; t, R_t^{\theta}, \eta, Z})$, not $V_{\infty}^P(\tilde{\theta}; \tau^{\tilde{\theta}; t, R_t^{\tilde{\theta}}, \eta, Z})$.

Moreover, if $(\eta^*, Z^*, \zeta^*) \in \mathcal{A}_{t, R_t^\theta}^{\infty, \theta}$ is optimal for $V_\infty^P(\theta; t)$, namely

$$V_\infty^P(\theta; t) = \mathbb{E}^{\zeta^*, Z^*} \left[\int_t^T [I^h(\zeta_r^*; r, Z_r^*) - \eta_r^*] dr - \sum_{i=1}^{\infty} c_{\tau_i^*}^P \mathbf{1}_{\{\tau_i^* < T\}} \right],$$

then $\tilde{\theta} = \zeta_{\tau_1^*}^*$ is optimal in (6.9), and (6.9) becomes

$$V_\infty^P(\theta; t) = \mathbb{E}^{\zeta^*, Z^*} \left[\int_t^{\tau_1^*} [I^h(\theta; r, Z_r^*) - \eta_r^*] dr + V_\infty^P(\zeta_{\tau_1^*}^*; \tau_1^*) - c_{\tau_1^*}^P \mathbf{1}_{\{\tau_1^* < T\}} \right].$$

This implies that

$$\mathbb{E}^{\zeta^*, Z^*} \left[V_\infty^P(\zeta_{\tau_1^*}^*; \tau_1^*) \right] = \mathbb{E}^{\zeta^*, Z^*} \left[\int_{\tau_1^*}^T [I^h(\zeta_r^*; r, Z_r^*) - \eta_r^*] dr - \sum_{i=2}^{\infty} c_{\tau_i^*}^P \mathbf{1}_{\{\tau_i^* < T\}} \right].$$

On the other hand, by (6.1) it is clear that, on $\{\tau_1^* < T\}$,

$$V_\infty^P(\zeta_{\tau_1^*}^*; \tau_1^*) \geq \mathbb{E}_{\mathcal{F}_{\tau_1^*}}^{\zeta^*, Z^*} \left[\int_{\tau_1^*}^T [I^h(\zeta_r^*; r, Z_r^*) - \eta_r^*] dr - \sum_{i=2}^{\infty} c_{\tau_i^*}^P \mathbf{1}_{\{\tau_i^* < T\}} \right].$$

Thus equality holds a.s. at above. That is, $(\eta^*, Z^*, \zeta^*)|_{[\tau_1^*, T]}$ remains optimal for the problem $V_\infty^P(\zeta_{\tau_1^*}^*; \tau_1^*)$. Repeat the arguments we see that $(\eta^*, Z^*, \zeta^*)|_{[\tau_i^*, T]}$ remains optimal for the problem $V_\infty^P(\zeta_{\tau_i^*}^*; \tau_i^*)$ on $\{\tau_i^* < T\}$ for all $i \geq 1$. \square

Remark 6.6. *Theorem 6.5 shows that the principal's problem $V_\infty^P(t)$ is time consistent at the times τ_i^* when the agents quit the job. We should note though, if the principal reconsiders the contract within the time period (τ_i^*, τ_{i+1}^*) , during which the agent is staying for the job, she may find (η^*, Z^*, ζ^*) not optimal. However, since the principal is required to commit to the contract, she is not allowed to fire the agent and sign a better contract during those time periods, so such time inconsistency for the principal is irrelevant in our model.*

7 Comparison with the standard contracts

In this section we investigate briefly the standard principal agent problem (2.19) with full commitment. In particular, the agent is not allowed to quit, rather than having no incentive to quit as in (2.20). We shall compare the principal's optimal values under different settings.

7.1 The dynamic value function for the standard principal agent problem

We first introduce the principal's dynamic value function corresponding to (2.19):

$$u_0^S(\theta; t, x) := \sup_{(\eta, Z) \in \mathcal{A}_{t, x}^\theta} J_P(\theta; t, \eta), \quad 0 \leq t < T, x \leq \bar{L}_t^\theta. \quad (7.1)$$

We then have the following results. Since the proofs are similar, actually easier than those in Sections 3 and 4, we omit them.

