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COMPARISON OF VISCOSITY SOLUTIONS OF SEMILINEAR PATH-DEPENDENT PDEs*

ZHENJIE REN†, NIZAR TOUZI‡, AND JIANFENG ZHANG§

Abstract. This paper provides a probabilistic proof of the comparison result for viscosity solutions of path-dependent semilinear PDEs. We consider the notion of viscosity solutions introduced in [I. Ekren, et al., *Ann. Probab.*, 42 (2014), pp. 204–236], which considers as test functions all those smooth processes which are tangent in mean. When restricted to the Markovian case, this definition induces a larger set of test functions and reduces to the notion of stochastic viscosity solutions analyzed in [E. Bayraktar and M. Sirbu, *Proc. Amer. Math. Soc.*, 140 (2012), pp. 3645–3654; *SIAM J. Control Optim.*, 51 (2013), pp. 4274–4294]. Our main result takes advantage of this enlargement of the test functions and provides an easier proof of comparison. This is most remarkable in the context of the linear path-dependent heat equation. As a key ingredient for our methodology, we introduce a notion of punctual differentiation, similar to the corresponding concept in the standard viscosity solutions [L. A. Caffarelli and X. Cabre, *Amer. Math. Soc. Colloq. Publ.*, 43, AMS, Providence, RI, 1995], and we prove that semimartingales are almost everywhere punctually differentiable. This smoothness result can be viewed as the counterpart of the Aleksandroff smoothness result for convex functions. A similar comparison result was established earlier in [I. Ekren et al., *Ann. Probab.*, 42 (2014), pp. 204–236]. The result of this paper is more general and, more importantly, the arguments that we develop do not rely on any representation of the solution.

Key words. viscosity solutions, optimal stopping, stochastic control, path-dependent PDEs

AMS subject classifications. 35D40, 35K10, 60H10, 60H30

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1. Introduction.

This paper provides a purely probabilistic wellposedness result for the semilinear path-dependent partial differential equation:

$$(1.1)\quad -\partial_t u - \frac{1}{2}\mathrm{Tr}\big[\sigma_t(\omega)\sigma_t^{\mathrm{T}}(\omega)\partial^2_{\omega\omega}u\big] - F_t\big(\omega, u_t(\omega), \sigma_t^T(\omega)\partial_\omega u_t(\omega)\big) = 0 \quad \text{on} \quad [0,T),$$

where $T > 0$ is a given terminal time, $\omega \in \Omega$ is a continuous path from $[0,T]$ to $\mathbb{R}^d$ starting from the origin, the diffusion coefficient σ is a mapping from $[0,T]\times\Omega$ to $\mathbb{R}^{d\times d}$ with σ^T denoting its transpose, and the nonlinearity F is a mapping from $(t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$ to $\mathbb{R}$. The unknown process $\{u_t(\omega), t\in[0,T]\}$ is required to be continuous in (t,ω). In the smooth case, the derivatives $\partial_t u$, $\partial_\omega u$, and $\partial^2_{\omega\omega}u$ are defined in accordance with the functional Itô formula introduced by Dupire [10] and further developed by Cont and Fournié [5, 6] (see also [23, 26, 32, 46]). We refer to Peng and Wang [37] for the study of smooth solutions of semilinear PPDEs. However, as is shown in the previous literature [12, 14, 15], such a smoothness requirement is rather exceptional, even in the case of the path-dependent heat equation, that is, $\sigma = I_d$ and $F \equiv 0$.

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†Université Paris-Dauphine, PSL Research University, CNRS, UMR [7534], Ceremade, 75016 Paris, France (ren@ceremade.dauphine.fr).

‡CMAP, Ecole Polytechnique Paris, Palaiseau, 91128, France (nizar.touzi@polytechinque.edu).

§University of Southern California, Department of Mathematics, Los Angeles, CA, 90089-2532 (jianfenz@usc.edu).

Our objective is to continue the development of the theory of viscosity solutions in this context. Viscosity solutions in finite-dimensional spaces, which are locally compact, were introduced by Crandall and Lions [8]; we refer to [9] and [19] for an overview. Extensions to infinite-dimensional spaces with special structure have also been established by Lions [27, 28, 29] and Swiech [48]. However, these extensions are not suitable for our purpose due to the following two reasons. First, the path space Ω is a Banach space when endowed with the $\mathbb{L}^\infty-$norm, and not a Hilbert space as assumed in the above literature. Second, the adaptedness requirement on the functions $u(t,\omega)$ is a special feature of our problem that is not addressed in the infinite-dimensional PDE literature.

Nonlinear path-dependent PDEs appear in various applications as the stochastic control of non-Markovian systems [14] and the corresponding stochastic differential games [38]. They are also intimately related to the backward stochastic differential equations introduced by Pardoux and Peng [33] and their extension to the second order in [4, 47]. Loosely speaking, backward SDEs can be viewed as Sobolev solutions of path-dependent PDEs, and our goal is to develop the alternative notion of viscosity solutions which is well-known to provide a suitable wellposedness and stability theory in the Markovian case $u(t,\omega) = u(t,\omega_t)$. We also refer to the recent applications in [20] to establish a representation of the solution of a class of equations (1.1) in terms of branching diffusions, and to [31] for the small time large deviation results of path-dependent diffusions.

The notion of viscosity solutions studied in this paper, as introduced in [12, 14, 15], consider smooth test processes which are tangent in mean, with respect to an appropriate class of probability measures, to the process of interest. This is in contrast with the Crandall and Lions [8] standard notion of viscosity solutions in finite-dimensional spaces where the test functions are tangent in the pointwise sense. In particular, when restricted to the Markovian case, our notion of viscosity solutions allows for a larger set of test functions, and in the case of the heat equation (or more general linear equation) case it reduces to the notion of stochastic viscosity solutions analyzed by Bayraktar and Sirbu [1, 2]. Consequently, the uniqueness may be easier with our notion, while existence is more restricted and may become harder. However, it was proved in the previous papers [12, 14, 15] that existence still holds true under this notion of viscosity solutions for a large class of equations. In particular, in the present semilinear case, the solution of backward SDEs provides a natural probabilistic representation for viscosity solution of path-dependent PDEs and thus extends the nonlinear Feynman–Kac formula of Pardoux and Peng [34] to the path-dependent case.

The main contribution of this paper is to provide a probabilistic proof of the comparison result for the path-dependent equation (1.1) which, in contrast with [12], does not rely on any representation of the solution. We also observe that the present comparison result is stronger than that of [12] as it holds in a larger class of processes and for a random and possibly degenerate diffusion coefficient σ (in [12], only $\sigma = I_d$ is considered). Our proof bypasses completely the delicate and deep Crandall and Ishii lemma (see Lemma 3.2 in [9]). In particular, our proof of comparison result for the path-dependent heat equation is elementary and does not require any penalization to address. (The standard comparison result for second order PDEs applies to a bounded domain, and the extension to an unbounded domain involves a penalization using the growth conditions.) In particular, the wellposedness of the path-dependent heat equation is a direct consequence of the equivalence between the viscosity subsolution and the submartingale properties.

Our arguments are inspired by the work of Caffarelli and Cabre [3]. By adapting the notion of punctual differentiation to our path-dependent framework, we prove an important smoothness result. Namely, we show that semimartingales are punctually differentiable Leb$\otimes\mathbb{P}$-a.e. This result can be viewed as the analogue of the Aleksandroff regularity result for convex functions. In the present semilinear case, an important property of our notion of viscosity solutions is that viscosity subsolutions (resp., supersolutions) are submartingales (resp., supermartingales) up to the addition of some absolutely continuous process. In particular, viscosity subsolutions and supersolutions are punctually differentiable Leb$\otimes\mathbb{P}$-a.e.

We shall remark that, while the framework of fully nonlinear path-dependent PDEs in [14, 15] covers the random coefficient σ here, their comparison result excludes this case due to their heavy reliance on the locally uniform smooth approximation of the viscosity solution. The definition of viscosity solutions here is slightly different from that in [14, 15] by considering even more test functions. This enlargement of test function class allows us to establish the punctual differentiation of viscosity solutions, which does not require the smooth approximation to be locally uniform. On a different perspective, the class of probability measures used to determine the test functions is nondominated in [14, 15], and consequently the corresponding convergence theorem requires very strong regularity of the involved processes. It is still unclear how to obtain the punctual differentiation of viscosity solutions, even for the present semilinear PPDE (1.1), if we use the nondominated class of probability measures as in [14, 15].

We note that the theory of PPDEs has received very strong attention in recent years. Lukoyanov [30] provided a well-posedness result for a viscosity solution of a first order path-dependent Hamilton–Jacobi equation, by using compactness arguments. Peng and Song [36] proposed a notion of a Sobolev solution of the path-dependent HJB equation as a reinterpretation of their G-BSDE [21], a similar notion to the second order BSDE of [47]. Cosso and Russo [7] introduced a notion of a strong viscosity solution as a limit of some appropriate classical solution in their sense. In a recent work, Ekren and Zhang [16] studied degenerate fully nonlinear PPDEs by restricting to solutions with appropriate piecewise Markovian structure. The notion of a viscosity solution studied in this paper was successfully developed in Ekren [11] for obstacle fully nonlinear PPDEs, in Keller [25] for semilinear nonlocal PPDEs, in Ren [39] for elliptic PPDEs, in Ren [40] for the existence by Perron's method, and in Zhang and Zhuo [49] and Ren and Tan [42] for the convergence of monotone approximation schemes for PPDEs.

We also emphasize that some results and arguments of this paper have been included in our survey paper [43], which was written subsequently to the first version of this paper. So the priority must go to the present paper. Moreover, in our very recent work [44], we have developed a new comparison result for degenerate fully nonlinear PPDEs with a uniform continuity condition on the nonlinearity (which is not needed in the present paper). The last result is obtained by a completely different method based on convenient approximation of the viscosity semisolutions. In particular, [44] does not make use of the notion of punctual differentiability which, as stated above, is still not understood in the nondominated framework.

Finally, the main wellposedness result of this paper states that the unique viscosity solution coincides with the unique solution of the corresponding backward SDE, which can be viewed as a Sobolev solution of the PPDE. This may seem to represent a strong argument against our notion of viscosity solutions. However, in the fully nonlinear case, we believe that viscosity solutions have more potential than Sobolev solutions, similar to the well-known situation in the PDE setting. *We believe that the* *results of this paper represent important steps toward a good understanding of viscosity solutions and the extension to the fully nonlinear case. In particular the general punctual differentiability result of this paper has a theoretical interest on its own.*

The rest of the paper is organized as follows. Section 2 introduces the set up of the problem, in particular the class of probability measures we will use. The notion of viscosity solution is defined in section 3. In particular, similar to the Crandall and Lions [8] standard notion of viscosity solutions, we show that our notion for path-dependent PDEs can be formulated equivalently in terms of the corresponding semijets. Section 4 is devoted to the main result of the paper: the comparison result of viscosity solutions of semilinear path-dependent PDEs. Then, section 5 proves briefly the existence of viscosity solutions by using the wellposedness of corresponding BSDEs. Finally section 6 completes the technical proofs.

2. Preliminaries. Throughout this paper let $T > 0$ be a given finite maturity, $\Omega := \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ be the set of continuous paths starting from the origin, and $\Theta := [0,T] \times \Omega$. We denote by B the canonical process on Ω, $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ the canonical filtration, $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times taking values in $[0,T]$, and $\mathbb{P}_0$ the Wiener measure on Ω. Moreover, let $\mathcal{T}^+$ denote the subset of $\tau \in \mathcal{T}$ taking values in $(0,T]$, and for $\text{H} \in \mathcal{T}$, let $\mathcal{T}_{\text{H}}$ and $\mathcal{T}_{\text{H}}^+$ be the subset of $\tau \in \mathcal{T}$ taking values in $[0,\text{H}]$ and $(0,\text{H}]$, respectively.

Following Dupire [10], we introduce the following pseudodistance on Θ:

$$\begin{aligned}\|\omega\|_t := \sup_{0\le s\le t} |\omega_s|, \quad & d\big((t,\omega),(t',\omega')\big)\\ & := |t-t'| + \|\omega_{t\wedge} - \omega'_{t'\wedge}\|_T \quad \text{for all } \ (t,\omega),(t',\omega') \in \Theta.\end{aligned}$$

We say a process valued in some metric space E is in $C^0(\Theta,E)$ whenever it is continuous with respect to d. Similarly, $\mathbb{L}^0(\mathcal{F}_t,E)$ and $\mathbb{L}^0(\mathbb{F},E)$ denote the set of $\mathcal{F}_t$-measurable random variables and $\mathbb{F}$-progressively measurable processes, respectively. We remark that $C^0(\Theta,E) \subset \mathbb{L}^0(\mathbb{F},E)$, and when $E = \mathbb{R}$, we shall omit it in this notation.

For any $A \in \mathcal{F}_T$, $\xi \in \mathbb{L}^0(\mathcal{F}_T,E)$, $X \in \mathbb{L}^0(\mathbb{F},E)$, and $(t,\omega) \in [0,T]\times\Omega$, define

$$\begin{aligned}&A^{t,\omega} := \{\omega' \in \Omega : \omega \otimes_t \omega' \in A\}, \quad \xi^{t,\omega}(\omega') := \xi(\omega\otimes_t\omega'), \quad X^{t,\omega}_s(\omega') := X_{t+s}(\omega\otimes_t\omega')\\ &\text{for all } \ s\in[0,T-t], \omega'\in\Omega, \ \text{ where } \ (\omega\otimes_t\omega')_r := \omega_r \mathbf{1}_{[0,t]}(r) + (\omega_t + \omega'_{r-t})\mathbf{1}_{(t,T]}(r),\\ &\quad 0 \le r \le T.\end{aligned}$$

Following the standard arguments of monotone class, we have the following simple results.

LEMMA 2.1. *Let $0 \le t \le s \le T$ and $\omega \in \Omega$. Then $A^{t,\omega} \in \mathcal{F}_{s-t}$ for all $A \in \mathcal{F}_s$, $\xi^{t,\omega} \in \mathbb{L}^0(\mathcal{F}_{s-t},E)$ for all $\xi \in \mathbb{L}^0(\mathcal{F}_s,E)$, $X^{t,\omega} \in \mathbb{L}^0(\mathbb{F},E)$ for all $X \in \mathbb{L}^0(\mathbb{F},E)$, and $\tau^{t,\omega} - t \in \mathcal{T}_{s-t}$ for all $\tau \in \mathcal{T}_s$.*

To study the semilinear PPDE (1.1), we need to introduce the diffusion coefficient σ. Throughout the paper, the following assumption will always be in force.

Assumption 2.2. The diffusion coefficient $\sigma : (t,\omega) \in \Theta \to \sigma_t(\omega) \in \mathbb{R}^{d\times d}$ is continuous in t, and Lipschitz continuous in ω uniformly in t, i.e.,

$$|\sigma_t(\omega) - \sigma_t(\omega')| \le C\|\omega-\omega'\|_t \quad \text{for all } \ t\in[0,T], \ \omega,\omega'\in\Omega, \ \text{ for some } \ C \ge 0.$$

Remark 2.3. Assumption 2.2 implies that σ is continuous in (t,ω), and thus $\mathbb{F}$-adapted. Also, we allow the parabolic PPDE (1.1) to be degenerate.

Our paper builds on the following result.

LEMMA 2.4. *For any bounded $\lambda \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d)$, the following SDE has a unique weak solution:*

$$dX_t = \sigma(t, X_\cdot)\big[dW_t + \lambda_t(X_\cdot)dt\big], \quad X_0 = 0, \tag{2.1}$$

where W is a Brownian motion. In the special case that $\lambda = 0$, the SDE has a unique strong solution. The solution will be denoted as $\mathbb{P}_{\sigma,\lambda}$, and $\mathbb{P}_\sigma := \mathbb{P}_{\sigma,0}$ when $\lambda = 0$.

