# Mean Field Game Master Equations with Anti-monotonicity Conditions 

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#### Abstract

It is well known that the monotonicity condition, either in Lasry-Lions sense or in displacement sense, is crucial for the global well-posedness of mean field game master equations, as well as for the uniqueness of mean field equilibria and solutions to mean field game systems. In the literature, the monotonicity conditions are always taken in a fixed direction. In this paper we propose a new type of monotonicity condition in the opposite direction, which we call the anti-monotonicity condition, and establish the global well-posedness for mean field game master equations with nonseparable Hamiltonians. Our anti-monotonicity condition allows our data to violate both the Lasry-Lions monotonicity and the displacement monotonicity conditions.


Keywords. Master equation, mean field games, Lasry-Lions monotonicity, displacement monotonicity, anti-monotonicity.

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## 1 Introduction

In this paper we consider the following second order master equation, arising from mean field games with common noise, with terminal condition $V(T, x, \mu)=G(x, \mu)$ :

$$
\begin{align*}
& \mathcal{L} V(t, x, \mu):=-\partial_{t} V-\frac{\widehat{\beta}^{2}}{2} \operatorname{tr}\left(\partial_{x x} V\right)+H\left(x, \mu, \partial_{x} V\right)-\mathcal{N} V=0, \quad \text { where } \\
& \begin{array}{l}
\mathcal{N} V(t, x, \mu):=\operatorname{tr}\left(\overline { \tilde { \mathbb { E } } } \left[\frac{\widehat{\beta}^{2}}{2} \partial_{\tilde{x}} \partial_{\mu} V(t, x, \mu, \tilde{\xi})-\partial_{\mu} V(t, x, \mu, \tilde{\xi})\left(\partial_{p} H\right)^{\top}\left(\tilde{\xi}, \mu, \partial_{x} V(t, \tilde{\xi}, \mu)\right)\right.\right. \\
\left.\left.\quad+\beta^{2} \partial_{x} \partial_{\mu} V(t, x, \mu, \tilde{\xi})+\frac{\beta^{2}}{2} \partial_{\mu \mu} V(t, x, \mu, \bar{\xi}, \tilde{\xi})\right]\right), \quad(t, x, \mu) \in[0, T) \times \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) .
\end{array} \tag{1.1}
\end{align*}
$$

Here $\beta \geq 0$ is a constant, $\widehat{\beta}^{2}:=1+\beta^{2}, \partial_{t}, \partial_{x}, \partial_{x x}$ are standard temporal and spatial derivatives, $\partial_{\mu}, \partial_{\mu \mu}$ are $W_{2}$-Wasserstein derivatives, $\tilde{\xi}$ and $\bar{\xi}$ are independent random variables with the same law $\mu$ and $\overline{\tilde{\mathbb{E}}}$ is the expectation with respect to their joint law. The theory of Mean Field Games (MFGs, for short), initiated independently by Caines-Huang-Malhamé [15] and Lasry-Lions [40], studies the asymptotic behavior of stochastic differential games with a large number of players interacting in certain symmetric way. We refer to Lions [41], Cardaliaguet [16], Bensoussan-Frehse-Yam [7], Carmona-Delarue [21, 22] and Cardaliaguet-Porretta [18] for a comprehensive exposition of the subject. First introduced by Lions [41], the master equation characterizes the value of the MFG, provided there is a unique mean field equilibrium. Roughly speaking, it plays the role of the HJB equation in the stochastic control theory.

The master equation (1.1) admits a unique local (in time) classical solution when the data $H$ and $G$ are sufficiently smooth, see e.g. Gangbo-Swiech [33], Bensoussan-Yam [10], Mayorga [42], Carmona-Delarue [22] and Cardaliaguet-Cirant-Porretta [17]. In particular, [17] studied the local well-posedness of the master equations not only for MFGs involving homogeneous minor players but also for MFGs with a major player. It is much more challenging to obtain a global classical solution, we refer to Buckdahn-Li-Peng-Rainer [14], Chassagneux-CrisanDelarue [23], Cardaliaguet-Delarue-Lasry-Lions [19], Carmona-Delarue [22], Gangbo-Meszaros-Mou-Zhang [32] and, in the realm of potential MFGs, Bensoussan-Graber-Yam [8, 9], GangboMeszaros [31]. We also refer to Mou-Zhang [43], Bertucci [12], and Cardaliaguet-Souganidis [20] for global weak solutions which require much weaker regularity on the data, and Bayraktar-Cohen [3], Bertucci-Lasry-Lions [13], Cecchin-Delarue [25], Bertucci [11] for classical or weak solutions of finite state mean field game master equations. All the above global well-posedness results, with the exception [14] that considers linear master equations and thus no control or game is involved, require certain monotonicity condition, which we explain next.

One typical condition, extensively used in the literature [ $3,11,12,13,19,20,22,23,43$ ], is
the well-known Lasry-Lions monotonicity condition: for a function $G: \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\mathbb{E}\left[G\left(\xi_{1}, \mathcal{L}_{\xi_{1}}\right)+G\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)-G\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-G\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)\right] \geq 0 \tag{1.2}
\end{equation*}
$$

for any square integrable random variables $\xi_{1}, \xi_{2}$. Another type of monotonicity condition,originating in Ahuja [1] and was later sparsely used in the literature, see Ahuja-Ren-Yang [2] and [8, 9, 31, 32], is the displacement (or weak) monotonicity,

$$
\begin{equation*}
\mathbb{E}\left[\left[\partial_{x} G\left(\xi_{1}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} G\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right]\left[\xi_{1}-\xi_{2}\right]\right] \geq 0 \tag{1.3}
\end{equation*}
$$

When $G$ is regular enough with bounded $\partial_{x x} G, \partial_{x \mu} G$, (1.2) and (1.3) are equivalent to the following inequalities, respectively: for all square integrable random variables $\xi, \eta$,

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} G\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \geq 0, \quad \tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} G\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right]+\mathbb{E}\left[\left\langle\partial_{x x} G\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle\right] \geq 0 \tag{1.4}
\end{equation*}
$$

where $(\tilde{\xi}, \tilde{\eta})$ is an independent copy of $(\xi, \eta)$. The monotonicity conditions are crucial for the uniqueness of the Nash equilibria of MFGs and thus the well-posedness of their master equations.

