# From finite population optimal stopping
# to mean field optimal stopping[*]

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## Abstract

This paper analyzes the convergence of the finite population optimal stopping problem towards the corresponding mean field limit. Building on the viscosity solution characterization of the mean field optimal stopping problem of our previous papers [51, 52], we prove the convergence of the value functions by adapting the Barles-Souganidis [1] monotone scheme method to our context. We next characterize the optimal stopping policies of the mean field problem by the accumulation points of the finite population optimal stopping strategies. In particular, if the limiting problem has a unique optimal stopping policy, then the finite population optimal stopping strategies do converge towards this solution. As a by-product of our analysis, we provide an extension of the standard propagation of chaos to the context of stopped McKean-Vlasov diffusions.

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## 1   Introduction

This paper is a continuation of our previous papers [51, 52], where we established the wellposedness, especially in the viscosity sense, for an obstacle equation on Wasserstein

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space derived from a mean field optimal stopping problem. In this paper we focus on the convergence of the corresponding $N$-player optimal stopping problem, the latter was investigated by Kobylanski, Quenez & Rouy-Mironescu [36] in a more general setting. We shall obtain both the convergence of the value functions and the propagation of chaos in terms of the (approximate) optimal trajectories.

In the context of mean field games, initiated independently by Lasry & Lions [42] and Caines, Huang & Malhamé [34], the convergence of the $N$-player game to the mean field game is one of the core issues in the field. When the mean field equilibrium is unique, typically under certain monotonicity conditions, one may use the popular master equation approach to obtain the convergence, see Cardaliaguet, Delarue, Lasry & Lions [8], followed by Bayraktar & Cohen [4], Bayraktar, Cecchin, Cohen & Delarue [2], Cardaliaguet [6], Cecchin & Pelino [14], Delarue-Lacker-Ramanan [17, 16], Gangbo & Meszaros [32], and Mou & Zhang [44], to mention a few. When there are multiple equilibria, there are numerous publications on the convergence issue, mainly on the propagation of chaos, see e.g. Camona & Delarue [10], Cecchin, Dai Pra, Fisher & Pelino [15], Djete [18, 19], Doncel, Gast & Gaujal [21], Feleqi [26], Fischer [27], Fischer & Silva [29], Lacker [38, 40], Lacker & Flem [41], Lasry & Lions [42], Lauriere & Tangpi [43], Nutz, San Martin & Tan [45], and Possamai & Tangpi [47]. We would also like to mention the set valued approach in Iseri & Zhang [35], where the object is the set of values over all equilibria.

In the context of mean field controls, an important stream of works focuses on the convergence issue, see e.g. Bayraktar & Chakraborty [3], Cardaliaguet, Daudin, Jackson & Souganidis [7], Cardaliaguet & Souganidis [9], Cavagnari, Lisini, Orrieri & Savare [12], Cecchin [13], Djete, Possamai & Tan [20], Fischer & Livieri [28], Fornasier, Lisini, Orrieri & Savare [30], Germain, Pham & Warin [33], and Lacker [39]. In particular, by utilizing some strong regularity of the value function, [33, 7] obtained certain rate of convergence. The work closest to ours is Gangbo, Mayorga & Swiech [31], which uses the uniqueness of viscosity solution (in certain sense) to prove the convergence of the value function in a setting without idiosyncratic but with common noise. We also mention the paper Talbi [50] by the first author which applies the same approach of this paper to finite dimensional approximation of mean field control problems, including the path dependent case.

The first goal of this paper is to apply the Barles & Sounganids [1] approach to prove the convergence of the value functions. To our best knowledge, this is the first work in the mean field literature to use the Barles-Sounganids approach. Roughly speaking, denoting by $V$ the value function of the mean field optimal stopping problem and $V^N$ the value

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function of the corresponding $N$-player problem, we shall prove that $\underline{V} := \liminf_{N\to\infty} V^N$ and $\overline{V} := \limsup_{N\to\infty} V^N$ are viscosity supersolution and subsolution, respectively, of our mean field obstacle equation. Then, it follows from the comparison principle of viscosity solutions, which was established in our previous paper [52], that $\underline{V} = \overline{V}$ and hence $\lim_{N\to\infty} V^N = V$. Note that our viscosity solution approach allows us to deal with value functions which are merely continuous. Consequently, besides the obvious difference that we consider mean field optimal stopping problem instead of mean field control problem, we require weaker regularity conditions for the coefficients than [7, 33].

Our convergence relies on the viscosity solution property of $V^N$ to a finite-dimensional PDE, as deduced from the general solution of [36] expressed in terms of a recursive sequence of Snell envelopes. We shall refer to this equation as a cascade obstacle problem due to its remarkable structure. One subtle issue is the choice of test functions for viscosity solutions of this finite-dimensional PDE. To adapt to our notion of viscosity solution for our mean field obstacle equation in [52], we only require test functions to be tangent to the super/subsolution through the mean, whereas the tangency is pointwise in the standard literature. Our definition is inspired by the the viscosity theory developed for path-dependent PDEs, see e.g. the series of papers by Ekren, Keller, Ren, Touzi & Zhang [22, 23, 24, 25, 48].

Our second main result is the propagation of chaos for stopped McKean-Vlasov diffusion (see e.g. Sznitman [49] for the case of classical diffusions). By using the convergence of the value functions, we establish three results: roughly speaking,

• Any optimal strategy of the mean field problem can be approximated by (approximate) optimal strategies of the $N$-player problems;

• Any limit point of the (approximate) optimal strategies of the $N$-player problem is optimal for the mean field problem;

• If the optimal strategy of the mean field problem is unique, then any (approximate) optimal strategies of the $N$-player problem converges to that unique optimal strategy of the mean field problem.

These results are consistent with the convergence results for mean field equilibria in the mean field game literature, see e.g. [38]. We shall remark that, for mean field games, the $N$-player problems are quite different between closed loop controls and open loop controls. However, for our mean field optimal stopping problem (and for mean field control problems), the open loop and the closed loop stopping strategies typically lead to the same value function. In particular, for the convenience of establishing the regularity of the value functions, we consider open loop stopping times for the $N$-player problems.

The paper is organized as follows. In Sections 2 and 3 we introduce the mean field optimal stopping problem and the finite population optimal stopping problem, respectively. In Section 4 we present the main results of the paper. Sections 5 and 6 are devoted to the proofs of the main results, one on the convergence of the value functions, and the other on the propagation of chaos. In Section 7 we study the multiple optimal stopping problem in details. Finally, we provide some technical proofs in Appendix.

**Notations.** We denote by $\mathcal{P}(\Omega, \mathcal{F})$ the set of probability measures on a measurable space $(\Omega, \mathcal{F})$, and $\mathcal{P}_2(\Omega, \mathcal{F})$ the subset of square integrable probability measures in $\mathcal{P}(\Omega, \mathcal{F})$, equipped with the 2-Wasserstein distance $\mathcal{W}_2$. When $\Omega$ is a topological space and $\mathcal{F}$ is the Borel $\sigma$-field, we simply denote them as $\mathcal{P}(\Omega)$ and $\mathcal{P}_2(\Omega)$. For a random variable $Z$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\mathbb{P}_Z := \mathbb{P} \circ Z^{-1}$ the law of $Z$ under $\mathbb{P}$. The space of $d \times d$—symmetric matrice is denoted by $\mathcal{S}_d$, with $\mathcal{S}_d^+$ the subset of non-negative matrices. For vectors $x, y \in \mathbb{R}^d$ and matrices $A, B \in \mathbb{R}^{d \times N}$, denote $x \cdot y := \sum_{i=1}^n x_i y_i$ and $A : B := \text{tr}(AB^\top)$. We shall also write "USC" (resp. "LSC") "upper (resp. lower) semi-continuous". Moreover, denote

$$\mathbf{S} := \mathbb{R}^d \times \{0, 1\}, \quad \mathbf{Q}_t := [t, T) \times \mathcal{P}_2(\mathbf{S}), \quad \overline{\mathbf{Q}}_t := [t, T] \times \mathcal{P}_2(\mathbf{S}), \quad t \in [0, T);$$

$$\|m\|_p^p := \int_{\mathbf{S}} |x|^p m(dy), \ m \in \mathcal{P}_2(\mathbf{S}), \quad \text{for any } p \geq 1;$$

and, for any $N \geq 1$,

$$\Lambda_t^N := [t, T) \times \mathbf{S}^N, \quad \overline{\Lambda}_t^N := [t, T] \times \mathbf{S}^N;$$

$$m^N(\mathbf{y}) := \frac{1}{N} \sum_{k=1}^N \delta_{y_k} \in \mathcal{P}(\mathbb{R}^d), \quad \text{for} \quad \mathbf{y} = (y_1, \cdots, y_N) \in \mathbf{S}^N;$$

$$\mathcal{P}^N(\mathbf{S}) := \{m_N(\mathbf{y}) : \mathbf{y} \in \mathbf{S}^N\}, \quad \mathbf{Q}_t^N := [t, T) \times \mathcal{P}^N(\mathbf{S}), \quad \overline{\mathbf{Q}}_t^N := [t, T] \times \mathcal{P}^N(\mathbf{S}); \quad (1.1)$$

$$\|\mathbf{x}\|_p^p := \frac{1}{N} \sum_{k=1}^N |x_k|^p, \ \mathbf{x} = (x_1, \cdots, x_N) \in \mathbb{R}^{d \times N},$$

## 2 The mean field optimal stopping problem

In this section we introduce the mean field optimal stopping problem and the related obstacle equation investigated in our previous works [51, 52].

### 2.1 The mean field optimal stopping problem

Let $T < \infty$, and denote $\Omega := C^0([-1, T], \mathbb{R}^d) \times \mathbb{I}^0([-1, T])$ the canonical space, with:

- $C^0([-1, T], \mathbb{R}^d)$ is the set of continuous paths from $[-1, T]$ to $\mathbb{R}^d$, constant on $[-1, 0)$;
- $\mathbb{I}^0([-1, T])$ is the set of non-increasing and càdlàg maps from $[-1, T]$ to $\{0, 1\}$, constant on $[-1, 0)$, and ending with value $0$ at $T$.

We recall that $\Omega$ is Polish under the Skorokhod distance. We also denote $Y := (X, I)$ the canonical process taking values in the state space $\mathbf{S}$, $\mathbb{F} = (\mathcal{F}_t)_{t \in [-1, T]}$ its canonical filtration, and the corresponding jump time of the survival process $I$:

$$\iota := \inf\{t \geq 0 : I_t = 0\}, \text{ so that } I_t := I_{0-} \mathbf{1}_{t < \iota} \text{ for all } t \in [-1, T]. \tag{2.1}$$

Since $I$ is càdlàg, $\iota$ is an $\mathbb{F}$−stopping time. We emphasise on the fact that the extension to $-1$ is arbitrary, as the extension of time to the left of the origin is only needed to allow for an immediate stop at time $t = 0$.

Let $(b, \sigma, f) : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbf{S}) \to \mathbb{R}^d \times \mathbb{S}_d^+ \times \mathbb{R}$, and $g : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$.

**Assumption 2.1** (i) $b, \sigma$ *are continuous in* $t$, *and uniformly Lipschitz continuous in* $(x, m)$. (ii) $f$ *is Borel measurable and has quadratic growth in* $x \in \mathbb{R}^d$, *and*

$$F(t, m) := \int_{\mathbb{R}^d} f(t, x, m) m(dx, 1) \text{ is continuous on } [0, T] \times \mathcal{P}_2(\mathbf{S}). \tag{2.2}$$

(iii) $g$ *is upper-semicontinuous and locally bounded.*

We also extend $g$ to $\mathcal{P}_2(\mathbf{S})$ by $g(m) := g(m(\cdot, \{0, 1\}))$. Let $V$ be the dynamic value function:

$$V(t, m) := \sup_{\mathbb{P} \in \mathcal{P}(t,m)} J(t, \mathbb{P}), \ (t, m) \in \overline{\mathbf{Q}}_0, \quad J(t, \mathbb{P}) := \int_t^T F(r, \mathbb{P}_{Y_r}) dr + g(\mathbb{P}_{Y_T}), \tag{2.3}$$

with $\mathcal{P}(t, m)$ the set of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ s.t. $\mathbb{P}_{Y_{t-}} = m$ and the processes:

$$M_{\cdot} := X_{\cdot} - \int_t^{\cdot} b(r, X_r, \mathbb{P}_{Y_r}) I_r dr \quad \text{and} \quad M_{\cdot} M_{\cdot}^{\intercal} - \int_t^{\cdot} \sigma^2(r, X_r, \mathbb{P}_{Y_r}) I_r dr \tag{2.4}$$

are $\mathbb{P}$−martingales on $[t, T]$, that is, for some $\mathbb{P}$−Brownian motion $W^{\mathbb{P}}$,

$$X_s = X_t + \int_t^s b(r, X_r, \mathbb{P}_{Y_r}) I_r dr + \sigma(r, X_r, \mathbb{P}_{Y_r}) I_r dW_r^{\mathbb{P}}, \ I_s = I_{t-} \mathbf{1}_{s < \iota}, \ \mathbb{P} - \text{a.s.} \tag{2.5}$$

A special element of $\mathcal{P}(t, m)$ is $\bar{\mathbb{P}} = \bar{\mathbb{P}}^{t,m}$ under which $X$ is unstopped. That is,

$$X_s = X_t + \int_t^s b(r, X_r, \bar{\mathbb{P}}_{Y_r}) I_r dr + \int_t^s \sigma(r, X_r, \bar{\mathbb{P}}_{Y_r}) I_r dW_r^{\bar{\mathbb{P}}}, \ I_s = I_{t-} \mathbf{1}_{[t,T)}(s), \ \bar{\mathbb{P}}\text{-a.s.} \tag{2.6}$$

Note that $Y_{\cdot} = Y_{\cdot \wedge \iota}$, and in particular $Y_T = Y_{\iota}$, $\mathbb{P}$−a.s. Moreover, from the definition of $F$ in (2.2), we have $\int_t^T F(r, \mathbb{P}_{Y_r}) dr = \mathbb{E}^{\mathbb{P}} \int_t^{\iota} f(r, X_r, \mathbb{P}_{Y_r}) dr$.

We observe that (2.3) admits an optimal control as $\mathcal{P}(t, m)$ is $\mathcal{W}_2$-compact by [51].

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## 2.2 Obstacle equation on Wasserstein space

We first recall some differential calculus tools on the Wasserstein space. We say that a function $U : \mathcal{P}_2(\mathbf{S}) \to \mathbb{R}$ has a functional linear derivative $\delta_m U : \mathcal{P}_2(\mathbf{S}) \times \mathbf{S} \to \mathbb{R}$ if

$$U(\tilde{m}) - U(m) = \int_0^1 \int_{\mathbf{S}} \delta_m U(l\tilde{m} + (1-l)m, y)(\tilde{m} - m)(dy)dl, \quad \text{for all } m, \tilde{m} \in \mathcal{P}_2(\mathbf{S}),$$

and $\delta_m U(m, \cdot)$ has quadratic growth, locally uniformly in $m$, so as to guarantee integrability in the last expression. As in [51], we denote

$$\delta_m U_i(t, m, x) := \delta_m U(t, m, x, i) \quad \text{for} \quad i \in \{0, 1\}, \qquad D_I U := \delta_m U_1 - \delta_m U_0,$$

and we introduce the measure flow generator of $X$

$$\mathbb{L}U(t, m) \quad := \quad \partial_t U(t, m) + \int_{\mathbb{R}^d} \mathcal{L}_x \delta_m U_1(t, m, x) m(dx, 1),$$

where $\mathcal{L}_x$ is the generator of $X$:

$$\mathcal{L}_x \delta_m U_1 \quad := \quad b \cdot \partial_x \delta_m U_1 + \frac{1}{2}\sigma^2 : \partial^2_{xx}\delta_m U_1.$$

We may write $\partial_\mu U_{1,1}(t, m, x) := \partial_x \delta_m U(t, m, x, 1)$ in the spirit of the Lions derivative, and we also introduce $\partial^2_{\mu\mu} U_{1,1}(t, m, x, \tilde{x}) := \partial_{\tilde{x}}\delta_m \partial_x \delta_m U(t, m, x, 1, \tilde{x}, 1) \in \mathbb{R}^{d \times d}$.

We next introduce a partial order $\preceq$ on $\mathcal{P}_2(\mathbf{S})$. We say that $m' \preceq m$ if $m'(dx, 1)$ is absolutely continuous with respect to $m(dx, 1)$ with density $p : \mathbb{R}^d \to [0, 1]$, i.e.,

$$m'(dx, 1) = p(x)m(dx, 1), \text{ and therefore } m'(dx, 0) = [1 - p(x)]m(dx, 1) + m(dx, 0), \quad (2.7)$$

In our context, $m_{t-} = \mathbb{P}_{(X_t, I_{t-})}$ and $m_t = \mathbb{P}_{(X_t, I_t)}$, with $\mathbb{P} \in \mathcal{P}(t, m)$, so that $m_t \preceq m_{t-}$ with conditional transition probability $p(x) = p_t(x) := \mathbb{P}(I_t = 1 \mid X_t = x, I_{t-} = 1)$. When $p = \mathbf{1}_A$, with $A \in \mathcal{B}(\mathbb{R}^d)$, we denote $m^A$ the measure defined by (2.7). We recall from [52] that the set $\{m' : m' \preceq m\}$ is compact. Moreover, we have the following simple lemma whose proof is postponed to Appendix.

**Lemma 2.2** *Let* $m \in \mathcal{P}_2(\mathbf{S})$ *such that* $x \in \mathbb{R}^d \to m\big((-\infty, x_1] \times \cdots \times (-\infty, x_d], \ 1\big)$ *is continuous. Then* $\{m^A : A \in \mathcal{B}(\mathbb{R}^d)\}$ *is* $\mathcal{W}_2$-*dense in* $\{m' : m' \preceq m\}$.

In [51, 52], we see that $V$ is associated with the obstacle problem on Wasserstein space:

$$\min \left\{ \min_{m' \in C_U(t,m)} \big[ -(\mathbb{L}U + F)(t, m')\big], \ (\mathbb{D}_I U)_*(t, m) \right\} = 0, \quad (t, m) \in \mathbf{Q}_0, \qquad (2.8)$$

with boundary condition $U|_{t=T} = g$. The function $(\mathbb{D}_I U)_*$ is the LSC envelope of

$$\mathbb{D}_I U : (t, m) \longmapsto \inf_{x \in \mathrm{Supp}(m(.,1))} D_I U(t, m, x),$$

which is upper semicontinuous, but may not be continuous in general, and the set

$$C_U(t, m) := \{ m' \preceq m : U(t, m') \geq U(t, m) \}, \quad (t, m) \in \mathbf{Q}_0,$$

indicates the set of positions at $t$ which improve $U$ by stopping the corresponding particles.