Proof. We first construct the solution by using the Girsanov transformation. First, thanks to the Lipschitz continuity of σ, let X^σ be the unique (strong) solution of the SDE

$$X^\sigma_\cdot = \int_0^\cdot \sigma_s(X^\sigma_\cdot)dB_s \ \ \mathbb{P}_0\text{-a.s. and denote} \tag{2.2}$$

$$B^\lambda := B - \int_0^\cdot \lambda_t(X^\sigma_\cdot)dt, \quad \mathbb{P}_{\sigma,\lambda} := \mathbb{P}_\lambda \circ (X^\sigma)^{-1},$$

where $\frac{d\mathbb{P}_\lambda}{d\mathbb{P}_0} := e^{\int_0^T \lambda_t(X^\sigma_\cdot)\cdot dB_t - \frac{1}{2}\int_0^T |\lambda_t(X^\sigma_\cdot)|^2 dt}$. Then $(B^\lambda, X^\sigma, \mathbb{P}_{\sigma,\lambda})$ is a weak solution of (2.1).

For the uniqueness, we follow the arguments in [24, Proposition 5.3.10]. Let $(W^i, X^i, \mathbb{P}^i)$, $i = 1, 2$, be two weak solutions to SDE (2.1), namely, W^i is a $\mathbb{P}^i$-Brownian motion and

$$dX^i_t = \sigma(t, X^i_\cdot)\big[dW^i_t + \lambda_t(X^i_\cdot)dt\big], \quad X^i_0 = 0, \quad \mathbb{P}^i\text{-a.s.}$$

Denote $\tilde{W}^i_t := \int_0^t [dW^i_s + \lambda_s(X^i_\cdot)ds]$, $\frac{d\tilde{\mathbb{P}}^i}{d\mathbb{P}^i} := M^i_T := e^{-\int_0^T \lambda_t(X^i_\cdot)\cdot dW^i_t - \frac{1}{2}\int_0^T |\lambda_t(X^i_\cdot)|^2 dt}$. Then $\tilde{W}^i$ is a $\tilde{\mathbb{P}}^i$-Brownian motion, and $X^i_t = \int_0^t \sigma(s, X^i_\cdot)d\tilde{W}^i_s$, $\tilde{\mathbb{P}}^i$-a.s. By the Lipschitz continuity of σ, the $\tilde{\mathbb{P}}^1$-distribution of $(\tilde{W}^1, X^1)$ is equal to the $\tilde{\mathbb{P}}^2$-distribution of $(\tilde{W}^2, X^2)$. Note that W^i and M^i_T are functions of $(\tilde{W}^i, X^i)$; we see that the $\tilde{\mathbb{P}}^1$-distribution of $(\tilde{W}^1, X^1, W^1, M^1_T)$ is equal to the $\tilde{\mathbb{P}}^2$-distribution of $(\tilde{W}^2, X^2, W^2, M^2_T)$. Now it follows from $d\mathbb{P}^i = (M^i_T)^{-1}d\tilde{\mathbb{P}}^i$ that the $\mathbb{P}^1$-distribution of X^1 is equal to the $\mathbb{P}^2$-distribution of X^2.

Finally, when $\lambda = 0$, since σ is uniformly Lipschitz continuous in ω, both the (strong) existence and uniqueness are obvious. ∎

For $\tau \in \mathcal{T}$ and $\omega \in \Omega$, let $\mathbb{P}^{\tau,\omega}$ be an regular conditional probability distribution (r.c.p.d.) of the probability measure $\mathbb{P}$ conditional to $\mathcal{F}_\tau$.

LEMMA 2.5. *Let M be a $\mathbb{P}$-martingale with continuous paths, $\mathbb{P}$-a.s. Then, for any $\tau \in \mathcal{T}$,*

$$\mathbb{P}\big[\Omega^0_\tau\big] = 1, \ \textit{where}\ \Omega^0_\tau := \big\{\omega : M^{\tau,\omega} \textit{ is a } \mathbb{P}^{\tau,\omega}\textit{-martingale}\big\}. \tag{2.3}$$

Proof. By standard approximation arguments, it is sufficient to prove that, for any $0 \le t_1 < t_2 \le T$ and any sequence $0 \le s_1 \le \cdots \le s_n \le t_1$ such that $(s_1, \ldots, s_n) \in \mathbb{Q}^n$, it holds that

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\big[(M^{\tau,\omega}_{t_2} - M^{\tau,\omega}_{t_1})\varphi(B_{s_1}, \ldots, B_{s_n})\big] = 0 \ \text{ for all } \ \varphi \in C_b(\mathbb{R}^n) \text{ and } \omega \in \Omega^n_\tau, \tag{2.4}$$

for some Ω^n_τ, such that $\mathbb{P}[\Omega^n_\tau] = 1$. Since $C_b(\mathbb{R}^n)$ is a separable space, there exists a countable set $(\psi^n_k)_{k\ge1}$ dense in $C_b(\mathbb{R}^n)$. By the tower property, we may find $\Omega^n_k \subset \Omega$ such that

$$\mathbb{E}^{\mathbb{P}^{\tau,\omega}}\big[(M^{\tau,\omega}_{t_2} - M^{\tau,\omega}_{t_1})\psi^n_k(B_{s_1}, \ldots, B_{s_n})\big] = 0 \ \text{ for all } \ \omega \in \Omega^n_k, \ \ \mathbb{P}[\Omega^n_k] = 1.$$

Then (2.4) holds on $\Omega^n_\tau := \cap_{k\ge1}\Omega^n_k$. ∎

For all $\tau \in \mathcal{T}$ and $\omega \in \Omega$, it is clear that $\sigma^{\tau,\omega}$ satisfies Assumption 2.2. Then, for any bounded $\lambda \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d)$, we may define from Lemma 2.4 a probability measure $\mathbb{P}_{\sigma^{\tau,\omega},\lambda^{\tau,\omega}}$. The next result compares this probability measure to the r.c.p.d. $\mathbb{P}^{\tau,\omega}_{\sigma,\lambda}$.

PROPOSITION 2.6. *Let $\lambda \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d)$ be bounded and $\tau \in \mathcal{T}$. Then for $\mathbb{P}_{\sigma,\lambda}$-a.e. ω, $\mathbb{P}^{\tau,\omega}_{\sigma,\lambda} = \mathbb{P}_{\sigma^{\tau,\omega},\lambda^{\tau,\omega}}$, namely, $\mathbb{E}^{\mathbb{P}^{\tau,\omega}_{\sigma,\lambda}}[\xi] = \mathbb{E}^{\mathbb{P}_{\sigma^{\tau,\omega},\lambda^{\tau,\omega}}}[\xi]$, for any bounded $\xi \in \mathbb{L}^0(\mathcal{F}_{T-\tau(\omega)})$.*

Proof. First, denote $M_t := B_t - \int_0^t (\sigma\lambda)_s(B_\cdot)ds$ and $N_t := M_t M_t^T - \int_0^t (\sigma\sigma^T)_s(B_\cdot)ds$. By the uniqueness of weak solution of SDE (2.1) we see that a probability measure $\mathbb{P}$ is equal to $\mathbb{P}_{\sigma,\lambda}$ if and only if both M and N are $\mathbb{P}$-martingales. Note that M and N are continuous. By Lemma 2.5, for $\mathbb{P}_{\sigma,\lambda}$-a.e. ω, it holds that $M^{\tau,\omega}_t = B_t - \int_0^t (\sigma\lambda)^{\tau,\omega}_s(B_\cdot)ds$ and $N^{\tau,\omega}_t = (M_t M_t^T)^{\tau,\omega}_t - \int_0^t (\sigma\sigma^T)^{\tau,\omega}_s(B_\cdot)ds$ are $\mathbb{P}^{\tau,\omega}_{\sigma,\lambda}$-martingales, which implies that $\mathbb{P}^{\tau,\omega}_{\sigma,\lambda} = \mathbb{P}_{\sigma^{\tau,\omega},\lambda^{\tau,\omega}}$. ∎

We now introduce an important family of probability measures on Ω: for $L \geq 0$ and $(t,\omega) \in \Theta$,

$$\begin{aligned} \mathcal{P}^{t,\omega}_L &:= \{\mathbb{P}_{\sigma^{t,\omega},\lambda} : \lambda \in \mathbb{L}_L(\mathbb{F})\}, \quad \text{where} \\ \mathbb{L}_L(\mathbb{F}) &:= \{\lambda \in \mathbb{L}^0(\mathbb{F}) : |\lambda| \leq L\}, \quad \text{and} \quad \mathcal{P}_L := \mathcal{P}^{0,0}_L, \end{aligned} \tag{2.5}$$

and the associated nonlinear expectations

$$\overline{\mathcal{E}}^{t,\omega}_L := \sup_{\mathbb{P}\in\mathcal{P}^{t,\omega}_L} \mathbb{E}^{\mathbb{P}}, \quad \underline{\mathcal{E}}^{t,\omega}_L := \inf_{\mathbb{P}\in\mathcal{P}^{t,\omega}_L} \mathbb{E}^{\mathbb{P}}, \quad \text{and} \quad \overline{\mathcal{E}}_L := \overline{\mathcal{E}}^{0,0}_L, \quad \underline{\mathcal{E}}_L := \underline{\mathcal{E}}^{0,0}_L. \tag{2.6}$$

We note that the family $\mathcal{P}_L$ depends on σ, which is fixed throughout the paper. For notational simplicity, we do not indicate this dependence explicitly. When $\sigma = I_d$, the nonlinear expectation $\overline{\mathcal{E}}_L$ is exactly the g-expectation of Peng [35] with $g = L|z|$.

Unlike [13, 14, 15], where mutually singular measures are considered for fully nonlinear PPDEs, here all measures $\mathbb{P} \in \cup_{L>0}\mathcal{P}_L$ are equivalent to $\mathbb{P}_\sigma$. In particular, for $\lambda \in \mathbb{L}_L(\mathbb{F})$ and $\xi \in \mathbb{L}^0(\mathcal{F}_T)$, we have, using the notation in (2.2),

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}[|\xi|] &= \mathbb{E}^{\mathbb{P}_\lambda}[|\xi(X^\sigma)|] = \mathbb{E}^{\mathbb{P}_0}[M^\lambda_T|\xi(X^\sigma)|] \\ &\leq C\Big(\mathbb{E}^{\mathbb{P}_0}[|\xi(X^\sigma)|^{1+\varepsilon}]\Big)^{\frac{1}{1+\varepsilon}} = C\Big(\mathbb{E}^{\mathbb{P}_\sigma}[|\xi|^{1+\varepsilon}]\Big)^{\frac{1}{1+\varepsilon}} \end{aligned}$$

for some constant $C = C_{L,\varepsilon}$. That is,

$$\overline{\mathcal{E}}_L[|\xi|] \leq C_{L,\varepsilon}\Big(\mathbb{E}^{\mathbb{P}_\sigma}[|\xi|^{1+\varepsilon}]\Big)^{\frac{1}{1+\varepsilon}}. \tag{2.7}$$

A direct consequence of this is the following convergence theorem, which makes some analysis in this paper much easier than that in [13, 14, 15].

PROPOSITION 2.7. *Assume $\xi_n, \xi \in \mathbb{L}^0(\mathcal{F}_T)$, $\xi_n \to \xi$ in probability $\mathbb{P}_\sigma$, and $\sup_n \mathbb{E}^{\mathbb{P}_\sigma}[|\xi_n|^{1+\varepsilon}] < \infty$ for some $\varepsilon > 0$. Then $\lim_{n\to\infty} \overline{\mathcal{E}}_L[|\xi_n - \xi|] = 0$ for all $L > 0$.*

3. Viscosity solutions of semilinear path-dependent PDEs. The objective of this paper is to study the semilinear path-dependent PDE (1.1), which we rewrite as

$$\begin{aligned} &-\mathcal{L}u_t(\omega) - F\big(t,\omega,u_t(\omega),\sigma^T_t(\omega)\partial_\omega u_t(\omega)\big) = 0, \quad (t,\omega) \in [0,T) \times \Omega, \\ &\text{where } \mathcal{L}u_t(\omega) := \partial_t u_t(\omega) + \frac{1}{2}\mathrm{Tr}[\sigma_t(\omega)\sigma^{\mathrm{T}}_t(\omega)\partial^2_{\omega\omega}u_t(\omega)], \end{aligned} \tag{3.1}$$

and the nonlinearity $F:(t,\omega,y,z)\in\Theta\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ is $\mathbb{F}$-progressively measurable in all variables.

Assumption 3.1. (i) F is uniformly L_0-Lipschitz continuous in (y,z) for some $L_0\ge 0$, i.e.,

$$|F(\cdot,y,z)-F(\cdot,y',z')|\le L_0\left(|y-y'|+|z-z'|\right)\quad\text{for all}\quad y,y'\in\mathbb{R},\ z,z'\in\mathbb{R}^d.$$

(ii) There exists $F^0\in C^0(\Theta)$ such that $|F(\cdot,0,0)|\le F^0$.

Remark 3.2. Unlike the papers on the general fully nonlinear cases (see, e.g., [15, 41, 44]), we do not impose any regularity assumption on the coefficients with respect to the variable (t,ω), thanks to the particular argument we develop here for the semilinear context.

3.1. Definition via test functions. Similar to [14], we introduce the following notion of smoothness of processes through Dupire's functional Itô formula. Let $\mathbb{S}^d$ denote the set of $d\times d$-symmetric matrices and we say a probability measure on $(\Omega,\mathcal{F}_T)$ is a semimartingale measure if B is a semimartingale under $\mathbb{P}$.

DEFINITION 3.3. *We say $u\in C^{1,2}(\Theta)$ if $u\in C^0(\Theta,\mathbb{R})$ and for all semimartingale measures $\mathbb{P}$,*

$$du=\partial_t u dt+\partial_\omega u\cdot dB_t+\frac{1}{2}\partial^2_{\omega\omega}u:d\langle B\rangle_t,\quad \mathbb{P}\text{-}a.s.$$

for some processes $\partial_t u\in C^0(\Theta,\mathbb{R})$, $\partial_\omega u\in C^0(\Theta,\mathbb{R}^d)$, $\partial^2_{\omega\omega}u\in C^0(\Theta,\mathbb{S}^d)$.

Remark 3.4. (i) By [14], for any $u\in C^{1,2}(\Theta)$, $\partial_t u,\partial_\omega u,\partial^2_{\omega\omega}u$ are unique and are naturally called the path-derivatives of u. We refer to [14] for various properties of these path-derivatives, including their relation with Dupire's original definition of path-derivatives. We note that [14] considers a smaller class of semimartingale measures $\mathbb{P}$ whose drift and diffusion characteristics are bounded. However, such a constraint does not cause any difference for our purpose and thus we do not introduce those constraints on $\mathbb{P}$ in this paper.

(ii) As we will see in next subsection, to study viscosity solutions it suffices to consider only linear processes for semilinear PPDEs (and paraboloid processes for fully nonlinear PPDEs). So technically speaking Definition 3.3 is needed only for classical solutions of PPDEs.

(iii) In our semilinear case, denote

$$\mathcal{L}u:=\partial_t u+\frac{1}{2}Tr(\sigma\sigma^T\partial^2_{\omega\omega}u). \tag{3.2}$$

Notice that all measures in $\cup_{L>0}\mathcal{P}_L$ are semimartingale measures, and then

$$du=\mathcal{L}u_t dt+\partial_\omega u\cdot dB_t,\quad \mathbb{P}\text{-a.s. for all }\mathbb{P}\in\cup_{L>0}\mathcal{P}_L.$$

Moreover, following the arguments in Lemma 2.5 and Proposition 2.6, we see that, for any $(t,\omega)\in\Theta$,

$$u^{t,\omega}_s-u_t(\omega)=\int_0^s(\mathcal{L}u)^{t,\omega}_r dr+\int_0^s(\partial_\omega u)^{t,\omega}_r\cdot dB_r, \tag{3.3}$$
$$s\in[0,T-t],\ \mathbb{P}\text{-a.s. for all }\mathbb{P}\in\cup_{L>0}\mathcal{P}^{t,\omega}_L.$$

(iv) Unlike [14, 15], where $\partial_t u$ and $\partial^2_{\omega\omega}u$ appear separately in PPDE, here since σ is given, we do not need to distinguish the two terms in (3.3) and need only to identify $\mathcal{L}u$.