When none of the monotonicity conditions holds, the MFG could have multiple equilibria, see e.g. Foguen Tchuendom [30], Cecchin-Dai Pra-Fisher-Pelino [24], Bayraktar-Zhang [6]. In this case, one approach is to consider a special type of equilibria, see e.g. [24], Delarue-Foguen Tchuendom [26], Cecchin-Delarue [25], Bayraktar-Cecchin-Cohen-Delarue [4, 5]. A larger literature is on the possible convergence of the equilibria for the $N$-player game, which is quite often unique because the corresponding Nash system is non-degenerate due to the presence of the individual noises, to the mean field equilibria (which may or may not be unique), see, e.g., [19, 22, 43], Delarue-Lacker-Ramanan [27, 28], Djete [29], Lacker [35, 36, 37, 38], Lacker-Flem [39], Nuts-San Martin-Tan [44]. Finally, we note that Iseri-Zhang [34] takes a quite different approach by investigating the set of game values over all mean field equilibria and establishes the dynamic programming principle and the convergence from the $N$-player game to the MFG.

We emphasize that the two inequalities in (1.4) share the same direction. Our goal of this paper is to propose a new type of monotonicity condition in the opposite direction, which we call anti-monotonicity condition, and establish the global well-posedness for the master equation (1.1), with possibly nonseparable Hamiltonian $H$. We remark that the mean field equilibrium is a fixed point, and the monotonicity conditions (1.4) were used to ensure the uniqueness of the fixed point. To motivate our anti-monotonicity condition, let us use a very simple example to illustrate the idea. Suppose that $f: \mathbb{R}^{1} \rightarrow \mathbb{R}^{1}$ is a continuously differentiable function and we are interested in its fixed point $x^{*}: f\left(x^{*}\right)=x^{*}$. When $f$ is decreasing, i.e., $f^{\prime} \leq 0$, clearly $f$ admits a unique fixed point $x^{*}$. When $f$ is increasing, in general neither the existence nor the uniqueness of $x^{*}$ is guaranteed. However, if $f$ is sufficiently monotone, in the sense that
$f^{\prime} \geq 1+\varepsilon$ for some $\epsilon>0$, then again $f$ has a unique fixed point $x^{*}$. While in complete different contexts, our conditions follow the same spirit. Roughly speaking, the standard monotonicity conditions (1.4) correspond to the case that $f$ is decreasing, while our new anti-monotonicity condition corresponds to the case $f$ is increasing, and for the same reason we will need to require our data to be sufficiently anti-monotone in appropriate sense.

To be precise, our anti-monotonicity condition takes the following form:

$$
\begin{gather*}
\tilde{\mathbb{E}}\left[\lambda_{0}\left\langle\partial_{x x} G\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle+\lambda_{1}\left\langle\partial_{x \mu} G\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right. \\
\left.+\left|\partial_{x x} G\left(\xi, \mathcal{L}_{\xi}\right) \eta\right|^{2}+\lambda_{2}\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} G\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}-\lambda_{3}|\eta|^{2}\right] \leq 0 \tag{1.5}
\end{gather*}
$$

for some appropriate constants $\lambda_{0}>0, \lambda_{1} \in \mathbb{R}, \lambda_{2}>0, \lambda_{3} \geq 0$. We remark that the inequality here takes the opposite direction to those in (1.4). In particular, the displacement monotonicity requires the convexity of $G$ in $x$, while here $G$ is typically concave in $x$, due to the first term in (1.5). This justifies the name of anti-monotonicity (and to have a better comparison with (1.4), we may also set $\lambda_{1}=1$ ). We also note that, considering the case $\lambda_{3}=0$, the second line of (1.5) is positive, this means that the first line of (1.5) should be sufficiently negative, which is exactly in the spirit that $G$ to be sufficiently anti-monotone.

To establish the global well-posedness of the master equation (1.1), we follow the strategy in [32], which consists of three steps. The key step of this approach is to show a priori that the anti-monotonicity propagates along the solution $V$. That is, under appropriate conditions, as long as $V(T, \cdot)=G$ is anti-monotone, then $V(t, \cdot)$ is anti-monotone for all $t$. The second step is to show that the anti-monotonicity of $V$ implies $\partial_{x} V$ is uniformly Lipschitz continuous in $(x, \mu)$, under $W_{2}$ in $\mu$. This, together with a representation formula established in [43], implies further the Lipschitz continuity under $W_{1}$. In the final step we show that the uniform Lipschitz continuity under $W_{1}$ enables us to extend a local classical solution to a global one.

There is a major technical difference from [32] though. The assumptions we impose for the propagation of anti-monotonicity prevents us from assuming uniform Lipschitz continuity of the data $G$ and $H$. Instead, we can only assume $\partial_{x} G, \partial_{x} H$ are uniformly Lipschitz. This has two consequences. First, the a priori estimate for the boundedness of $\partial_{x x} V$, which is crucial for the global well-posedness of the master equation and is pretty easy to obtain under the conditions in [32], becomes very subtle. In fact, we need some serious efforts to obtain this estimate. Moreover, unlike in [32], under our conditions the solution $V$ will not be Lipschitz continuous. Instead, we can only expect the Lipschitz continuity of $\partial_{x} V$. Therefore, we will actually consider the vector master equation of $\vec{U}:=\partial_{x} V$ and establish its global well-posedness first. Once we obtain $\vec{U}$, then it is immediate to solve the original master equation (1.1) for $V$.