Theorem 7.1. *Let Assumption 2.2 hold.*

(i) For each $\theta \in \Theta$ and $t < T$, $u_0^S(\theta; t, \cdot)$ is decreasing in $x \in (-\infty, \bar{L}_t^\theta]$. Consequently,

$$V_0^{S,P}(\theta; t) = u_0^S(\theta; t, R_t^\theta), \quad \text{provided } R_t^{\theta_0} \leq \bar{L}_t^{\theta_0}. \quad (7.2)$$

(ii) For any $t_1, t_2 < T$ and $x_1 \leq \bar{L}_{t_1}^\theta$, $x_2 \leq \bar{L}_{t_2}^\theta$, we have

$$|u_0^S(\theta; t_1, x_1) - u_0^S(\theta; t_2, x_2)| \leq C \left[\sqrt{|x_1 - x_2|} + \sqrt{\rho(|t_1 - t_2|) - x_1|t_1 - t_2|} \right]. \quad (7.3)$$

(iii) $u_0^S(\theta; \cdot, \cdot)$ has boundary conditions:

$$u_0^S(\theta; t, \bar{L}_t^\theta) = -C_0(T - t), \quad \forall t \in [0, T]; \quad u_0^S(\theta; T-, x) = 0, \quad \forall x < 0. \quad (7.4)$$

Moreover, for $x \rightarrow -\infty$, u_0 satisfies the following growth condition: for $\forall t < T$, $x \leq \bar{L}_t^\theta$,

$$-C_0(T - t) \leq u_0^S(\theta; t, x) \leq C\sqrt{\rho(T - t) - x(T - t)}, \quad (7.5)$$

(iv) $u_0^S(\theta; \cdot, \cdot)$ is the unique viscosity solution of the following HJB equation with boundary conditions (7.4) and growth conditions (7.5):

$$\mathcal{L}^\theta u_0^S(\theta; t, x) = 0, \quad 0 \leq t < T, x \leq \bar{L}_t^\theta. \quad (7.6)$$

Similarly to (5.1) and (5.2), we now consider the principal agent problem where the agent is allowed to quit, however, after seeing n agents' quitting, the principal will offer only committed contracts so that the new agent cannot quit anymore: for $n = 1, 2, \dots$,

$$\begin{aligned} \bar{u}_{n-1}^S(t) &:= \sup_{\theta \in \Theta} u_{n-1}^S(\theta; t, R_t^\theta) - c_t^P; \\ u_n^S(\theta; t, x) &:= \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^\theta} \mathbb{E}^{\theta, Z} \left[\int_t^{\tau^{\theta; t, x, \eta, Z}} [I^h(\theta; s, Z_s) - \eta_s] ds + \bar{u}_{n-1}^S(\tau^{\theta; t, x, \eta, Z}) \mathbf{1}_{\{\tau^{\theta; t, x, \eta, Z} < T\}} \right]; \\ V_n^{S,P}(\theta; t) &:= \sup_{\eta \in \mathcal{A}_t^P} \mathbb{E}^{\theta, Z^{\theta; t, \eta}} \left[\int_t^{\tau^{\theta; t, \eta}} [I^h(\theta; s, Z_s^{\theta; t, \eta}) - \eta_s] ds + \bar{u}_{n-1}^S(\tau^{\theta; t, \eta}) \mathbf{1}_{\{\tau^{\theta; t, \eta} < T\}} \right], \\ &\text{subject to } V_1^A(\theta; t, \eta) \geq R_t^\theta. \end{aligned} \quad (7.7)$$

Then Theorem 5.1 remain true for u_n^S, \bar{u}_n^S and $V_n^{S,P}$, after obvious modifications. Moreover, for the same function u_∞ in (6.1), following similar arguments as in Proposition 6.2 we have