## 2.3 Viscosity solutions

For $\delta > 0$ and $(t, m) \in \mathbf{Q}_0$, we introduce the neighborhood

$$\mathcal{N}_\delta(t, m) := \big\{ (s, \tilde{m}) : s \in [t, t + \delta], \mathbb{P} \in \mathcal{P}(t, m), \ \tilde{m} \in \{ \mathbb{P}_{Y_{s-}}, \mathbb{P}_{Y_s} \} \big\}.$$

Note that $\mathcal{N}_\delta(t, m)$ is compact, as the closure of a càdlàg $\mathcal{P}_2(\mathbf{S})$-valued graph. For a locally bounded function $U : \overline{\mathbf{Q}}_0 \longrightarrow \mathbb{R}$, we introduce its LSC and USC envelopes relatively to $\mathcal{P}(t, m)$, $U_*$ and $U^*$ respectively:

$$U_*(t, m) := \liminf_{(s, \tilde{m}) \to (t, m)} U(s, \tilde{m}), \quad U^*(t, m) := \limsup_{(s, \tilde{m}) \to (t, m)} U(s, \tilde{m}), \quad \text{for all} \quad (t, m) \in \overline{\mathbf{Q}}_0,$$

where the limits are taken on all sequences $\{t_n, m_n\}_{n \geq 1}$ converging to $(t, m)$, with $(t_n, m_n) \in \mathcal{N}_{T-t}(t, m)$ for all $n$. Moreover, the smooth functions are in the following sense.

**Definition 2.3** Let $C_2^{1,2}(\overline{\mathbf{Q}}_t)$ denote the set of continuous functions $U : \overline{\mathbf{Q}}_t \to \mathbb{R}$ such that
- $\partial_t U, \delta_m U, \partial_x \delta_m U_1, \partial_{xx}^2 \delta_m U_1$ exist, and are continuous in all variables,
- $\partial_{xx}^2 \delta_m U_1$ is bounded in $x$, locally uniformly in $(s, m)$,
- $\partial_{\mu\mu}^2 U_{1,1}$ exists and $(s, m, x) \mapsto \partial_{\mu\mu}^2 U_{1,1} U(s, m, x, x)$ is continuous and has quadratic growth in $x$, locally uniformly in $(s, m)$.

We then introduce the sets of test functions

$$\overline{\mathcal{A}} U(t, m) := \Big\{ \varphi \in C_2^{1,2}(\overline{\mathbf{Q}}_t) : (\varphi - U_*)(t, m) = \max_{\mathcal{N}_\delta(t, m)} (\varphi - U_*) \text{ for some } \delta > 0 \Big\},$$

$$\underline{\mathcal{A}} U(t, m) := \Big\{ \varphi \in C_2^{1,2}(\overline{\mathbf{Q}}_t) : (\varphi - U^*)(t, m) = \min_{\mathcal{N}_\delta(t, m)} (\varphi - U^*) \text{ for some } \delta > 0 \Big\}.$$

**Definition 2.4** Let $U : \overline{\mathbf{Q}}_0 \to \mathbb{R}$ be locally bounded.

(i) $U$ is a viscosity supersolution of (2.8) if, for any $(t, m) \in \mathbf{Q}_0$ and $m' \preceq m$,

$$U_*(t, m) \geq U_*(t, m') \text{ and } -(\mathbb{L}\varphi + F)(t, m) \geq 0, \text{ for all } \varphi \in \overline{\mathcal{A}} U(t, m). \tag{2.9}$$

(ii) $U$ is a viscosity subsolution of (2.8) if, for any $(t,m) \in \mathbf{Q}_0$ s.t. $C_{U^*}(t,m) = \{m\}$,

$$\min\{-(\mathbb{L}\varphi + F), (\mathbb{D}_I\varphi)_*\}(t,m) \leq 0, \quad \text{for all } \varphi \in \underline{\mathcal{A}}U(t,m). \tag{2.10}$$

(iii) $U$ is a viscosity solution of (2.8) if it is a viscosity supersolution and subsolution.

**Remark 2.5** In the present paper, the set $C_2^{1,2}(\overline{\mathbf{Q}}_t)$ has a different definition from the one we used in our previous work [52], as we additionally require the existence of the second order derivative $\partial_{\mu\mu}^2 U_{1,1}$. However, we can see that the viscosity theory implied by requiring such regularity for the test functions is equivalent to the one developed in [52]; in particular, the test functions involved in the proof of our comparison result [52, Theorem 3.11] are infinitely differentiable, so that the validity of this result does not depend on the definition of $C_2^{1,2}(\overline{\mathbf{Q}}_t)$ we set. ∎

Recall the assumptions for our comparison result [52, Theorem 3.11]. Together with [52, Theorem 3.9] and Remark 2.5, this shows that $V$ is the unique viscosity solution of (2.8).

**Assumption 2.6** (i) $b$ and $\sigma$ can be extended to $\mathcal{P}_1(\mathbf{S})$ under $\mathcal{W}_1$ continuously, and are uniformly Lipschitz continuous in $(x,m)$ under $\mathcal{W}_1$;

(ii) $f$ and $g$ extend continuously to $\mathcal{P}_1(\mathbf{S})$ and have linear growth in $(x,m)$, under $\mathcal{W}_1$;

(iii) $f$ is uniformly continuous in $(t,x,m)$, under $\mathcal{W}_2$ for $m$; and all first and second order derivatives of $\sigma(t,x,m)$ with respect to $x$ and those of $\delta_m\sigma(t,x,m,\tilde{x},\tilde{i})$ with respect to $(x,\tilde{x})$ are bounded, continuous in $t$, and Lipschitz continuous in $(x,\tilde{x},m)$, under $\mathcal{W}_1$ for $m$.

# 3 The $N$-player optimal stopping problem

In this section we introduce a general $N$-player optimal stopping problem, following Kobylanski, Quenez & Rouy-Mironescu [36]. In the symmetric case, this reduces to the $N$-player problem corresponding to the mean field problem (2.3), as we will see in Subsection 3.3.

## 3.1 The general $N$-player optimal stopping problem

Let $(\Omega^0, \mathcal{F}_T^0, \mathbb{F}^0, \mathbb{P}^0)$ be a filtered probability space, equipped with a sequence of $d$-dimensional Brownian motions $\{W^k\}_{k \geq 1}$. For each $N \geq 1$, let $\mathbb{F}^N$ denote the canonical filtration of $\{W^k\}_{1 \leq k \leq N}$, $\mathcal{T}_{t,T}^N$ the set of $[t,T]-$valued $\mathbb{F}^N-$stopping times for all $t \in [0,T]$, $\boldsymbol{\mathcal{T}}_{t,T}^N := (\mathcal{T}_{t,T}^N)^N$, $[N] := \{1, \ldots, N\}$. We emphasize that, unlike the weak formulation in the previous section, here we use strong formulation. In particular, the stopping times in $\mathcal{T}_{t,T}^N$ are

adapted to the Brownian motion filtration, in the spirit of open loop controls. Moreover, in this section $N$ is fixed, but in the next section we will send $N$ to $\infty$.

Let $(b_k, \sigma_k, f_k) : [0, T] \times \mathbb{R}^{d \times N} \longrightarrow \mathbb{R}^d \times \mathcal{S}_d^+(\mathbb{R}) \times \mathbb{R}$, $k \in [N]$, and $g_N : \mathbb{R}^{d \times N} \to \mathbb{R}$ satisfy:

**Assumption 3.1** (i) *The functions $b_k, \sigma_k$, $k \in [N]$, are continuous in $t$ and uniformly Lipschitz continuous in $\mathbf{x}$ (with the Lipschitz constant possibly depending on $N$);*
(ii) *The functions $f_k$, $k \in [N]$, and $g_N$ are continuous in $t$ and uniformly continuous in $\mathbf{x}$ (with the uniform continuity possibly depending on $N$).*

Fix $t \in [0, T)$ and $\mathbf{y} = (\mathbf{x}, \mathbf{i}) \in \mathbf{S}^N$. For $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_N) \in \mathcal{T}_{t,T}^N$, let $\mathbf{X}^{\boldsymbol{\tau}} = \mathbf{X}^{N,\boldsymbol{\tau}} = (X^{\boldsymbol{\tau},1}, \ldots, X^{\boldsymbol{\tau},N})$ and $\mathbf{I}^{\boldsymbol{\tau}} = \mathbf{I}^{N,\boldsymbol{\tau}} = (I^{\boldsymbol{\tau},1}, \ldots, I^{\boldsymbol{\tau},N})$ solve the following system of SDEs:

$$X_s^{\boldsymbol{\tau},k} = x_k + \int_t^s b_k(r, \mathbf{Y}_r^{\boldsymbol{\tau}}) I_r^{\boldsymbol{\tau},k} dr + \int_t^s \sigma_k(r, \mathbf{Y}_r^{\boldsymbol{\tau}}) I_r^{\boldsymbol{\tau},k} dW_r^k, \ I_s^{\boldsymbol{\tau},k} = i_k \mathbf{1}_{\tau_k > s}, \quad \mathbb{P}^0\text{-a.s.} \ (3.1)$$

for $k \in [N]$, where $\mathbf{Y}^{\boldsymbol{\tau}} = \mathbf{Y}^{N,\boldsymbol{\tau}} := (\mathbf{X}^{\boldsymbol{\tau}}, \mathbf{I}^{\boldsymbol{\tau}})$. Here and throughout the paper, the dependence of $\mathbf{Y}$ on $N$ is omitted for notational simplicity. Moreover, in light of (2.6), let $\bar{\mathbf{Y}} = (\bar{\mathbf{X}}, \bar{\mathbf{I}})$ denote the unstopped version, namely $\tau_k = T$, or equivalently $\bar{I}_s^k = i_k$, for all $k \in [N]$.

We now consider the problem

$$v^N(t, \mathbf{y}) := \sup_{\boldsymbol{\tau} \in \mathcal{T}_{t,T}^N} J_N(t, \mathbf{y}, \boldsymbol{\tau}), \quad (t, \mathbf{y}) \in \overline{\Lambda}_0^N,$$

$$\text{where} \quad J_N(t, \mathbf{y}, \boldsymbol{\tau}) := \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\left[\frac{1}{N} \sum_{k=1}^N \int_t^T f_k(r, \mathbf{Y}_r^{\boldsymbol{\tau}}) I_r^{\boldsymbol{\tau},k} dr + g_N(\mathbf{Y}_T^{\boldsymbol{\tau}})\right]. \tag{3.2}$$

Here the subscript $(t, \mathbf{y})$ in $\mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}[\cdot] := \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}[\cdot | \mathbf{Y}_{t-}^{\boldsymbol{\tau}} = \mathbf{y}]$ indicates the initial condition.

## 3.2 Viscosity solutions with tangency in mean

The function $v^N$ is associated with the following partial differential equation on $\overline{\Lambda}_0^N$:

$$\min\{-\partial_t u(t, \mathbf{y}) - \mathbf{i} \cdot (\mathcal{L}u + \mathbf{f})(t, \mathbf{y}), u(t, \mathbf{y}) - \max_{\mathbf{i}' < \mathbf{i}} u(t, \mathbf{x}, \mathbf{i}')\} = 0, \tag{3.3}$$

with boundary condition $u|_{t=T} = u|_{\mathbf{i}=0} = g_N$, where $\mathbf{i}' < \mathbf{i}$ means $i_k' \leq i_k$, $k \in [N]$, and $\mathbf{i} \neq \mathbf{i}'$, $\mathbf{f} := (f_1, \cdots, f_N)$, and

$$\mathcal{L}\varphi := \left(b_k \cdot \partial_{x_k} \varphi + \frac{1}{2}\sigma_k^2 : \partial_{x_k x_k}^2 \varphi\right)_{k \in [N]} \quad \text{for all } \varphi \in C^2(\mathbb{R}^{d \times N}). \tag{3.4}$$

We call (3.3) *cascade obstacle equation*. We consider the notion of viscosity solutions introduced by Ekren, Keller, Touzi & Zhang [23] where, for any function $u : \overline{\Lambda}_0^N \to \mathbb{R}$, the

corresponding set of test functions at some point $(t, \mathbf{y}) \in \overline{\Lambda}_0^N$ is defined by a tangency mean rather than the standard pointwise tangency of Crandall & Lions:

$$\overline{\mathcal{A}}u(t, \mathbf{y}) := \left\{ \phi \in C^{1,2}(\overline{\Lambda}_t^N) : \exists \mathrm{H} \in \mathcal{T}_{t,T}^+, \ (\phi - u)(t, \mathbf{y}) = \max_{\theta \in \mathcal{T}_{t,T}^N} \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0} \left[ (\phi - u)(\theta \wedge \mathrm{H}, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}}) \right] \right\},$$
$$\underline{\mathcal{A}}u(t, \mathbf{y}) := \left\{ \phi \in C^{1,2}(\overline{\Lambda}_t^N) : \exists \mathrm{H} \in \mathcal{T}_{t,T}^+, \ (\phi - u)(t, \mathbf{y}) = \min_{\theta \in \mathcal{T}_{t,T}^N} \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0} \left[ (\phi - u)(\theta \wedge \mathrm{H}, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}}) \right] \right\},$$

$$(3.5)$$

with $\mathcal{T}_{t,T}^+ := \{ \mathrm{H} \in \mathcal{T}_{t,T}^N : \mathrm{H} > t \}$, and $C^{1,2}(\overline{\Lambda}_t^N)$ denoting the set of continuous test functions $\phi$ such that: for any $\mathbf{i} = (i_1, \cdots, i_N)$ and for all $k \in [N]$ with $i_k = 1$, the derivatives $\partial_t \phi(s, \mathbf{x}, \mathbf{i}), \partial_{x_k} \phi(s, \mathbf{x}, \mathbf{i}), \partial_{x_k, x_k} \phi(s, \mathbf{x}, \mathbf{i})$ exist and are continuous for $(s, \mathbf{x}) \in [t, T] \times \mathbb{R}^{d \times N}$.

**Definition 3.2** *Let* $u : \overline{\Lambda}_0^N \longrightarrow \mathbb{R}$.

(i) $u$ *is a viscosity supersolution of* (3.3) *if, for all* $(t, \mathbf{y}) \in \overline{\Lambda}_0^N$,

$$\min\{-\partial_t \phi(t, \mathbf{y}) - (\mathcal{L}\phi + \mathbf{f})(t, \mathbf{y}) \cdot \mathbf{i}, u(t, \mathbf{y}) - \max_{\mathbf{i}' < \mathbf{i}} u(t, \mathbf{x}, \mathbf{i}')\} \geq 0 \quad \text{for all } \phi \in \overline{\mathcal{A}}u(t, \mathbf{y}).$$

(ii) $u$ *is a viscosity subsolution of* (3.3) *if, for all* $(t, \mathbf{y}) \in \Lambda_0^N$,

$$\min\{-\partial_t \phi(t, \mathbf{y}) - (\mathcal{L}\phi + \mathbf{f})(t, \mathbf{y}) \cdot \mathbf{i}, u(t, \mathbf{y}) - \max_{\mathbf{i}' < \mathbf{i}} u(t, \mathbf{x}, \mathbf{i}')\} \leq 0 \quad \text{for all } \phi \in \underline{\mathcal{A}}u(t, \mathbf{y}).$$

(iii) $u$ *is a viscosity solution of* (3.3) *if it is a viscosity supersolution and subsolution.*

We then have the following result, whose proof is postponed to §7.

**Theorem 3.3** *Under Assumption 3.1, the function* $v^N$ *in* (3.2) *is the unique continuous viscosity solution of* (3.3) *with boundary conditions* $v^N|_{t=T} = v^N|_{\mathbf{i}=0} = g$.

**Remark 3.4** We note that a viscosity solution in our sense is always a viscosity solution in Crandall-Lions sense, as the latter involves a smaller class of test functions. In particular, the comparison principle and the uniqueness of viscosity solution in the latter sense implies the same results in our sense. Our definition turns out to be more convenient for our proof of the convergence results in the next section. ∎

### 3.3 The symmetric case

Recall the functions $b, \sigma, f, g$ in §2 and the notation $m^N$ in (1.1). In this subsection we consider the symmetric case:

$$\varphi_k(t, \mathbf{y}) = \varphi(t, x_k, m^N(\mathbf{y})), \text{ for } \varphi = b, \sigma, f, \quad \text{and} \quad g_N(\mathbf{y}) = g(m^N(\mathbf{y})). \tag{3.6}$$

We first have the following simple lemma, whose proof is postponed to §7.

**Lemma 3.5** (i) *Under Assumption 2.1* (i) *and* (3.6), *there is a constant* $C \geq 0$, *independent of* $N$, *such that, for* $(t, \mathbf{y}) \in \Lambda_0^N$, $\boldsymbol{\tau} \in \mathcal{T}_{t,T}^N$, $k \in [N]$, *and* $\theta_1, \theta_2 \in \mathcal{T}_{[t,T]}^N$ *with* $\theta_1 \leq \theta_2$,

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} |X_s^{\boldsymbol{\tau},k}|^2\Big] \leq C(1 + |x_k|^2 + \|\mathbf{x}\|_2^2); \tag{3.7}$$

$$\mathbb{E}\Big[\sup_{\theta_1 \leq s \leq \theta_2} |X_s^{\boldsymbol{\tau},k} - X_{\theta_1}^{\boldsymbol{\tau},k}|^2\Big] \leq C(1 + |x_k|^2 + \|\mathbf{x}\|_2^2)\|\theta_2 - \theta_1\|_\infty. \tag{3.8}$$

(ii) *If we assume further that Assumption 2.6* (i) *holds, then*

$$\mathbb{E}\Big[\sup_{t \leq s \leq T} |X_s^{\boldsymbol{\tau},k}|\Big] \leq C(1 + |x_k| + \|\mathbf{x}\|_1); \tag{3.9}$$

$$\mathbb{E}\Big[\sup_{\theta_1 \leq s \leq \theta_2} |X_s^{\boldsymbol{\tau},k} - X_{\theta_1}^{\boldsymbol{\tau},k}|\Big] \leq C(1 + |x_k| + \|\mathbf{x}\|_1)\sqrt{\|\theta_2 - \theta_1\|_\infty}. \tag{3.10}$$

Next, under (3.6) one can easily see that $v^N$ is also symmetric in $\mathbf{y}$, that is, there exists a function $V^N : \overline{\mathbf{Q}}_0^N \to \mathbb{R}$ such that

$$v^N(t, \mathbf{y}) = V^N(t, m^N(\mathbf{y})). \tag{3.11}$$

**Definition 3.6** *Under* (3.6), *we call a function* $U^N : \overline{\mathbf{Q}}_0^N \to \mathbb{R}$ *a symmetric viscosity solution (resp. subsolution, supersolution) of* (3.3) *if* $u^N(t, \mathbf{y}) := U^N(t, m^N(\mathbf{y}))$ *is a viscosity solution (resp. subsolution, supersolution) of* (3.3) *in the sense of Definition 3.2.*

The following result establishes stronger regularity than Theorem 3.3 and its proof is also postponed to §7.

**Theorem 3.7** *Let Assumptions 2.1, 2.6* (i)-(ii) *and* (3.6) *hold.*
(i) *The function* $V^N$ *in* (3.11) *is the unique continuous symmetric viscosity solution of* (3.3).
(ii) *There exist a constant* $C$, *independent of* $N$, *such that*

$$|V^N(t, m_N)| \leq C(1 + \|m_N\|_1). \tag{3.12}$$

(iii) *For any* $R > 0$, *there exist a modulus of continuity function* $\rho_R$, *which may depend on* $R$ *but not on* $N$, *such that, for any* $(t, m_N), (\tilde{t}, \tilde{m}_N) \in \overline{\mathbf{Q}}_0^N$ *with* $\|m_N\|_2, \|\tilde{m}_N\|_2 \leq R$,

$$|V^N(t, m_N) - V^N(\tilde{t}, \tilde{m}_N)| \leq \rho_R\big(\mathcal{W}_2(m_N, \tilde{m}_N) + |\tilde{t} - t|\big). \tag{3.13}$$

Moreover, similar to Carmona & Delarue [11, Vol. 1, Propositions 5.35 & 5.91] we have the following result, and we shall provide a proof in Appendix.