The rest of the paper is organized as follows. In Section 2 we review the setting in [32] and
introduce our problem. In Section 3 we introduce the new notion of anti-monotonicity and present the technical conditions used in the paper. In Section 4 we show a priori the crucial propagation of the anti-monotonicity. Section 5 is devoted to the a priori uniform Lipschitz estimate of $\partial_{x} V$ in $\mu$, first under $W_{2}$ and then under $W_{1}$. In Section 6 we provide the a priori estimate for $\partial_{x x} V$. Finally in Section 7 we establish the global well-posedness of the master equation (1.1).

## 2 The setting

Throughout the paper we will use the setting in [32]. We review it briefly in this section and refer to [32] for more details.

We consider the following product filtered probability space on $[0, T]$ :

$$
\Omega:=\Omega_{0} \times \Omega_{1}, \quad \mathbb{F}:=\left\{\mathcal{F}_{t}\right\}_{0 \leq t \leq T}:=\left\{\mathcal{F}_{t}^{0} \otimes \mathcal{F}_{t}^{1}\right\}_{0 \leq t \leq T}, \quad \mathbb{P}:=\mathbb{P}_{0} \otimes \mathbb{P}_{1}, \quad \mathbb{E}:=\mathbb{E}^{\mathbb{P}}
$$

Here, for $\omega=\left(\omega^{0}, \omega^{1}\right) \in \Omega, B^{0}(\omega)=B^{0}\left(\omega^{0}\right)$ and $B(\omega)=B\left(\omega^{1}\right)$ are independent $d$-dimensional Brownian motions; $\mathbb{F}^{0}=\left\{\mathcal{F}_{t}^{0}\right\}$ is generated by $B^{0}$; and $\mathbb{F}^{1}=\left\{\mathcal{F}_{t}^{1}\right\}$ is generated by $B$ and $\mathcal{F}_{0}^{1}$, where we assume $\mathcal{F}_{0}^{1}$ has no atom. Let $\left(\tilde{\Omega}_{1}, \tilde{\mathbb{F}}^{1}, \tilde{B}, \tilde{\mathbb{P}}_{1}\right)$ be a copy of the filtered probability space $\left(\Omega_{1}, \mathbb{F}^{1}, B, \mathbb{P}_{1}\right)$ and define the larger filtered probability space by

$$
\tilde{\Omega}:=\Omega \times \tilde{\Omega}_{1}, \quad \tilde{\mathbb{F}}=\left\{\tilde{\mathcal{F}}_{t}\right\}_{0 \leq t \leq T}:=\left\{\mathcal{F}_{t} \otimes \tilde{\mathcal{F}}_{t}^{1}\right\}_{0 \leq t \leq T}, \quad \tilde{\mathbb{P}}:=\mathbb{P} \otimes \tilde{\mathbb{P}}_{1}, \quad \tilde{\mathbb{E}}:=\mathbb{E}^{\tilde{\mathbb{P}}}
$$

Given an $\mathcal{F}_{t}$-measurable random variable $\xi(\tilde{\omega})=\varphi\left(\omega^{0}, \omega^{1}\right), \tilde{\omega}=\left(\omega^{0}, \omega^{1}, \tilde{\omega}^{1}\right) \in \tilde{\Omega}$, we see that $\tilde{\xi}(\tilde{\omega}):=\varphi\left(\omega^{0}, \tilde{\omega}^{1}\right)$ is a conditionally independent copy of $\xi$, conditional on $\mathcal{F}_{t}^{0}$ under $\tilde{\mathbb{P}}$. When two conditionally independent copies are needed, we let $\left(\bar{\Omega}_{1}, \overline{\mathbb{F}}^{1}, \bar{B}, \overline{\mathbb{P}}_{1}\right)$ be another copy of $\left(\Omega_{1}, \mathbb{F}^{1}, B, \mathbb{P}_{1}\right)$, and enlarge the joint product space further:

$$
\overline{\tilde{\Omega}}:=\Omega \times \tilde{\Omega}_{1} \times \bar{\Omega}_{1}, \overline{\tilde{\mathbb{F}}}=\left\{\overline{\tilde{\mathcal{F}}}_{t}\right\}_{0 \leq t \leq T}:=\left\{\mathcal{F}_{t} \otimes \tilde{\mathcal{F}}_{t}^{1} \otimes \overline{\mathcal{F}}_{t}^{1}\right\}_{0 \leq t \leq T}, \overline{\tilde{\mathbb{P}}}:=\mathbb{P} \otimes \tilde{\mathbb{P}}_{1} \otimes \overline{\mathbb{P}}_{1}, \overline{\tilde{\mathbb{E}}}:=\mathbb{E}^{\overline{\tilde{\mathbb{P}}}}
$$

Throughout the paper we will use the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. However, when conditionally independent copies of random variables or processes are needed, we will tacitly use their extensions to the larger space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})(\overline{\tilde{\Omega}}, \overline{\tilde{\mathbb{F}}}, \overline{\tilde{\mathbb{P}}}, \overline{\tilde{\mathbb{E}}})$ without mentioning.