Proposition 7.2. *Under Assumptions 2.2 and 3.1, we have $|u_n^S - u_\infty| \leq \frac{C}{n}$.*

Remark 7.3. We may also define a value function u_0 to characterize V_0^P . Denote

$$\begin{aligned} \overline{\mathcal{A}}_{t,x}^\theta &:= \left\{ (\eta, Z) \in \mathcal{A}_{t,x}^\theta : \mathcal{X}_s^{\theta;t,x,\eta,Z} \geq R_s^\theta, s \in [t, T], \text{ a.s.} \right\}; \quad \overline{R}_t^\theta := \inf \left\{ x : \overline{\mathcal{A}}_{t,x}^\theta \neq \emptyset \right\}; \\ u_0(\theta; t, x) &:= \sup_{(\eta, Z) \in \overline{\mathcal{A}}_{t,x}^\theta} J_P(\theta; t, \eta), \quad t < T, x \in [\overline{R}_t^\theta, \underline{L}_t^\theta]. \end{aligned} \quad (7.8)$$

Then one can easily show that $u_0 \leq u_0^S$, and u_0 is decreasing in x , and thus $V_0^P(\theta; t) = u_0(\theta; t, \overline{R}_t^\theta)$. However, it seems hard to establish the regularity of u_0 and its boundary condition at \overline{R}^θ . Nevertheless, we have the desired regularity for $V_0^P(\theta; t)$ in (2.23).

7.2 Some comparisons

In this subsection we compare the principal's optimal utilities under different settings. First, recall (2.21), it is obvious that

$$V_0^{S,P}(\theta; t) \geq V_0^P(\theta; t), \quad \text{and hence} \quad u_n^S(\theta; t, x) \geq u_n(\theta; t, x) \text{ for all } \theta \in \Theta, (t, x) \in D_\theta.$$

That is, the principal would endure some loss by providing the agent the incentives to stay in the contract.

We next consider the case that $\Theta = \{\theta_0\}$ is a singleton, namely there is only one type of agent in the market, as in most publications in the literature. Recall Proposition 5.2.

Proposition 7.4. *Let Assumptions 2.2 and 3.1 hold and $\Theta = \{\theta_0\}$. Then $u_n^S \downarrow u_\infty$ and $V_n^{S,P} \downarrow V_\infty^P$, as $n \rightarrow \infty$.*

Proof We first show that

$$u_1^S(\theta_0; t, x) \leq u_0^S(\theta_0; t, x). \quad (7.9)$$

This, together with (7.7) and Proposition 7.2, implies recursively $u_n^S \downarrow u_\infty$ and $V_n^{S,P} \downarrow V_\infty^P$.

Indeed, in this case by (7.7) we have

$$u_1^S(\theta_0; t, x) = \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^{\theta_0}} \mathbb{E}^{\theta_0, Z} \left[\int_t^{\tau_1} [I^h(\theta_0; s, Z_s) - \eta_s] ds + [u_0^S(\theta_0; \tau_1, R_{\tau_1}^{\theta_0}) - c_{\tau_1}^P] \mathbf{1}_{\{\tau_1 < T\}} \right] \quad (7.10)$$

On the other hand, similarly to (4.26) u_0^S satisfies the dynamic programming principle:

$$u_0^S(\theta_0; t, x) = \sup_{(\eta, Z) \in \mathcal{A}_{t,x}^{\theta_0}} \mathbb{E}^{\theta_0, Z} \left[\int_t^{\tau_1} [I^h(\theta_0; s, Z) - \eta_s] ds + u_0^S(\theta_0; \tau_1, \underline{L}_{\tau_1}^{\theta_0}) \mathbf{1}_{\{\tau_1 < T\}} \right]. \quad (7.11)$$