**Lemma 3.8** *Assume* $\varphi \in C_2^{1,2}(\overline{\mathbf{Q}}_0)$ *and* $\phi(t, \mathbf{y}) := \varphi\big(t, m^N(\mathbf{y})\big)$. *Then, for all* $(s, \mathbf{y}) \in \overline{\Lambda}_0^N$ *and* $k \in [N]$ *such that* $i_k = 1$,

$$\partial_{x_k}\phi(s, \mathbf{y}) = \frac{1}{N}\partial_x \delta_m \varphi_1(s, m^N(\mathbf{y}), x_k),$$
$$\partial_{x_k x_k}^2 \phi(s, \mathbf{y}) = \frac{1}{N}\partial_{xx}^2 \delta_m \varphi_1(s, m^N(\mathbf{y}), x_k) + \frac{1}{N^2}\partial_{\mu\mu}^2 \varphi_{1,1}(s, m^N(\mathbf{y}), x_k, x_k). \tag{3.14}$$

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# 4 Main results

Our first main result states the convergence of the value function of the multiple optimal stopping problem to the corresponding mean field problem.

**Theorem 4.1** *Let Assumptions 2.1 and 2.6 hold, and $V^N$ and $V$ be respectively the value functions of the multiple and mean field optimal stopping problems (3.2)-(3.6)-(3.11) and (2.3). Then $V^N$ converges to $V$, i.e.*

$$\lim_{\substack{N \to \infty,\, s \to t \\ m_N \in \mathcal{P}^N(\mathbf{S}) \xrightarrow{\mathcal{W}_2} m}} V^N(s, m_N) = V(t, m), \quad \text{for all } (t, m) \in \overline{\mathbf{Q}}_0. \tag{4.1}$$

The above theorem relies heavily on the following result. For a sequence of functions $U^N : \overline{\mathbf{Q}}_0^N \to \mathbb{R}$ which are locally bounded functions, uniformly in $N$, we denote

$$\underline{U}(t, m) := \liminf_{\substack{N \to \infty,\, s \to t \\ m_N \in \mathcal{P}^N(\mathbf{S}) \xrightarrow{\mathcal{W}_2} m}} U^N(s, m_N), \quad \text{and} \quad \overline{U}(t, m) := \limsup_{\substack{N \to \infty,\, s \to t \\ m_N \in \mathcal{P}^N(\mathbf{S}) \xrightarrow{\mathcal{W}_2} m}} U^N(s, m_N), \tag{4.2}$$

for all $(t, m) \in \overline{\mathbf{Q}}_0$. We then have

**Theorem 4.2** *Let Assumption 2.1 and (3.6) hold and $U^N : \overline{\mathbf{Q}}_0^N \to \mathbb{R}$ be a sequence of continuous and locally bounded functions, uniformly in $N$.*
*(i) If each $U^N$ is a viscosity subsolution of (3.3), then $\overline{U}$ is a viscosity subsolution of (2.8).*
*(ii) If each $U^N$ is a viscosity supersolution of (3.3), and $\underline{U}$ is continuous, then $\underline{U}$ is a viscosity supersolution of (2.8).*

The above two theorems are proved in §5. We now turn to the convergence of the optimal strategies. Our second main result is as follows.

**Theorem 4.3** *Let Assumptions 2.1 and 2.6 hold. Fix $(t, m) \in \mathbf{Q}_0$ and $\mathbf{y}^N \in \mathbf{S}^N$, $N \geq 1$, such that $m^N(\mathbf{y}^N) \xrightarrow{\mathcal{W}_2} m$. Recall (3.1) and assume (3.6) holds.*
*(i) Denote $m^N(\mathbf{Y}) := \frac{1}{N} \sum_{k=1}^N \delta_{Y^k} \in \mathcal{P}_2(\Omega)$. Assume $\hat{\boldsymbol{\tau}}^N \in \mathcal{T}_{t,T}^N$ is $\varepsilon_N$-optimal for (3.2) for all $N \geq 1$, with $\varepsilon_N \to 0$. Then, $\{\mathbb{P}^0 \circ (m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}))^{-1}\}_{N \geq 1}$ is tight, and all its accumulation points are supported on the set of the optimal controls for (2.3).*
*(ii) Let $\mathbb{P}^* \in \mathcal{P}(t, m)$ be optimal for (2.3). Then there exist $\varepsilon_N \to 0$ and $\hat{\boldsymbol{\tau}}^N \in \mathcal{T}_{t,T}^N$ which is $\varepsilon_N$-optimal for (3.2) such that $m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}) \xrightarrow{\mathcal{W}_2} \mathbb{P}^*$, $\mathbb{P}^0$-a.s.*
*(iii) Assume further that (2.3) has a unique optimal control $\mathbb{P}^*$. Then for any sequence of $\varepsilon_N$-optimal stopping strategies $\{\hat{\boldsymbol{\tau}}^N\}_{N \geq 1}$ for (3.2), $m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}) \xrightarrow{\mathcal{W}_2} \mathbb{P}^*$, $\mathbb{P}^0$-a.s.*

Since $N$ varies here, we use the superscript $^N$ in $\hat{\boldsymbol{\tau}}^N$ to indicate its dependence on $N$.

Theorem 4.3 relies heavily on the following result concerning the propagation of chaos for arbitrary stopping strategies. Both results are proved in §6.

**Theorem 4.4** *Let Assumption 2.1 hold, and fix $(t,m) \in \mathbf{Q}_0$ and $\mathbf{y}^N \in \mathbf{S}^N$, $N \geq 1$, such that $m^N(\mathbf{y}^N) \overset{\mathcal{W}_2}{\longrightarrow} m$. Recall (3.1) and assume (3.6) holds.*
*(i) For any $\boldsymbol{\tau}^N \in \mathcal{T}_{t,T}^N$, the sequence $\left\{ \mathbb{P}^0 \circ \left( m^N(\mathbf{Y}^{\boldsymbol{\tau}^N}) \right)^{-1} \right\}_{N \geq 1}$ is tight, and all its accumulation points are supported on $\mathcal{P}(t,m)$.*
*(ii) Assume further that $b, \sigma$ are uniformly Lipschitz continuous in $m$ under $\mathcal{W}_1$. Then, for any $\mathbb{P} \in \mathcal{P}(t,m)$, there exists $\boldsymbol{\tau}^N \in \mathcal{T}_{t,T}^N$, $N \geq 1$, such that $m^N(\mathbf{Y}^{\boldsymbol{\tau}^N}) \overset{\mathcal{W}_2}{\longrightarrow} \mathbb{P}$, $\mathbb{P}^0$-a.s.*

## 5 Convergence of the value function

In this section we prove Theorems 4.1 and 4.2.

**Proof of Theorem 4.1** (given Theorem 4.2). First, for any $(t,m) \in \overline{\mathbf{Q}}_0$ and $(t^N, m_N(\mathbf{y}^N)) \to (t,m)$ as $N \to \infty$, the sequence $\{V^N(t^N, m_N(\mathbf{y}^N))\}_{N \geq 1}$ is bounded by (3.12). Then the corresponding functions $\overline{V}, \underline{V}$ defined as in (4.2) are finite, and are continuous by Theorem 3.7 (iii). In particular, $\overline{V}(T, \cdot) = \underline{V}(T, \cdot) = g$. Applying Theorem 4.2 we see that $\overline{V}$ and $\underline{V}$ are a viscosity subsolution and supersolution of (2.8). Then it follows from the comparison principle [52, Theorem 3.11] that $\underline{V} \geq \overline{V}$. As the converse inequality holds by definition, this shows that $\underline{V} = \overline{V}$, and by uniqueness, they are equal to $V$ and thus (4.1) holds. ∎

**Proof of Theorem 4.2.** Without loss of generality, we prove the theorem only at $t = 0$.

(i) Fix $m \in \mathcal{P}_2(\mathbf{S})$ and $\varphi \in \underline{\mathcal{A}\overline{U}}(0,m)$ with corresponding $\delta_0 \in (0,T)$. We assume w.l.o.g. (see [52, Remark 3.2]) that

$$(0,m) \text{ is a strict minimum of } (\varphi - \overline{U}) \text{ on } \mathcal{N}_{\delta_0}(0,m), \text{ with } C_{\overline{U}}(0,m) = \{m\}. \tag{5.1}$$

Let $(t^N, \mathbf{y}^N)$ be such that

$$(t^N, m^N(\mathbf{y}^N)) \longrightarrow (0,m), \quad \text{and} \quad U^N(t^N, m^N(\mathbf{y}^N)) \longrightarrow \overline{U}(0,m), \quad \text{as} \quad N \to \infty. \tag{5.2}$$

We also introduce the functions $\phi^N(s, \mathbf{y}) := \varphi(s, m^N(\mathbf{y}))$ for all $N \geq 1$ and $(s, \mathbf{y}) \in \overline{\Lambda}_{t^N}^N$.

**Step 1:** We prove that, without loss of generality, $\{\mathbf{y}^N = (\mathbf{x}^N, \mathbf{i}^N)\}_{N \geq 1}$ may be taken s.t.

$$U^N(t^N, \mathbf{y}^N) - \max_{\mathbf{i}' < \mathbf{i}^N} U^N(t^N, \mathbf{x}^N, \mathbf{i}') > 0 \quad \text{for all } N \geq 1. \tag{5.3}$$

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Indeed, for each $N \geq 1$, if $\mathbf{i}^N = 0$, then the maximum is equal to $-\infty$ and thus (5.3) is trivially satisfied. Now assume $\mathbf{i}^N \neq 0$. Then there exists $\mathbf{i}^{N,*} \leq \mathbf{i}^N$ attaining the maximum $\max_{\mathbf{i}' \leq \mathbf{i}^N} U^N(t^N, \mathbf{x}^N, \mathbf{i}')$. This implies that $\mathbf{y}^{N,*} := (\mathbf{x}^N, \mathbf{i}^{N,*})$ satisfies (5.3).

It remains to show that $\mathbf{y}^{N,*}$ also satisfies (5.2). First, as $m^N(\mathbf{y}^N)$ converges, the first marginal of $m^N(\mathbf{y}^{N,*})$ converges. Next, since its second marginal is a measure on $\{0, 1\}$, the sequence $\{m^N(\mathbf{y}^{N,*})\}_{N \geq 1}$ is tight, and we may thus extract a subsequence (still denoted the same) converging to some $m^* \preceq m$. Then, noting that $U^N(t^N, \mathbf{y}^N) \leq U^N(t^N, \mathbf{y}^{N,*})$, by taking the $\liminf_{N \to \infty}$ we obtain

$$
\begin{aligned}
\overline{U}(0, m) &= \lim_{N \to \infty} U^N(t^N, m^N(\mathbf{y}^N)) \leq \liminf_{N \to \infty} U^N(t^N, m(\mathbf{y}^{N,*})) \\
&\leq \limsup_{N \to \infty} U^N(t^N, m(\mathbf{y}^{N,*})) \leq \overline{U}(0, m^*),
\end{aligned}
$$

Then $m^* \in C_{\overline{U}}(0, m)$, and therefore $m^* = m$ by (5.1), so that equality holds everywhere and and $U^N(t^N, m^N(\mathbf{y}^{N,*})) \xrightarrow[N \to \infty]{} \overline{U}(0, m)$. Consequently, $\{\mathbf{y}^{N,*}\}_{N \geq 1}$ satisfies (5.2), and we may replace $\{\mathbf{y}^N\}_{N \geq 1}$ with $\{\mathbf{y}^{N,*}\}_{N \geq 1}$, which satisfies in addition (5.3).

**Step 2:** Fix $\delta \in (0, \frac{\delta_0}{3})$. By standard optimal stopping theory, there exists $\theta_\delta^N \in \mathcal{T}_{t^N, T}$ s.t.

$$
\mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ (\phi^N - U^N)(\theta_\delta^N \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta_\delta^N \wedge \mathrm{H}_\delta^N}^N) \right] = \inf_{\theta \in \mathcal{T}_{t^N, T}} \mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ (\phi^N - U^N)(\theta \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}_\delta^N}^N) \right],
$$

where $\bar{\mathbf{Y}}^N := (\bar{\mathbf{X}}^N, \bar{\mathbf{i}}^N)$ is the unstopped version of (3.1) starting from $(t^N, \mathbf{y}^N)$, and

$$
\mathrm{H}_\delta^N := \inf \left\{ s \geq t^N : \mathcal{W}_2 (m^N(\bar{\mathbf{Y}}_s), m^N(\mathbf{y}^N)) = 2\delta \right\} \wedge (t^N + 2\delta).
$$

Since $\{U^N\}_{N \geq 1}$ is locally bounded, we may assume w.l.o.g. that $N$ is large enough and $\delta$ is small enough so that $\{(\phi^N - U^N)(\theta_\delta^N \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta_\delta^N \wedge \mathrm{H}_\delta^N}^N)\}_{N \geq 1}$ is uniformly bounded.

**Step 3:** We next show that, for any $\delta$,

$$
\limsup_{N \to \infty} \mathbb{P}^0 \left( \theta_\delta^N < \mathrm{H}_\delta^N \right) > 0. \tag{5.4}
$$

Indeed, assume to the contrary that $\limsup_{N \to \infty} \mathbb{P}^0 \left( \theta_\delta^N < \mathrm{H}_\delta^N \right) = 0$. Then

$$
\begin{aligned}
(\varphi - \overline{U})(0, m) &= \lim_{N \to \infty} (\varphi - U^N)(t^N, m^N(\mathbf{y}^N)) \\
&\geq \limsup_{N \to \infty} \mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ (\varphi - U^N)(\theta_\delta^N \wedge \mathrm{H}_\delta^N, m^N(\bar{\mathbf{Y}}_{\theta_\delta^N \wedge \mathrm{H}_\delta^N}^N)) \right] \\
&\geq \liminf_{N \to \infty} \mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ (\varphi - U^N)(\theta_\delta^N \wedge \mathrm{H}_\delta^N, m^N(\bar{\mathbf{Y}}_{\theta_\delta^N \wedge \mathrm{H}_\delta^N}^N)) \right] \\
&= \liminf_{N \to \infty} \mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ \left\{ (\varphi - U^N)(\mathrm{H}_\delta^N, m^N(\bar{\mathbf{Y}}_{\mathrm{H}_\delta^N}^N))(1 - \mathbf{1}_{\{\theta_\delta^N < \mathrm{H}_\delta^N\}}) \right. \right. \tag{5.5} \\
&\qquad\qquad \left. \left. + (\varphi - U^N)(\theta_\delta^N, m^N(\bar{\mathbf{Y}}_{\theta_\delta^N}^N)) \mathbf{1}_{\{\theta_\delta^N < \mathrm{H}_\delta^N\}} \right\} \right] \\
&= \liminf_{N \to \infty} \mathbb{E}_{t^N, \mathbf{y}^N}^{\mathbb{P}^0} \left[ (\varphi - \mathrm{U}^N)(\mathrm{H}_\delta^N, m^N(\bar{\mathbf{Y}}_{\mathrm{H}_\delta^N}^N)) \right],
\end{aligned}
$$

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where we use the uniform boundedness of $\{(\phi^N - U^N)(\theta \wedge \mathrm{H}^N_\delta, \bar{\mathbf{Y}}^N_{\theta \wedge \mathrm{H}^N_\delta})\}_{N \geq 1}$.

Recall that $m^N(\bar{\mathbf{Y}}^N) \in \mathcal{P}_2(\Omega)$, $\mathbb{P}^0$-a.s. Since $(t^N, m^N(\mathbf{y}^N)) \underset{N \to \infty}{\longrightarrow} (0, m)$ and $\bar{\mathbf{Y}}^N$ is an unstopped diffusion whose coefficients satisfy the usual Lipschitz conditions, by classical propagation of chaos (see Oelschläger [46]) and compactness of $[0, T]$, again after possibly passing to a subsequence, we see that $\nu^N := \mathbb{P}^0 \circ (\mathrm{H}^N_\delta, m^N(\bar{\mathbf{Y}}^N))^{-1}$ converges weakly to some element of $\nu \in \mathcal{P}_2([0, T] \times \mathcal{P}_2(\Omega))$. Let $(\tau, \lambda)$ denote the canonical mapping on $[0, T] \times \mathcal{P}_2(\Omega)$. Recall that $Y$ is the canonical mapping on $\Omega$, and since $\bar{\mathbf{Y}}^N$ is unstopped, then $Y$ is unstopped, $m^N(\bar{\mathbf{Y}}^N)(\omega^0)$-a.s., for $\mathbb{P}^0$-a.e. $\omega^0 \in \Omega^0$. This implies that $Y$ is unstopped, $\lambda(\overline{\omega})$-a.s., for $\nu$-a.e. $\overline{\omega} \in \mathcal{P}_2([0, T] \times \mathcal{P}_2(\Omega))$, and thus the mapping $(\tau, \lambda) \mapsto \lambda_{Y_\tau}$ (the $\lambda$-distribution of the random variable $Y_\tau$) is continuous, $\nu$-a.s. Then, by the definition of $\overline{U}$, together with Fatou's lemma, we deduce from (5.5) that

$$(\varphi - \overline{U})(0, m) \geq \liminf_{N \to \infty} \mathbb{E}^{\nu^N}\Big[(\varphi - U^N)(\tau, \lambda_{Y_\tau})\Big] \geq \mathbb{E}^{\nu}\Big[(\varphi - \overline{U})(\tau, \lambda_{Y_\tau})\Big]. \tag{5.6}$$

Thus there exists $\bar{\omega} \in [0, T] \times \mathcal{P}_2(\Omega)$ such that $Y$ is unstopped under $\lambda(\bar{\omega})$, and

$$(\varphi - \overline{U})(0, m) \quad \geq \quad (\varphi - \overline{U})(\tau(\bar{\omega}), [\lambda(\bar{\omega})]_{Y_{\tau(\bar{\omega})}}).$$

Moreover, we may choose $\bar{\omega}$ so that

$$\mathcal{W}_2\big([\lambda(\bar{\omega})]_{Y_{\tau(\bar{\omega})}}, m\big) \vee \tau(\bar{\omega}) \leq 3\delta < \delta_0, \quad \mathcal{W}_2\big([\lambda(\bar{\omega})]_{Y_{\tau(\bar{\omega})}}, m\big) + \tau(\bar{\omega}) \geq \delta,$$

as we have by the definition of $\mathrm{H}^N_\delta$, for all $N \geq 1$ that

$$\mathcal{W}_2\big(\lambda_{Y_\tau}, m^N(\mathbf{y}^N)\big) \vee (\tau - t^N) \leq 2\delta \leq \mathcal{W}_2\big(\lambda_{Y_\tau}, m^N(\mathbf{y}^N)\big) + (\tau - t^N), \quad \nu^N\text{-a.s.},$$

and for sufficiently large $N$:

$$\mathcal{W}_2\big(\lambda_{Y_\tau}, m\big) \vee \tau \leq 3\delta \quad \text{and} \quad \delta \leq \mathcal{W}_2\big(\lambda_{Y_\tau}, m\big) + \tau, \quad \nu^N - \text{a.s.} \tag{5.7}$$

By the continuity of $\mathcal{W}_2$ and $\bar{\omega} \longmapsto \lambda_{Y_\tau}(\bar{\omega})$, the last inequalities exhibit a fixed closed support for $\nu^N$, which is then inherited by the weak limit $\nu$. Note further $\lambda_{Y_{t-}} = m$, we deduce that $(0, m) \neq (\tau(\bar{\omega}), [\lambda(\bar{\omega})]_{Y_{\tau(\bar{\omega})}}) \in \mathcal{N}_{\delta_0}(0, m)$. This contradicts with (5.1) that $(0, m)$ is a strict minimum on $\mathcal{N}_{\delta_0}(0, m)$. Thus, (5.4) holds true.

