Probability Theory and Stochastic Modelling 86

Jianfeng Zhang

Backward Stochastic Differential Equations

From Linear to Fully Nonlinear Theory

Springer

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Volume 86

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To my family

张发生　　赵玉铭
何茶娥　　牛福华
Ying　　*Albert*

Preface

Why this book was written: Initiated by Pardoux & Peng [167], the theory of Backward Stochastic Differential Equations has been extensively studied in the past decades, and their applications have been found in many areas. While there are a few excellent monographs and book chapters on the subject, see, e.g., El Karoui & Mazliak [80], Peng [175], Ma & Yong [148] (on forward-backward SDEs), and Pardoux & Rascanu [170], there is an increasing demand for a textbook which is accessible to graduate students and junior researchers interested in this important and fascinating area. In the meantime, there is a strong need for a book that includes the more recent developments, e.g., the fully nonlinear theory and path-dependent PDEs. The aim in this book is to introduce up-to-date developments in the field in a systematic and "elementary" way.

There is often a trade-off between generality and clarity. While it is convenient to have the most general results for the purpose of direct applications, in many situations, one may need the ideas rather than the results. The high technicality involved due to the generality may unfortunately obscure the key ideas, even for experts. In this book, the focus is on ideas, so that readers may have a comprehensive taste of the main results, the required conditions, and the techniques involved. Almost all results in the book have been proven from scratch, and the arguments have been made to look as natural as possible. As such, the generality has been sacrificed in many places.

Who is it for: Ph.D. students and junior researchers majoring in Stochastic Analysis are the main target audience. However, it is the author's hope that it (at least Part I) proves useful for graduate students majoring in Engineering and Quantitative Finance. The material from Part I of the book was used for regular Ph.D. courses at USC, with students majoring in various fields.

The last part of the book on fully nonlinear theory is more advanced. The material has been used for a special topics course at USC as well as for some short courses in other places. It is hoped that junior researchers interested in this area will find it helpful.

Prerequisites: A solid knowledge of graduate-level Stochastic Calculus and Real Analysis is required, and basic knowledge on second-order PDEs and financial derivatives will also be helpful. However, the book has been written to be as self-contained as possible (except some limited material in Part III), with the more advanced prerequisite material presented in the Chapter 1. Therefore, readers with less knowledge but with a good sense of general mathematics theory may also find the book accessible, by skipping some technical proofs when necessary.

Structure of the book: The book is divided into three main parts: basic theory of SDEs and BSDEs, further theory of BSDEs, and more recent developments in fully nonlinear theory. References for related topics are given at the end of each chapter.

Part I is basic and is more or less mature. It starts with the basics of stochastic calculus, such as stochastic integration and martingale representation theorem, which can be viewed as linear SDEs and linear Backward SDEs, respectively. In contrast with the fully nonlinear theory in Part III, these materials can be viewed as the linear theory. Then, the general (nonlinear) SDEs and Backward SDEs are dealt with in the same spirit. In particular, BSDE theory is described as a semilinear theory because it is associated with semilinear PDE in the Markovian case and semilinear path-dependent PDE in the non-Markovian case. Such a connection in the Markovian case is established rigorously in this part, and the non-Markovian case is studied in Part III.

Part II covers three important extensions of the theory: reflected BSDEs, BSDEs with quadratic growth in the Z component, and coupled forward-backward SDEs. This part can be expanded drastically, for example, Peng's g-expectation, BSDEs with non-smooth coefficients, reflected BSDEs with two barriers and Dynkin games, BSDEs with general constraints, infinite-dimensional BSDEs, BSDEs with random or infinite horizon and their connection with elliptic PDEs, weak solutions of BSDEs, and backward stochastic PDEs, and BSDEs with jumps, to mention a few. These topics are not covered here, following the aim to keep the book within a reasonable size. However, they are briefly discussed for the benefit of the readers.

Part III has been developing very dynamically in recent years. It covers three topics: nonlinear expectation, path dependent PDEs, and second-order BSDEs, together with a preparation chapter on weak formulation upon which all the three subjects are built. The theory is far from mature. Nevertheless, it is intrinsically a continuation of Parts I and II, and we have received strong feedback to provide an introduction on its recent developments.

The book is aimed at theory rather than application. While there are numerous publications on applications of BSDEs, including a few excellent books (see, e.g., Yong & Zhou [242], Pham [190], Cvitanic & Zhang [52], Touzi [227], Crepey [43], Delong [60], and Carmona [29]), the author's opinion is that most fall into three categories: pricing and hedging financial derivatives, stochastic optimization and games, and connections with nonlinear PDEs. These applications are scattered in this book, mainly to motivate the theories.

Another major topic in the field is probabilistic numerical methods for nonlinear PDEs. The book provides only a very brief introduction to it. It is believed that this deserves a separate research monograph.

Exercises are an important part of the book. There are mainly four types of problems: (i) technical preparations for some main results, (ii) alternative proofs of the main results, (iii) extensions of some results proved in the book, and (iv) practice of some important techniques used in the book. Hints and/or solutions to some selective problems are available in the author's website: http://www-bcf.usc.edu/~jianfenz/.

A few additional notes: As already mentioned, clarity has been given precedence over generalization. There are occasions where we impose conditions stronger than necessary so as to focus on the main ideas and make the proofs more accessible to the readers. So generalized results which require rather sophisticated techniques are not included, but some of them are listed in the Bibliographical Notes section for interested readers.

To make the book more readable, for most proofs, only one-dimensional notation has been used. For results where the dimension is crucial, it is mentioned explicitly.

The field has grown rapidly. It is impossible to exhaust the references. The author admits that they have inevitably missed many very important and highly relevant works.

Acknowledgments A very large portion of my research was related to the material presented in this book. I take this opportunity to thank all my teachers, collaborators, students, and friends who have helped me along my career. In particular, I am very grateful to the following people who have had the greatest impact on my academic career: Jaksa Cvitanic, Jin Ma, Shige Peng, and Nizar Touzi.

While some presentations in the book might be new, all credits of the results should go to the original papers, and thus I am indebted to all the authors whose results and/or ideas I have borrowed from. I am grateful to several reviewers who provided many constructive suggestions and corrected numerous typos in the earlier versions of the book. Of course, I am solely responsible for the remaining errors in the book, and I would truly appreciate any comments or suggestions the readers may have.

I am grateful for the support from the Springer staff, especially Donna Chernyk for her endless patience throughout the writing of this book. The book was partially supported by the National Science Foundation grants DMS 1008873 and 1413717. I am also grateful for the support of my home institution: the University of Southern California.

Finally, my foremost gratitude goes to my family, for their understanding, support, and love.

Los Angeles, CA, USA Jianfeng Zhang
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