We introduce the sets of test processes for subsolutions and supersolutions:

$$
\begin{aligned}
(3.4)\quad &\underline{\mathcal{A}}_L u_t(\omega)\\
&:= \Big\{\varphi \in C^{1,2}(\Theta) : (\varphi - u)_t(\omega) = \min_{\tau\in\mathcal{T}_{\rm H}} \underline{\mathcal{E}}_L^{t,\omega}\big[(\varphi-u)^{t,\omega}_\tau\big] \text{ for some } {\rm H}\in\mathcal{T}^+_{T-t}\Big\},\\
&\overline{\mathcal{A}}_L u_t(\omega)\\
&:= \Big\{\varphi \in C^{1,2}(\Theta) : (\varphi - u)_t(\omega) = \max_{\tau\in\mathcal{T}_{\rm H}} \overline{\mathcal{E}}_L^{t,\omega}\big[(\varphi-u)^{t,\omega}_\tau\big] \text{ for some } {\rm H}\in\mathcal{T}^+_{T-t}\Big\}.
\end{aligned}
$$

The stopping time H implies that the test processes are locally bounded at (t,ω), and in particular the integrability in (3.4) will always be guaranteed. For a test function $\varphi \in \underline{\mathcal{A}}_L u_t(\omega)\cup\overline{\mathcal{A}}_L u_t(\omega)$, we shall refer to a corresponding H as its localizing time. Note that in our definition, a test function is tangent to u at a point (t,ω) in mean value (under a family of probability measures), which is different from the corresponding notion in Crandall and Lions [8].

DEFINITION 3.5 (viscosity solution of path-dependent PDE). *Let $u \in C^0(\Theta,\mathbb{R})$.*

(i) *u is a $\mathcal{P}_L$-viscosity supersolution (resp., subsolution) of* (3.1) *if for any $(t,\omega)\in [0,T)\times\Omega$,*

$$
\begin{aligned}
&-\mathcal{L}\varphi_t(\omega) - F\big(t,\omega,u_t(\omega),\sigma_t^T(\omega)\partial_\omega\varphi_t(\omega)\big)\\
&\qquad \geq \ \textit{(resp., } \leq\textit{) } 0 \ \textit{ for all } \ \varphi\in\overline{\mathcal{A}}_L u_t(\omega) \ \textit{(resp., } \underline{\mathcal{A}}_L u_t(\omega)\textit{)}.
\end{aligned}
$$

(ii) *A $\mathcal{P}_L$-viscosity solution of* (3.1) *is both a $\mathcal{P}_L$-subsolution and a $\mathcal{P}_L$-supersolution.*

Remark 3.6. (i) In [14, 15], a larger and nondominated set $\overline{\mathcal{P}}_L$ which consists of mutually singular probability measures is used. The corresponding sets of test functions $\underline{\mathcal{A}}_L^{\overline{\mathcal{P}}_L}u$ and $\overline{\mathcal{A}}_L^{\overline{\mathcal{P}}_L}u$ are smaller there, and consequently a viscosity solution here must be a viscosity solution in the sense of [14], but not vice versa in general. Therefore, by putting more test functions in this paper, we are helping for the proof of uniqueness.

(ii) When $\sigma = I_d$ but under the above $\overline{\mathcal{P}}_L$-definition, the wellposedness of the semilinear PPDE (3.1) is achieved in [14, 15] by using a different approach. However, the general case with random σ and under $\overline{\mathcal{P}}_L$-definition does not fall into the framework of this paper and does not satisfy the sufficient conditions for the comparison principle in [15], and its wellposedness is still open.

3.2. Equivalent definition via semijets. Following the standard theory of viscosity solutions for PDEs, we may define viscosity solutions via semijets. In light of Remark 3.4 (iii) and (iv), let

$$
(3.5)\qquad Q^{\alpha,\beta}(t,\omega) := \alpha t + \beta\cdot\omega_t, \quad \alpha\in\mathbb{R},\ \beta\in\mathbb{R}^d, \text{ and } (t,\omega)\in\Theta.
$$

DEFINITION 3.7 (semijets). *For $u\in C^0(\Theta,\mathbb{R})$, define subjet and superjet of u at (t,ω):*

$$
\begin{aligned}
\underline{\mathcal{J}}_L u_t(\omega) &:= \big\{(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}^d :\ Q^{a,\beta}\in\underline{\mathcal{A}}_L u_t(\omega)\big\};\\
\overline{\mathcal{J}}_L u_t(\omega) &:= \big\{(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}^d :\ Q^{a,\beta}\in\overline{\mathcal{A}}_L u_t(\omega)\big\}.
\end{aligned}
$$

Moreover, $cl(\underline{\mathcal{J}}_L u_t(\omega))$ and $cl(\overline{\mathcal{J}}_L u_t(\omega))$ denote their closures.

Remark 3.8. In the fully nonlinear case, one has to distinguish $\partial_t u$ and $\partial^2_{\omega\omega} u$, and accordingly one needs to introduce paraboloid processes:

$$Q^{\alpha,\beta,\gamma}(t,\omega) := \alpha t + \beta\cdot\omega_t + \frac{1}{2}\gamma\omega_t\cdot\omega_t, \quad \alpha\in\mathbb{R},\ \beta\in\mathbb{R}^d,\ \gamma\in\mathbb{R}^{d\times d} \text{ and } (t,\omega)\in\Theta.$$

See [43] for more details. In the present semilinear case, one can easily show that the linear processes (3.5) is sufficient for our purpose.

PROPOSITION 3.9. *For $u \in C^0(\Theta,\mathbb{R})$ and $(t,\omega)\in[0,T)\times\Omega$, the following are equivalent:*

(i) *u is a $\mathcal{P}_L$-viscosity subsolution of the path-dependent PDE* (3.1) *at (t,ω);*
(ii) *$-\alpha - F\big(t,\omega,u_t(\omega),\sigma_t^T(\omega)\beta\big)\le 0$ for all $(\alpha,\beta)\in\underline{\mathcal{J}}_L u_t(\omega)$;*
(iii) *$-\alpha - F\big(t,\omega,u_t(\omega),\sigma_t^T(\omega)\beta\big)\le 0$ for all $(\alpha,\beta)\in cl(\underline{\mathcal{J}}_L u_t(\omega))$.*

Proof. Since $Q^{\alpha,\beta}\in C^{1,2}(\Theta,\mathbb{R})$, clearly (i) implies (ii). Now assume (ii) holds true. For any $(\alpha,\beta)\in cl(\underline{\mathcal{J}}_L u_t(\omega))$, there exist $(\alpha_n,\beta_n)\in\underline{\mathcal{J}}_L u_t(\omega)$ such that $(\alpha_n,\beta_n)\to(\alpha,\beta)$. By (ii) we have $-\alpha_n - F\big(t,\omega,u_t(\omega),\sigma_t^T(\omega)\beta_n\big)\le 0$. Sending $n\to\infty$ we prove (iii).

It remains to prove that (iii) implies (i). Let $(t,\omega)\in[0,T)\times\Omega$ and $\varphi\in\underline{\mathcal{A}}_L u_t(\omega)$ with localizing time $\textsc{h}\in\mathcal{T}^+_{T-t}$. Without loss of generality, we take $(t,\omega)=(0,0)$ and $(\varphi-u)_0=0$. Denote

$$\alpha := \mathcal{L}\varphi_0, \quad \beta := \partial_\omega\varphi_0. \tag{3.6}$$

For any $\varepsilon>0$, since $\sigma\in C^0(\Theta)$ and φ is smooth, by otherwise choosing a smaller $\textsc{h}$ we may assume

$$|\sigma_t-\sigma_0|\le 1, \quad |\mathcal{L}\varphi_t-\alpha|\le\varepsilon, \quad |\partial_\omega\varphi_t-\beta|\le\varepsilon, \quad 0\le t\le\textsc{h}.$$

Denote $\alpha_\varepsilon := \alpha + [1+L(1+|\sigma_0|)]\varepsilon$. Then, for all $\tau\in\mathcal{T}_{\textsc{h}}$,

$$\begin{aligned}
&(Q^{\alpha_\varepsilon,\beta}-u)_0 - \underline{\mathcal{E}}_L\big[(Q^{\alpha_\varepsilon,\beta}-u)_\tau\big] = \overline{\mathcal{E}}_L\big[(u-u_0-Q^{\alpha_\varepsilon,\beta})_\tau\big]\\
&\le \overline{\mathcal{E}}_L\big[(u-\varphi)_\tau\big] + \overline{\mathcal{E}}_L\big[(\varphi-\varphi_0-Q^{\alpha_\varepsilon,\beta})_\tau\big]\\
&\le \overline{\mathcal{E}}_L\Big[\int_0^\tau(\mathcal{L}\varphi_s-\alpha_\varepsilon)ds + \int_0^\tau(\partial_\omega\varphi_s-\beta)\cdot dB_s\Big],
\end{aligned}$$

where the last inequality holds thanks to the fact that $\varphi\in\underline{\mathcal{A}}_L u_0$. Note that, for any $\lambda\in\mathbb{L}_L(\mathbb{F})$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}\Big[\int_0^\tau(\mathcal{L}\varphi_s-\alpha_\varepsilon)ds + \int_0^\tau(\partial_\omega\varphi_s-\beta)\cdot dB_s\Big]\\
&= \mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}\Big[\int_0^\tau(\mathcal{L}\varphi_s-\alpha)ds + \int_0^\tau(\partial_\omega\varphi_s-\beta)\cdot\sigma_s\lambda_s ds - [1+L(1+|\sigma_0|)]\varepsilon\tau\Big]\\
&\le \mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}\Big[\int_0^\tau[\varepsilon+\varepsilon L(1+|\sigma_0|)]ds - [1+L(1+|\sigma_0|)]\varepsilon\tau\Big] = 0.
\end{aligned}$$

By the arbitrariness of $\lambda\in\mathbb{L}_L(\mathbb{F})$, we see that

$$(Q^{\alpha_\varepsilon,\beta}-u)_0 - \underline{\mathcal{E}}_L\big[(Q^{\alpha_\varepsilon,\beta}-u)_\tau\big] \le \overline{\mathcal{E}}_L\Big[\int_0^\tau(\mathcal{L}\varphi_s-\alpha_\varepsilon)ds + (\partial_\omega\varphi_s-\beta)\cdot dB_s\Big]\le 0.$$

That is, $(\alpha_\varepsilon,\beta)\in\underline{\mathcal{J}}_L u_0$. Then, $(\alpha,\beta)\in cl(\underline{\mathcal{J}}_L u_0)$, and (iii) provides $-\alpha - F(0,0,u_0,\sigma_0^T\beta)\le 0$. By (3.6), this exactly means (i). ∎

The following simple results will be useful later.

PROPOSITION 3.10. *Let $u, u' \in C^0(\Theta, \mathbb{R})$ and $(t,\omega) \in \Theta$.*
(i) *$(\alpha, \beta) \in \underline{\mathcal{J}}_L u_t(\omega)$ if and only if $(-\alpha, -\beta) \in \overline{\mathcal{J}}_L(-u)_t(\omega)$.*
(ii) *If $(\alpha,\beta) \in \underline{\mathcal{J}}_L u_t(\omega)$, $(\alpha',\beta') \in \underline{\mathcal{J}}_L u'_t(\omega)$, then $(\alpha+\alpha', \beta+\beta') \in \underline{\mathcal{J}}_L(u+u')_t(\omega)$. Moreover, the results remain true if we replace the semijets with their closures.*

Proof. Item (i) is obvious, and we can easily extend the results from semijets to their closures. It remains to prove (ii). Indeed, by definition, there exists a common $\mathrm{H} \in \mathcal{T}^+_{T-t}$ such that

$$u_t(\omega) \geq \overline{\mathcal{E}}_L[(u^{t,\omega} - Q^{\alpha,\beta})_\tau], \quad u'_t(\omega) \geq \overline{\mathcal{E}}_L[((u')^{t,\omega} - Q^{\alpha',\beta'})_\tau] \text{ for all } \tau \in \mathcal{T}_{\mathrm{H}}.$$

Then, by the sublinearity of $\overline{\mathcal{E}}_L$ we have

$$\begin{aligned}(u+u')_t(\omega) &\geq \overline{\mathcal{E}}_L\Big[(u^{t,\omega} - Q^{\alpha,\beta})_\tau + ((u')^{t,\omega} - Q^{\alpha',\beta'})_\tau\Big] \\ &= \overline{\mathcal{E}}_L\Big[([u+u']^{t,\omega} - Q^{\alpha+\alpha',\beta+\beta'})_\tau\Big].\end{aligned}$$

This means that $(\alpha+\alpha', \beta+\beta') \in \underline{\mathcal{J}}_L(u+u')_t(\omega)$. ∎

3.3. Punctual differentiability. For $L \geq L_0$, note that $(\mathcal{L}u_t(\omega), \partial_\omega u_t(\omega)) \in cl(\underline{\mathcal{J}}_L u_t(\omega))$, whenever $u \in C^{1,2}(\Theta, \mathbb{R})$. Moreover, similar to [14], and also combining the arguments in Proposition 3.9, one can easily show that u is a classical subsolution at (t,ω) if and only if u is a viscosity subsolution at (t,ω).

Following Caffarelli and Cabre [3], we introduce a notion of differentiation which is weaker than the path-derivatives and will be crucial for the proof of our main comparison result.

DEFINITION 3.11. *Let $\varphi \in \mathbb{L}^0(\mathbb{F})$. We say φ is $\mathcal{P}_L$-punctually $C^{1,2}$ at (t,ω) if*

$$\mathcal{J}_L\varphi_t(\omega) := \mathrm{cl}\Big(\underline{\mathcal{J}}_L\varphi_t(\omega)\Big) \cap \mathrm{cl}\Big(\overline{\mathcal{J}}_L\varphi_t(\omega)\Big) \neq \emptyset.$$

The following result is straightforward.

PROPOSITION 3.12. *Let $u \in C^0(\Theta, \mathbb{R})$.*
(i) *If $u \in C^{1,2}(\Theta,\mathbb{R})$, then u is $\mathcal{P}_L$-punctually $C^{1,2}$ at all (t,ω) with $(\mathcal{L}u_t(\omega), \partial_\omega u_t(\omega)) \in \mathcal{J}_L u_t(\omega)$;*
(ii) *If u is $\mathcal{P}_L$-punctually $C^{1,2}$ at (t,ω) and is a $\mathcal{P}_L$-viscosity solution (resp., subsolution, supersolution) of the path-dependent PDE (3.1) at (t,ω), then for any $(\alpha,\beta) \in \mathcal{J}_L u_t(\omega)$ we have*

$$-\alpha - F\left(t,\omega,u_t(\omega), \sigma_t^T(\omega)\beta\right) = \text{(resp., } \leq, \geq\text{) } 0.$$

4. Comparison result. We first introduce some notation for appropriate spaces.

- $\mathbb{S}_2^{t,\omega} := \big\{Y \in \mathbb{L}^0(\mathbb{F}) : Y$ continuous in t, $\mathbb{P}_{\sigma^{t,\omega}}$-a.s. and $\mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega}}}\big[\sup_{s\leq T-t}|Y_s|^2\big] < \infty\big\}$; $\mathbb{S}^2 := \mathbb{S}_2^{0,0}$;
- $C_2^0(\Theta) := \big\{u \in C^0(\Theta) : u^{t,\omega} \in \mathbb{S}_2^{t,\omega}$ for all $(t,\omega) \in \Theta\big\}$; in particular, it follows from Assumption 2.2 and standard estimates for SDEs that $\sigma \in C_2^0(\Theta)$;
- $\mathbb{H}^2 := \big\{Z \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d) : \mathbb{E}^{\mathbb{P}_\sigma}\big[\int_0^T |\sigma_s^T Z_s|^2 ds\big] < \infty\big\}$;
- $\mathbb{I}^2 := \big\{K \in \mathbb{S}^2 : K$ is increasing, $\mathbb{P}_\sigma$-a.s. and $K_0 = 0\big\}$.