We next introduce the Wasserstein space and differential calculus on Wasserstein space. Let $\mathcal{P}:=\mathcal{P}\left(\mathbb{R}^{d}\right)$ be the set of all probability measures on $\mathbb{R}^{d}$ and, for any $q \geq 1$, let $\mathcal{P}_{q}$ denote the set of $\mu \in \mathcal{P}$ with finite $q$-th moment. For any sub- $\sigma$-field $\mathcal{G} \subset \mathcal{F}_{T}$ and $\mu \in \mathcal{P}_{q}$, we denote the set of $\mathbb{R}^{d}$-valued, $\mathcal{G}$-measurable, and $q$-integrable random variables $\xi$ by $\mathbb{L}^{q}(\mathcal{G})$; and the set of $\xi \in \mathbb{L}^{q}(\mathcal{G})$ such that the law $\mathcal{L}_{\xi}=\mu$ by $\mathbb{L}^{q}(\mathcal{G} ; \mu)$. For any $\mu, \nu \in \mathcal{P}_{q}$, the $W_{q}$-Wasserstein distance between them is defined as follows:

$$
W_{q}(\mu, \nu):=\inf \left\{\left(\mathbb{E}\left[|\xi-\eta|^{q}\right]\right)^{\frac{1}{q}}: \text { for all } \xi \in \mathbb{L}^{q}\left(\mathcal{F}_{T} ; \mu\right), \eta \in \mathbb{L}^{q}\left(\mathcal{F}_{T} ; \nu\right)\right\}
$$

For a $W_{2}$-continuous functions $U: \mathcal{P}_{2} \rightarrow \mathbb{R}$, its Wasserstein gradient, also called Lions-derivative, takes the form $\partial_{\mu} U:(\mu, \tilde{x}) \in \mathcal{P}_{2} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ and satisfies:

$$
\begin{equation*}
U\left(\mathcal{L}_{\xi+\eta}\right)-U(\mu)=\mathbb{E}\left[\left\langle\partial_{\mu} U(\mu, \xi), \eta\right\rangle\right]+o\left(\|\eta\|_{2}\right), \forall \xi \in \mathbb{L}^{2}\left(\mathcal{F}_{T} ; \mu\right), \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}\right) . \tag{2.1}
\end{equation*}
$$

Let $\mathcal{C}^{0}\left(\mathcal{P}_{2}\right)$ denote the set of $W_{2}$-continuous functions $U: \mathcal{P}_{2} \rightarrow \mathbb{R}$. For $k=1,2$, we introduce $\mathcal{C}^{k}\left(\mathcal{P}_{2}\right)$, which are referred to as functions of full $\mathcal{C}^{k}$ regularity in [21, Theorem 4.17], as follows. By $\mathcal{C}^{1}\left(\mathcal{P}_{2}\right)$, we mean the space of functions $U \in \mathcal{C}^{0}\left(\mathcal{P}_{2}\right)$ such that $\partial_{\mu} U$ exists and is continuous on $\mathcal{P}_{2} \times \mathbb{R}^{d}$, it is uniquely determined by (2.1). Similarly, $\mathcal{C}^{2}\left(\mathcal{P}_{2}\right)$ stands for the set of functions $U \in \mathcal{C}^{1}\left(\mathcal{P}_{2}\right)$ such that $\partial_{\tilde{x} \mu} U, \partial_{\mu \mu} U$ exist and are continuous on $\mathcal{P}_{2} \times \mathbb{R}^{d}$ and $\mathcal{P}_{2} \times \mathbb{R}^{2 d}$ respectively. Let $\mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$ denote the set of continuous functions $U: \mathbb{R}^{d} \times \mathcal{P}_{2} \rightarrow \mathbb{R}$ satisfying $\partial_{x} U, \partial_{x x} U$ exist and are joint continuous on $\mathbb{R}^{d} \times \mathcal{P}_{2}, \partial_{\mu} U, \partial_{x \mu} U, \partial_{\tilde{x} \mu} U$ exist and are continuous on $\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}$, and $\partial_{\mu \mu} U$ exists and is continuous on $\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{2 d}$. Finally, we fix the state space

$$
\Theta:=[0, T] \times \mathbb{R}^{d} \times \mathcal{P}_{2}
$$

for our master equation, and let $\mathcal{C}^{1,2}(\Theta)$ denote the set of continuous functions $U \in \Theta \rightarrow \mathbb{R}$ which has the following continuous derivatives: $\partial_{t} U, \partial_{x} U, \partial_{x x} U, \partial_{\mu} U, \partial_{x \mu} U, \partial_{\tilde{x} \mu} U, \partial_{\mu \mu} U$.

One crucial property of $U \in \mathcal{C}^{1,2}(\Theta)$ functions is the Itô formula. For $i=1,2$, let $d X_{t}^{i}:=$ $b_{t}^{i} d t+\sigma_{t}^{i} d B_{t}+\sigma_{t}^{i, 0} d B_{t}^{0}$, where $b^{i}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ and $\sigma^{i}, \sigma^{i, 0}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d \times d}$ are $\mathbb{F}$-progressively measurable and bounded (for simplicity), and $\rho_{t}:=\mathcal{L}_{X_{t}^{2} \mid \mathcal{F}_{t}^{0}}$, then we have