**Step 4:** We prove that $\phi^N$ is a test function of $U^N$ at some point. Introduce

$$\mathcal{Z}^N_s := \underset{\theta \in \mathcal{T}^N_{t^N, T}}{\mathrm{ess\,inf}} \ \mathbb{E}^{\mathbb{P}^0}\big[(\phi^N - U^N)(s, \bar{\mathbf{Y}}^{t^N, \mathbf{y}^N}_s)|\mathcal{F}^N_s\big] \text{ for all } s \in [t^N, T],$$

i.e., the $\mathbb{P}^0$-lower Snell envelope of $s \mapsto (\phi^N - U^N)(s, \bar{\mathbf{Y}}_s^{t^N, \mathbf{y}^N})$. Note that $\mathcal{Z}^N$ is a $\mathbb{P}^0$-submartingale. By (5.4), for all $\delta > 0$ and $N \geq 1$, we may find $\omega^{\delta, N} \in \Omega^0$ such that $\theta_\delta^N(\omega^{\delta, N}) < \mathrm{H}_\delta^N(\omega^{\delta, N})$ and, denoting $(t_\delta^N, \mathbf{y}_\delta^N) := (\theta^{\delta, N}(\omega^{\delta, N}), \bar{\mathbf{Y}}_{\theta_\delta^N(\omega^{\delta, N})}^{t^N, \mathbf{y}^N}(\omega^{\delta, N}))$,

$$(\phi^N - U^N)(t_\delta^N, \mathbf{y}_\delta^N) = \mathcal{Z}_{t_\delta^N}^N(\omega^{\delta, N}) \leq \mathbb{E}^{\mathbb{P}^0}[\mathcal{Z}_{\theta \wedge \mathrm{H}_\delta^N}^N] \leq \mathbb{E}^{\mathbb{P}^0}[(\phi^N - U^N)(\theta \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}_\delta^N}^{t^N, y^N})],$$

for all $\theta \in \mathcal{T}_{t_\delta^N, T}$. Thus we have

$$(\phi^N - U^N)(t_\delta^N, \mathbf{y}_\delta^N) = \min_{\theta \in \mathcal{T}_{t_\delta^N, T}} \mathbb{E}_{t_\delta^N, \mathbf{y}_\delta^N}^{\mathbb{P}^0} [(\phi^N - U^N)(\theta \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}_\delta^N})]. \tag{5.8}$$

Therefore, since $\phi^N \in C^{1,2}(\overline{\Lambda}_{t^N}^N)$ by Lemma 3.8, we have $\phi^N \in \underline{\mathcal{A}}^N U^N(t_\delta^N, \mathbf{y}_\delta^N)$, and thus the subsolution property of $U^N$ provides,

$$\min\left\{ -\partial_t \phi^N(t_\delta^N, \mathbf{y}_\delta^N) - (\mathcal{L}\phi^N + \mathbf{f}^N)(t_\delta^N, \mathbf{y}_\delta^N) \cdot \mathbf{i}^N, U^N(t_\delta^N, \mathbf{y}_\delta^N) - \max_{\mathbf{i}' < \mathbf{i}^N} U^N(t_\delta^N, \mathbf{x}_\delta^N, \mathbf{i}') \right\} \leq 0. \tag{5.9}$$

**Step 5:** We finally derive the viscosity subsolution property of $\overline{U}$. First observe that, since $t_\delta^N = \theta_\delta^N(\omega^{\delta, N}) < \mathrm{H}_\delta^N(\omega^{\delta, N})$, we have $(t_\delta^N, \mathbf{y}_\delta^N) \xrightarrow{\delta \to 0} (t^N, \mathbf{y}^N)$. Since $\phi^N$, $\mathcal{L}\phi^N + \mathbf{f}^N$ and $U^N$ are continuous, sending $\delta$ to 0 in (5.9) provides

$$\min\left\{ -\partial_t \phi^N(t^N, \mathbf{y}^N) - (\mathcal{L}\phi^N + \mathbf{f}^N)(t^N, \mathbf{y}^N) \cdot \mathbf{i}^N, U^N(t^N, \mathbf{y}^N) - \max_{\mathbf{i}' < \mathbf{i}^N} U^N(t^N, \mathbf{x}^N, \mathbf{i}') \right\} \leq 0.$$

By (5.3), this implies

$$-\partial_t \phi^N(t^N, \mathbf{y}^N) - (\mathcal{L}\phi^N + \mathbf{f}^N)(t^N, \mathbf{y}^N) \cdot \mathbf{i}^N \leq 0, \quad \text{for all } N \geq 1. \tag{5.10}$$

Thus, noting that $i_k^N$ takes only values 0 and 1, by (3.14) and (3.6) we have

$$-\partial_t \phi^N(t^N, \mathbf{y}^N) - (\mathcal{L}\phi^N + \mathbf{f}^N)(t^N, \mathbf{y}^N) \cdot \mathbf{i}^N = -\mathbb{L}\varphi(t^N, m^N(\mathbf{y}^N)) \tag{5.11}$$

$$-\frac{1}{N}\sum_{k=1}^N f(t^N, x_k^N, m^N(\mathbf{y}^N))i_k^N - \frac{1}{N^2}\sum_{k=1}^N \sigma^2 : \partial_{\mu\mu}^2 \varphi_{1,1}(t^N, m^N(\mathbf{y}^N), x_k^N, x_k^N)i_k^N$$

$$= -[\mathbb{L}\varphi + F](t^N, m^N(\mathbf{y}^N)) - \frac{1}{N}\int_{\mathbb{R}^d} \sigma^2 : \partial_{\mu\mu}^2 \varphi_{1,1}(t^N, m^N(\mathbf{y}^N), x, x)m^N(\mathbf{y}^N)(dx, 1),$$

where $\sigma(s, \tilde{m}, x, x) := \sigma(s, x, \tilde{m})$. Note that, as $N \to \infty$,

$$\int_{\mathbb{R}^d} \sigma^2 : \partial_{\mu\mu}^2 \varphi_{1,1}(t^N, m^N(\mathbf{y}^N), x, x)m^N(\mathbf{y}^N)(dx, 1) \to \int_{\mathbb{R}^d} \sigma^2 : \partial_{\mu\mu}^2 \varphi_{1,1}(0, m, x, x)m(dx, 1),$$

since $(t^N, m^N(\mathbf{y}^N)) \to (0, m)$ and the mapping $(s, \tilde{m}) \mapsto \int_{\mathbb{R}^d} \sigma^2 : \partial_{\mu\mu}^2 \varphi_{1,1}(s, \tilde{m}, x, x)\tilde{m}(dx, 1)$ is continuous. Then (5.10) and (5.11) imply that

$$-(\mathbb{L}\varphi + F)(t^N, m^N(\mathbf{y}^N)) + \mathcal{O}(\frac{1}{N}) \leq 0.$$

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Sending $N \to \infty$ and noticing that $\mathbb{L}\varphi + F$ is continuous, we conclude the viscosity subsolution property of $\overline{U}$ at $(0, m)$.

(ii) We now prove the viscosity supersolution property. Fix $\varphi \in \overline{\mathcal{A}}\underline{U}(0, m)$ with corresponding $\delta_0$ and s.t. $(0, m)$ is a strict maximizer of $\varphi - \underline{U}$ on $\mathcal{N}_{\delta_0}(0, m)$. With the same procedure and notations as in (i) (in particular, we rewrite (5.5) with reversed inequalities and switch $\limsup$ and $\liminf$), we may find $(t_\delta^N, m^N(\mathbf{y}_\delta^N)) \xrightarrow[(\delta, N) \to (0, \infty)]{} (0, m)$ s.t.

$$(\phi^N - U^N)(t_\delta^N, \mathbf{y}_\delta^N) = \max_{\theta \in \mathcal{T}_{t_\delta^N, T}} \mathbb{E}_{t_\delta^N, \mathbf{y}_\delta^N}^{\mathbb{P}^0} \left[ (\phi^N - U^N)(\theta \wedge \mathrm{H}_\delta^N, \bar{\mathbf{Y}}_{\theta \wedge \mathrm{H}_\delta^N}) \right].$$

Thus $\phi^N \in \overline{\mathcal{A}}^N U^N(t_\delta^N, \mathbf{y}_\delta^N)$ and the viscosity supersolution property provides

$$\min \left\{ -(\mathbb{L}\varphi + F)(t_\delta^N, m^N(\mathbf{y}_\delta^N)) + \mathcal{O}\left(\frac{1}{N}\right), \right.$$
$$\left. U^N(t_\delta^N, m^N(\mathbf{y}_\delta^N)) - \max_{\mathbf{i}' < \mathbf{i}_\delta^N} U^N(t_\delta^N, m^N(\mathbf{x}_\delta^N, \mathbf{i}')) \right\} \geq 0. \tag{5.12}$$

As $(t_\delta^N, m^N(\mathbf{y}_\delta^N)) \xrightarrow[(\delta, N) \to (0, \infty)]{} (0, m)$, sending $(\delta, N) \longrightarrow (0, \infty)$ in (5.12) provides the first part of the viscosity supersolution property:

$$-(\mathbb{L}\varphi + F)(0, m) \geq 0.$$

It remains to show that $\underline{U}$ is nondecreasing for $\preceq$. We first assume that

$$m(dx, 1) \text{ satisfies the required regularity in Lemma 2.2.} \tag{5.13}$$

For any $A \in \mathcal{B}(\mathbb{R}^d)$, by considering the corresponding cdf, we easily see that

$$(m^N)^A(\mathbf{y}^N) := \frac{1}{N} \sum_{j=1}^N \delta_{(x_k^N, i_k^N \mathbf{1}_A(x_k^N))} \xrightarrow[N \to \infty]{} m^A.$$

By (5.12), we have $U^N(t^N, m^N(\mathbf{y}^N)) \geq U^N(t^N, (m^N)^A(\mathbf{y}^N))$ for all $N \geq 1$. Taking the $\liminf$ and using the fact that $U^N(t^N, m^N(\mathbf{y}^N)) \xrightarrow[N \to \infty]{} \underline{U}(0, m)$, we deduce $\underline{U}(0, m) \geq \underline{U}(0, m^A)$. Given the arbitrariness of $A$ and by Lemma 2.2, we deduce from the continuity of $\underline{U}$ that $\underline{U}(0, m) \geq \underline{U}(0, m')$ for all $m' \preceq m$, under the condition (5.13).

Now for general $m$, there exist $m_n \in \mathcal{P}_2(\mathbf{S})$ such that $\lim_{n \to \infty} W_2(m_n, m) = 0$ and each $m_n$ satisfies (5.13). Then $\underline{U}(0, m_n) \geq \underline{U}(0, m_n')$ for all $m_n' \preceq m_n$. Recall (2.7) and let $m' \preceq m$ with transition function $p$. We first assume $p$ is continuous. Then, letting $m_n' \preceq m_n$ have the same $p$, we have $\lim_{n \to \infty} W_2(m_n', m') = 0$. Thus, the continuity of $\underline{U}$ implies $\underline{U}(0, m) \geq \underline{U}(0, m')$. Finally, by [52, Lemma 3.8] we extend the inequality to all $m' \preceq m$, with possibly discontinuous $p$. ■

# 6 Propagation of chaos for stopped diffusions

In this section we prove Theorems 4.3 and 4.4. We first use the latter to prove Theorem 4.3. Recall (2.3), (3.2), and that $\lambda$ denotes the canonical map on $\mathcal{P}_2(\Omega)$.

**Proof of Theorem 4.3.** (i) Let $\hat{\nu}^N := \mathbb{P}^0 \circ (m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}))^{-1}$. By Theorem 4.4 (i) and by otherwise considering a subsequence, we may assume $\hat{\nu}^N$ converges weakly to some $\hat{\nu}$ supported on $\mathcal{P}(t,m)$. For each $N \geq 1$, since $\hat{\boldsymbol{\tau}}^N$ is $\varepsilon_N$-optimal for (3.2), we have

$$v^N(t, \mathbf{y}^N) \leq J_N(t, \mathbf{y}^N, \hat{\boldsymbol{\tau}}^N) + \varepsilon_N = \mathbb{E}^{\mathbb{P}^0}\big[J(t, m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}))\big] + \varepsilon_N = \mathbb{E}^{\hat{\nu}^N}[J(t,\lambda)] + \varepsilon_N. \quad (6.1)$$

For any $R > 0$, let $J_R(t,\lambda)$ be defined by truncating $F$ and $g$ with $R$ in (2.3), and define $J_{N,R}$ similarly by truncation in (3.2). Then

$$\lim_{N\to\infty} \mathbb{E}^{\hat{\nu}^N}[J_R(t,\lambda)] = \mathbb{E}^{\hat{\nu}}[J_R(t,\lambda)]. \quad (6.2)$$

By (3.7), we have the following uniform estimate:

$$
\begin{aligned}
\Big|\mathbb{E}^{\hat{\nu}^N}[J_R(t,\lambda)] - \mathbb{E}^{\hat{\nu}^N}[J(t,\lambda)]\Big| &= \Big|J_{N,R}(t,\mathbf{y}^N,\hat{\boldsymbol{\tau}}^N) - J_N(t,\mathbf{y}^N,\hat{\boldsymbol{\tau}}^N)\Big| \\
&\leq C\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^T \big(1 + |X_r^{\hat{\boldsymbol{\tau}}^N,k}| + \|\mathbf{X}_r^{\hat{\boldsymbol{\tau}}^N}\|_1\big)\mathbf{1}_{\{C(1+|X_r^{\hat{\boldsymbol{\tau}}^N,k}|+\|\mathbf{X}_r^{\hat{\boldsymbol{\tau}}^N}\|_1)\geq R\}}dr \\
&\quad + \|\mathbf{X}_T^{\hat{\boldsymbol{\tau}}^N}\|_1 \mathbf{1}_{\{C\|\mathbf{X}_T^{\hat{\boldsymbol{\tau}}^N}\|_1\}\geq R\}}\Big] \\
&\leq \frac{C}{R}\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^T [1 + |X_r^{\hat{\boldsymbol{\tau}}^N,k}|^2 + \|\mathbf{X}_r^{\hat{\boldsymbol{\tau}}^N}\|_1^2]dr + \|\mathbf{X}_T^{\hat{\boldsymbol{\tau}}^N}\|_1^2\Big] \\
&\leq \frac{C}{R}\Big[\frac{1}{N}\sum_{k=1}^N [1 + |x_k^N|^2] + \|\mathbf{x}^N\|_1^2\Big] \leq \frac{C}{R}[1 + \|\mathbf{x}^N\|_2^2].
\end{aligned}
$$

Thus it is clear that

$$\lim_{R\to\infty}\sup_{N\geq 1}\Big|\mathbb{E}^{\hat{\nu}^N}[J_R(t,\lambda)] - \mathbb{E}^{\hat{\nu}^N}[J(t,\lambda)]\Big| = \lim_{R\to\infty}\Big|\mathbb{E}^{\hat{\nu}}[J_R(t,\lambda)] - \mathbb{E}^{\hat{\nu}}[J(t,\lambda)]\Big| = 0.$$

Then by (6.2) we have $\lim_{N\to\infty}\mathbb{E}^{\hat{\nu}^N}[J(t,\lambda)] = \mathbb{E}^{\hat{\nu}}[J(t,\lambda)]$. Send $N \to 0$ in (6.1), note that $\varepsilon_N \to 0$ and by Theorem 4.1 $v^N(t,\mathbf{y}^N) \xrightarrow[N\to\infty]{} V(t,m)$, we obtain $V(t,m) \leq \mathbb{E}^{\hat{\nu}}[J(t,\lambda)]$. Since $\hat{\nu}$ is supported on $\mathcal{P}(t,m)$, we have $V(t,m) \geq J(t,\lambda)$, $\hat{\nu}$-a.s., and therefore $V(t,m) = J(t,\lambda)$, $\hat{\nu}$-a.s. This proves that $\hat{\nu}$ is supported on the set of optimal controls.

(ii) By Theorem 4.4 (ii), there exists a sequence $\hat{\boldsymbol{\tau}}^N$, $N \geq 1$, such that $m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}) \xrightarrow{\mathcal{W}_2} \mathbb{P}^*$, $\mathbb{P}^0$-a.s. Then, by the continuity of $F$ and $g$ and following similar arguments as in (i),

$$\mathbb{E}^{\mathbb{P}^0}\big[J(t, m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}))\big] \to J(t,\mathbb{P}^*) = V(t,m). \quad (6.3)$$

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Introduce
$$\varepsilon_N := V^N(t, m^N(\mathbf{y}^N)) - \mathbb{E}^{\mathbb{P}^0}\left[J(t, m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}))\right].$$

By definition $\hat{\boldsymbol{\tau}}^N$ is $\varepsilon_N$-optimal for (3.2), and by Theorem 4.1 and (6.3), we get $\varepsilon_N \to 0$.

(iii) Assume by contradiction that there exist $\varepsilon_N \to 0$ but $m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N})$ does not converge to $\mathbb{P}^*$. Then there exists $c > 0$ such that, by otherwise choosing a subsequence, $\mathbb{P}_0\big(\mathcal{W}_2(m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}), \mathbb{P}^*) > c\big) > c$. By Theorem 4.3 (i), again by otherwise choosing a subsequence, we may assume $\mathbb{P}^0 \circ \big(m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N})\big)^{-1}$ converges weakly to some $\nu \in \mathcal{P}_2(\mathcal{P}_2(\Omega))$ whose support is on the singleton $\{\mathbb{P}^*\}$, then we must have $m^N(\mathbf{Y}^{\hat{\boldsymbol{\tau}}^N}) \to \mathbb{P}^*$, in probability $\mathbb{P}^0$. This is a desired contradiction. ∎

**Proof of Theorem 4.4.** Again for simplicity we assume $t = 0$.

(i) For each $N \geq 1$, denote $\nu^N := \mathbb{P}^0 \circ (m^N(\mathbf{Y}^{\boldsymbol{\tau}^N}))^{-1} \in \mathcal{P}_2(\mathcal{P}_2(\Omega))$.