4.1. Main result. The main focus of this paper is the following comparison result.

THEOREM 4.1. *Let Assumption* 3.1 *hold true and $u, v \in C_2^0(\Theta)$ be $\mathcal{P}_L$-viscosity subsolution and supersolution, respectively, of PPDE* (3.1) *for some $L \geq L_0$. If $u_T \leq v_T$ on Ω, then $u \leq v$ on Θ.*

A similar result in the case of $\sigma = I_d$ was proved in [12]. Their proof is based on the construction of a regular approximation of the BSDE representation of the solution. Also, the comparison result in the fully nonlinear case addressed in [15] is crucially based on an approximation by finite-dimensional partial differential equations induced by conveniently freezing the path-dependency. With this approximation in hand, the comparison result is proved by building on the corresponding classical results in the PDE literature.

The main contribution of this paper is to provide an alternative proof which does not rely on any representation of the solution and which does not appeal to the corresponding PDE literature. We also observe that the comparison result of Theorem 4.1 allows for a random and possibly degenerate diffusion coefficient σ. Our proof of the comparison result is new and is even relevant in the Markovian case which reduces to a PDE in a finite-dimensional space. Notice that in the last context, any test function $\phi(t, x)$ which is pointwise tangent from below to a function $f(t, x)$ at point (t^*, x^*) induces a test process $\varphi_t(\omega) := \phi(t, \omega_t)$ which lies in $\underline{\mathcal{A}}_L u_{t^*}(\omega^*)$ with $u_t(\omega) := f(t, \omega_t)$, whenever $\omega^*_{t^*} = x^*$. In general, the opposite direction is not true, even for a Markovian test process $\varphi_t(\omega) = \varphi(t, \omega_t)$ in $\underline{\mathcal{A}}_L u_t(\omega)$. This shows that our definition of viscosity solutions involves a larger class of test functions than the standard Crandall–Lions notion of viscosity solutions in finite-dimensional spaces. Consequently, the comparison result has more chances under our definition, and we may hope to have an easier proof. We believe that the present proof achieves this goal. This is definitely true in the linear case which is isolated in Subsection 4.5.

4.2. Martingale representation and optimal stopping problem. In this subsection, we state the results of the martingale representation under $\mathbb{P}_\sigma$ and the related optimal stopping problem, which is the keystone for our comparison of viscosity solutions. We remark that, due to the possible degeneracy of σ, the martingale representation theorem here does not follow directly from the standard one under Brownian motion, but instead follows from Theorem III.4.29 in [22] together with Lemma 2.4. For readers' convenience we shall provide an alternative proof in the appendix.

THEOREM 4.2 (martingale representation). *$\mathbb{P}_\sigma$ satisfies the martingale representation property. That is, for any $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{P}_\sigma)$, $\xi = \mathbb{E}^{\mathbb{P}_\sigma}[\xi] + \int_0^T Z_t \cdot dB_t$, $\mathbb{P}_\sigma$-a.s. for some $Z \in \mathbb{H}^2$.*

Moreover Z is unique in the sense that $\mathbb{E}^{\mathbb{P}_\sigma}[\int_0^T |\sigma_t^T(\tilde{Z}_t - Z_t)|^2 dt] = 0$ for any $\tilde{Z} \in \mathbb{H}^2$ satisfying the above representation property.

COROLLARY 4.3. *Let $\lambda \in \mathbb{L}_L(\mathbb{F})$ and $M \in \mathbb{S}^2$. Then M is a $\mathbb{P}_{\sigma,\lambda}$-martingale if and only if*

$$dM_t = Z_t \cdot \Big[dB_t - \sigma_t \lambda_t dt\Big], \quad t \in [0, T], \quad \mathbb{P}_\sigma\text{-a.s. for some } \; Z \in \mathbb{H}^2.$$

Let $\mathrm{H} \in \mathcal{T}^+$ and $X \in \mathbb{L}^0(\mathbb{F})$ be a process with continuous sample paths. Consider the optimal stopping problem under dominated nonlinear expectation:

$$V_t(\omega) := \sup_{\tau \in \mathcal{T}} \overline{\mathcal{E}}_L^{t,\omega}\big[X^{t,\omega}_{\tau \wedge (\mathrm{H}^{t,\omega} - t)}\big] \;\; \text{for all} \;\; (t, \omega) \in \Theta. \tag{4.1}$$

THEOREM 4.4 (optimal stopping problem). *Let $L > 0$ and $X \in \mathbb{L}^0(\mathbb{F})$ such that $X_{\cdot\wedge \mathrm{H}} \in \mathbb{S}^2$. Then, there exists an $\mathbb{F}$-adapted and $\mathbb{P}_\sigma$-a.s. continuous process Y satisfying*

(i) *there exists $\tau^* \in \mathcal{T}_{\mathrm{H}}$ such that $\tau^* = \inf\{t : Y_t = X_t\}$, $\mathbb{P}_\sigma$-a.s. and $Y_0 = \overline{\mathcal{E}}_L[X_{\tau^*}]$;*

(ii) *for all $\tau \in \mathcal{T}_{\mathrm{H}}$, we have $Y_\tau = V_\tau$, $\mathbb{P}_\sigma$-a.s.; in particular, $Y_0 = V_0$;*

(iii) *there exist $\mathbb{P}^* \in \mathcal{P}_L$, $\mathbb{P}^*$-martingale M starting from 0, and $K \in \mathbb{I}^2$ such that*

$$Y = Y_0 + M - K \quad \text{and} \quad \int (Y - X) dK = 0, \quad \mathbb{P}_\sigma\text{-a.s.}$$

Remark 4.5. Although the results in Theorem 4.4 look very much the same as in standard literature (see, e.g., [18]), there is a very subtle measurability issue involved here. In the standard literature, one typically uses the augmented filtration. However, due to the pathwise nature of this paper, in Theorem 4.4 we require X, Y, and τ^* to be adapted to the natural filtration $\mathbb{F}$. We shall provide a detailed proof in the appendix.

We further remark that the optimal stopping problem always plays a crucial role in the analysis on the viscosity solutions to path-dependent PDEs. In the recent work [41], the authors randomized the time-variable in the definition of the viscosity solution. By doing so and applying the classical dynamic programming argument, they obtained the desired optimal stopping results similar to the points (i), (ii) above in the general fully nonlinear case. However, it does not automatically lead to the point (iii) in the randomized setting, while the representation in (iii) is important for our argument in this paper for the semilinear case.

DEFINITION 4.6 (Snell envelope). *The process Y introduced in Theorem 4.4 is called a Snell envelope of the stopped process $X_{\cdot\wedge \mathrm{H}}$ and denoted $\mathrm{Snell}(X_{\cdot\wedge \mathrm{H}}) := Y$. The stopping time $\tau^* \in \mathcal{T}_{\mathrm{H}}$ is called an optimal stopping rule.*

4.3. Pathwise semimartingales. In this subsection, let $u \in \mathbb{L}^0(\mathbb{F})$ such that all the (nonlinear) expectations involved below exist. Similar to a standard semimartingale under a fixed probability measure $\mathbb{P}$, we say u is an $\overline{\mathcal{E}}_L$-submartingale (resp., supermartingale) if

(4.2)
$$u_t \le (\text{resp.}, \ \ge) \ \overline{\mathcal{E}}_L[u_\tau|\mathcal{F}_t] := \operatorname*{ess-sup}_{\mathbb{P}\in\mathcal{P}_L} \mathbb{E}^{\mathbb{P}}[u_\tau|\mathcal{F}_t], \ \mathbb{P}_\sigma\text{-a.s. for any } t, \tau \in \mathcal{T} \text{ with } \tau \ge t.$$

As viscosity solutions are pathwise defined, we extend this notion in a pathwise manner.

DEFINITION 4.7. (i) *We say u is a pathwise $\mathbb{P}_\sigma$-submartingale (resp., supermartingale) if*

$$u_t(\omega) \le (\textit{resp.}, \ \ge) \ \mathbb{E}^{\mathbb{P}_\sigma^{t,\omega}}[u^{t,\omega}_\tau] \quad \textit{for any } (t,\omega) \in \Theta \textit{ and } \tau \in \mathcal{T}_{T-t}.$$

(ii) *We say u is a pathwise $\overline{\mathcal{E}}_L$-submartingale (resp., supermartingale) if*

$$u_t(\omega) \le (\textit{resp.}, \ \ge) \ \overline{\mathcal{E}}^{t,\omega}_L[u^{t,\omega}_\tau] \quad \textit{for any } (t,\omega) \in \Theta \textit{ and } \tau \in \mathcal{T}_{T-t}.$$

Remark 4.8. By Proposition 2.6 and the definition of r.c.p.d., it is clear that a pathwise $\mathbb{P}_\sigma$-submartingale is a $\mathbb{P}_\sigma$-submartingale. Similar statements hold for supermartingales.

PROPOSITION 4.9. *Assume $u \in \mathbb{S}^2$ is a pathwise $\overline{\mathcal{E}}_L$-submartingale. Then,*

(i) *u is an $\overline{\mathcal{E}}_L$-submartingale;*

(ii) *there exists $\mathbb{P}^* \in \mathcal{P}_L$ such that u is a $\mathbb{P}^*$-submartingale.*

4.4. A fundamental lemma. The following result shows how to find a point of tangency in mean. This replaces the local compactness in the standard Crandall–Lions theory of viscosity solutions.

LEMMA 4.10. *Assume* $u \in \mathbb{L}^0(\mathbb{F})$ *satisfying* $u_{\cdot\wedge \mathrm{H}} \in \mathbb{S}^2$ *and* $u_0 > \overline{\mathcal{E}}_L[u_{\mathrm{H}}]$ *for some* $\mathrm{H} \in \mathcal{T}^+$. *Then there exists* $\omega^* \in \Omega$ *and* $t^* < \mathrm{H}(\omega^*)$ *such that* $0 \in \underline{\mathcal{A}}_L u_{t^*}(\omega^*)$.

Proof. Define the optimal stopping value V by (4.1) with $X := u$. Let $\tau^* \in \mathcal{T}_{\mathrm{H}}$ be the optimal stopping rule. Since by Theorem 4.4(i) and (ii) we have $\overline{\mathcal{E}}_L[u_{\tau^*}] = V_0 \geq u_0 > \overline{\mathcal{E}}_L[u_{\mathrm{H}}]$ and $\mathbb{P}_\sigma\big[u_{\tau^*} = V_{\tau^*}\big] = 1$. It follows that $\mathbb{P}_\sigma\big[u_{\tau^*} = V_{\tau^*}, \tau^* < \mathrm{H}\big] > 0$, and then there exists $\omega^* \in \Omega$ such that $t^* := \tau^*(\omega^*) < \mathrm{H}(\omega^*)$ and $u_{t^*}(\omega^*) = V_{t^*}(\omega^*)$. By the definition of V and $\underline{\mathcal{A}}_L u$, this means that (t^*, ω^*) is the desired point. ∎

4.5. Comparison result for the heat equation. For $F = 0$, the equation reduces to

$$-\mathcal{L}u(t,\omega) = 0 \quad t < T, \quad \omega \in \Omega. \tag{4.3}$$

Our objective is to provide an easy proof of the comparison result of Theorem 4.1, which requires standard tools from stochastic analysis. For simplicity, we specialize the comparison Theorem 4.1 to the case $L = 0$ and call the corresponding viscosity solution a $\mathbb{P}_\sigma$-viscosity solution. We emphasize that the set of test processes is the largest possible with $L = 0$.

THEOREM 4.11. *For a process* $u \in C^0_2(\Theta)$, *the following are equivalent:*

(i) *u is a pathwise $\mathbb{P}_\sigma$-submartingale (resp. supermartingale);*

(ii) *u is a $\mathbb{P}_\sigma$-viscosity subsolution (resp., supersolution) of the path-dependent heat equation* (4.3).

Proof. (i) $\Longrightarrow$ (ii) Assume to the contrary that $-c := \mathcal{L}\varphi_t(\omega) < 0$ for some $(t,\omega) \in \Theta$ and $\varphi \in \underline{\mathcal{A}}_0 u_t(\omega)$ with localizing time $\mathrm{H} \in \mathcal{T}^+$. Without loss of generality, $(t,\omega) = (0,0)$. Note that

$$(\varphi - u)_0 \leq \mathbb{E}^{\mathbb{P}_\sigma}\big[(\varphi - u)_\tau\big] \quad \text{for all} \quad \tau \in \mathcal{T}_{\mathrm{H}}.$$

Denote $\tau := \inf\{t : \mathcal{L}\varphi_t \geq -\frac{c}{2}\} \wedge \mathrm{H} \in \mathcal{T}^+$. Then, by (ii), we obtain the following desired contradiction:

$$0 \;\geq\; u_0 - \mathbb{E}^{\mathbb{P}_\sigma}\big[u_\tau\big] \geq \varphi_0 - \mathbb{E}^{\mathbb{P}_\sigma}\big[\varphi_\tau\big] \;=\; \mathbb{E}^{\mathbb{P}_\sigma}\Big[-\int_0^\tau \mathcal{L}\varphi_s ds\Big] \;\geq\; \frac{c}{2}\mathbb{E}^{\mathbb{P}_\sigma}[\tau] > 0.$$

(ii) $\Longrightarrow$ (i) It is easy to verify that $u^\varepsilon_t(\omega) := u_t(\omega) + \varepsilon t$ is a $\mathbb{P}_\sigma$-viscosity subsolution to the equation:

$$-\mathcal{L}u^\varepsilon_t(\omega) + \varepsilon \leq 0.$$

We now show that u^ε is a pathwise $\mathbb{P}_\sigma$-submartingale. Suppose to the contrary that there exists a point (t,ω) at which the supermartingale property fails, and set $(t,\omega) = (0,0)$ without loss of generality. Then, there exists a stopping time $\mathrm{H} \in \mathcal{T}^+_T$ such that $u^\varepsilon_0 > \mathbb{E}^{\mathbb{P}_\sigma}[u^\varepsilon_{\mathrm{H}}]$. By Lemma 4.10, there exists (t^*, ω^*) such that $0 \in \underline{\mathcal{A}}_0 u^\varepsilon_{t^*}(\omega^*)$, and it follows from the $\mathbb{P}_\sigma$-viscosity subsolution property of u^ε that $\varepsilon \leq 0$, which is the required contradiction.

Hence, u^ε is a pathwise $\mathbb{P}_\sigma$-submartingale, namely, $u_t(\omega) + \varepsilon t \leq \mathbb{E}^{\mathbb{P}^{\sigma t,\omega}}[u^{t,\omega}_\tau + \varepsilon(\tau + t)]$ for all $\tau \in \mathcal{T}_{T-t}$. Sending $\varepsilon \to 0$, we obtain immediately that u is a pathwise $\mathbb{P}_\sigma$-submartingale. ∎

Theorem 4.11 leads immediately to the comparison result.

THEOREM 4.12. *Let $u, v \in C_2^0(\Theta)$ be a $\mathbb{P}_\sigma$-viscosity subsolution and a $\mathbb{P}_\sigma$-viscosity supersolution, respectively, of path-dependent heat equation* (4.3). *If $u_T \le v_T$ on Ω, then $u \le v$ on Θ.*

Remark 4.13. By Theorem 4.11 we see that our notion of a $\mathbb{P}_\sigma$-viscosity solution reduces to the notion of a stochastic viscosity solution introduced by Bayraktar and Sirbu [1, 2] in the Markovian case.

Remark 4.14. (i) Theorem 4.11 also provides the unique solution of the heat equation. Indeed it implies that a pathwise $\mathbb{P}_\sigma$-martingale is a viscosity solution. Since the final value is fixed by the boundary condition ξ, we are naturally lead to the candidate solution $u(t,\omega) := \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega}}}\big[\xi^{t,\omega}\big]$. Therefore, if this process is in $C_2^0(\Theta)$, it is the unique viscosity solution of the heat equation.

(ii) For the heat equation, we can in fact prove the comparison principle without requiring the continuity (in ω) of the viscosity semisolutions.

4.6. Partial comparison. We next return to the general semilinear PPDE (3.1). The following partial comparison result, as in [12] and [14], is a crucial step for our proof of the comparison result.