$$
\begin{align*}
& d U\left(t, X_{t}^{1}, \rho_{t}\right)=\left[\partial_{t} U+\partial_{x} U \cdot b_{t}^{1}+\frac{1}{2} \operatorname{tr}\left(\partial_{x x} U\left[\sigma_{t}^{1}\left(\sigma_{t}^{1}\right)^{\top}+\sigma_{t}^{1,0}\left(\sigma_{t}^{1,0}\right)^{\top}\right]\right)\right]\left(t, X_{t}^{1}, \rho_{t}\right) d t \\
& +\partial_{x} U\left(t, X_{t}^{1}, \rho_{t}\right) \cdot \sigma_{t}^{1} d B_{t}+\left(\sigma_{t}^{1,0}\right)^{\top} \partial_{x} U\left(t, X_{t}^{1}, \rho_{t}\right) \cdot d B_{t}^{0} \\
& \left.+\operatorname{tr}\left(\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{\mu} U\left(t, X_{t}^{1}, \rho_{t}, \tilde{X}_{t}^{2}\right)\left(\tilde{b}_{t}^{2}\right)^{\top}\right]\right) d t+\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left(\tilde{\sigma}_{t}^{2,0}\right)^{\top} \partial_{\mu} U\left(t, X_{t}^{1}, \rho_{t}, \tilde{X}_{t}^{2}\right)\right]\right] \cdot d B_{t}^{0}  \tag{2.2}\\
& +\operatorname{tr}\left(\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{x} \partial_{\mu} U\left(t, X_{t}^{1}, \rho_{t}, \tilde{X}_{t}^{2}\right) \sigma_{t}^{1,0}\left(\tilde{\sigma}_{t}^{2,0}\right)^{\top}+\frac{1}{2} \partial_{\tilde{x}} \partial_{\mu} U\left(t, X_{t}^{1}, \rho_{t}, \tilde{X}_{t}^{2}\right)\left[\tilde{\sigma}_{t}^{2}\left(\tilde{\sigma}_{t}^{2}\right)^{\top}+\tilde{\sigma}_{t}^{2,0}\left(\tilde{\sigma}_{t}^{2,0}\right)^{\top}\right]\right]\right. \\
& \left.\quad+\frac{1}{2} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{\mu \mu} U\left(t, X_{t}^{1}, \rho_{t}, \tilde{X}_{t}^{2}, \bar{X}_{t}^{2}\right) \tilde{\sigma}_{t}^{2,0}\left(\bar{\sigma}_{t}^{2,0}\right)^{\top}\right]\right) d t .
\end{align*}
$$

See, e.g., [22, Theorem 4.17], [14, 23]). Here $\mathcal{L}_{X_{t}^{2} \mid \mathcal{F}_{t}^{0}}$ stands for the conditional law of $X_{t}^{2}$ given $\mathcal{F}_{t}^{0}$, and $\tilde{\mathbb{E}}_{\mathcal{F}_{t}}$ and $\overline{\tilde{\mathbb{E}}}_{\mathcal{F}_{t}}$ are the conditional expectations given $\mathcal{F}_{t}$ corresponding to the probability measures $\tilde{\mathbb{P}}$ and $\overline{\mathbb{P}}$ respectively. Throughout the paper, the elements of $\mathbb{R}^{d}$ are viewed as column vectors; $\partial_{x} U, \partial_{\mu} U \in \mathbb{R}^{d}$ are also column vectors; $\partial_{x \mu} U:=\partial_{x} \partial_{\mu} U:=\partial_{x}\left[\left(\partial_{\mu} U\right)^{\top}\right] \in \mathbb{R}^{d \times d}$, where ${ }^{\top}$ denotes the transpose, and similarly for the other second order derivatives; both the notations "." and $\langle\cdot, \cdot\rangle$ denote the inner product of column vectors.

We finally introduce the mean field system related to the master equation (1.1). It either
takes the form of forward backward McKean-Vlasov SDEs on $\left[t_{0}, T\right]$ : given $t_{0}$ and $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{t_{0}}\right)$,

$$
\begin{align*}
& X_{t}^{\xi}=\xi-\int_{t_{0}}^{t} \partial_{p} H\left(X_{s}^{\xi}, \rho_{s}, Z_{s}^{\xi}\right) d s+B_{t}^{t_{0}}+\beta B_{t}^{0, t_{0}}, \quad B_{t}^{t_{0}}:=B_{t}-B_{t_{0}}, B_{t}^{0, t_{0}}:=B_{t}^{0}-B_{t_{0}}^{0} \\
& Y_{t}^{\xi}=G\left(X_{T}^{\xi}, \rho_{T}\right)+\int_{t}^{T} \widehat{L}\left(X_{s}^{\xi}, \rho_{s}, Z_{s}^{\xi}\right) d s-\int_{t}^{T} Z_{s}^{\xi} \cdot d B_{s}-\int_{t}^{T} Z_{s}^{0, \xi} \cdot d B_{s}^{0}  \tag{2.3}\\
& \text { where } \quad \widehat{L}(x, \mu, p):=p \cdot \partial_{p} H(x, \mu, p)-H(x, \mu, p), \quad \rho_{t}:=\rho_{t}^{\xi}:=\mathcal{L}_{X_{t}^{\xi} \mid \mathcal{F}_{t}^{0}},
\end{align*}
$$

or take the form of forward backward stochastic PDE system on $\left[t_{0}, T\right]$ : denoting $\widehat{\beta}^{2}:=1+\beta^{2}$,

$$
\begin{align*}
d \rho(t, x)= & {\left[\frac{\widehat{\beta}^{2}}{2} \operatorname{tr}\left(\partial_{x x} \rho(t, x)\right)+\operatorname{div}\left(\rho(t, x) \partial_{p} H\left(x, \rho(t, \cdot), \partial_{x} u(t, x)\right)\right)\right] d t-\beta \partial_{x} \rho(t, x) \cdot d B_{t}^{0} ; } \\
d u(t, x)= & v(t, x) \cdot d B_{t}^{0}-\left[\operatorname{tr}\left(\frac{\widehat{\beta}^{2}}{2} \partial_{x x} u(t, x)+\beta \partial_{x} v^{\top}(t, x)\right)-H\left(x, \rho(t, \cdot), \partial_{x} u(t, x)\right)\right] d t ;  \tag{2.4}\\
& \rho\left(t_{0}, \cdot\right)=\mathcal{L}_{\xi}, \quad u(T, x)=G(x, \rho(T, \cdot)),
\end{align*}
$$