Note that $R_{\tau_1}^{\theta_0} > R_{\tau_1}^{\theta_0} - c_{\tau_1}^{\theta_0} = \underline{L}_{\tau_1}^{\theta_0}$ and u_0 is decreasing in x , then

$$u_0^S(\theta_0; \tau_1, R_{\tau_1}^{\theta_0}) - c_{\tau_1}^P \leq u_0^S(\theta_0; \tau_1, \underline{L}_{\tau_1}^{\theta_0}), \quad (7.12)$$

which implies (7.9) immediately, where the loss is due to the costs $c_{\tau_1}^{\theta_0}$ and $c_{\tau_1}^P$. \square

Remark 7.5. By Propositions 5.2 and 7.4 we see that, in the case $\Theta = \{\theta_0\}$, it is most desirable for the principal to offer committed contracts, where the agent is not allowed to quit, and receives the optimal utility $V_0^{S,P}$. However, if the principal cannot force the agent to commit, she would still prefer to offer a standard contract than a self-enforcing one, being aware that she might need to hire a new agent if the current one quits the job (or improve the contract by a retention offer when the agent threatens to quit).

We should note though that, in practice, if the agent is asked to commit to the contract for the whole period, he might ask for a larger individual reservation R^θ . This will of course change our analysis.

We now turn to the case that there are multiple (or even infinitely many) types of agents in the market, which is one of the main features of our model and is often the case in practice. In this case, the principal may have the opportunity to choose a cheaper agent when the current one quits the job. If this gain is larger than the costs c^{θ_0}, c^P induced by the agent's quitting, namely (again this is impossible when $\Theta = \{\theta_0\}$ as we see in (7.12))

$$\bar{u}_0(\tau_1) = \sup_{\theta \in \Theta} u_0(\theta; \tau_1, R_{\tau_1}^\theta) - c_{\tau_1}^P > u_0(\theta_0; \tau_1, R_{\tau_1}^{\theta_0} - c_{\tau_1}^{\theta_0}),$$

then the principal may design the contract to induce the agent to quit. The following example shows that this can indeed be the case.

Example 7.6. Set $T = 1$, $\Theta = \{\theta_0, \theta_1\} \subset (0, \infty)$, $f(\theta; t, \eta) = -e^{-\eta}$, $h(\theta; t, \alpha) = \frac{1}{2\theta}\alpha_t^2$, and $R_0^{\theta_0} = R_0^{\theta_1} = x_0 := -2e^{-C_0}$, where $C_0 > 0$ is the fixed constant for \mathcal{A}^P . These determine the value $u_0^S(\theta; 0, x_0)$ and assume without loss of generality that $u_0^S(\theta_0; 0, x_0) \geq u_0^S(\theta_1; 0, x_0)$. Set further that $c^\theta \equiv e^{-C_0}$, $c^P \equiv 0$, and, for some constant $n \geq 1$,

$$\begin{aligned} R_t^{\theta_0} &:= [x_0 + 3e^{-C_0}t]\mathbf{1}_{[0, \frac{1}{2})}(t) - e^{-C_0}[1-t]\mathbf{1}_{[\frac{1}{2}, 1]}(t); \\ R_t^{\theta_1} &:= [x_0 - nt]\mathbf{1}_{[0, \frac{1}{2})}(t) + 2[x_0 - \frac{n}{2}](1-t)\mathbf{1}_{[\frac{1}{2}, 1]}(t). \end{aligned}$$

Then, $\lim_{n \rightarrow \infty} u_1^S(\theta_0; 0, x_0) = \infty$, and consequently, when n is large enough, we have

$$\max_{\theta \in \Theta} V_0^{S,P}(\theta; 0) = u_0^S(\theta_0; 0, x_0) < u_1^S(\theta_0; 0, x_0) \leq \max_{\theta \in \Theta} V_1^{S,P}(\theta; 0). \quad (7.13)$$

Proof First, it is clear that $R_t^{\theta_1} \leq R_t^{\theta_0} \leq -e^{-C_0}(1-t) = \bar{L}_t^{\theta_0} = \bar{L}_t^{\theta_1}$. Note that

$$\underline{L}_t^{\theta_0} = [-2e^{-C_0} + 3e^{-C_0}t]\mathbf{1}_{[0, \frac{1}{2})}(t) - e^{-C_0}[1-t]\mathbf{1}_{[\frac{1}{2}, 1]}(t) - e^{-C_0}.$$