PROPOSITION 4.15. *In the setting of Theorem* 4.1, *if in addition $v \in C^{1,2}(\Theta)$, then $u \le v$ on Θ.*

Proof. We report the proof from [12] for completeness. First, by possibly transforming the problem to the comparison of $\tilde u_t := e^{\lambda t}u_t$ and $\tilde v_t := e^{\lambda t}v_t$, it follows from the Lipschitz property of the nonlinearity F in y that we may assume without loss of generality that F is decreasing in y.

Suppose to the contrary that $c := (u-v)_t(\omega) > 0$ at some point $(t,\omega) \in [0,T)\times\Omega$. Without loss of generality assume $(t,\omega) = (0,0)$. Let $c_0 := \frac{c}{2T}$, and define $X_s := (u-v)_s^+ + c_0 s$, $s \in [0,T]$. Clearly $X \in C_2^0(\Theta)$. Since $(u-v)_T \le 0$, it follows that $X_0 > \overline{\mathcal{E}}_L[X_T]$. By Lemma 4.10, we may find a point (t^*,ω^*) such that $t < T$ and $0 \in \underline{\mathcal{A}}_L X_{t^*}(\omega^*)$. In particular, this implies that

$$-(u-v)_{t^*}^+(\omega^*) - c_0 t^* \le \underline{\mathcal{E}}_L\big[-\big\{(u-v)^+\big\}_{T-t^*}^{t^*,\omega^*} - c_0 T\big] \;=\; -c_0 T,$$

and thus $(u-v)_{t^*}^+(\omega^*) \ge c_0(T-t^*) > 0$. Therefore, $(u-v)_{t^*}^+(\omega^*) = (u-v)_{t^*}(\omega^*) > 0$. Then, since $(u-v)^+ \ge u-v$, we deduce from $0 \in \underline{\mathcal{A}}_L X_{t^*}(\omega^*)$ that

$$(\varphi-u)_{t^*}(\omega^*) \le \underline{\mathcal{E}}_L\big[(\varphi-u)_\tau^{t^*,\omega^*}\big] \;\text{ for all } \tau \in \mathcal{T}_{T-t^*}, \text{ where } \varphi_s(\omega) := v_s(\omega) - c_0 s.$$

Since $v \in C^{1,2}(\Theta)$, this means that $\varphi \in \underline{\mathcal{A}}_L u_{t^*}(\omega^*)$. Note that $\mathcal{L}\varphi = \mathcal{L}v - c_0$ and $\partial_\omega\varphi = \partial_\omega v$. Then, since u is a viscosity subsolution and v is a classical supersolution, we deduce that

$$\begin{aligned}
0 &\ge \{-\mathcal{L}\varphi - F(.,u,\sigma^T\partial_\omega\varphi)\}(t^*,\omega^*)\\
&= c_0 + \{-\mathcal{L}v - F(.,u,\sigma^T\partial_\omega v)\}(t^*,\omega^*)\\
&\ge c_0 + \{F(.,v,\sigma^T\partial_\omega v) - F(.,u,\sigma^T\partial_\omega v)\}(t^*,\omega^*) \;\ge\; c_0,
\end{aligned}$$

where the last inequality follows from the nonincrease of F in y and the fact that $u_{t^*}(\omega^*) \ge v_{t^*}(\omega^*)$. Since $c_0 > 0$, this is the required contradiction. ∎

4.7. Punctual differentiability of viscosity semisolutions.

LEMMA 4.16. *Let Assumption* 3.1 *hold, and for some* $L \geq L_0$, *let* $u \in C_2^0(\Theta)$ *be an* L*-subsolution of PPDE* (3.1). *Then, the process* $\hat{u} := u + \int_0^{\cdot}(L_0|u_s| + F_s^0 + 1)ds$ *is a pathwise* $\overline{\mathcal{E}}_L$*-submartingale.*

Proof. Suppose to the contrary that $\hat{u}_t(\omega) > \overline{\mathcal{E}}_L^{t,\omega}[\hat{u}_{\mathrm{H}}^{t,\omega}]$ for some $(t,\omega) \in [0,T) \times \Omega$ and $\mathrm{H} \in \mathcal{T}_{T-t}^+$. Then, it follows from Lemma 4.10 that there exist $\omega^* \in \Omega$ and $t^* \in [t, t + \mathrm{H}(\omega^*))$ such that $0 \in \underline{\mathcal{A}}_L \hat{u}_{t^*}(\omega^*)$, i.e., there exists $\mathrm{H}' \in \mathcal{T}_{T-t^*}^+$ such that $-\hat{u}_{t^*}(\omega^*) \leq \underline{\mathcal{E}}_L^{t^*,\omega^*}[-\hat{u}_\tau^{t^*,\omega^*}]$ for all $\tau \in \mathcal{T}_{\mathrm{H}'}$. Rewriting it we have

$$-u_{t^*}(\omega^*) \leq \underline{\mathcal{E}}_L^{t^*,\omega^*}\Big[\varphi_\tau - u_\tau^{t^*,\omega^*}\Big] \text{ for all } \tau \in \mathcal{T}_{\mathrm{H}'},$$
$$\text{where } \varphi_t := -\int_0^t \big(L_0|u_s| + (F^0)_s + 1\big)ds.$$

Then $\varphi \in \underline{\mathcal{A}}_L u_{t^*}(\omega^*)$, as $\varphi \in C^{1,2}(\Theta)$ with $\mathcal{L}\varphi_{t^*}(\omega^*) = -L_0|u_{t^*}(\omega^*)| - F_{t^*}^0(\omega^*) - 1$ and $\partial_\omega \varphi_{t^*}(\omega^*) = 0$. The contradiction now follows from the viscosity subsolution of u and Assumption 3.1:

$$\begin{aligned}0 &\geq -\mathcal{L}\varphi_{t^*}(\omega^*) - F_{t^*}(\omega^*, u_{t^*}(\omega^*), \sigma_{t^*}^T(\omega^*)\partial_\omega\varphi_{t^*}(\omega^*))\\ &= L_0|u_{t^*}(\omega^*)| + F_{t^*}^0(\omega^*) + 1 - F_{t^*}(\omega^*, u_{t^*}(\omega^*), 0) \geq F_{t^*}^0(\omega^*) + 1 - F_{t^*}(\omega^*, 0, 0) \geq 1.\end{aligned}$$

∎

Unlike the heat equation case, this property and the corresponding statement for a viscosity supersolution v do not induce the comparison principle directly. Our main idea is the following.

PROPOSITION 4.17. *Assume* u *is a* $\mathbb{P}_\sigma$*-semimartingale with decomposition:* $du_t = Z_t \cdot dB_t + dA_t$, *where* $Z \in \mathbb{H}^2$ *and* $A \in \mathbb{L}^0(\mathbb{F})$ *is continuous and has finite variation,* $\mathbb{P}_\sigma$*-a.s. Then there exist a Borel set* $\mathbb{T}^u \subset [0,T]$ *and* $\Omega_t^u \in \mathcal{F}_t$ *for each* $t \in \mathbb{T}^u$ *such that, for any* $L > 0$,

$$\text{(4.4)} \qquad \begin{aligned}&Leb(\mathbb{T}^u) = T,\ \mathbb{P}_\sigma(\Omega_t^u) = 1,\\ &\quad \textit{and } u \textit{ is } \mathcal{P}_L\textit{-punctually } C^{1,2} \textit{ at } (t,\omega) \textit{ for all } t \in \mathbb{T}^u,\ \omega \in \Omega_t^u.\end{aligned}$$

Proof. Denote

$$\zeta_t := \overline{\lim_{0\downarrow h \in \mathbb{Q}}} \frac{1}{h}\int_t^{t+h} |\sigma_s^T Z_s - \sigma_t^T Z_t| ds,$$
$$\dot{A}_t^+ := \overline{\lim_{0\downarrow h\in\mathbb{Q}}} \frac{1}{h}[A_{t+h} - A_t],\ \ \dot{A}_t^- := \underline{\lim_{0\downarrow h\in\mathbb{Q}}} \frac{1}{h}[A_{t+h} - A_t].$$

Note that the processes $\zeta, \dot{A}^+$, and $\dot{A}_t^-$ are $\mathbb{F}^+$-measurable (with possible values ∞ and $-\infty$). Denote

$$\text{(4.5)} \qquad \begin{aligned}\Omega_0 &:= \Big\{\omega \in \Omega : \textstyle\int_0^T |\sigma_t^T Z_t(\omega)| dt < \infty,\\ &\qquad\qquad \text{and } A \text{ is continuous and has finite variation on } [0,T]\Big\};\\ \Theta_0 &:= \Big\{(t,\omega) \in [0,T) \times \Omega : \zeta_t(\omega) = 0, \dot{A}_t^+(\omega) = \dot{A}_t^-(\omega) \in \mathbb{R}\Big\} \in \mathcal{B}\big([0,T]\big) \times \mathcal{F}_T.\end{aligned}$$

Then $\mathbb{P}_\sigma(\Omega_0) = 1$, and, by the Lebesgue differentiation theorem (see, e.g., [45, Theorem 7.7, p. 139]), Leb$[t : (t,\omega) \in \Theta_0] = T$ for all $\omega \in \Omega_0$. Applying the Fubini theorem there exists $\mathbb{T}^u \subset [0,T]$ such that

$$\text{Leb}[\mathbb{T}^u] = T \text{ and } \mathbb{P}_\sigma[\Omega_t^1] = 1 \text{ for all } t \in \mathbb{T}^u, \text{ where } \Omega_t^1 := \{\omega \in \Omega : (t,\omega) \in \Theta_0\}. \tag{4.6}$$

Note that $\Omega_t^1 \in \mathcal{F}_{t+} \subset \mathcal{F}_t^*$, thanks to Proposition 6.2 in the appendix. Moreover, for any $t \in \mathbb{T}^u$, by Proposition 2.6 one can easily see that there exists $\Omega_t^2 \in \mathcal{F}_t$ such that

$$\begin{gathered} \mathbb{P}_\sigma[\Omega_t^2] = 1 \text{ and } du_s^{t,\omega} = Z_s^{t,\omega} \cdot dB_s + dA_s^{t,\omega}, \\ 0 \le s \le T-t, \ \mathbb{P}_{\sigma^{t,\omega}}\text{-a.s. for all } \omega \in \Omega_t^2. \end{gathered} \tag{4.7}$$

Now define $\Omega_t := \Omega_t^1 \cap \Omega_t^2 \cap \Omega_0 \in \mathcal{F}_t^*$ for all $t \in \mathbb{T}^u$, and then we may find $\Omega_t^u \subset \Omega_t$ such that

$$\Omega_t^u \in \mathcal{F}_t, \quad \mathbb{P}_\sigma[\Omega_t^u] = 1, \quad \text{for all } t \in \mathbb{T}^u. \tag{4.8}$$

Define $\dot{A}_t(\omega) := \dot{A}_t^+(\omega) = \dot{A}_t^-(\omega)$ for $(t,\omega) \in \Theta_0$. We claim that $(\dot{A}_t(\omega), Z_t(\omega)) \in \mathcal{J}_L u_t(\omega)$ for all $t \in \mathbb{T}^u$, $\omega \in \Omega_t^u$ and $L > 0$. Without loss of generality, we shall only show that

$$(\dot{A}_t(\omega) + \varepsilon, Z_t(\omega)) \in \underline{\mathcal{J}}_L u_t(\omega) \quad \text{for any} \quad \varepsilon > 0. \tag{4.9}$$

Indeed, fix $t \in \mathbb{T}^u$ and $\omega \in \Omega_t^u$. First, since $A(\omega)$ is continuous, we have

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} |\sigma_s^T Z_s(\omega) - \sigma_t^T Z_t(\omega)| ds = 0, \ \lim_{h\downarrow 0} \frac{1}{h}[A_{t+h}(\omega) - A_t(\omega)] = \dot{A}_t(\omega).$$

Next, set $\delta := \frac{\varepsilon}{2L(1+|Z_t(\omega)|}$. By Lemma 6.4 in the appendix, there exists $\textsc{h} \in \mathcal{T}_{T-t}$ such that

$$\begin{aligned} \textsc{h} = \inf\Big\{ s > 0 : \int_0^s |(\sigma^T Z)_r^{t,\omega} - (\sigma^T Z)_t(\omega)| dr \ge \delta s, \text{ or } |\sigma_s^{t,\omega} - \sigma_t(\omega)| \ge \delta, \\ \text{or } A_s^{t,\omega} - A_t(\omega) \ge \Big(\dot{A}_t(\omega) + \frac{\varepsilon}{2}\Big) s \Big\} \ \wedge \ (T-t), \quad \mathbb{P}_{\sigma^{t,\omega}}\text{-a.s.} \end{aligned}$$

By (4.5) we see that $\textsc{h} > 0$ and thus $\textsc{h} \in \mathcal{T}_{T-t}^+$. For any $\lambda \in \mathbb{L}_L(\mathbb{F})$ and $\tau \in \mathcal{T}_{\textsc{h}}$, by (4.7) we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[u_\tau^{t,\omega} - Q_\tau^{\dot{A}_t(\omega)+\varepsilon, Z_t(\omega)}\Big] - u_t(\omega) \\ &= \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[u_\tau^{t,\omega} - u_0^{t,\omega} - (\dot{A}_t(\omega)+\varepsilon)\tau - Z_t(\omega)\cdot B_\tau\Big] \\ &= \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[\int_0^\tau [Z_s^{t,\omega} - Z_t(\omega)] \cdot dB_s + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega)+\varepsilon)\tau\Big] \\ &= \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[\int_0^\tau [Z_s^{t,\omega} - Z_0^{t,\omega}] \cdot (\sigma_s^{t,\omega}\lambda_s) ds + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega)+\varepsilon)\tau\Big] \\ &\le \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[L\int_0^\tau |(\sigma^T Z)_s^{t,\omega} - (\sigma^T Z)_t(\omega)| ds \\ &\qquad + L|Z_t(\omega)| \int_0^\tau |\sigma_s^{t,\omega} - \sigma_t(\omega)| ds + (A_\tau^{t,\omega} - A_0^{t,\omega}) - (\dot{A}_t(\omega)+\varepsilon)\tau\Big] \\ &\le \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega},\lambda}}\Big[L\delta\tau + L|Z_t(\omega)|\delta\tau + (\dot{A}_t(\omega) + \frac{\varepsilon}{2})\tau - (\dot{A}_t(\omega)+\varepsilon)\tau\Big] = 0. \end{aligned}$$

Then (4.9) follows from the arbitrariness of λ and τ. ∎

4.8. Comparison result for general semilinear PPDEs. We are now ready for the key step for the proof of Theorem 4.1. We observe that this statement is an adaptation of the approach of Caffarelli and Cabre [3] to the comparison in the context of the standard Crandall–Lions theory of viscosity solutions in finite dimensional spaces. See [3, Theorem 5.3, p. 45].