**Step 1.** We first show that $\{\nu^N\}_{N \geq 1}$ is uniformly integrable, that is,
$$\lim_{R \to \infty} \sup_{N \geq 1} \mathbb{E}^{\nu^N}\left[\|\lambda\|_2 \mathbf{1}_{\{\|\lambda\|_2 \geq R\}}\right] = 0. \tag{6.4}$$

Indeed, for any $R > 0$, by (3.7) we have
$$\mathbb{E}^{\nu^N}\left[\|\lambda\|_2 \mathbf{1}_{\{\|\lambda\|_2 \geq R\}}\right] \leq \frac{1}{R}\mathbb{E}^{\nu^N}\left[\|\lambda\|_2^2\right] = \frac{1}{R}\mathbb{E}^{\mathbb{P}^0}\left[\frac{1}{N}\sum_{k=1}^{N} \sup_{0 \leq t \leq T} |Y_t^{\boldsymbol{\tau}^{N,k}}|^2\right]$$
$$\leq \frac{C}{R}\left[1 + \frac{1}{N}\sum_{k=1}^{N}[|x_k^N|^2 + \|\mathbf{x}^N\|_2^2]\right] \leq \frac{C}{R}\left[1 + \|\mathbf{y}^N\|_2^2\right].$$

Since $\mathcal{W}_2\big(m^N(\mathbf{y}^N), m\big) \xrightarrow[N \to \infty]{} 0$, we have $\|\mathbf{y}^N\|_2 \to \|m\|_2$, and thus $\{\|\mathbf{y}^N\|_2\}_{N \geq 1}$ is bounded. This implies (6.4) immediately.

**Step 2.** We next prove the tightness of $\{\nu^N\}_{N \geq 1}$. By Lacker [37, Corollary B.1], given (6.4), it suffices to show that the mean measures $\{\mathbb{E}^{\mathbb{P}^0}\big[m^N(\mathbf{Y}^{\boldsymbol{\tau}^N})\big]\}_{N \geq 1} \subset \mathcal{P}_2(\Omega)$ is tight. Here, for a random measure $\tilde{m} : \Omega^0 \to \mathcal{P}_2(\Omega)$, the mean measure $\mathbb{E}^{\mathbb{P}^0}[\tilde{m}]$ is defined by
$$\langle \mathbb{E}^{\mathbb{P}^0}[\tilde{m}], \varphi \rangle := \mathbb{E}^{\mathbb{P}^0}\big[\langle \tilde{m}, \varphi \rangle\big] \quad \text{for all } \varphi \in C_b^0(\Omega).$$

To see this, we first remark that, the tightness of the joint measure is equivalent to the tightness of the marginals. For the first marginal, by (3.8) we have
$$\sup_{N \geq 1} \sup_{\tau \in \mathcal{T}_{0,T}} \left\langle \mathbb{E}^{\mathbb{P}^0}\big[m^N(\mathbf{Y}^{\boldsymbol{\tau}^N})\big], |X_{(\tau+\delta) \wedge T} - X_\tau|^2 \right\rangle = \sup_{N \geq 1} \sup_{\tau \in \mathcal{T}_{0,T}} \frac{1}{N}\sum_{k=1}^{N} \mathbb{E}^{\mathbb{P}^0}\left[|X_{(\tau+\delta) \wedge T}^{\boldsymbol{\tau}^{N,k}} - X_\tau^{\boldsymbol{\tau}^{N,k}}|^2\right] \leq C\delta.$$

Then it follows from Aldous' criterion (see Billingsley [5, Theorem 16.10]) that the first marginal of $\{\mathbb{E}^{\mathbb{P}^0}\big[m^N(\mathbf{Y}^{\boldsymbol{\tau}^N})\big]\}_{N \geq 1}$ is tight. Moreover, since the second marginals are measures on $\mathbb{I}^0([-1, T])$, which are in continuous bijection with measures on $[0, T]$, and thus are tight. Therefore, $\{\mathbb{E}^{\mathbb{P}^0}\big[m^N(\mathbf{Y}^{\boldsymbol{\tau}^N})\big]\}_{N \geq 1}$ and hence $\{\nu^N\}_{N \geq 1}$ are tight.

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**Step 3.** We now show that $\nu$ is supported on $\mathcal{P}(0, m)$, or equivalently, that $\lambda \in \mathcal{P}(0, m)$, $\nu$-a.s. Since $\mathcal{W}_2(m^N(\mathbf{y}^N), m) \xrightarrow[N\to\infty]{} 0$, we have $\lambda_{0-} = m$, $\nu$-a.s. Recall (2.4), we may consider equivalently the martingale problem. That is, for any $\psi \in C_b^2(\mathbb{R}^d)$, we want to show that

$$M_s^{\psi} := \psi(X_s) - \int_0^s \mathcal{L}_x \psi(r, X_r, \lambda_r) I_r dr \text{ is a } \lambda\text{-martingale, for } \nu\text{-a.e. } \lambda. \tag{6.5}$$

For this purpose, fix $0 \le s_1 < s_2 \le T$ and $h_{s_1} \in C_b^0(\Omega)$ that is $\mathcal{F}_{s_1}$-measurable. Note that

$$\mathbb{E}^{\nu^N}\left[\left|\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right| \wedge 1\right] \le \mathbb{E}^{\nu^N}\left[\left|\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right|^2\right]$$

$$= \mathbb{E}^{\mathbb{P}^0}\left[\left|\langle m^N(\mathbf{Y}^{\tau^N}), h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right|^2\right]$$

$$= \mathbb{E}^{\mathbb{P}^0}\left[\left|\frac{1}{N}\sum_{k=1}^N h_{s_1}(Y^{\tau^N,k})\left[\psi(X_{s_2}^{\tau^N,k}) - \psi(X_{s_1}^{\tau^N,k}) - \int_{s_1}^{s_2} \mathcal{L}_x \psi(r, X_r^{\tau^N,k}, m^N(\mathbf{Y}_r^{\tau^N})) I_r^{\tau^N,k} dr\right]\right|^2\right].$$

By (3.1) and (3.6), it follows from the Itô formula that

$$\mathbb{E}^{\nu^N}\left[\left|\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right| \wedge 1\right]$$

$$\le \mathbb{E}^{\mathbb{P}^0}\left[\left|\frac{1}{N}\sum_{k=1}^N h_{s_1}(Y^{\tau^N,k})\int_{s_1}^{s_2} \partial_x \psi(X_r^{\tau^N,k})\sigma(r, X_r^{\tau^N,k}, m^N(\mathbf{Y}_r^{\tau^N})) I_r^{\tau^N,k} dW_r\right|^2\right]$$

$$= \frac{1}{N^2}\sum_{k=1}^N \mathbb{E}^{\mathbb{P}^0}\left[\left|h_{s_1}(Y^{\tau^N,k})\right|^2 \int_{s_1}^{s_2}\left|\partial_x \psi(X_r^{\tau^N,k})\sigma(r, X_r^{\tau^N,k}, m^N(\mathbf{Y}_r^{\tau^N})) I_r^{\tau^N,k}\right|^2 dr\right]$$

$$\le \frac{C}{N^2}\sum_{k=1}^N \mathbb{E}^{\mathbb{P}^0}\left[\int_{s_1}^{s_2}\left|\sigma(r, X_r^{\tau^N,k}, m^N(\mathbf{Y}_r^{\tau^N}))\right|^2 dr\right] \le \frac{C}{N},$$

where the last inequality thanks to (3.7). Note that $\nu^N$ converges to $\nu$ weakly, and the mapping $\lambda \in \mathcal{P}_2(\Omega) \longmapsto \left|\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right| \wedge 1$ is bounded and continuous. Then

$$\mathbb{E}^{\nu}\left[\left|\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle\right| \wedge 1\right] = 0.$$

This implies that, for any desired $\psi$, $h_{s_1}$ and $s_1 < s_2$,

$$\langle \lambda, h_{s_1}[M_{s_2}^{\psi} - M_{s_1}^{\psi}]\rangle = 0, \quad \text{for } \nu\text{-a.e. } \lambda. \tag{6.6}$$

Since $M^{\psi}$ is continuous in $s$, and the spaces of the above $\psi$ and $h_{s_1}$ are separable, then (6.6) implies (6.5). Therefore, $\lambda \in \mathcal{P}(0, m)$, $\nu$-a.s.

(ii) Without loss of generality, assume $(\Omega^0, \mathcal{F}_T^0, \mathbb{F}^0, \mathbb{P}^0)$ is sufficiently large so that there exist a sequence of i.i.d. processes $\{(I^{0,k}, W^k)\}_{k \ge 1}$ with $\mathbb{P}^0_{(I^{0,k}, W^k)} = \mathbb{P}_{(I, W^{\mathbb{P}})|I_{0-}=1}$. In particular, $W^k$ is a $\mathbb{P}^0$-Brownian motion and $I_{0-}^{0,k} = 1$. Now for given $\mathbf{y}^N$, define

$$I_t^{N,k} := I_t^{0,k} \mathbf{1}_{\{i_k=1\}}; \quad \tau_k^N := \inf\{t \ge 0 : I_t^{N,k} = 0\}, \quad \boldsymbol{\tau}^N := (\tau_1^N, \cdots, \tau_N^N). \tag{6.7}$$

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We shall prove in four steps that the above $\tau^N$ satisfies the requirement.

**Step 1.** We first assume

$$\int_{\mathbf{S}} |x|^3 m(dy) < \infty, \quad \text{and} \quad \lim_{N \to \infty} \mathcal{W}_3(m^N(\mathbf{y}^N), m) = 0. \tag{6.8}$$

Denote $m_t := \mathbb{P}_{Y_t}$ and introduce

$$\tilde{X}_t^{N,k} = x_k^N + \int_0^t b(r, \tilde{X}_r^{N,k}, m_r) I_r^{N,k} dr + \int_0^t \sigma(r, \tilde{X}_r^{N,k}, m_r) I_r^{N,k} dW_r^k. \tag{6.9}$$

Note that $(W^k, I^{N,k}, \tilde{X}^{N,k})$, $k \in [N]$, are independent. In this step we show that

$$\lim_{N \to \infty} \sup_{0 \le t \le T} \mathbb{E}^{\mathbb{P}^0} \left[ \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), m_t) \right] = 0, \quad \text{for all } t, \text{ where } \quad \tilde{\mathbf{Y}}^{N,k} := (\tilde{X}^{N,k}, I^{N,k}). \tag{6.10}$$

To see this, let $E_k^N \in \mathcal{F}_0^0$, $k \in [N]$ be a partition of $\Omega^0$ such that $\mathbb{P}^0(E_k^N) = \frac{1}{N}$. Denote

$$(\hat{W}^N, \hat{X}_t^N, \hat{I}_t^N) := \sum_{k=1}^N (W^k, \tilde{X}_t^{N,k}, I_t^{N,k}) \mathbf{1}_{E_k^N}. \tag{6.11}$$

By [44, Lemma 8.4], and denoting $\hat{Y}^N := (\hat{X}_t^N, \hat{I}_t^N)$, we have

$$\mathbb{E}^{\mathbb{P}^0} \left[ \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), \mathbb{P}_{\hat{Y}_t^N}^0) \right] \le \frac{C}{N^{\frac{2}{d \vee 3}}} \left( \mathbb{E}^{\mathbb{P}^0}[|\hat{Y}_t^N|^3] \right)^{\frac{2}{3}}.$$

By standard SDE estimate and noting that $\mathcal{W}_3(m^N(\mathbf{y}^N), m) \to 0$, we may estimate $\mathbb{E}^{\mathbb{P}^0}[|\hat{Y}_t^N|^3]$ and thus obtain

$$\mathbb{E}^{\mathbb{P}^0} \left[ \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), \mathbb{P}_{\hat{Y}_t^N}^0) \right] \le \frac{C(1 + \|\mathbf{x}^N\|_3^2)}{N^{\frac{2}{d \vee 3}}} \le \frac{C_m}{N^{\frac{2}{d \vee 3}}}, \tag{6.12}$$

for some constant $C_m$, which may depend on $m$, but not on $N$.

Next, for any bounded function $\varphi \in C^0(\Omega)$. Denote

$$v(i) := \mathbb{E}^{\mathbb{P}}[\varphi(W^{\mathbb{P}}, I)|I_{0-} = i], \quad i = 0, 1.$$

Note that $(W^k, \tilde{X}_t^{N,k}, I_t^{N,k})$ is independent of $\mathcal{F}_0^0$. Then, by (6.7),

$$\mathbb{E}^{\mathbb{P}^0}[\varphi(\hat{W}^N, \hat{I}^N)] = \frac{1}{N} \sum_{k=1}^N \mathbb{E}^{\mathbb{P}^0}[\varphi(W^k, I^{N,k})] = \frac{1}{N} \sum_{k=1}^N \left[ v(1) \mathbf{1}_{\{i_k=1\}} + v(0) \mathbf{1}_{\{i_k=0\}} \right]$$

$$\longrightarrow v(1) \mathbb{P}(I_{0-} = 1) + v(0) \mathbb{P}(I_{0-} = 0) = \mathbb{E}^{\mathbb{P}}[\varphi(W^{\mathbb{P}}, I)], \tag{6.13}$$

as $N \to \infty$. That is, $\mathbb{P}^0 \circ (\hat{W}^N, \hat{I}^N)^{-1} \to \mathbb{P} \circ (W^{\mathbb{P}}, I)^{-1}$ weakly. Since $I$ is bounded, by [11, Theorem 5.5] we have

$$\lim_{N \to \infty} \mathcal{W}_2 \left( \mathbb{P}^0 \circ (\hat{W}^N, \hat{I}^N)^{-1}, \ \mathbb{P} \circ (W^{\mathbb{P}}, I)^{-1} \right) = 0.$$

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Moreover, since $\lim_{N\to\infty} \mathcal{W}_2\big(m^N(\mathbf{y}^N), m)\big) = 0$, then

$$\lim_{N\to\infty} \mathcal{W}_2\Big(\mathbb{P}^0 \circ (\hat{X}_0^N, \hat{W}^N, \hat{I}^N)^{-1}, \ \mathbb{P} \circ (X_0, W^{\mathbb{P}}, I)^{-1}\Big) = 0.$$

One may easily verify that

$$\hat{X}_t^N = \hat{X}_0^N + \int_0^t b(r, \hat{X}_r^N, m_r)\hat{I}_r^N dr + \int_0^t \sigma(r, \hat{X}_r^N, m_r)\hat{I}_r^N d\hat{W}_r^N.$$

Then, by comparing this with (2.5), $\lim_{N\to\infty} \sup_{0\le t\le T} \mathcal{W}_2\big(\mathbb{P}^0_{\hat{Y}_t^N}, m_t\big) = 0$. This, together with (6.12), implies (6.10) immediately.

**Step 2.** In this step we show that, again under the additional condition (6.8),

$$\lim_{N\to\infty} \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{\tau^N}), m_t)\Big] = 0. \tag{6.14}$$

Recall (6.7) and (3.6), and compare (3.1) and (6.9). By the Lipschitz continuity of $b, \sigma$, especially the $\mathcal{W}_1$-Lipschitz continuity in $m$, it follows from standard SDE estimates that

$$\mathbb{E}^{\mathbb{P}^0}\Big[|X_t^{\tau^N,k} - \tilde{X}_t^{N,k}|^2\Big] \le C\mathbb{E}^{\mathbb{P}^0}\Big[\int_0^t \mathcal{W}_1^2(m^N(\mathbf{Y}_s^{\tau^N}), m_s)ds\Big].$$

Then

$$\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{\tau^N}), m_t)\Big] \le C\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{\tau^N}), m^N(\tilde{\mathbf{Y}}_t^N)) + \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), m_t))\Big]$$

$$\le C\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N |X_t^{\tau^N,k} - \tilde{X}_t^{N,k}|^2 + \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), m_t))\Big]$$

$$\le C\mathbb{E}^{\mathbb{P}^0}\Big[\int_0^t \mathcal{W}_1^2(m^N(\mathbf{Y}_s^{\tau^N}), m_s)ds + \mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), m_t))\Big]$$

Apply the Gronwall inequality, we obtain

$$\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{\tau^N}), m_t)\Big] \le C\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\tilde{\mathbf{Y}}_t^N), m_t))\Big].$$

This, together with (6.10), implies (6.14) immediately.

**Step 3.** We now prove the result without assuming (6.8). For any $R > 0$, let $\phi_R(x)$ denote the truncation function such that $x$ is truncated by $R$. Denote $m^R := m \circ (\phi_R(X_0), I_{0-})^{-1}$, $\mathbf{y}^{N,R} := (\phi_R(x_k^N), i_k^N)_{1\le k\le N}$. Then it is clear that

$$\lim_{R\to\infty} \mathcal{W}_2(m^R, m) = 0, \quad \lim_{R\to\infty} \sup_{N\ge 1} \frac{1}{N}\sum_{k=1}^N |\phi_R(x_k^N) - x_k^N|^2 = 0, \tag{6.15}$$

$$\lim_{N\to\infty} \mathcal{W}_3(m^N(\mathbf{y}^{N,R}), m^R) = 0, \quad \text{for all } R > 0. \tag{6.16}$$

Introduce $\mathbb{P}^R$ and $\mathbf{Y}^{R,\tau}$ in an obvious way. By (6.14) and (6.16) we have

$$\lim_{N\to\infty} \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{R,\tau^N}), \mathbb{P}_{Y_t}^R)\Big] = 0. \tag{6.17}$$

Moreover, by standard SDE estimates one can easily see that

$$\mathcal{W}_2(\mathbb{P}_{Y_t}^R, m_t) \le C\mathcal{W}_2(m^R, m),$$

$$\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_2^2\big(m^N(\mathbf{Y}_t^{R,\tau^N}), m^N(\mathbf{Y}_t^{\tau^N})\big)\Big] \le \frac{1}{N}\sum_{k=1}^N \mathbb{E}^{\mathbb{P}^0}\Big[\big|X_t^{R,\tau^N,k} - X_t^{\tau^N,k}\big|^2\Big]$$

$$\le \frac{C}{N}\sum_{k=1}^N \big|\phi_R(x_k^N) - x_k^N\big|^2.$$

Then it follows from (6.15) that, for any $R > 0$,

$$\limsup_{N\to\infty} \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2\big(m^N(\mathbf{Y}_t^{\tau^N}), m_t\big)\Big] \le \limsup_{N\to\infty} C \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2\big(m^N(\mathbf{Y}_t^{\tau^N}), m^N(\mathbf{Y}_t^{R,\tau^N})\big)$$

$$+ \mathcal{W}_1^2(m^N(\mathbf{Y}_t^{R,\tau^N}), \mathbb{P}_{Y_t}^R) + \mathcal{W}_2^2(\mathbb{P}_{Y_t}^R, m_t)\Big]$$

$$= C \sup_{0\le t\le T} \mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1^2(m^N(\mathbf{Y}_t^{R,\tau^N}), \mathbb{P}_{Y_t}^R)\Big].$$

This, together with (6.17), implies (6.14), without assuming (6.8).