where the solution triple $(\rho, u, v)$ is $\mathbb{F}^{0}$-progressively measurable and $\rho(t, \cdot, \omega)$ is a (random) probability measure. The systems (2.3) and (2.4) connect to the master equation (1.1) as follows: provided all the equations are well-posed and in particular (1.1) has a classical solution $V$, then

$$
\begin{equation*}
Y_{t}^{\xi}=V\left(t, X_{t}^{\xi}, \rho_{t}\right), \quad Z_{t}^{\xi}=\partial_{x} V\left(t, X_{t}^{\xi}, \rho_{t}\right), \quad \text { and } \quad u(t, x, \omega)=V(t, x, \rho(t, \cdot, \omega)) \tag{2.5}
\end{equation*}
$$

It is already well known that, c.f. [22], if the master equation (1.1) has a classical solution $V$ with bounded derivatives, then we can get existence and uniqueness of the mean field equilibrium, and the equilibrium of the corresponding $N$-player game will converge to the mean field equilibrium. Therefore, we shall only focus on the global well-posedness of the master equation (1.1).

We conclude this section with the strategy in [32] for the global well-posedness of (1.1). We will follow the same strategy in this paper, except that we shall replace the monotonicity condition with the anti-monotonicity condition:

Step 1. Introduce appropriate monotonicity condition on data which ensure the propagation of the monotonicity along any classical solution to the master equation.

Step 2. Show that the monotonicity of $V(t, \cdot, \cdot)$ implies an (a priori) uniform Lipschitz continuity of $V$ in the measure variable $\mu$.

Step 3. Combine the local well-posedness of classical solutions and the above uniform Lipschitz continuity to obtain the global well-posedness of classical solutions.

## 3 Assumptions and anti-monotonicity conditions

In this section, we introduce the following notations. For any $A \in \mathbb{R}^{d \times d}$,

$$
\begin{gathered}
\underline{\kappa}(A):=\inf _{|x|=1}\langle A x, x\rangle=\text { the smallest eigenvalue of } \frac{1}{2}\left[A+A^{\top}\right], \quad \bar{\kappa}(A):=\sup _{|x|=1}\langle A x, x\rangle ; \\
\underline{\kappa}^{\prime}(A):=\text { the smallest real part of eigenvalues of } A ; \\
|A|:=\sup _{|x|=|y|=1}\langle A x, y\rangle .
\end{gathered}
$$

It is obvious that, for any $A, A_{1}, A_{2} \in \mathbb{R}^{d \times d}$ and $x \in \mathbb{R}^{d}$,

$$
\begin{equation*}
|\cdot| \text { is a norm on } \mathbb{R}^{d \times d}, \quad\left|A_{1} A_{2}\right| \leq\left|A_{1}\right|\left|A_{2}\right|, \quad|A x| \leq|A||x|, \tag{3.2}
\end{equation*}
$$

$$
\text { and, when } A \text { is symmetric, } \underline{\kappa}^{\prime}(A)=\underline{\kappa}(A) \text { and }|A|=|\underline{\kappa}(A)| \vee|\bar{\kappa}(A)| \text {. }
$$

### 3.1 Regularity assumptions

We first specify some technical assumptions on $G$ and $H$.
Assumption 3.1 (i) $H \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)$ and there exist constants $\bar{L}_{x p}^{H}, \bar{L}_{x x}^{H}, L_{2}^{H}>0$ such that

$$
\begin{equation*}
\left|\partial_{x p} H\right| \leq \bar{L}_{x p}^{H}, \quad\left|\partial_{x x} H\right| \leq \bar{L}_{x x}^{H}, \quad\left|\partial_{p p} H\right|,\left|\partial_{x \mu} H\right|,\left|\partial_{p \mu} H\right| \leq L_{2}^{H} . \tag{3.3}
\end{equation*}
$$

(ii) $H \in \mathcal{C}^{3}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)$, and

$$
\begin{gathered}
\partial_{x} H, \partial_{p} H, \partial_{x x} H, \partial_{x p} H, \partial_{p p} H, \partial_{x x p} H, \partial_{x p p} H, \partial_{p p p} H \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right), \\
\partial_{\mu} H, \partial_{x \mu} H, \partial_{p \mu} H, \partial_{x p \mu} H, \partial_{p p \mu} H \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{2 d}\right),
\end{gathered}
$$

where all the second and higher order derivatives of $H$ involved above are uniformly bounded.
Assumption 3.2 (i) $G \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$, and there exist constants $\bar{L}_{x x}^{G}, L_{2}^{G}>0$ such that

$$
\begin{equation*}
\left|\partial_{x x} G\right| \leq \bar{L}_{x x}^{G}, \quad\left|\partial_{x \mu} G\right| \leq L_{2}^{G} \tag{3.4}
\end{equation*}
$$

(ii) $\partial_{x} G, \partial_{x x} G \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$, and $\partial_{\mu} G, \partial_{x \mu} G \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)$, and all the second and higher order derivatives of $G$ involved here are uniformly bounded.

Here the spaces $\mathcal{C}^{2}, \mathcal{C}^{3}$ are defined in the same manner as $\mathcal{C}^{1,2}(\Theta)$. Note that at above we do not require the first order derivatives to be uniformly bounded. In fact, the condition (3.18) below does not allow $\partial_{x} H$ to be bounded.