PROPOSITION 4.18. *Let Assumption* 3.1 *hold true,* $L \ge L_0$*, and* $u, v \in C_2^0(\Theta)$ *be the* $\mathcal{P}_L$*-viscosity subsolution and supersolution of* (3.1)*. Then,* $w := u - v$ *is an* L*-viscosity subsolution of*

$$-\mathcal{L}w(t,\omega) - L|w_t(\omega)| - L|\sigma_t^T(\omega)\partial_\omega w_t(\omega)| \le 0. \tag{4.10}$$

Before we prove this proposition, we use it to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.18, functional $u - v$ is a $\mathcal{P}_L$-viscosity subsolution of PPDE (4.10). Clearly, 0 is a classical supersolution of the same equation. Since $(u - v)_T \le 0$, we conclude from the partial comparison Proposition 4.15 that $u - v \le 0$ on Θ. ∎

Proof of Proposition 4.18. Without loss of generality, we only check the viscosity subsolution property at $(t, \omega) = (0, 0)$. For an arbitrary $(\alpha, \beta) \in \underline{\mathcal{J}}_L w_0$, we want to show that

$$-\alpha - L|w_0| - L|\sigma_0^T\beta| \le 0. \tag{4.11}$$

1. By definition, there exists $\mathrm{H} \in \mathcal{T}^+$ such that

$$w_0 = \max_{\tau \in \mathcal{T}_{\mathrm{H}}} \overline{\mathcal{E}}_L\big[(w - Q^{\alpha,\beta})_\tau\big].$$

Fix $\delta > 0$. By otherwise choosing a smaller H, we may assume without loss of generality that

$$|\varphi_t - \varphi_0| \le \delta \quad \text{for} \quad \varphi = B, \sigma, u, v. \tag{4.12}$$

Recall Definition 4.6 and introduce the processes $X := w - Q^{\alpha+\delta,\beta}$ and $Y := \mathrm{Snell}(X_{\cdot\wedge \mathrm{H}})$. Clearly,

$$\overline{\mathcal{E}}_L\left[X_{\mathrm{H}}\right] < w_0 = X_0 \le Y_0 \quad \text{and} \quad Y_{\mathrm{H}} = X_{\mathrm{H}}, \quad \mathbb{P}_\sigma\text{-a.s.} \tag{4.13}$$

Then, it follows from (4.13) and Theorem 4.4(iii) that there exists $\mathbb{P}^* \in \mathcal{P}_L$ and $K \in \mathbb{I}^2$ such that

$$0 > \overline{\mathcal{E}}_L\left[Y_{\mathrm{H}} - Y_0\right] \ge \mathbb{E}^{\mathbb{P}^*}\left[Y_{\mathrm{H}} - Y_0\right] = -\mathbb{E}^{\mathbb{P}^*}\left[K_{\mathrm{H}}\right] = -\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{\mathrm{H}} \mathbf{1}_{\{Y_t = X_t\}} dK_t\right].$$

We shall prove in step 3 below that

$$K \quad \text{is absolutely continuous, } \mathbb{P}_\sigma\text{-a.s.} \tag{4.14}$$

Then, noticing that $\mathbb{P}^*$ is equivalent to $\mathbb{P}_\sigma$, we deduce from the previous inequalities that

$$\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{\mathrm{H}} \mathbf{1}_{\{Y_t = X_t\}} \dot{K}_t dt\right] > 0 \quad \text{and thus} \quad \mathrm{Leb} \otimes \mathbb{P}_\sigma\big[t < \mathrm{H}, Y_t = X_t\big] > 0, \tag{4.15}$$

where $\dot{K}$ is the derivative of K. Moreover, combining Lemma 4.16, Remark 4.8, and Proposition 4.9, we see that $\hat{u}$, and hence u, is a $\mathbb{P}_\sigma$-semimartingale. Then by Proposition 4.17, there exist measurable sets $\mathbb{T}^u \subset [0,T]$ and $\Omega_t^u \in \mathcal{F}_t$ for each $t \in \mathbb{T}^u$ such that (4.4) holds. Similarly, we may find $\mathbb{T}^v$ and Ω_t^v such that (4.4) holds for v as well. Then (4.15) leads to $\mathrm{Leb} \otimes \mathbb{P}_\sigma\big[t \in [0, \textsc{h}) \cap \mathbb{T}^u \cap \mathbb{T}^v, Y_t = X_t\big] > 0$, and thus there exists $t^* \in \mathbb{T}^v \cap \mathbb{T}^u$ such that $\mathbb{P}_\sigma\big[t^* < \textsc{h}, Y_{t^*} = X_{t^*}\big] > 0$, which implies further that, recalling the V defined in (4.1) and Theorem 4.4(i),

$$\mathbb{P}_\sigma\Big[\Omega_{t^*}^u \cap \Omega_{t^*}^v \cap \{t^* < \textsc{h}, Y_{t^*} = X_{t^*}\} \cap \{Y_{t^*} = V_{t^*}\}\Big] \ > \ 0.$$

Therefore, there exists $\omega^* \in \Omega$ such that

$$\begin{aligned} &\text{both } u \text{ and } v \text{ are } \mathcal{P}_L\text{-punctually } C^{1,2} \text{ at } (t^*, \omega^*), \\ &t^* < \textsc{h}(\omega^*) \text{ and } X_{t^*}(\omega^*) = \sup\nolimits_{\tau\in\mathcal{T}} \overline{\mathcal{E}}_L^{t^*,\omega^*}\big[X^{t^*,\omega^*}_{\tau\wedge(\textsc{h}^{t^*,\omega^*} - t^*)}\big]. \end{aligned} \tag{4.16}$$

2. Let $(\alpha^u, \beta^u) \in \mathcal{J}_L u(t^*, \omega^*) \subset cl(\overline{\mathcal{J}}_L u_{t^*}(\omega^*))$ and $(\alpha^v, \beta^v) \in \mathcal{J}_L v(t^*, \omega^*) \subset cl(\underline{\mathcal{J}}_L v_{t^*}(\omega^*))$. Then $(\alpha^u - \delta, \beta^u) \in \overline{\mathcal{J}}_L u_{t^*}(\omega^*)$ and $(\alpha^v + \delta, \beta^v) \in \underline{\mathcal{J}}_L v(t^*, \omega^*)$. Applying Proposition 3.10, we have

$$(\alpha', \beta') \in \overline{\mathcal{J}}_L X_{t^*}(\omega^*), \quad \text{where} \quad \alpha' := \alpha^u - \alpha^v - \alpha - 3\delta,\ \beta' := \beta^u - \beta^v - \beta. \tag{4.17}$$

Choose $\lambda \in \mathbb{L}_L(\mathbb{F})$ such that $(\sigma^T)^{t^*,\omega^*}\beta' \cdot \lambda = L|(\sigma^T)^{t^*,\omega^*}\beta'|$. Then, for any $\varepsilon > 0$, letting $\textsc{h}' \le \textsc{h}^{t^*,\omega^*} - t^*$ be a common localizing time satisfying $|\sigma_t^{t^*,\omega^*} - \sigma_{t^*}(\omega^*)| \le \varepsilon$ for $0 \le t \le \textsc{h}'$, we have

$$\begin{aligned} X_{t^*}(\omega^*) &\le \underline{\mathcal{E}}_L\Big[X_{\textsc{h}'}^{t^*,\omega^*} - Q_{\textsc{h}'}^{\alpha',\beta'}\Big] \le \mathbb{E}^{\mathbb{P}_{\sigma^{t^*,\omega^*},\lambda}}\Big[X_{\textsc{h}'}^{t^*,\omega^*} - Q_{\textsc{h}'}^{\alpha',\beta'}\Big] \\ &= \mathbb{E}^{\mathbb{P}_{\sigma^{t^*,\omega^*},\lambda}}\Big[X_{\textsc{h}'}^{t^*,\omega^*} - \alpha'\textsc{h}' - \int_0^{\textsc{h}'} L|(\sigma^T)_t^{t^*,\omega^*}\beta'|dt\Big] \\ &\le \overline{\mathcal{E}}_L^{t^*,\omega^*}[X_{\textsc{h}'}^{t^*,\omega^*}] - (\alpha' + L|\sigma_{t^*}^T(\omega^*)\beta'| - L\varepsilon|\beta'|)\mathbb{E}^{\mathbb{P}_{\sigma^{t^*,\omega^*},\lambda}}[\textsc{h}']. \end{aligned}$$

With the optimality in (4.16), this implies that $\alpha' + L|\sigma_{t^*}^T(\omega^*)\beta'| - L\varepsilon|\beta'| \le 0$, and by the arbitrariness of $\varepsilon > 0$, we get $\alpha' + L|\sigma_{t^*}^T(\omega^*)\beta'| \le 0$. Moreover, applying Proposition 3.9, the semiviscosity properties of u and v lead to

$$-\alpha^u - F_{t^*}(\omega^*, u_{t^*}(\omega^*), \sigma_{t^*}^T(\omega^*)\beta^u) \le 0, \quad -\alpha^v - F({t^*}(\omega^*, v_{t^*}(\omega^*), \sigma_{t^*}^T(\omega^*)\beta^v) \ge 0.$$

Then, recalling (4.17) and by (4.12),

$$\begin{aligned} 0 &\le \alpha^u + F_{t^*}(\omega^*, u_{t^*}(\omega^*), \sigma_{t^*}^T(\omega^*)\beta^u) - \alpha^v - F_{t^*}(\omega^*, v_{t^*}(\omega^*), \sigma_{t^*}^T(\omega^*) \\ &\qquad - \alpha' - L|\sigma_{t^*}^T(\omega^*)\beta'| \\ &\le \alpha + 3\delta + L|w_{t^*}(\omega^*)| + L|\sigma_{t^*}^T(\omega^*)\beta| \le \alpha + L|w_0| + L|\sigma_0^T\beta| + (3 + L + L|\beta|)\delta. \end{aligned}$$

Now sending $\delta \to 0$, we obtain (4.11).

3. It remains to prove (4.14). By Proposition 4.16 and Remark 4.8, we know the process $\hat{u}$ is an $\overline{\mathcal{E}}_L$-submartingale. Then it follows from Proposition 4.9 and Corollary 4.3 that there exist $\lambda^u \in \mathbb{L}_L(\mathbb{F})$ and $K^u \in \mathbb{I}^2$ such that $d\hat{u}_t = Z_t^u \cdot [dB_t - \sigma_t\lambda_t^u dt] + dK_t^u$, $\mathbb{P}_\sigma$-a.s. and thus

$$du_t = Z_t^u \cdot dB_t - [\sigma_t\lambda_t^u \cdot Z_t^u - L_0|u_t| - F_t^0 - 1]dt + dK_t^u, \quad \mathbb{P}_\sigma\text{-a.s.}$$

Similarly, $dv_t = Z_t^v \cdot dB_t + [-\sigma_t\lambda_t^v \cdot Z_t^v + L_0|v_t| + F_t^0 + 1]dt - dK_t^v$, $\mathbb{P}_\sigma$-a.s. for some $\lambda^v \in \mathbb{L}_L(\mathbb{F})$ and $K^v \in \mathbb{I}^2$. Thus, with appropriately defined processes Z^X, σ^X and the λ^* corresponding to $\mathbb{P}^*$,

$$dX_t = Z_t^X \cdot [dB_t - \sigma_t\lambda_t^* dt] - \sigma_t^X dt + d(K_t^u + K_t^v), \quad \mathbb{P}_\sigma\text{-a.s.} \tag{4.18}$$

Now for any $0 \le s \le t \le T$, define $\tau_s := \inf\{t \ge s \wedge \textsc{h} : X_t = Y_t\}$. Recalling Theorem 4.4(iii), we have $K_{\tau_s} = K_{s\wedge\textsc{h}}$, $\mathbb{P}_\sigma$-a.s. Then, by (4.18) we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^*}\Big[K_{t\wedge\textsc{h}} - K_{s\wedge\textsc{h}}\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big] = \mathbb{E}^{\mathbb{P}^*}\Big[K_{t\wedge\textsc{h}} - K_{\tau_s}\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big]\\
&= \mathbb{E}^{\mathbb{P}^*}\Big[Y_{\tau_s} - Y_{t\wedge\textsc{h}}\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big] \le \mathbb{E}^{\mathbb{P}^*}\Big[X_{\tau_s} - X_{t\wedge\textsc{h}}\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big]\\
&= \mathbb{E}^{\mathbb{P}^*}\Big[\int_{\tau_s}^{t\wedge\textsc{h}}[\sigma_r^X dr - dK_r^u - dK_r^v]\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big] \le \mathbb{E}^{\mathbb{P}^*}\Big[\int_{s\wedge\textsc{h}}^{t\wedge\textsc{h}}|\sigma_r^X|dr\Big|\mathcal{F}_{s\wedge\textsc{h}}\Big].
\end{aligned}$$

This implies that $dK_t \le |\sigma_t^X|dt$, $\mathbb{P}^*$-a.s. and hence also $\mathbb{P}_\sigma$-a.s. ∎

5. Existence. To construct a viscosity solution to a semilinear path-dependent PDE, we need to introduce BSDEs. For $(t,\omega) \in \Theta$, $\tau \in \mathcal{T}_{T-t}$, and $\xi \in \mathbb{L}^2(\mathcal{F}_\tau, \mathbb{P}_{\sigma^{t,\omega}})$, consider the BSDE:

$$Y_s = \xi + \int_s^\tau F_r^{t,\omega}(B_\cdot, Y_r, (\sigma^T)_r^{t,\omega}Z_r)dr - \int_s^\tau Z_r \cdot dB_r, \quad 0 \le s \le \tau, \mathbb{P}_{\sigma^{t,\omega}}\text{-a.s.} \tag{5.1}$$

By Assumption 3.1 and Theorem 4.2, additionally assuming that

$$\mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega}}}\Big[\int_0^{T-t} F_s^{t,\omega}(B,0,0)^2 ds\Big] < \infty$$

for all $(t,\omega) \in \Theta$, one may easily prove by standard arguments that the above BSDE admits a unique $\mathbb{F}$-measurable solution, denoted as $(\mathcal{Y}^{t,\omega}(\tau,\xi), \mathcal{Z}^{t,\omega}(\tau,\xi))$. Now, fix $\xi \in \mathbb{L}^0(\mathcal{F}_T)$ such that $\xi^{t,\omega} \in \mathbb{L}^2(\mathcal{F}_{T-t}, \mathbb{P}_{\sigma^{t,\omega}})$ for any $(t,\omega) \in \Theta$, and define

$$u(t,\omega) := \mathcal{Y}_0^{t,\omega}(T-t, \xi^{t,\omega}). \tag{5.2}$$

The conditions here are slightly weaker than those of [14, section 4.2], but the arguments are more or less the same.

THEOREM 5.1. *Let Assumption* 3.1 *hold true. Assume F is continuous in t and $u \in C_2^0(\Theta)$. Then u is an L-viscosity solution of PPDE* (3.1) *for any $L \ge L_0$.*

Proof. Since $u \in C_2^0(\Theta)$, following similar arguments as in [14, section 4.2], (5.2) implies the dynamic programming principle: given $(t,\omega) \in \Theta$ and $\tau \in \mathcal{T}_{T-t}$,

$$u(t,\omega) = \mathcal{Y}_t^{t,\omega}(\tau, u_\tau^{t,\omega}). \tag{5.3}$$

Without loss of generality, we check only the viscosity subsolution property at $(0,0)$. Assume not, and then there exists $\varphi \in \underline{\mathcal{A}}_L u_0$ with localizing time $\textsc{h}$ such that $-c := \mathcal{L}\varphi_0 + F_0(u_0, \sigma_0^T\partial_\omega\varphi_0) < 0$. By continuity there exists $\tau \in \mathcal{T}_{\textsc{h}}^+$ such that $\mathcal{L}\varphi_t + F_t(u_t, \sigma_t^T\partial_\omega\varphi_t) \le -\frac{c}{2}$ for $0 \le t \le \tau$. Note that $u_t = \mathcal{Y}_t^{0,0}(\tau, u_\tau)$ and denote $Z_t := \mathcal{Z}^{0,0}(\tau, u_\tau)$. Then, by (5.3) and the functional Itô formula (3.3),

$$\begin{aligned}
&[\varphi - u]_\tau - [\varphi - u]_0\\
&\quad= \int_0^\tau [\mathcal{L}\varphi_t + F_t(u_t, \sigma_t^T Z_t)]dt + \int_0^\tau [\partial_\omega\varphi_t - Z_t]\cdot dB_t\\
&\quad\le \int_0^\tau \Big[-\frac{c}{2} + F_t(u_t, \sigma_t^T Z_t) - F_t(u_t, \sigma_t^T\partial_\omega\varphi_t)\Big]dt + \int_0^\tau [\partial_\omega\varphi_t - Z_t]\cdot dB_t\\
&\quad= \int_0^\tau \Big[-\frac{c}{2} - [\partial_\omega\varphi_t - Z_t]\cdot\sigma_t\lambda_t\Big]dt + \int_0^\tau [\partial_\omega\varphi_t - Z_t]\cdot dB_t,\ \mathbb{P}_\sigma\text{-a.s.},
\end{aligned}$$

where $\lambda \in \mathbb{L}_{L_0}(\mathbb{F})$. Note that $\mathbb{P}_{\sigma,\lambda}$ and $\mathbb{P}_\sigma$ are equivalent. This implies

$$[\varphi - u]_\tau - [\varphi - u]_0 \le -\frac{c}{2}\tau + \int_0^\tau [\partial_\omega \varphi_t - Z_t]\cdot[dB_t - \sigma_t\lambda_t dt],\ \mathbb{P}_{\sigma,\lambda}\text{-a.s.}$$

Thus, noting that $L \ge L_0$ and that $dB_t - \sigma_t\lambda_t dt$ is a $\mathbb{P}_{\sigma,\lambda}$-martingale,

$$[\varphi - u]_0 \ge \mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}\Big[[\varphi - u]_\tau + \frac{c}{2}\tau\Big] > \mathbb{E}^{\mathbb{P}_{\sigma,\lambda}}\Big[[\varphi - u]_\tau\Big] \ge \underline{\mathcal{E}}_L\Big[[\varphi - u]_\tau\Big],$$

contradicting the fact that $\varphi \in \underline{\mathcal{A}}_L u_0$. ∎

The following proposition gives a sufficient condition so that $u \in C_2^0(\Theta)$. The proof follows from standard BSDE estimates (see, e.g., [14, section 4.2]), and thus is omitted.