**Step 4.** Finally we prove the convergence of the distribution of the processes. First, compare (3.1) and (6.9) again, by (6.14) we obtain immediately that

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^0}\Big[\sup_{0\le t\le T} |X_t^{\tau^N,k} - \tilde{X}_t^{N,k}|^2\Big] = 0. \tag{6.18}$$

Next, for any bounded function $\varphi \in C^0(\Omega)$, similarly to (6.13), we can show

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^0}\Big[\mathbb{E}^{m^N(\tilde{\mathbf{Y}}^N)}[\varphi(Y)]\Big] = \lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N [\varphi(\tilde{Y}^{N,k})]\Big] = \mathbb{E}^{\mathbb{P}}[\varphi(Y)].$$

This, together with (6.18), implies

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^0}\Big[\mathbb{E}^{m^N(\mathbf{Y}^{\tau^N})}[\varphi(Y)]\Big] = \mathbb{E}^{\mathbb{P}}[\varphi(Y)]. \tag{6.19}$$

On the other hand, by (i) of this theorem, we know $\big\{\mathbb{P}^0 \circ \big(m^N(\mathbf{Y}^{\tau^N})\big)^{-1}\big\}_{N\ge 1}$ is tight. By (6.19) clearly $\mathbb{P}$ is its unique accumulation point. Then, similarly to the arguments in Theorem 4.3 (iii), we obtain $m^N(\mathbf{Y}^{\tau^N}) \xrightarrow{\mathcal{W}_2} \mathbb{P}$, $\mathbb{P}^0$-a.s.. ∎

# 7 Further study on the multiple optimal stopping problem

## 7.1 The viscosity solution property

In this subsection we prove Theorem 3.3. Recall the notations in Subsection 3.1 and 3.2, in particular the unstopped process $\bar{\mathbf{Y}}$. As standard, the viscosity property relies on the following dynamic programming principle.

**Proposition 7.1** *Under Assumption 3.1, for all $(t, \mathbf{y}) \in \Lambda_0^N$, we have*

$$v^N(t, \mathbf{y}) = \sup_{\theta \in \mathcal{T}_{t,T}^N} \mathbb{E}_{t,\mathbf{y}} \Big[ \frac{1}{N} \sum_{k=1}^N \int_t^\theta f^k(r, \bar{\mathbf{X}}_r) \bar{I}_r^k dr + \max_{\mathbf{i}' < \mathbf{i}} v^N(\theta, \bar{\mathbf{X}}_\theta, \mathbf{i}') \Big]. \tag{7.1}$$

*In particular, by setting $\theta = t$, we see that $v^N$ is non-decreasing in $\mathbf{i}$:*

$$v^N(t, \mathbf{y}) \geq v^N(t, \mathbf{x}, \mathbf{i}'), \quad \text{for all} \quad \mathbf{i}' < \mathbf{i}. \tag{7.2}$$

**Proof** First, by Kobylanski, Quenez & Rouy-Mironescu [36, Theorem 3.1][1], we have

$$
\begin{aligned}
v^N(t, \mathbf{y}) &= \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}_{t,\mathbf{y}} \Big[ \max_{l \in [N]} \sup_{\boldsymbol{\tau} \in \mathcal{T}_{\theta,T}^{N-1}} \mathbb{E}_{\theta, \mathbf{Y}_\theta^{-l}} \Big[ \frac{1}{N} \sum_{k=1}^N \int_t^T f^k(r, \mathbf{X}_r^{\boldsymbol{\tau} \otimes_l \theta}) \bar{I}_r^k dr + g(\mathbf{X}_T^{\boldsymbol{\tau} \otimes_l \theta}) \Big] \Big] \\
&= \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}_{t,\mathbf{y}} \Big[ \frac{1}{N} \sum_{k=1}^N \int_t^\theta f^k(r, \bar{\mathbf{X}}_r) \bar{I}_r^k dr \\
&\quad + \max_{l \in [N]} \sup_{\boldsymbol{\tau} \in \mathcal{T}_{\theta,T}^{N-1}} \mathbb{E}_{\theta, \mathbf{Y}_\theta^{-l}} \Big[ \frac{1}{N} \sum_{k=1}^N \int_\theta^T f^k(r, \mathbf{X}_r^{\boldsymbol{\tau} \otimes_l \theta}) \bar{I}_r^k dr + g(\mathbf{X}_T^{\boldsymbol{\tau} \otimes_l \theta}) \Big] \Big],
\end{aligned}
$$

where

$$\boldsymbol{\tau} \otimes_l \theta := (\tau_1, \dots, \tau_{l-1}, \theta, \tau_{l+1}, \dots, \tau_N), \quad \mathbf{i}^{-l} := (i_1, \dots, i_{l-1}, 0, i_{l+1}, \dots, i_N), \tag{7.3}$$

and $\mathbf{Y}^{-l} := (\bar{\mathbf{X}}, \mathbf{i}^{-l})$. By the Markov property, we have

$$\sup_{\boldsymbol{\tau} \in \mathcal{T}_{\theta,T}^{N-1}} \mathbb{E}_{\theta, \mathbf{Y}_\theta^{-l}} \Big[ \frac{1}{N} \sum_{k=1}^N \int_\theta^T f^k(r, \mathbf{X}_r^{\boldsymbol{\tau} \otimes_l \theta}) \bar{I}_r^k dr + g(\mathbf{X}_T^{\boldsymbol{\tau} \otimes_l \theta}) \Big] = v^N(\theta, \bar{\mathbf{X}}_\theta, \mathbf{i}^{-l}).$$

Finally, by induction $v^N$ is nondecreasing in $\mathbf{i}$ for the partial order on $\{0, 1\}^N$, and thus

$$\max_{l \in [N]} v^N(\theta, \bar{\mathbf{X}}_\theta, \mathbf{i}^{-l}) = \max_{\mathbf{i}' < \mathbf{i}} v^N(\theta, \bar{\mathbf{X}}_\theta, \mathbf{i}'). \qquad \blacksquare$$

---

[1] Although Kobylanski, Quenez & Rouy-Mironescu assume nonnegative reward processes throughout their paper, the validity of their result in our context is easy to check.

**Proof of Theorem 3.3.** We proceed by induction on the cardinality $|A(\mathbf{i})|$ of the set

$$A(\mathbf{i}) := \{k \in [N] : i_k = 1\}. \tag{7.4}$$

As $(t, \mathbf{x}) \longmapsto g_N(t, \mathbf{x})$ is continuous, it suffices to prove that whenever $v^N(t, \mathbf{x}, \mathbf{i})$ is continuous in $(t, \mathbf{x})$ for $|A(\mathbf{i})| \leq n$, we may conclude for $|A(\mathbf{i})| = n + 1$ that $v^N(\cdot, \mathbf{i})$ is the unique continuous viscosity solution of (3.3). Note that $\bar{I}_r^k \equiv 1$ for $k \in A(\mathbf{i})$ and $\bar{I}_r^k \equiv 0$ for $k \notin A(\mathbf{i})$. To see this, we rewrite (7.1), emphasizing again that the process $\bar{\mathbf{X}}$ is unstopped, thus reducing (7.5) to a standard optimal stopping problem:

$$v^N(t, \mathbf{x}, \mathbf{i}) = \sup_{\theta \in \mathcal{T}_{t,T}^N} \mathbb{E}_{t,\mathbf{y}} \Big[ \frac{1}{N} \sum_{k \in A(\mathbf{i})} \int_t^\theta f^k(r, \bar{\mathbf{X}}_r) dr + \hat{v}^N(\theta, \bar{\mathbf{X}}_\theta, \mathbf{i}) \Big], \quad \hat{v}^N(t, \mathbf{x}, \mathbf{i}) := \sup_{\mathbf{i}' < \mathbf{i}} v^N(t, \mathbf{x}, \mathbf{i}'). \tag{7.5}$$

Notice that $\hat{v}^N(t, \mathbf{x}, \mathbf{i})$ is continuous in $(t, \mathbf{x})$, as $|A(\mathbf{i}')| \leq n$. Then it follows from Ekren [22] that $v^N(\cdot, \cdot, \mathbf{i})$ is a continuous viscosity solution of (3.3) in the sense of Definition 3.2.[2] Moreover, note that a viscosity sub/super solution in the sense of Definition 3.2 implies it is a viscosity sub/super solution in the classical Crandall-Lions sense. Then the comparison principle for viscosity semi-solutions in Crandall-Lions sense implies that in our sense, see Remark 3.4, and hence $v^N$ is the unique viscosity solution to (3.3). ∎

## 7.2 The symmetric case

We now prove Lemma 3.5 and Theorem 3.7 in the symmetric case (3.6).

**Proof of Lemma 3.5.** (i) Recall (3.1). For $k \in [N]$, by standard SDE estimates we have

$$\mathbb{E}^{\mathbb{P}^0}[|X_s^{\boldsymbol{\tau},k}|^2] \leq \mathbb{E}^{\mathbb{P}^0}[\sup_{t \leq r \leq s} |X_r^{\boldsymbol{\tau},k}|^2] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[ |x_k|^2 + \int_t^s (1 + \|\mathbf{X}_r^{\boldsymbol{\tau}}\|_2^2) dr \Big]. \tag{7.6}$$

Then

$$\mathbb{E}^{\mathbb{P}^0}[\|\mathbf{X}_s^{\boldsymbol{\tau}}\|_2^2] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[ \|\mathbf{x}\|_2^2 + 1 + \int_t^s \|\mathbf{X}_r^{\boldsymbol{\tau}}\|_2^2 dr \Big].$$

Applying Gronwall inequality, we have $\sup_{t \leq s \leq T} \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_2 \leq C(1 + \|\mathbf{x}\|_2)$. Plug this into the second inequality of (7.6), we obtain (3.7) immediately.

Moreover, for any $\theta_1, \theta_2 \in \mathcal{T}_{[t,T]}^N$ with $\theta_1 \leq \theta_2$, by standard SDE estimates we have

$$\mathbb{E}^{\mathbb{P}^0}[\sup_{\theta_1 \leq s \leq \theta_2} |X_s^{\boldsymbol{\tau},k} - X_{\theta_1}^{\boldsymbol{\tau},k}|^2] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[ \int_{\theta_1}^{\theta_2} (1 + |X_r^{\boldsymbol{\tau},k}|^2 + \|\mathbf{X}_r^{\boldsymbol{\tau}}\|^2) dr \Big]$$

$$\leq C\|\theta_2 - \theta_1\|_\infty \mathbb{E}^{\mathbb{P}^0}\Big[ 1 + \sup_{t \leq r \leq T} |X_r^{\boldsymbol{\tau},k}|^2 + \frac{1}{N} \sum_{l=1}^N \sup_{t \leq r \leq T} |X_r^{\boldsymbol{\tau},l}|^2 \Big].$$

---

[2]Here again, we notice that our conditions are slightly different from Ekren and that the viscosity solution property is easily checked to be valid in our context.

Plug (3.7) into it, we obtain (3.8).

(ii) We now assume Assumption 2.6 (i) also holds true. Let $\delta > 0$ be a small number which will be specified later, and consider a partition $t = t_0 < \cdots < t_n = T$ such that $\Delta t_j \leq \delta$ for all $j$. For each $j = 1, \cdots, n$ and $k = 1, \cdots, N$, by standard SDE estimates we have, for a generic constant $C$ which is independent of $N$ and $\delta$,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} |X_s^{\boldsymbol{\tau},k}|\Big] &\leq C\mathbb{E}^{\mathbb{P}^0}\Big[|X_{t_j}^{\boldsymbol{\tau},k}| + \int_{t_j}^{t_{j+1}} \big(1 + |X_s^{\boldsymbol{\tau},k}| + \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\big)ds \\
&\quad + \big(\int_{t_j}^{t_{j+1}} \big(1 + |X_s^{\boldsymbol{\tau},k}| + \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\big)^2 ds\big)^{\frac{1}{2}}\Big] \\
&\leq C\mathbb{E}^{\mathbb{P}^0}\Big[|X_{t_j}^{\boldsymbol{\tau},k}| + 1 + \sqrt{\delta} \sup_{t_j \leq s \leq t_{j+1}} [|X_s^{\boldsymbol{\tau},k}| + \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1]\Big]
\end{aligned}
$$

Set $\delta := \frac{1}{9C^2}$ for the above $C$, we have

$$
\frac{2}{3}\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} |X_s^{\boldsymbol{\tau},k}|\Big] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[|X_{t_j}^{\boldsymbol{\tau},k}| + 1\Big] + \frac{1}{3}\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big]. \tag{7.7}
$$

This implies that

$$
\frac{2}{3}\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^{N} \sup_{t_j \leq s \leq t_{j+1}} |X_s^{\boldsymbol{\tau},k}|\Big] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[\|\mathbf{X}_{t_j}^{\boldsymbol{\tau}}\|_1 + 1\Big] + \frac{1}{3}\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big],
$$

and thus

$$
\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big] \leq \mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^{N} \sup_{t_j \leq s \leq t_{j+1}} |X_s^{\boldsymbol{\tau},k}|\Big] \leq C\mathbb{E}^{\mathbb{P}^0}\Big[\|\mathbf{X}_{t_j}^{\boldsymbol{\tau}}\|_1 + 1\Big].
$$

By induction on $j = 0, \cdots, n-1$, one can easily obtain

$$
\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t \leq s \leq T} \|\mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big] \leq C(1 + \|\mathbf{x}\|_1).
$$

Plugging this into (7.7) and by induction on $j = 0, \cdots, n-1$ again, we prove (3.9). Moreover, (3.10) follows similar arguments as for (3.8). ∎

**Proof of Theorem 3.7.** It is obvious that Assumptions 2.1, 2.6 and (3.6) imply Assumption 3.1. In particular, the uniform continuity of $g^N$ comes from the fact that $g$ is uniformly continuous on all bounded parts of $\mathcal{P}_2(\mathbb{R}^d)$, which are $\mathcal{W}_1$-compact. Then (i) follows directly from Theorem 3.3.

To see (ii), assume $m_N = m^N(\mathbf{y})$ and recall (3.2) and (3.11). By the linear growth of $f$ and $g$ and by (3.7), for any $\boldsymbol{\tau}$ we have

$$
\begin{aligned}
\big|J_N(t, \mathbf{y}, \boldsymbol{\tau})\big| &\leq \mathbb{E}^{\mathbb{P}^0}\Big[\frac{C}{N}\sum_{k=1}^{N}\int_t^T \big(1 + |X_r^{\boldsymbol{\tau},k}| + \|\mathbf{X}_r^{\boldsymbol{\tau}}\|_1\big)dr + 1 + \|\mathbf{X}_T^{\boldsymbol{\tau}}\|_1\Big] \\
&\leq C + \frac{C}{N}\sum_{k=1}^{N}\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t \leq r \leq T} \|\mathbf{X}_r^{\boldsymbol{\tau}}\|_1\Big] \leq C(1 + \|\mathbf{x}\|_1) \leq C(1 + \|m_N\|_1).
\end{aligned}
$$

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Then by the arbitrariness of $\boldsymbol{\tau}$ we prove (3.12).

It remains to prove (iii). We proceed in three steps.

**Step 1.** By Birkhoff's theorem, the Kantorovitch solution of the optimal transport problem in the finite discrete case is a Monge solution, i.e. the optimal coupling measure is concentrated on a graph, see e.g. Villani [53] p5. Then, for any $t \in [0, T]$, $\mathbf{y}, \tilde{\mathbf{y}} \in \mathbf{S}^N$:

$$\mathcal{W}_2^2(m^N(\mathbf{y}), m^N(\tilde{\mathbf{y}})) = \min_{\pi \in \mathfrak{S}_N} \frac{1}{N} \sum_{k=1}^{N} |y_k - \tilde{y}_{\pi(k)}|^2,$$

where $\mathfrak{S}_N$ is the set of permutations on $[N]$. By possibly reordering the $y_k$'s, we may then assume without loss of generality that

$$\mathcal{W}_2(m^N(\mathbf{y}), m^N(\tilde{\mathbf{y}})) = \|\mathbf{y} - \tilde{\mathbf{y}}\|_2. \tag{7.8}$$

Given $\boldsymbol{\tau}$, let $\mathbf{Y}^{\boldsymbol{\tau}}$ and $\tilde{\mathbf{Y}}^{\boldsymbol{\tau}}$ be defined by (3.1) with initial condition $\mathbf{y}$ and $\tilde{\mathbf{y}}$, respectively, and denote $\Delta \mathbf{Y}^{\boldsymbol{\tau}} := \mathbf{Y}^{\boldsymbol{\tau}} - \tilde{\mathbf{Y}}^{\boldsymbol{\tau}}$, $\Delta \mathbf{y} := \mathbf{y} - \tilde{\mathbf{y}}$. Recall that $\boldsymbol{\tau}$ is adapted to the Brownian filtration, namely in the spirit of open loop controls, then

$$\Delta I_s^{\boldsymbol{\tau},k} = \Delta i_k \mathbf{1}_{\{\tau_k > s\}} \quad \text{and thus} \quad \|\Delta \mathbf{I}_t^{\boldsymbol{\tau}}\|_2 \leq \|\Delta \mathbf{i}\|_2, \quad \forall t \in [0, T]. \tag{7.9}$$

In this step we shall prove further that

$$\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N} \sum_{k=1}^{N} \sup_{t \leq s \leq T} |\Delta X_s^{\boldsymbol{\tau},k}|\Big] \leq C(1 + \|\mathbf{x}\|_2)\|\Delta \mathbf{y}\|_2. \tag{7.10}$$

Indeed, when $i_k = 1$ and $\tilde{i}_k = 0$, we have $\tilde{\mathbf{X}}_s^{\boldsymbol{\tau},k} \equiv \tilde{x}_k$, and thus, by (3.10),

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t \leq s \leq T} |\Delta X_s^{\boldsymbol{\tau},k}|\Big] &\leq C \mathbb{E}^{\mathbb{P}^0}\Big[|\Delta x_k| + \sup_{t \leq s \leq T} |X_s^{\boldsymbol{\tau},k} - x_k|\Big] \\
&\leq C(1 + |x_k| + \|\mathbf{x}\|_1 + |\Delta x_k|). \tag{7.11}
\end{aligned}$$

Similarly, when $i_k = 0$ and $\tilde{i}_k = 1$, the above still holds true. Moreover, when $\Delta i_k = 0$, we have $\Delta I_s^{\boldsymbol{\tau},k} \equiv 0$. Let $\delta$ and $t = t_0 < \cdots < t_n = T$ be as in the proof of Lemma 3.5 (ii). Then, for each $j = 0, \cdots, n-1$, similar to (7.7) we have

$$\frac{2}{3} \mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} |\Delta X_s^{\boldsymbol{\tau},k}|\Big] \leq \frac{1}{3} \mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} \|\Delta \mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big] + C \mathbb{E}^{\mathbb{P}^0}\Big[|\Delta X_{t_j}^{\boldsymbol{\tau},k}|\Big].$$

This, together with (7.11) which is for all $k$ such that $\Delta i_k \neq 0$, implies that

$$\begin{aligned}
\frac{2}{3} \mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N} \sum_{k=1}^{N} \sup_{t_j \leq s \leq t_{j+1}} |\Delta X_s^{\boldsymbol{\tau},k}|\Big] &\leq \frac{1}{3} \mathbb{E}^{\mathbb{P}^0}\Big[\sup_{t_j \leq s \leq t_{j+1}} \|\Delta \mathbf{X}_s^{\boldsymbol{\tau}}\|_1\Big] + C \mathbb{E}^{\mathbb{P}^0}\Big[\|\Delta X_{t_j}^{\boldsymbol{\tau}}\|_1\Big] \\
&\quad + \frac{C}{N} \sum_{k=1}^{N} (1 + |x_k| + \|\mathbf{x}\|_1 + |\Delta x_k|)|\Delta i_k|
\end{aligned}$$

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Then one can easily get

$$\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \sup_{t\leq s\leq T}|\Delta X_s^{\tau,k}|\Big] \leq C\Big[\|\Delta\mathbf{y}\|_1 + (\|\mathbf{x}\|_2 + \|\Delta\mathbf{x}\|_2)\|\Delta\mathbf{i}\|_2\Big],$$

which implies (7.10) immediately.