Remark 3.3 Under Assumption 3.2-(i), we see that $\partial_{x} G$ is uniformly Lipschitz continuous in $\mu$ under $W_{1}$ on $\mathbb{R}^{d} \times \mathcal{P}_{2}$ with Lipschitz constant $L_{2}^{G}$. This implies further the Lipschitz continuity of $\partial_{x} G$ in $\mu$ under $W_{2}$ on $\mathbb{R}^{d} \times \mathcal{P}_{2}$, and we denote the Lipschitz constant by $\tilde{L}_{2}^{G} \leq L_{2}^{G}$ :

$$
\tilde{\mathbb{E}}\left[\left[\partial_{x \mu} G(x, \mu, \tilde{\xi}) \tilde{\eta}\right]\right] \leq \tilde{L}_{2}^{G}\left(\mathbb{E}\left[|\eta|^{2}\right]\right)^{\frac{1}{2}}, \quad \forall \xi \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}, \mu\right), \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)
$$

### 3.2 Monotonicity and anti-monotoncity conditions

Under the above regularity conditions on the data $G$ and $H$, the MFG may still have multiple mean field equilibria over a long time duration and thus the global well-posedness of classical solutions for the master equations can fail. Therefore, some structural conditions on $G, H$ are needed in order to guarantee its global well-posedness. The typical structural conditions assumed in the literature are two types of monotonicity conditions, i.e., the Lasry-Lions monotonicity condition and the displacement monotonicity condition.

Definition 3.4 Let $U: \mathbb{R}^{d} \times \mathcal{P}_{2} \rightarrow \mathbb{R}$ be such that $U \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$.
(i) $U$ is called Lasry-Lions monotone, if for any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \geq 0 . \tag{3.5}
\end{equation*}
$$

(ii) $U$ is called displacement monotone if for any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle+\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle\right] \geq 0 \tag{3.6}
\end{equation*}
$$

(iii) $U$ is called displacement semi-monotone if, for some $\lambda \in \mathbb{R}$ and for any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle+\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle\right]-\lambda \mathbb{E}\left[|\eta|^{2}\right] \geq 0 \tag{3.7}
\end{equation*}
$$

Here, as in Section 2, ( $\tilde{\xi}, \tilde{\eta})$ is an independent copy of $(\xi, \eta)$. We remark that the displacement semi-monotonicity is obviously weaker than the displacement monotonicity (3.6), and when $\partial_{x x} U$ is bounded, it is also weaker than the Lasry-Lions monotonicity (3.5).

Remark 3.5 The above formulations of the monotonicity conditions are convenient for our purpose. For $U \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$, (3.5) and (3.6) are equivalent to (1.2) and (1.3), respectively, which appear more often in the literature. See [32, Remark 2.4].

We next turn to the monotonicity conditions for the Hamiltonian $H$. In the literature, the Lasry-Lions monotonicity has only been proposed for the separable Hamiltonians, i.e., $H(x, \mu, p)=$ $H_{0}(x, p)-F(x, \mu)$ and $F$ satisfies (1.2). In [32], a notion of displacement monotonicity for nonseparable $H$ was proposed to study the well-posedness of the master equation (1.1).

Definition 3.6 Let $H$ be a Hamiltonian satisfying Assumption 3.1(i) and $H$ is strictly convex in p. We say that $H$ is displacement monotone if: for any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$ and any bounded Lipschitz continuous function $\varphi \in C^{1}\left(\mathbb{R}^{d} ; \mathbb{R}^{d}\right)$,

$$
\begin{align*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} H\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}, \varphi(\xi)\right) \tilde{\eta}+\partial_{x x} H\left(\xi, \mathcal{L}_{\xi}, \varphi(\xi)\right) \eta, \eta\right\rangle\right. \\
\left.+\frac{1}{4}\left|\left(\partial_{p p} H\left(\xi, \mathcal{L}_{\xi}, \varphi(\xi)\right)\right)^{-\frac{1}{2}} \tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{p \mu} H\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}, \varphi(\xi)\right) \tilde{\eta}\right]\right|^{2}\right] \leq 0 . \tag{3.8}
\end{align*}
$$

Remark 3.7 (i) The above definition of displacement monotonicity for non-separable Hamiltonians is not really used in the rest of the paper except for the comparison with the new notion of anti-monotonicity introduced below. We refer to [32, Proposition 3.7] for another equivalent definition of the above one.
(ii) The function $\varphi(\xi)$ in (3.8) is chosen to be $\partial_{x} V\left(t, \xi, \mathcal{L}_{\xi}\right)$ in the proof of the propagation of the displacement monotonicity (3.6) along $V(t, \cdot)$ in [32]. Since $\partial_{x} V$ is not priorily known, the displacement monotonicity (3.8) is made for any desirable function $\varphi$.
(iii) When $H$ is non-separable, it still remains a challenge to find appropriate conditions on $H$ so that the Lasry-Lions monotonicity (3.5) could propagate along the solution $V(t, \cdot)$.

Finally we introduce the anti-monotonicity condition, which is the main structural condition in this paper and serves as an alternative sufficient condition for the global wellposedness of the master equation. Denote

$$
\begin{equation*}
D_{4}:=\left\{\vec{\lambda}=\left(\lambda_{0}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right): \lambda_{0}>0, \lambda_{1} \in \mathbb{R}, \lambda_{2}>0, \lambda_{3} \geq 0\right\} . \tag{3.9}
\end{equation*}
$$

Definition 3.8 Let $U \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$ and $\vec{\lambda} \in D_{4}$. We say $U$ is $\vec{\lambda}$-anti-monotone if,

$$
\begin{gather*}
(\text { AntiMon })_{\xi}^{\vec{\lambda}} U(\eta, \eta):=\tilde{\mathbb{E}}\left[\lambda_{0}\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle+\lambda_{1}\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right.  \tag{3.10}\\
\left.+\left|\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta\right|^{2}+\lambda_{2}\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}-\lambda_{3}|\eta|^{2}\right] \leq 0, \quad \forall \xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)
\end{gather*}
$$

Remark 3.9 (i) The main feature of (3.10) is that the direction of the inequality is opposite to those in Definition 3.4. In particular, (3.10) implies the Lasry-Lions anti-monotonicity, i.e.