PROPOSITION 5.2. *Let F, ξ be uniformly continuous in ω, F continuous in t. Then $u \in C_2^0(\Theta)$.*

6. Appendix.

6.1. Martingale representation. Recall (2.2), denote $X := X^\sigma$ for notational simplicity, and define $\mathbb{F}^X := \{\mathcal{F}_t^X, 0 \le t \le T\}$ to be the filtration generated by the process X.

LEMMA 6.1. *For any $\eta \in \mathbb{L}^1(\mathcal{F}_T^X, \mathbb{P}_0)$, we have $\mathbb{E}^{\mathbb{P}_0}[\eta|\mathcal{F}_t^X] = \mathbb{E}^{\mathbb{P}_0}[\eta|\mathcal{F}_t]$, $\mathbb{P}_0$-a.s.*

Proof. Denote $\mathcal{G}_t := \sigma\{B_s - B_t : s \ge t\}$. Since X is a strong solution, we see that $\mathcal{F}_t^X \subset \mathcal{F}_t$ and $\mathcal{F}_T^X \subset \mathcal{F}_t^X \vee \mathcal{G}_t$. In particular, $\mathcal{F}_t^X$ and $\mathcal{F}_t$ are independent of $\mathcal{G}_t$ under $\mathbb{P}_0$. Then, $\mathbb{E}^{\mathbb{P}_0}[\mathbf{1}_E\mathbf{1}_{E'}|\mathcal{F}_t^X] = \mathbf{1}_E\mathbb{P}_0[E'] = \mathbb{E}^{\mathbb{P}_0}[\mathbf{1}_E\mathbf{1}_{E'}|\mathcal{F}_t]$ for any $E \in \mathcal{F}_t^X, E' \in \mathcal{G}_t$. Now the result follows from the standard argument of monotone class theorem. ∎

We next establish the martingale representation property for $\mathbb{P}_\sigma$.

Proof of Theorem 4.2. Uniqueness is obvious. To prove existence, by standard approximation arguments, we may assume without loss of generality that ξ is Lipschitz continuous in ω. Denote

$$u(t,\omega) := \mathbb{E}^{\mathbb{P}_{\sigma^{t,\omega}}}[\xi^{t,\omega}] = \mathbb{E}^{\mathbb{P}_0}\Big[\xi^{t,\omega}(X^{t,\omega})\Big],\ \text{ where }\ X_s^{t,\omega} = \int_0^s \sigma_r^{t,\omega}(X_\cdot^{t,\omega})dB_r,\ \mathbb{P}_0\text{-a.s.}$$

Since σ is Lipschitz continuous in ω, one can show that u is uniformly Lipschitz continuous in ω and, by Proposition 2.6 with $\lambda = 0$, u is a $\mathbb{P}_\sigma$-martingale. The rest of the proof organizes into three steps.

1. We first assume σ is constant and show that the above Z exists and is bounded. Indeed, by standard approximation again, we may assume $\xi = g(B_{t_1}, \cdots, B_{t_n})$ for some $0 < t_1 < \cdots < t_n \le T$ and smooth g. Then one can easily see that $u(t,\omega) = v(t, B_{t_1}, \cdots, B_{t_i}, B_t)$, $t_i \le t < t_{i+1}$, for some smooth function v. Applying Itô's formula we obtain the representation with $Z_t = Dv(t, B_{t_1}, \cdots, B_{t_i}, B_t)$, where Dv is the gradient in terms of the last variable B_t. It is straightforward to check that Dv is bounded by the Lipschitz constant of ξ, which implies the boundedness of Z.

2. We now prove the general case. Denote $\tilde{\xi} := \xi(X_\cdot)$, $\tilde{u} := u(X_\cdot)$, and $\tilde{\sigma} := \sigma(X_\cdot)$. It follows from Lemma 6.1 that $\tilde{u}$ is a $(\mathbb{P}_0, \mathbb{F})$-martingale. By the standard martingale representation theorem under $\mathbb{P}_0$, there exists $\tilde{Z}$ such that $\mathbb{E}^{\mathbb{P}_0}\big[\int_0^T |\tilde{Z}_t|^2 dt\big] < \infty$ and $d\tilde{u}_t = \tilde{Z}_t \cdot dB_t$, $\mathbb{P}_0$-a.s. We claim that

$$\tilde{Z} = \tilde{\sigma}^T\zeta \quad \text{for some } \zeta \in \mathbb{L}^0(\mathbb{F}, \mathbb{R}^d). \tag{6.1}$$

Then $d\tilde{u}_t = \zeta_t \cdot dX_t$, and thus $d\langle\tilde{u}, X\rangle_t = \tilde{\sigma}_t\tilde{\sigma}_t^T\zeta_t dt$, $\mathbb{P}_0$-a.s. Rewrite $\sigma = P\sigma^* Q$, where

P, Q take values in orthogonal matrices and $\sigma^* = \mathrm{diag}[a_1, cds, a_d]$ is a diagonal matrix valued process. Denote $\tilde{P} := P(X)$, and similarly for other terms. Since $\langle \tilde{u}, X\rangle \in \mathbb{L}^0(\mathbb{F}^X)$, we see that $\tilde{\sigma}\tilde{\sigma}^T\zeta = \tilde{P}(\tilde{\sigma}^*)^2\tilde{P}^T\zeta \in \mathbb{L}^0(\mathbb{F}^X)$, and thus $(\tilde{\sigma}^*)^2\tilde{P}^T\zeta \in \mathbb{L}^0(\mathbb{F}^X)$. Denote $\tilde{P}^T\zeta := [\zeta_1, \cdots, \zeta_d]^T$, and let ζ' be determined by

$$\tilde{P}^T\zeta' := [\zeta_1\mathbf{1}_{\{\tilde{a}_1\neq 0\}}, \cdots, \zeta_d\mathbf{1}_{\{\tilde{a}_d\neq 0\}}]^T.$$

Then one can easily check that

- $\tilde{\sigma}^*\tilde{P}^T\zeta' = [\tilde{a}_1\zeta_1\mathbf{1}_{\{\tilde{a}_1\neq 0\}}, \cdots, \tilde{a}_d\zeta_d\mathbf{1}_{\{\tilde{a}_d\neq 0\}}]^T = \tilde{\sigma}^*\tilde{P}^T\zeta$ and thus $\tilde{\sigma}\zeta' = \tilde{\sigma}\zeta$;
- $\tilde{a}_i^2\zeta_i\mathbf{1}_{\{\tilde{a}_i\neq 0\}} \in \mathbb{L}^0(\mathbb{F}^X)$, then $\zeta_i\mathbf{1}_{\{\tilde{a}_i\neq 0\}} \in \mathbb{L}^0(\mathbb{F}^X)$, thus $\tilde{P}^T\zeta' \in \mathbb{L}^0(\mathbb{F}^X)$, and hence $\zeta' \in \mathbb{L}^0(\mathbb{F}^X)$.

The second property implies that $\zeta' = Z(X)$ for some $Z \in \mathbb{L}^0(\mathbb{F})$. Then, by the first property,

$$d\tilde{u}_t = \zeta'_t \cdot dX_t, \quad \mathbb{P}_0\text{-a.s.} \quad \text{and thus} \quad du_t = Z_t \cdot dB_t, \quad \mathbb{P}_\sigma\text{-a.s.}$$

3. It remains to prove the claim (6.1). Consider the decomposition $\tilde{Z} = \tilde{\sigma}^T\zeta + \eta$, where $\tilde{\sigma}\eta = 0$, and let us prove that $\eta = 0$, $\mathbb{P}_0$-a.s. For this purpose, let $n > 0$, $h := \frac{T}{n}$, $t_i := ih$, $i = 0, \cdots, n$, and denote $\bar{\eta}_i := h^{-1}\mathbb{E}^{\mathbb{P}_0}\big[\int_{t_i}^{t_{i+1}}\eta_s ds|\mathcal{F}_{t_i}\big]$, $\bar{\sigma}_i := h^{-1}\mathbb{E}^{\mathbb{P}_0}\big[\int_{t_i}^{t_{i+1}}\tilde{\sigma}_s ds|\mathcal{F}_{t_i}\big]$, $i = 0, \ldots, n-1$. Then,

$$\mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T |\eta_t|^2 dt\Big] = \mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T \tilde{Z}_t \cdot \eta_t dt\Big] \;=\; \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}_0}\Big[\int_{t_i}^{t_{i+1}} \tilde{Z}_t \cdot \bar{\eta}_i dt\Big] + R_1^n,$$

where $R_1^n \longrightarrow 0$ as $n \to \infty$. Denoting $B_s^t := B_s - B_t$, it follows from the Itô isometry that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T |\eta_t|^2 dt\Big] \\
&\quad= \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}_0}\Big[\bar{\eta}_i \cdot B_{t_{i+1}}^{t_i}\int_{t_i}^{t_{i+1}} \tilde{Z}_t \cdot dB_t\Big] + R_1^n \\
&\quad= \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}_0}\Big[\big(\tilde{u}_{t_{i+1}} - \tilde{u}_{t_i}\big)\bar{\eta}_i \cdot B_{t_{i+1}}^{t_i}\Big] + R_1^n \\
&\quad= \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}_0}\Big[\big(u_{t_{i+1}}(X) - u_{t_i}(X)\big)\bar{\eta}_i \cdot B_{t_{i+1}}^{t_i}\Big] + R_1^n \\
&\quad= \sum_{i=0}^{n-1}\mathbb{E}^{\mathbb{P}_0}\Big[\big(u_{t_{i+1}}(X \otimes_{t_i} \bar{\sigma}_i B^{t_i}) - \mathbb{E}^{\mathbb{P}_0}[u_{t_{i+1}}(X \otimes_{t_i} \bar{\sigma}_i B^{t_i})|\mathcal{F}_{t_i}]\big)\bar{\eta}_i \cdot B_{t_{i+1}}^{t_i}\Big] + R_2^n,
\end{aligned}$$

where we used the fact that $B_{t_{i+1}}^{t_i}$ and $\mathcal{F}_{t_i}$ are $\mathbb{P}_0$-independent. By the uniform Lipschitz continuity of u, we see that $R_2^n \longrightarrow 0$ as $n \to \infty$. We further decompose $\bar{\eta}_i = \bar{\sigma}_i^T\varepsilon_i + \hat{\eta}_i$, where $\bar{\sigma}_i\hat{\eta} = 0$. Note that, conditionally on $\mathcal{F}_{t_i}$, $\bar{\sigma}_i B^{t_i}$ and $\hat{\eta}_i \cdot B_{t_{i+1}}^{t_i}$ are $\mathbb{P}_0$-independetnt. Then

$$\mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T |\eta_t|^2 dt\Big] = R_2^n + \sum_{i=1}^n r_i^n,$$

where $r_i^n := \mathbb{E}^{\mathbb{P}_0}\Big[\big(u_{t_{i+1}}(X \otimes_{t_i} \bar{\sigma}_i B^{t_i}) - \mathbb{E}^{\mathbb{P}_0}[u_{t_{i+1}}(X \otimes_{t_i} \bar{\sigma}_i B^{t_i})|\mathcal{F}_{t_i}]\big)\bar{\sigma}_i^T\varepsilon_i \cdot B_{t_{i+1}}^{t_i}\Big].$

We now analyze r_i^n. By step 1, there exists γ bounded by the Lipschitz constant of $u_{t_{i+1}}$ (in terms of ω) such that $u_{t_{i+1}}(X \otimes_{t_i} \bar\sigma_i B^{t_i}) - \mathbb{E}^{\mathbb{P}_0}[u_{t_{i+1}}(X \otimes_{t_i} \bar\sigma_i B^{t_i})|\mathcal{F}_{t_i}] = \int_{t_i}^{t_{i+1}} \gamma_t \cdot \bar\sigma_i dB_t^{t_i}$. Then

$$|r_i^n| = \Big|\mathbb{E}^{\mathbb{P}_0}\Big[\int_{t_i}^{t_{i+1}} \gamma_t dt \cdot \bar\sigma_i \bar\sigma_i^T \varepsilon_i\Big]\Big| = \Big|\mathbb{E}^{\mathbb{P}_0}\Big[\int_{t_i}^{t_{i+1}} \gamma_t dt \cdot \bar\sigma_i \bar\eta_i\Big]\Big| \le Ch\mathbb{E}^{\mathbb{P}_0}\big[|\bar\sigma_i\bar\eta_i|\big].$$

Since $\tilde\sigma\eta = 0$, we have $0 = \mathbb{E}^{\mathbb{P}_0}[\int_{t_i}^{t_{i+1}} \tilde\sigma_t \eta_t dt|\mathcal{F}_{t_i}] = \bar\sigma_i\bar\eta_i + \mathbb{E}^{\mathbb{P}_0}[\int_{t_i}^{t_{i+1}}([\tilde\sigma_t - \bar\sigma_i]\eta_t + \bar\sigma_i[\eta_t - \bar\eta_i])dt|\mathcal{F}_{t_i}]$. Thus, noting that $\sigma \in C_2^0(\Theta) \subset \mathbb{S}^2$ and $\mathbb{E}^{\mathbb{P}_0}\big[\int_0^T |\eta_t|^2 dt\big] < \infty$,

$$\begin{aligned}\sum_{i=1}^n |r_i^n| &\le C\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_0}\Big[\int_{t_i}^{t_{i+1}}\big|[\tilde\sigma_t - \bar\sigma_i]\eta_t + \bar\sigma_i[\eta_t - \bar\eta_i]\big|dt\Big]\\ &\le C\Big(\sum_{i=1}^n \mathbb{E}^{\mathbb{P}_0}\Big[\int_{t_i}^{t_{i+1}}[|\tilde\sigma_t - \bar\sigma_i|^2 + |\eta_t - \bar\eta_i|^2]dt\Big]\Big)^{\frac12},\end{aligned}$$

which tends to zero, as $n \to \infty$. This implies that $\mathbb{E}^{\mathbb{P}_0}\Big[\int_0^T |\eta_t|^2 dt\Big] = 0$ and thus proves (6.1). ∎

6.2. Some measurability issues. As a preparation for the nonlinear optimal stopping problem, we investigate a subtle but crucial measurability issue. Denote

$$\mathbb{F}^* := \mathbb{P}_\sigma\text{-augmentation of } \mathbb{F} \quad\text{and}\quad \mathcal{T}^* := \text{the set of } \mathbb{F}^*\text{-stopping times}, \tag{6.2}$$

and recall that $\mathbb{F}$ is the natural filtration of B. We start with the Blumenthal 0-1 law under $\mathbb{P}_\sigma$.