**Step 2.** In this step we prove the claimed regularity of $v_N$ in terms of $\mathbf{y}$. Fix $R > 1$ and let $\tilde{R} > R$ be a large number which will be specified later. Denote

$$D_{\tilde{R}} := [0,T] \times \{(x,m) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbf{S}) : |x| + \|m\|_2 \leq \tilde{R}\}.$$

Note in particular that any bounded subset of $\mathcal{P}_2(\mathbf{S})$ is tight in $\mathcal{P}_1(\mathbf{S})$. Then $f$ and $g$ are uniformly continuous for $\mathcal{W}_1$ on $D_{\tilde{R}}$, and let $\tilde{\rho}_{\tilde{R}}$ denote their common modulus of continuity function. Then, by the linear growth of $f$ and $g$,

$$|f(t,x,m) - f(t,\tilde{x},\tilde{m})| \leq \tilde{\rho}_{\tilde{R}}\big(|\Delta x| + \mathcal{W}_1(m,\tilde{m})\big)$$
$$+ C(1 + |x| + \|m\|_1)\mathbf{1}_{\{|x| + \|m\|_2 \geq \tilde{R}\}} + C(1 + |\tilde{x}| + \|\tilde{m}\|_1)\mathbf{1}_{\{|\tilde{x}| + \|\tilde{m}\|_2 \geq \tilde{R}\}}\Big]$$
$$\leq \tilde{\rho}_{\tilde{R}}\big(|\Delta x| + \mathcal{W}_1(m,\tilde{m})\big) + \frac{C}{\tilde{R}}(1 + |x|^2 + \|m\|_2^2 + |\tilde{x}|^2 + \|\tilde{m}\|_2^2);$$
$$|g(m) - g(\tilde{m})| \leq \tilde{\rho}_{\tilde{R}}\big(\mathcal{W}_1(m,\tilde{m})\big) + \frac{C}{\tilde{R}}(1 + \|m\|_2^2 + \|\tilde{m}\|_2^2).$$

We now fix $t \in [0,T]$, $\mathbf{y}, \tilde{\mathbf{y}} \in \mathbf{S}^N$ with $\|\mathbf{y}\|_2, \|\tilde{\mathbf{y}}\|_2 \leq R$. By (3.9) we have

$$\mathbb{E}^{\mathbb{P}^0}\Big[\big|f(s,X_s^{\tau,k},m^N(\mathbf{Y}_s^{\tau})) - f(s,\tilde{X}_s^{\tau,k},m^N(\tilde{\mathbf{Y}}_s^{\tau}))\big|\Big]$$
$$\leq \mathbb{E}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}}(|\Delta X_s^{\tau,k}| + \mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))$$
$$+ \frac{C}{\tilde{R}}\big(1 + |X_s^{\tau,k}|^2 + \|m^N(\mathbf{Y}_s^{\tau})\|_2^2 + |\tilde{X}_s^{\tau,k}|^2 + \|m^N(\tilde{\mathbf{Y}}_s^{\tau})\|_2^2\big)\Big]$$
$$\leq \mathbb{E}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}}(|\Delta X_s^{\tau,k}| + \mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))\Big] + \frac{C(1 + R^2)}{\tilde{R}}; \qquad (7.12)$$
$$\mathbb{E}^{\mathbb{P}^0}\Big[\big|g(m^N(\mathbf{Y}_T^{\tau})) - g(m^N(\tilde{\mathbf{Y}}_T^{\tau}))\big|\Big] \leq \mathbb{E}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}}(\mathcal{W}_1(m^N(\mathbf{Y}_T^{\tau}),m^N(\tilde{\mathbf{Y}}_T^{\tau})))\Big] + \frac{C(1 + R^2)}{\tilde{R}}.$$

Then

$$\Big|J_N(t,\mathbf{y},\tau) - J_N(t,\tilde{\mathbf{y}},\tau)\Big| \leq \mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^{\tau_k} \tilde{\rho}_{\tilde{R}}(|\Delta X_s^{\tau,k}| + \mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))ds$$
$$+ \tilde{\rho}_{\tilde{R}}\big(\mathcal{W}_1(m^N(\mathbf{Y}_T^{\tau}),m^N(\tilde{\mathbf{Y}}_T^{\tau})))\Big] + \frac{C(1 + R^2)}{\tilde{R}}.$$

Now for any $\varepsilon > 0$, choose $\tilde{R} = \tilde{R}_\varepsilon := \frac{2C(1+R^2)}{\varepsilon}$. By (7.9) and (7.10), and observing that

$$\mathbb{E}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}_\varepsilon}\big(\mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))\Big] \qquad\qquad\qquad\qquad\qquad (7.13)$$
$$\leq \tilde{\rho}_{\tilde{R}_\varepsilon}(\gamma) + \frac{1}{\sqrt{\gamma}}\sqrt{\mathbb{E}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}_\varepsilon}^2\big(\mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))\Big]\mathbb{E}^{\mathbb{P}^0}\Big[\mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}),m^N(\tilde{\mathbf{Y}}_s^{\tau})))\Big]}$$

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for all $\gamma > 0$ and $s \in [t, T]$, one can easily see that there exists $\delta > 0$, depending on $\tilde{\rho}_{\tilde{R}_\varepsilon}$ and hence on $\varepsilon$, but not on $N$ or $\tau$, such that, whenever $\|\Delta \mathbf{y}\|_2 < \delta$,

$$\mathbb{E}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^{\tau_k} \tilde{\rho}_{\tilde{R}_\varepsilon}(|\Delta X_s^{\tau,k}| + \mathcal{W}_1(m^N(\mathbf{Y}_s^{\tau}), m^N(\tilde{\mathbf{Y}}_s^{\tau})))ds + \tilde{\rho}_{\tilde{R}_\varepsilon}\big(\mathcal{W}_1(m^N(\mathbf{Y}_T^{\tau}), m^N(\tilde{\mathbf{Y}}_T^{\tau}))\big)\Big] \le \frac{\varepsilon}{2}.$$

Thus $\big|J_N(t, \mathbf{y}, \tau) - J_N(t, \tilde{\mathbf{y}}, \tau)\big| \le \varepsilon$. By the arbitrariness of $\tau$, we obtain

$$\big|v_N(t, \mathbf{y}) - v_N(t, \tilde{\mathbf{y}})\big| \le \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2, \|\tilde{\mathbf{y}}\|_2 \le R, \ \|\Delta \mathbf{y}\|_2 < \delta.$$

This implies (3.13) when $t = \tilde{t}$, that is, for a modulus of continuity function $\rho_R'$,

$$\big|v_N(t, \mathbf{y}) - v_N(t, \tilde{\mathbf{y}})\big| \le \rho_R'(\mathcal{W}_2(m^N(\mathbf{y}), m^N(\tilde{\mathbf{y}}))), \quad \text{whenever} \quad \|\mathbf{y}\|_2, \|\tilde{\mathbf{y}}\|_2 \le R. \quad (7.14)$$

**Step 3.** Fix $t < \tilde{t}$ and $\mathbf{y} \in \mathbf{S}^N$ with $\|\mathbf{y}\|_2 \le R$. For any $\tau \in \mathcal{T}_{\tilde{t},T}^N$, by (3.2) we have

$$J_N(\tilde{t}, \mathbf{y}, \tau) - v_N(t, \mathbf{y}) \le J_N(\tilde{t}, \mathbf{y}, \tau) - J_N(t, \mathbf{y}, \tau)$$

$$= J_N(\tilde{t}, \mathbf{y}, \tau) - \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^{\tilde{t}} f(r, X_r^{\tau,k}, m^N(\mathbf{Y}_r^{\tau}))I_r^{\tau,k}dr + J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)\Big]$$

$$\le J_N(\tilde{t}, \mathbf{y}, \tau) - \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)\Big] + \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\frac{C}{N}\sum_{k=1}^N \int_t^{\tilde{t}} \big(1 + |X_r^{\tau,k}| + \|\mathbf{Y}_r^{\tau}\|_1\big)dr\Big]$$

$$\le J_N(\tilde{t}, \mathbf{y}, \tau) - \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)\Big] + C(1 + R)\sqrt{\tilde{t} - t},$$

where the last inequality is due to (3.7). Again let $\tilde{R} > R$ be a large number which will be specified later. By (7.14), (3.12), and (3.7) we have

$$J_N(\tilde{t}, \mathbf{y}, \tau) - \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)\Big]$$

$$\le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\mathcal{W}_2(m^N(\mathbf{y}), m^N(\mathbf{Y}_{\tilde{t}}^{\tau}))) + \big(|J_N(\tilde{t}, \mathbf{y}, \tau)| + |J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)|\big)\mathbf{1}_{\{\|\mathbf{Y}_{\tilde{t}}^{\tau}\|_2 > \tilde{R}\}}\Big]$$

$$\le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\|\mathbf{Y}_{\tilde{t}}^{\tau} - \mathbf{y}\|_2) + \frac{C}{\tilde{R}}\big(|J_N(\tilde{t}, \mathbf{y}, \tau)|^2 + |J_N(\tilde{t}, \mathbf{Y}_{\tilde{t}}^{\tau}, \tau)|^2 + \|\mathbf{Y}_{\tilde{t}}^{\tau}\|_2^2\big)\Big]$$

$$\le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\|\mathbf{Y}_{\tilde{t}}^{\tau} - \mathbf{y}\|_2) + \frac{C}{\tilde{R}}\big(\|\mathbf{y}\|_2^2 + \|\mathbf{Y}_{\tilde{t}}^{\tau}\|_2^2\big)\Big]$$

$$\le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\|\mathbf{Y}_{\tilde{t}}^{\tau} - \mathbf{y}\|_2)\Big] + \frac{C(1 + R^2)}{\tilde{R}}.$$

Then

$$J_N(\tilde{t}, \mathbf{y}, \tau) - v_N(t, \mathbf{y}) \le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\|\mathbf{Y}_{\tilde{t}}^{\tau} - \mathbf{y}\|_2)\Big] + \frac{C(1 + R^2)}{\tilde{R}} + C[1 + R]\sqrt{\tilde{t} - t}.$$

Now for any $\varepsilon > 0$, set $\tilde{R} = \tilde{R}_\varepsilon := \frac{3C(1+R^2)}{\varepsilon}$, and $\delta > 0$ small enough. Then for $\tilde{t} - t \le \delta$,

$$J_N(\tilde{t}, \mathbf{y}, \tau) - v_N(t, \mathbf{y}) \le \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\rho_{\tilde{R}}'(\|\mathbf{Y}_{\tilde{t}}^{\tau} - \mathbf{y}\|_2)\Big] + \frac{2\varepsilon}{3}.$$

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Since $\boldsymbol{\tau} \in \mathcal{T}_{\tilde{t},T}^N$, we have $\tau_k \geq \tilde{t}$ for all $k$, then $\mathbf{I}_{\tilde{t}-}^{\boldsymbol{\tau}} = \mathbf{i}$. Thus, by (3.8),

$$\mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\|\mathbf{Y}_{\tilde{t}}^{\boldsymbol{\tau}} - \mathbf{y}\|_2\Big] = \mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\|\mathbf{X}_{\tilde{t}}^{\boldsymbol{\tau}} - \mathbf{x}\|_2\Big] \leq C(1+R)(\tilde{t}-t) \leq C(1+R)\delta.$$

Using an estimate similar to (7.13), one can then see that, for $\delta > 0$ sufficiently small, which may depend on $R$ and $\tilde{R}_\varepsilon$ and hence on $\varepsilon$, but not on $\boldsymbol{\tau}$ or $N$, we have

$$J_N(\tilde{t}, \mathbf{y}, \boldsymbol{\tau}) - v_N(t, \mathbf{y}) \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta.$$

Then, by the arbitrariness of $\boldsymbol{\tau} \in \mathcal{T}_{\tilde{t},T}^N$, we have

$$v_N(\tilde{t}, \mathbf{y}) - v_N(t, \mathbf{y}) \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta. \tag{7.15}$$

On the other hand, for any $\boldsymbol{\tau} \in \mathcal{T}_{t,T}^N$, note that $\boldsymbol{\tau} \vee \tilde{t} \in \mathcal{T}_{\tilde{t},T}^N$. Then similarly we have

$$J_N(t, \mathbf{y}, \boldsymbol{\tau} \vee \tilde{t}) - v_N(\tilde{t}, \mathbf{y}) \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta. \tag{7.16}$$

Note also that,

$$\mathcal{W}_1\big(m^N(\mathbf{X}_{\boldsymbol{\tau}}^{\boldsymbol{\tau}}), m^N(\mathbf{X}_{\boldsymbol{\tau}\vee\tilde{t}}^{\boldsymbol{\tau}})\big) \leq \|\mathbf{X}_{\boldsymbol{\tau}}^{\boldsymbol{\tau}} - \mathbf{X}_{\boldsymbol{\tau}\vee\tilde{t}}^{\boldsymbol{\tau}}\|_1 \leq C \sup_{t\leq s\leq\tilde{t}} \|\mathbf{X}_s^{\boldsymbol{\tau}} - \mathbf{x}\|_1.$$

Then, by (3.9), (3.10), and the last estimate in (7.12), we have

$$\left|J_N(t, \mathbf{y}, \boldsymbol{\tau}) - J_N(t, \mathbf{y}, \boldsymbol{\tau} \vee \tilde{t})\right|$$

$$= \left|\mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_{\tau_k}^{\tau_k\vee\tilde{t}} f(r, X_r^{\boldsymbol{\tau},k}, m^N(\mathbf{Y}_r^{\boldsymbol{\tau}}))dr + g\big(m^N(\mathbf{X}_{\boldsymbol{\tau}}^{\boldsymbol{\tau}})\big) - g\big(m^N(\mathbf{X}_{\boldsymbol{\tau}\vee\tilde{t}}^{\boldsymbol{\tau}})\big)\Big]\right|$$

$$\leq C\mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\frac{1}{N}\sum_{k=1}^N \int_t^{\tilde{t}} \big(1 + |X_r^{\boldsymbol{\tau},k}| + \|\mathbf{Y}_r^{\boldsymbol{\tau}}\|_1\big)dr + \tilde{\rho}_{\tilde{R}}(\mathcal{W}_1(m^N(\mathbf{X}_{\boldsymbol{\tau}}^{\boldsymbol{\tau}}), m^N(\mathbf{X}_{\boldsymbol{\tau}\vee\tilde{t}}^{\boldsymbol{\tau}})))\Big] + \frac{C(1+R^2)}{\tilde{R}}$$

$$\leq C\mathbb{E}_{t,\mathbf{y}}^{\mathbb{P}^0}\Big[\tilde{\rho}_{\tilde{R}}\big(C \sup_{t\leq s\leq\tilde{t}} \|\mathbf{X}_s^{\boldsymbol{\tau}} - \mathbf{x}\|_1\big)\Big] + \frac{C(1+R^2)}{\tilde{R}} + C[1+R]\sqrt{\tilde{t}-t}.$$

This, together with (7.16), implies that, for a possibly smaller $\delta$,

$$J_N(t, \mathbf{y}, \boldsymbol{\tau}) - v_N(\tilde{t}, \mathbf{y}) \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta.$$

Since $\boldsymbol{\tau} \in \mathcal{T}_{t,T}^N$ is arbitrary, we have

$$v_N(t, \mathbf{y}) - v_N(\tilde{t}, \mathbf{y}) \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta.$$

Then by (7.15) we obtain

$$\left|v_N(t, \mathbf{y}) - v_N(\tilde{t}, \mathbf{y})\right| \leq \varepsilon, \quad \text{whenever} \quad \|\mathbf{y}\|_2 \leq R, \quad \tilde{t}-t \leq \delta.$$

This, together with (7.14) and (7.8), proves (3.13). ∎

## 7.3 Construction of an optimal stopping strategy

In this subsection we come back to the general case in Subsection 3.1. We first recall (7.4), and note that, by (7.2),

$$\max_{\mathbf{i}' < \mathbf{i}} v^N(t, \mathbf{x}, \mathbf{i}') = \max_{k \in A(\mathbf{i})} v^N(t, \mathbf{x}, \mathbf{i}^{-k}) \tag{7.17}$$

We shall use the above relation to construct an optimal $\hat{\boldsymbol{\tau}} \in \mathcal{T}_{0,T}^N$ for the problem (3.2). We also refer to [51, Section 4.3] for some heuristic discussions on the optimal stopping policy on the mean field problem (2.3).

Fix $\mathbf{y} \in \mathbf{S}^N$ and assume for simplicity $t = 0$. We set

$$\hat{\boldsymbol{\tau}} := \boldsymbol{\tau}^{(|A(\mathbf{i})|)}, \tag{7.18}$$

where $\boldsymbol{\tau}^{(j)} \in \mathcal{T}_{0,T}^N$, $j = 0, \cdots, |A(\mathbf{i})|$, are constructed as follows.

**Step 0 (Initialization).** Set $\tau_k^{(0)} := T\mathbf{1}_{A(\mathbf{i})}(k)$. That is, $\mathbf{Y}^{\boldsymbol{\tau}^{(0)}} = \bar{\mathbf{Y}}$ is the unstopped process. Moreover, set $A_0 := A(\mathbf{i})$ and $\tau_0^* := 0$.

We then stop the particles in $A(\mathbf{i})$ recursively: for $j = 0, \cdots, |A(\mathbf{i})| - 1$:

**Step 1.** The next particle is stopped at: denoting $\mathbf{I}^A := (1_A(1), \cdots, 1_A(N))$ for $A \subset [N]$,

$$\tau_{j+1}^* := \inf \left\{ s \geq \tau_j^* : v^N(s, \mathbf{Y}_s^{\boldsymbol{\tau}^{(j)}}) = \max_{\mathbf{i}' < \mathbf{I}^{A_j}} v^N(s, \mathbf{X}_s^{\boldsymbol{\tau}^{(j)}}, \mathbf{i}') \right\}. \tag{7.19}$$

We remark that it is possible that $\tau_{j+1}^* = \tau_j^*$, namely two (or more) particles are stopped at the same time.

**Step 2.** By (7.17), we may pick an index for the particle stopped at $\tau_{j+1}^*$:

$$\kappa_{j+1}^* := \min \left\{ k \in A_j : v^N(s, \mathbf{X}_s^{\boldsymbol{\tau}^{(j)}}, (\mathbf{I}^{A_j})^{-k}) = v^N(s, \mathbf{Y}_s^{\boldsymbol{\tau}^{(j)}}) \right\}, \quad A_{j+1} := A_j \backslash \{\kappa_{j+1}^*\}, \tag{7.20}$$

where $\mathbf{I}^{-k}$ is defined in (7.3), and $A_{j+1} \in \mathcal{F}_{\tau_{j+1}^*}^N$ is random. Note that there might be multiple $k$ satisfying the above requirement and serving for our purpose. We choose the smallest one just for convenience.