Set

$$\eta_t := C_0\mathbf{1}_{[0, \frac{1}{2})}(t) + [C_0 - \ln 3]\mathbf{1}_{[\frac{1}{2}, 1]}(t), \quad \tilde{\eta} \equiv -\ln(n + 4e^{-C_0}).$$

Then one can easily see that $(\eta, 0) \in \mathcal{A}_{0,x_0}^{\theta_0}$ with $\tau^{\theta_0;0,x_0,\eta,0} = \frac{1}{2}$, and $(\tilde{\eta}, 0) \in \mathcal{A}_{\frac{1}{2},x_0-\frac{n}{2}}^{\theta_1}$. Thus

$$\begin{aligned} u_1(\theta_0; 0, x_0) &\geq J_1(\theta_0; 0, x_0, \eta, Z) = \int_0^{\frac{1}{2}} [\theta_0 Z_s - \eta_s] ds + \bar{u}_0\left(\frac{1}{2}\right) \\ &\geq -\frac{1}{2}C_0 + u_0\left(\theta_1; \frac{1}{2}, R_{\frac{1}{2}}^{\theta_1}\right) = -\frac{1}{2}C_0 + u_0\left(\theta_1; \frac{1}{2}, x_0 - \frac{n}{2}\right); \\ u_0\left(\theta_1; \frac{1}{2}, x_0 - \frac{n}{2}\right) &\geq J_0\left(\theta_1; \frac{1}{2}, \tilde{\eta}, \tilde{Z}\right) = \int_{\frac{1}{2}}^1 [\ln(n + 4e^{-C_0})] ds = \frac{1}{2} \ln(n + 4e^{-C_0}). \end{aligned}$$

Therefore,

$$u_1(\theta_0; 0, x_0) \geq \frac{1}{2} \ln(n + 4e^{-C_0}) - \frac{1}{2}C_0 \rightarrow \infty, \quad \text{as } n \rightarrow \infty. \quad \square$$

Remark 7.7. *In Example 7.6, the agent θ_1 becomes cheaper when time t evolves (before $\frac{1}{2}$). While it is desirable for the principal to hire agent θ_0 at initial time 0, it becomes obvious that at later time the agent θ_1 is more desirable. Since in our model the principal is not allowed to fire the agent θ_0 , she may design the contract to induce the agent θ_0 to quit. This will provide the principal the opportunity to hire the cheaper agent θ_1 at a later time and obtain a larger utility.*

8 Summary

In this paper we study a principal agent problem with moral hazard in finite time horizon. Our model has the following features.

- The principal is committed to the contract but the agent is allowed to quit.
- When the current agent quits the job, both the principal and the agent would incur some costs, and the principal would rehire a new agent, satisfying the agent's individual rationality constraint at that time.
- There are a family of agents in the market with different types. The principal knows the type of the agents and at any time the principal hires only one agent.
- Each agent's dynamic individual rationality is given exogenously and is deterministic.

We characterize the principal's dynamic optimal utility function through a system of HJB equations, parametrized by the agent's type. These functions could be discontinuous at the boundary. A few findings are interesting from the practical perspective.

- The self-enforcing contracts can only be sub-optimal for the principal. That is, it is not desirable for the principal to disincentivize the agent from quitting.
- In some markets, the standard optimal contract for committed agents may also be sub-optimal. That is, the principal may prefer the agent to quit so that she can hire a cheaper agent from the market. In this case she would design the contract to induce the agent to quit. This is mainly due to the presence of various types of agents.
- By assuming uniform lower bound of the cost c^θ , the agents won't quit too frequently and actually the principal will only see finitely many quittings.
- The principal's problem is time consistent in the sense that, the optimal contract remains optimal at any optimal quitting time of the agents.

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