PROPOSITION 6.2 (Blumenthal's 0-1 law). *Under Assumption* 2.2*, for any bounded* $\xi \in \mathcal{F}_{t+}$, $\mathbb{E}^{\mathbb{P}_\sigma}[\xi|\mathcal{F}_t] = \xi$, $\mathbb{P}_\sigma$*-a.s. Consequently, the augmented filtration* $\mathbb{F}^*$ *is right continuous.*

Proof. Denote again $X := X^\sigma$ and $\tilde\xi := \xi(X)$. Clearly $\tilde\xi \in \mathcal{F}_{t+}$, and by the Blumenthal 0-1 law under $\mathbb{P}_0$, we have $\mathbb{E}^{\mathbb{P}_0}[\tilde\xi|\mathcal{F}_t] = \tilde\xi$, $\mathbb{P}_0$-a.s. Since $\tilde\xi \in \mathbb{L}^1(\mathcal{F}_T^X, \mathbb{P}_0)$, applying Lemma 6.1 we see that $\mathbb{E}^{\mathbb{P}_0}[\tilde\xi|\mathcal{F}_t^X] = \tilde\xi$, $\mathbb{P}_0$-a.s., which exactly means $\mathbb{E}^{\mathbb{P}_\sigma}[\xi|\mathcal{F}_t] = \xi$, $\mathbb{P}_\sigma$-a.s. ∎

Following the arguments in [17], we have the following.

PROPOSITION 6.3. *Let* $\tau \in \mathcal{T}^*$ *be previsible, namely, there exist* $\tau_n \in \mathcal{T}^*$ *such that* $\tau_n < \tau$ *and* $\tau_n \uparrow \tau$*. Then there exists* $\bar\tau \in \mathcal{T}$ *such that* $\bar\tau = \tau$, $\mathbb{P}_\sigma$*-a.s.*

Proof. Denote by $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0\le t\le T}$ the right filtration. For each $n \ge 1$ and $r \in \mathbb{Q}\cap[0,T]$, denote $E_r^n := \{\tau_n < r\} \in \mathcal{F}_r^*$. Then there exists $\tilde E_r^n \in \mathcal{F}_r$ such that $\tilde E_r^n \subset E_r^n$ and $\mathbb{P}_\sigma(E_r^n \backslash \tilde E_r^n) = 0$. Note that E_r^n is decreasing in n and increasing in r; without loss of generality we may assume that $\tilde E_r^n$ has the same monotonicity. Define

$$\tilde\tau_n := \inf\{r \in \mathbb{Q}\cap[0,T] : \omega \in \tilde E_r^n\} \wedge T, \quad \tilde\tau := \lim_{n\to\infty}\tilde\tau_n.$$

One can easily check that $\tilde\tau_n$ and $\tilde\tau$ are $\mathbb{F}^+$-stopping times, $\tilde\tau_n \uparrow \tilde\tau$, and $\mathbb{P}_\sigma(\tilde\tau = \tau) = 1$. To construct the desired $\mathbb{F}$-stopping time, we modify $\tilde\tau_n$ and $\tilde\tau$ as follows.

$$\bar\tau_n := \Big(\tilde\tau_n \mathbf{1}_{\{\tilde\tau_n < \tilde\tau\}} + T\mathbf{1}_{\{\tilde\tau_n = \tilde\tau\}}\Big) \wedge \Big(T - \frac{1}{n}\Big), \quad \bar\tau := \lim_{n\to\infty}\bar\tau_n.$$

It is clear that $\bar\tau_n$ are also $\mathbb{F}^+$-stopping times, $\bar\tau_n \uparrow \bar\tau$, $\bar\tau \geq \tilde\tau$, and $\mathbb{P}_\sigma(\bar\tau = \tau) = 1$. Moreover, for each n, on $\{\tilde\tau_n < \tilde\tau\}$ we have $\bar\tau_n = \tilde\tau_n \wedge (T - \frac{1}{n}) < \tilde\tau \leq \bar\tau$; and on $\{\tilde\tau_n = \tilde\tau\}$, we have $\tilde\tau_m = \tilde\tau$ for all $m \geq n$, thus $\bar\tau_m = T - \frac{1}{m}$, $\bar\tau = T$, and therefore $\bar\tau_n = T - \frac{1}{n} < \bar\tau$. So in both cases we have $\bar\tau_n < \bar\tau$. Then $\{\bar\tau \leq t\} = \cap_{n\geq 1}\{\bar\tau_n < t\} \in \mathcal{F}_t$ for all $t \leq T$, that is, $\bar\tau$ is an $\mathbb{F}$-stopping time. ∎

LEMMA 6.4. *Assume $X \in \mathbb{L}^0(\mathbb{F})$ is continuous (in t), $\mathbb{P}_\sigma$-a.s. Then there exists $\tau \in \mathcal{T}$ such that $\tau = \inf\{t : X_t = 0\} \wedge T$, $\mathbb{P}_\sigma$-a.s.*

Proof. If $X_0 = 0$, $\tau := 0$ satisfies all the requirement. We thus assume $X_0 \neq 0$. Set $E := \{\omega : X(\omega) \text{ continuous on } [0,T]\}$ and $\hat X := X\mathbf{1}_E + \mathbf{1}_{E^c}$. Then $\hat X \in \mathbb{L}^0(\mathbb{F}^*)$ is continuous for all ω and $\hat X_0 \neq 0$. Denote $\hat\tau := \inf\{t : \hat X_t = 0\} \wedge T \in \mathcal{T}^*$ and $\hat\tau_n := \inf\{t : |\hat X_t| \leq \frac{1}{n}\} \wedge (T - \frac{1}{n}) \in \mathcal{T}^*$. Clearly $\hat\tau_n < \hat\tau$ and $\hat\tau_n \uparrow \hat\tau$. By Proposition 6.3, there exists $\tau \in \mathcal{T}$ such that $\hat\tau = \tau$, $\mathbb{P}_\sigma$-a.s. Note that $\tau = \inf\{t : X_t = 0\} \wedge T$ on $\{\hat\tau = \tau\} \cap E$. Since $\mathbb{P}_\sigma[\hat\tau = \tau] = \mathbb{P}_\sigma[E] = 1$, this concludes the proof. ∎

6.3. Optimal stopping under $\overline{\mathcal{E}}_L$. The next result is a BSDE characterization of the nonlinear expectation $\overline{\mathcal{E}}_L$, which extends the g-expectation of Peng [35] to general σ.

PROPOSITION 6.5. *Let $\xi \in \mathbb{L}^2(\mathcal{F}_T, \mathbb{P}_\sigma)$ and $\tau \in \mathcal{T}$.*

(i) *For any $\lambda \in \mathbb{L}^0(\mathbb{F})$ bounded, $\mathbb{E}^{\mathbb{P}^{\tau,\omega}_{\sigma,\lambda}}[\xi^{\tau,\omega}] = Y^\lambda_\tau(\omega)$ for $\mathbb{P}_\sigma$-a.e. ω, where*

$$Y^\lambda_t = \xi + \int_t^T Z_s \cdot \sigma_s \lambda_s ds - \int_t^T Z_s \cdot dB_s, \quad \mathbb{P}_\sigma\text{-a.s.}$$

(ii) *For any $L > 0$, $\overline{\mathcal{E}}^{\tau,\omega}_L[\xi^{\tau,\omega}] = Y_\tau(\omega)$ for $\mathbb{P}_\sigma$-a.e. ω, where*

$$Y_t = \xi + \int_t^T L|\sigma_s^T Z_s| ds - \int_t^T Z_s \cdot dB_s, \quad \mathbb{P}_\sigma\text{-a.s.}$$

Proof. (i) The result follows directly from the definition of $\mathbb{P}_{\sigma,\lambda}$ and Proposition 2.6.

(ii) Following Proposition 2.6, for $\mathbb{P}_\sigma$-a.e. ω, we have $Y^{\tau,\omega}_t = \tilde Y_t$, $0 \leq t \leq \tilde T := T - \tau(\omega)$, $\mathbb{P}_{\sigma^{\tau,\omega}}$-a.s., where $\tilde Y$ is the solution to the following shifted BSDE:

$$\tilde Y_t = \xi^{\tau,\omega} + \int_t^{\tilde T} L|(\sigma^{\tau,\omega})_s^T \tilde Z_s| ds - \int_t^{\tilde T} \tilde Z_s \cdot dB_s, \quad 0 \leq t \leq \tilde T, \quad \mathbb{P}_{\sigma^{\tau,\omega}}\text{-a.s.}$$

Clearly, we have $\tilde Y_0 = \overline{\mathcal{E}}^{\tau,\omega}_L[\xi^{\tau,\omega}]$, and therefore, $Y_\tau(\omega) = \overline{\mathcal{E}}^{\tau,\omega}_L[\xi^{\tau,\omega}]$, $\mathbb{P}_\sigma$-a.s. ∎

As an application of Proposition 6.5, we study the optimal stopping problem under $\overline{\mathcal{E}}_L$ via reflected BSDE under $\mathbb{P}_\sigma$:

$$\text{(6.3)} \quad \begin{cases} Y_t = X_{\textsc{h}} + \displaystyle\int_t^{\textsc{h}} L|\sigma_s^T Z_s| ds - \int_t^{\textsc{h}} Z_s \cdot dB_s + K_{\textsc{h}} - K_t; \\ Y \geq X, \quad (Y_t - X_t) dK_t = 0; \end{cases} \quad 0 \leq t \leq \textsc{h}, \quad \mathbb{P}_\sigma\text{-a.s.}$$

Here the component K of the solution triplet (Y, Z, K) is by definition nondecreasing with $K_0 = 0$. Given the martingale representation Theorem 4.2, it follows from standard arguments (see, e.g., [18]) that (6.3) has a unique solution $(Y, Z, K) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{I}^2$, restricted on $[0, \textsc{h}]$.

We are now ready to establish the nonlinear Snell envelope theory.

Proof of Theorem 4.4. (i) Since X and Y are continuous, $\mathbb{P}_\sigma$-a.s., applying Lemma 6.4 we have $\tau^* \in \mathcal{T}$ such that $\tau^* = \inf\{t : Y_t = X_t\} \wedge \text{H}$, $\mathbb{P}_\sigma$-a.s. Moreover, since $Y_{\text{H}} = X_{\text{H}}$, it is clear that $Y_{\tau^*} = X_{\tau^*}$, $\mathbb{P}_\sigma$-a.s. To see the optimality of τ^*, we first note that $Y > X$ in $[0, \tau^*)$. Then it follows from the minimum condition in (6.3) that $K = 0$ in $[0, \tau^*)$. Thus RBSDE (6.3) becomes a standard BSDE on $[0, \tau^*]$. Now it follows from Proposition 6.5(ii) that $Y_0 = \overline{\mathcal{E}}_L[Y_{\tau^*}] = \overline{\mathcal{E}}_L[X_{\tau^*}]$.

(ii) We first show that $V_0 = Y_0$. For any $\tau \in \mathcal{T}_{\text{H}}$, by Proposition 6.5(ii) $\overline{\mathcal{E}}_L[X_\tau] = Y_0^\tau$, where

$$Y_t^\tau = X_\tau + \int_t^\tau L|\sigma_s^T Z_s^\tau| ds - \int_t^\tau Z_s^\tau \cdot dB_s, \quad 0 \le t \le \tau,\ \mathbb{P}_\sigma\text{-a.s.}$$

Noting that $X_\tau \le Y_\tau$, it follows from the comparison principle of BSDEs that $Y_0^\tau \le Y_0$. Then $V_0 \le Y_0$. On the other hand, by (i) we have $Y_0 = \overline{\mathcal{E}}_L[X_{\tau^*}] \le V_0$. So $Y_0 = V_0$.

For the general case, following Proposition 2.6, for any $\tau \in \mathcal{T}_{\text{H}}$ and $\mathbb{P}_\sigma$-a.e. ω, we have $Y_t^{\tau,\omega} = \tilde{Y}_t$, $0 \le t \le \tilde{\text{H}} := \text{H}^{\tau,\omega} - \tau(\omega)$, $\mathbb{P}_{\sigma^{\tau,\omega}}$-a.s., where $\tilde{Y}$ is the solution to the following shifted RBSDE:

$$\begin{cases} \tilde{Y}_t = X_{\tilde{\text{H}}}^{\tau,\omega} + \displaystyle\int_t^{\tilde{\text{H}}} L|(\sigma^{\tau,\omega})_s^T \tilde{Z}_s| ds - \int_t^{\tilde{\text{H}}} \tilde{Z}_s \cdot dB_s + \tilde{K}_{\tilde{\text{H}}} - \tilde{K}_t; \\ \tilde{Y} \ge X^{\tau,\omega}, \quad (\tilde{Y}_t - X_t^{\tau,\omega}) d\tilde{K}_t = 0; \end{cases} \quad 0 \le t \le \tilde{\text{H}}, \quad \mathbb{P}_{\sigma^{\tau,\omega}}\text{-a.s.}$$

Then the above arguments (for $t = 0$) imply that $V_\tau(\omega) = \tilde{Y}_0$, and therefore, $V_\tau = Y_\tau$, $\mathbb{P}_\sigma$-a.s.

(iii) Take $\mathbb{P}^* := \mathbb{P}_{\sigma,\lambda^*}$, where λ^* is so that $(\lambda^*)^T \sigma^T Z = L|\sigma^T Z|$ holds, completing the proof. ∎

We remark that the optimal stopping problem here relies on the convergence Proposition 2.7 implicitly; more precisely, the wellposedness of RBSDE (6.3) relies on the dominated convergence theorem under $\mathbb{P}_\sigma$. In [13] the class $\mathcal{P}_L$ is nondominated and we do not have this type of convergence theorem. Consequently, the optimal stopping problem in [13] is technically much more involved than here. As an application of RBSDE, we may prove Proposition 4.9.

Proof of Proposition 4.9. (i) For any $\tau \in \mathcal{T}$ such that $\tau \ge t$, consider the BSDE:

$$Y_s = u_\tau + \int_s^\tau L|\sigma_r^T Z_r| dr - \int_s^\tau Z_r \cdot dB_r, \quad 0 \le s \le \tau, \quad \mathbb{P}_\sigma\text{-a.s.}$$

One may easily show that $Y_t = \overline{\mathcal{E}}_L\big[u_\tau \big| \mathcal{F}_t\big]$, $\mathbb{P}_\sigma$-a.s. By (ii) of Proposition 6.5, we have $Y_t(\omega) = \overline{\mathcal{E}}_L^{t,\omega}\big[u_{\tau^{t,\omega}}^{t,\omega}\big]$ for $\mathbb{P}_\sigma$-a.e. ω. Since u is a pathwise $\overline{\mathcal{E}}_L$-submartingale and $\tau^{t,\omega} \in \mathcal{T}_{T-t}$, we obtain that $u_t(\omega) \le \overline{\mathcal{E}}_L^{t,\omega}\big[u_{\tau^{t,\omega}}^{t,\omega}\big] = \overline{\mathcal{E}}_L\big[u_\tau \big| \mathcal{F}_t\big](\omega)$, $\mathbb{P}_\sigma$-a.s. Therefore, u is an $\overline{\mathcal{E}}_L$-submartingale.

(ii) Consider the following RBSDE with upper barrier:

$$\begin{cases} Y_t = u_T + \displaystyle\int_t^T L|\sigma_s^T Z_s| ds - \int_t^T Z_s \cdot dB_s - K_T + K_t; \\ Y_t \le u_t, \quad (u_t - Y_t) dK_t = 0; \end{cases} \quad 0 \le t \le T, \mathbb{P}_\sigma\text{-a.s.}$$

Similar to Theorem 4.4, one can show that $Y_t = \text{ess}-\inf_{\tau \in \mathcal{T}, \tau \ge t} \overline{\mathcal{E}}_L[u_\tau | \mathcal{F}_t]$, $\mathbb{P}_\sigma$-a.s. Since u is an $\overline{\mathcal{E}}_L$-submartingale, we get $\overline{\mathcal{E}}_L[u_\tau | \mathcal{F}_t] \ge u_t$, $\mathbb{P}_\sigma$-a.s. for all $\tau \in \mathcal{T}_{T-t}$, and thus $Y \ge u$. On the other hand, by definition $Y \le u$. Hence, $u = Y$. Further, take $\mathbb{P}^* := \mathbb{P}_{\sigma,\lambda^*}$, where λ^* is so that $(\lambda^*)^T \sigma^T Z = L|\sigma^T Z|$ holds. Then the desired result follows. ∎

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