**Step 3.** We then update the vector of the stopping times:

$$\tau_{\kappa_{j+1}^*}^{(j+1)} := \tau_{j+1}^*, \quad \text{and} \quad \tau_k^{(j+1)} := \tau_k^{(j)}, \text{ for all } k \neq \kappa_{j+1}^*.$$

**Theorem 7.2** *Under Assumption 3.1, the $\hat{\boldsymbol{\tau}}$ constructed in (7.18) is an optimal stopping policy for the problem (3.2).*

**Proof** We prove by induction on $|A(\mathbf{i})|$. When $|A(\mathbf{i})| = 0$, all particles are already stopped. Then it is trivial that $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau}^{(0)} = (0, \cdots, 0)$ is optimal. Assume the result is true for all $\mathbf{i}'$ with $|A(\mathbf{i}')| \leq n$, and now consider $\mathbf{i}$ such that $|A(\mathbf{i})| = n + 1$.

Recall (7.1) and note that $\mathbf{Y}^{\boldsymbol{\tau}^{(0)}} = \bar{\mathbf{Y}}$, $\mathbf{I}^{A_0} = \mathbf{i}$. Then it follows from the standard optimal stopping theory that $\tau_1^*$ constructed in (7.19) is an optimal stopping strategy for (7.1) with $t = 0$. That is,

$$
\begin{aligned}
v^N(0, \mathbf{y}) &= \mathbb{E}_{0, \mathbf{y}}\Big[\frac{1}{N}\sum_{k=1}^N \int_0^{\tau_1^*} f^k(r, \bar{\mathbf{X}}_r)\bar{I}_r^k dr + \max_{\mathbf{i}' < \mathbf{i}} v^N(\tau_1^*, \bar{\mathbf{X}}_{\tau_1^*}, \mathbf{i}')\Big] \\
&= \mathbb{E}_{0, \mathbf{y}}\Big[\frac{1}{N}\sum_{k \in A_0} \int_0^{\tau_1^*} f^k(r, \mathbf{X}_r^{\boldsymbol{\tau}^{(0)}}) dr + v^N(\tau_1^*, \mathbf{X}_{\tau_1^*}^{\boldsymbol{\tau}^{(0)}}, \mathbf{I}^{A_1})\Big],
\end{aligned}
$$

where the last equality follows from (7.20). Our construction also induces that

$$
\mathbf{X}_r^{\hat{\boldsymbol{\tau}}} = \mathbf{X}_r^{\boldsymbol{\tau}^{(j)}}, \quad \tau_j^* \leq r \leq \tau_{j+1}^*, \quad \text{and} \quad \mathbf{I}_r^{\hat{\boldsymbol{\tau}}} = \mathbf{I}^{A_j}, \quad \tau_j^* \leq r < \tau_{j+1}^*.
$$

Then

$$
v^N(0, \mathbf{y}) = \mathbb{E}_{0, \mathbf{y}}\Big[\frac{1}{N}\sum_{k=1}^N \int_0^{\tau_1^*} f^k(r, \mathbf{X}_r^{\hat{\boldsymbol{\tau}}})I_r^{\hat{\boldsymbol{\tau}}, k} dr + v^N(\tau_1^*, \mathbf{X}_{\tau_1^*}^{\hat{\boldsymbol{\tau}}}, \mathbf{I}^{A_1})\Big].
$$

Since $|A_1| = n$, by induction assumption we have

$$
v^N(\tau_1^*, \mathbf{X}_{\tau_1^*}^{\hat{\boldsymbol{\tau}}}, \mathbf{I}^{A_1}) = \mathbb{E}_{\tau_1^*, \mathbf{X}_{\tau_1^*}^{\hat{\boldsymbol{\tau}}}, \mathbf{I}^{A_1}}\Big[\frac{1}{N}\sum_{k=1}^N \int_{\tau_1^*}^T f^k(r, \mathbf{X}_r^{\hat{\boldsymbol{\tau}}})I_r^{\hat{\boldsymbol{\tau}}, k} dr + g_N(\mathbf{X}_T^{\hat{\boldsymbol{\tau}}})\Big],
$$

thus proving the optimality of $\hat{\boldsymbol{\tau}}$:

$$
v^N(0, \mathbf{y}) = \mathbb{E}_{0, \mathbf{y}}\Big[\frac{1}{N}\sum_{k=1}^N \int_0^T f^k(r, \mathbf{X}_r^{\hat{\boldsymbol{\tau}}})I_r^{\hat{\boldsymbol{\tau}}, k} dr + g_N(\mathbf{X}_T^{\hat{\boldsymbol{\tau}}})\Big]. \qquad \blacksquare
$$

# 8  Appendix

In this appendix, we present a few technical proofs.

**Proof of Lemma 2.2.** Let $m' \preceq m$ with corresponding $p(x)$ as in (2.7). For any $n \geq 1$, let $\{O_k^n\}_{k \geq 1}$ be a partition of $\mathbb{R}^d$ such that, for each $k \geq 1$, $|x - \tilde{x}| \leq \frac{1}{n}$ for all $x, \tilde{x} \in O_k^n$. By the required continuity of $m(dx, 1)$, one may easily construct $A_k^n \subset O_k^n$ such that

$$
m(A_k^n, 1) = \int_{O_k^n} p(x) m(dx, 1).
$$

Set $A^n := \cup_{k \geq 1} A_k^n$ and fix an $x_k^n \in A_k^n$. Then, for any $\varphi : \mathbf{S} \to \mathbb{R}$ which is Lipschitz continuous with Lipschitz constant 1, we have: denoting $\Delta \varphi(x) := \varphi(x, 1) - \varphi(x, 0)$,

$$
\int_{\mathbf{S}} \varphi(\mathbf{y}) m^{A^n}(dy) - \int_{\mathbf{S}} \varphi(\mathbf{y}) m'(dy)
$$

$$
= \int_{A^n} \varphi(x, 1) m(dx, 1) + \int_{(A^n)^c} \varphi(x, 0) m(dx, 1) + \int_{\mathbb{R}^d} \varphi(x, 0) m(dx, 0)
$$

$$
- \int_{\mathbb{R}^d} \varphi(x, 1) p(x) m(dx, 1) - \int_{\mathbb{R}^d} \Big( \varphi(x, 0)[1 - p(x)] m(dx, 1) + \varphi(x, 0) m(dx, 0) \Big)
$$

$$
= \int_{A^n} \Delta \varphi(x) m(dx, 1) - \int_{\mathbb{R}^d} \Delta \varphi(x) p(x) m(dx, 1)
$$

$$
= \sum_{k \geq 1} \Big( \int_{A_k^n} \Delta \varphi(x) m(dx, 1) - \int_{O_k^n} \Delta \varphi(x) p(x) m(dx, 1) \Big)
$$

$$
= \sum_{k \geq 1} \Big( \int_{A_k^n} (\Delta \varphi(x) - \Delta \varphi(x_k^n)) m(dx, 1) - \int_{O_k^n} (\Delta \varphi(x) - \Delta \varphi(x_k^n)) p(x) m(dx, 1) \Big)
$$

$$
\leq \frac{C}{n} \sum_{k \geq 1} \Big( \int_{A_k^n} m(dx, 1) + \int_{O_k^n} p(x) m(dx, 1) \Big) \leq \frac{C}{n}.
$$

This implies that $\mathcal{W}_1(m^{A^n}, m') \leq \frac{C}{n}$, and thus $m^{A^n} \to m'$ weakly as $n \to \infty$. Moreover, it is obvious that $\{m^{A^n}\}_{n \geq 1}$ are uniformly square integrable, then it follows from [11, Theorem 5.5] that $\lim_{n \to \infty} \mathcal{W}_2(m^{A^n}, m') = 0$. ∎

**Proof of Lemma 3.8.** First, let $h \in \mathbb{R}^d$ and denote $\mathbf{y} +_k h \in \mathbf{S}^N$ as the perturbation of $\mathbf{y}$ by replacing $x_k$ with $x_k + h$. Set $m_h^{l,k} := l m^N(\mathbf{y} +_k h) + (1 - l) m^N(\mathbf{y})$. Since $\varphi \in C_2^{1,2}(\mathbf{Q}_0)$ and $i_k = 1$, we have

$$
\begin{aligned}
\phi(s, \mathbf{y} +_k h) - \phi(s, \mathbf{y}) &= \varphi(s, m^N(\mathbf{y} +_k h)) - \varphi(s, m^N(\mathbf{y})) \\
&= \int_0^1 \int_{\mathbf{S}} \delta_m \varphi(s, m_h^{l,k}, y) d(m^N(\mathbf{y} +_k h) - m^N(\mathbf{y}))(y) dl \\
&= \frac{1}{N} \int_0^1 \Big[ \delta_m \varphi(s, m_h^{l,k}, (x_k + h, 1)) - \delta_m \varphi(s, m_h^{l,k}, (x_k, 1)) \Big] dl \\
&= \frac{1}{N} \int_0^1 \int_0^1 h \cdot \partial_x \delta_m \varphi_1(s, m_h^{l,k}, x_k + \tilde{l} h) d\tilde{l} dl.
\end{aligned}
$$

Note that $x_k + \tilde{l} h \to x_k$ and $m_h^{l,k} \xrightarrow[h \to 0]{} m^N(\mathbf{y})$, uniformly in $\tilde{l}, l \in [0, 1]$, as $|h| \to 0$. Then

$$
\phi(s, \mathbf{y} +_k h) - \phi(s, \mathbf{y}) = h \cdot \frac{1}{N} \partial_x \delta_m \varphi_1(s, m^N(\mathbf{y}), x_k) + o(|h|).
$$

This implies immediately the first equality in (3.14).

Similarly, one can show that

$$
\partial_x \delta_m \{ \varphi_1(s, m^N(\mathbf{y} +_k h), x_k) - \varphi_1(s, m^N(\mathbf{y}), x_k) \} = \frac{1}{N} \partial_{\mu\mu}^2 \varphi_{1,1}(s, m^N(\mathbf{y}), x_k, x_k) h + o(|h|).
$$

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Then

$$\partial_{x_k}\big\{\phi(s,\mathbf{y}+_k h)-\phi(s,\mathbf{y})\big\} = \frac{1}{N}\partial_x\delta_m\big\{\varphi_1(s,m^N(\mathbf{y}+_k h),x_k+h)-\varphi_1(s,m^N(\mathbf{y}),x_k)\big\}$$

$$= \frac{1}{N}\partial^2_{xx}\delta_m\varphi_1(s,m^N(\mathbf{y}),x_k)h + \frac{1}{N^2}\partial^2_{\mu\mu}\varphi_{1,1}(s,m^N(\mathbf{y}),x_k,x_k)h + o(|h|).$$

This implies the second equality in (3.14). ■

# References

[1] Guy Barles and Panagiotis E Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic analysis*, 4(3):271–283, 1991.

[2] Erhan Bayraktar, Alekos Cecchin, Asaf Cohen, and François Delarue. Finite state mean field games with Wright–Fisher common noise as limits of N-player weighted games. *Mathematics of Operations Research*, 2022.

[3] Erhan Bayraktar and Prakash Chakraborty. Mean field control and finite dimensional approximation for regime-switching jump diffusions. *ArXiv:2109.09134*, 2021.

[4] Erhan Bayraktar and Asaf Cohen. Analysis of a finite state many player game using its master equation. *SIAM Journal on Control and Optimization*, 56(5):3538–3568, 2018.

[5] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[6] Pierre Cardaliaguet. The convergence problem in mean field games with local coupling. *Applied Mathematics & Optimization*, 76(1):177–215, 2017.

[7] Pierre Cardaliaguet, Samuel Daudin, Joe Jackson, and Panagiotis Souganidis. An algebraic convergence rate for the optimal control of Mckean-Vlasov dynamics. *ArXiv:2203.14554*, 2022.

[8] Pierre Cardaliaguet, François Delarue, Jean-Michel Lasry, and Pierre-Louis Lions. *The Master Equation and the Convergence Problem in Mean Field Games:(AMS-201)*, volume 201. Princeton University Press, 2019.

[9] Pierre Cardaliaguet and Panagiotis Souganidis. Regularity of the value function and quantitative propagation of chaos for mean field control problems. *ArXiv:2204.01314*, 2022.

[10] René Carmona and François Delarue. Probabilistic analysis of mean-field games. *SIAM Journal on Control and Optimization*, 51(4):2705–2734, 2013.

[11] René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, 2018.

[12] Giulia Cavagnari, Stefano Lisini, Carlo Orrieri, and Giuseppe Savaré. Lagrangian, Eulerian and Kantorovich formulations of multi-agent optimal control problems: Equivalence and gamma-convergence. *Journal of Differential Equations*, 322:268–364, 2022.

[13] Alekos Cecchin. Finite state N-agent and mean field control problems. *ESAIM: Control, Optimisation and Calculus of Variations*, 27:31, 2021.

[14] Alekos Cecchin and Guglielmo Pelino. Convergence, fluctuations and large deviations for finite state mean field games via the master equation. *Stochastic Processes and their Applications*, 129(11):4510–4555, 2019.

[15] Alekos Cecchin, Paolo Dai Pra, Markus Fischer, and Guglielmo Pelino. On the convergence problem in mean field games: a two state model without uniqueness. *SIAM Journal on Control and Optimization*, 57(4):2443–2466, 2019.

[16] François Delarue, Daniel Lacker, and Kavita Ramanan. From the master equation to mean field game limit theory: a central limit theorem. *Electronic Journal of Probability*, 24:1–54, 2019.

[17] François Delarue, Daniel Lacker, and Kavita Ramanan. From the master equation to mean field game limit theory: Large deviations and concentration of measure. *The Annals of Probability*, 48(1):211–263, 2020.

[18] Mao Fabrice Djete. Mean field games of controls: on the convergence of Nash equilibria. *ArXiv:2006.12993*, 2020.

[19] Mao Fabrice Djete. Large population games with interactions through controls and common noise: convergence results and equivalence between open-loop and closed-loop controls. *ArXiv:2108.02992*, 2021.

[20] Mao Fabrice Djete, Dylan Possamaï, and Xiaolu Tan. McKean–Vlasov optimal control: limit theory and equivalence between different formulations. *Mathematics of Operations Research*, 2022.

[21] Josu Doncel, Nicolas Gast, and Bruno Gaujal. Discrete mean field games: Existence of equilibria and convergence. *ArXiv:1909.01209*, 2019.

[22] Ibrahim Ekren. Viscosity solutions of obstacle problems for fully nonlinear path-dependent PDEs. *Stochastic Processes and their Applications*, 127(12):3966–3996, 2017.

[23] Ibrahim Ekren, Christian Keller, Nizar Touzi, Jianfeng Zhang, et al. On viscosity solutions of path dependent PDEs. *Annals of Probability*, 42(1):204–236, 2014.

[24] Ibrahim Ekren, Nizar Touzi, Jianfeng Zhang, et al. Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I. *Annals of Probability*, 44(2):1212–1253, 2016.

[25] Ibrahim Ekren, Nizar Touzi, Jianfeng Zhang, et al. Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II. *Annals of Probability*, 44(4):2507–2553, 2016.

[26] Ermal Feleqi. The derivation of ergodic mean field game equations for several populations of players. *Dynamic Games and Applications*, 3(4):523–536, 2013.

[27] Markus Fischer. On the connection between symmetric $n$-player games and mean field games. *The Annals of Applied Probability*, 27(2):757–810, 2017.

[28] Markus Fischer and Giulia Livier. Continuous time mean-variance portfolio optimization through the mean field approach. *ESAIM: Probability & Statistics*, 20, 2016.

[29] Markus Fischer and Francisco J Silva. On the asymptotic nature of first order mean field games. *Applied Mathematics & Optimization*, 84(2):2327–2357, 2021.

[30] Massimo Fornasier, Stefano Lisini, Carlo Orrieri, and Giuseppe Savaré. Mean-field optimal control as gamma-limit of finite agent controls. *European Journal of Applied Mathematics*, 30(6):1153–1186, 2019.

[31] Wilfrid Gangbo, Sergio Mayorga, and Andrzej Swiech. Finite dimensional approximations of Hamilton–Jacobi–Bellman equations in spaces of probability measures. *SIAM Journal on Mathematical Analysis*, 53(2):1320–1356, 2021.

[32] Wilfrid Gangbo and Alpár R Mészáros. Global well-posedness of master equations for deterministic displacement convex potential mean field games. *ArXiv:2004.01660*, 2020.

[33] Maximilien Germain, Huyên Pham, and Xavier Warin. Rate of convergence for particles approximation of PDEs in Wasserstein space. *ArXiv:2103.00837*, 2021.

[34] Minyi Huang, Roland P Malhamé, and Peter Caines. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[35] Melih Iseri and Jianfeng Zhang. Set values for mean field games. *ArXiv:2107.01661*, 2021.

[36] Magdalena Kobylanski, Marie-Claire Quenez, and Elisabeth Rouy-Mironescu. Optimal multiple stopping time problem. *The Annals of Applied Probability*, 21(4):1365–1399, 2011.

[37] Daniel Lacker. Mean field games via controlled martingale problems: existence of Markovian equilibria. *Stochastic Processes and their Applications*, 125(7):2856–2894, 2015.

[38] Daniel Lacker. A general characterization of the mean field limit for stochastic differential games. *Probability Theory and Related Fields*, 165(3):581–648, 2016.

[39] Daniel Lacker. Limit theory for controlled McKean–Vlasov dynamics. *SIAM Journal on Control and Optimization*, 55(3):1641–1672, 2017.

[40] Daniel Lacker. On the convergence of closed-loop Nash equilibria to the mean field game limit. *The Annals of Applied Probability*, 30(4):1693–1761, 2020.

[41] Daniel Lacker and Luc Le Flem. Closed-loop convergence for mean field games with common noise. *ArXiv:2107.03273*, 2021.

[42] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[43] Mathieu Lauriere and Ludovic Tangpi. Convergence of large population games to mean field games with interaction through the controls. *SIAM Journal on Mathematical Analysis*, 54(3):3535–3574, 2022.

[44] Chenchen Mou and Jianfeng Zhang. Wellposedness of second order master equations for mean field games with nonsmooth data. *Memoirs of the AMS, accepted*, 2020.

[45] Marcel Nutz, Jaime San Martin, and Xiaowei Tan. Convergence to the mean field game limit: a case study. *The Annals of Applied Probability*, 30(1):259–286, 2020.

[46] Karl Oelschlager. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *The Annals of Probability*, pages 458–479, 1984.

[47] Dylan Possamaï and Ludovic Tangpi. Non-asymptotic convergence rates for mean-field games: weak formulation and McKean–Vlasov BSDEs. *ArXiv:2105.00484*, 2021.

[48] Zhenjie Ren, Nizar Touzi, and Jianfeng Zhang. An overview of viscosity solutions of path-dependent PDEs. *Stochastic Analysis and Applications 2014*, pages 397–453, 2014.

[49] Alain-Sol Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX—1989*, pages 165–251. Springer, 1991.

[50] Mehdi Talbi. A finite-dimensional approximation for viscosity solutions of partial differential equations on Wasserstein space. *Preprint*, 2022.

[51] Mehdi Talbi, Nizar Touzi, and Jianfeng Zhang. Dynamic programming equation for the mean field optimal stopping problem. *ArXiv:2103.05736*, 2021.

[52] Mehdi Talbi, Nizar Touzi, and Jianfeng Zhang. Viscosity solutions for obstacle problems on Wasserstein space. *ArXiv:2203.17162*, 2022.

[53] Cédric Villani. *Topics in optimal transportation*, volume 58. American Mathematical Soc., 2021.