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \leq 0 \tag{3.11}
\end{equation*}
$$

for the case that $\lambda_{0}=\lambda_{3}=0$ and $\lambda_{1}=\lambda_{2}=1$. In fact, in this case the condition (3.10) is stronger than (3.11) and we interpret it as $U$ is sufficiently Lasry-Lions anti-monotone:

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \leq-\tilde{\mathbb{E}}\left[\left|\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta\right|^{2}+\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}\right] \leq 0 \tag{3.12}
\end{equation*}
$$

Similarly, in the case $\lambda_{0}=\lambda_{1}=\lambda_{2}=1$ and $\lambda_{3}=0$, we see that (3.10) implies $U$ is sufficiently displacement anti-monotone:

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle+\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \leq-\tilde{\mathbb{E}}\left[\left|\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta\right|^{2}+\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}\right] \leq 0 \tag{3.13}
\end{equation*}
$$

Note that the concavity of $U$ in $x$ could help in (3.13), while in (3.6) its convexity is helpful.
(ii) The inequality (3.10) implies the displacement semi-anti-monotonicity, i.e.

$$
\begin{equation*}
\tilde{\mathbb{E}}\left[\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle+\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right] \leq \lambda_{3} \tilde{\mathbb{E}}\left[|\eta|^{2}\right] \tag{3.14}
\end{equation*}
$$

if $\vec{\lambda} \in D_{4}, \lambda_{0}=\lambda_{1}=1$ and $\lambda_{3} \geq 0$. Note that the condition (3.14) is weaker than (3.13) for the case. We recall that in the literature a function $u: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is said to be semi-concave, or $\lambda$-concave, if $\partial_{x x} u \leq \lambda I_{d}$ for some constant $\lambda>0$, where $I_{d}$ is the $d \times d$ identity matrix. We follow the same spirit to call $U \vec{\lambda}$-anti-monotone if $U$ satisfies (3.10).

We next provide an example which is $\vec{\lambda}$-anti-monotone.
Example 3.10 Let $d=1$ and consider the function: for some constants $a_{0}, a_{1}$,

$$
U(x, \mu)=\frac{a_{0}}{2}|x|^{2}+a_{1} x \int_{\mathbb{R}} y \mu(d y), \quad(x, \mu) \in \mathbb{R} \times \mathcal{P}_{2} .
$$

It is clear that $\partial_{x x} U=a_{0}$ and $\partial_{x \mu} U=a_{1}$.
(i) For any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$, we have

$$
\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right]=a_{1}|\mathbb{E}[\eta]|^{2}
$$

So $U$ is Lasry-Lions monotone if $a_{1} \geq 0$, and Lasry-Lions anti-monotone if $a_{1} \leq 0$.
(ii) Similarly we have

$$
\tilde{\mathbb{E}}\left[\left\langle\partial_{x x} U\left(\xi, \mathcal{L}_{\xi}\right) \eta, \eta\right\rangle+\left\langle\partial_{x \mu} U\left(\xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right]=a_{0} \mathbb{E}\left[|\eta|^{2}\right]+a_{1}|\mathbb{E}[\eta]|^{2}
$$

Then one can easily check that $U$ is displacement monotone if $a_{0} \geq 0, a_{1} \geq-a_{0}$, and displacement anti-monotone if $a_{0} \leq 0, a_{1} \leq-a_{0}$.
(iii) For any $\vec{\lambda} \in D_{4}$, we have

$$
(\text { AntiMon })_{\xi}^{\vec{\lambda}} U(\eta, \eta):=\left[\lambda_{0} a_{0}+\left|a_{0}\right|^{2}-\lambda_{3}\right] \mathbb{E}\left[|\eta|^{2}\right]+\left[\lambda_{1} a_{1}+\lambda_{2}\left|a_{1}\right|^{2}\right]|\mathbb{E}[\eta]|^{2} .
$$

Then $U$ is $\vec{\lambda}$-anti-monotone if:

$$
\lambda_{0} a_{0}+\left|a_{0}\right|^{2}-\lambda_{3} \leq 0, \quad \lambda_{0} a_{0}+\left|a_{0}\right|^{2}-\lambda_{3} \leq-\left[\lambda_{1} a_{1}+\lambda_{2}\left|a_{1}\right|^{2}\right],
$$

which is equivalent to:

$$
\lambda_{3} \geq \max \left(\lambda_{0} a_{0}+\left|a_{0}\right|^{2}, \quad \lambda_{0} a_{0}+\left|a_{0}\right|^{2}+\lambda_{1} a_{1}+\lambda_{2}\left|a_{1}\right|^{2}\right) .
$$

In particular, if we set $\lambda_{0}=\lambda_{1}=\lambda_{2}=1, \lambda_{3}=0$, and $-1 \leq a_{0}, a_{1} \leq 0$, we see that $U$ is $\vec{\lambda}$-anti-monotone.

Remark 3.11 Let $U \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right)$ and $\vec{\lambda} \in D_{4}$.
(i) When $\lambda_{0}=0,(3.10)$ is equivalent to the following integral form: for any $\xi_{1}, \xi_{2} \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{align*}
& \lambda_{1} \mathbb{E}\left[U\left(\xi_{1}, \mathcal{L}_{\xi_{1}}\right)+U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)-U\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-U\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)\right] \\
+ & \mathbb{E}\left[\left|\partial_{x} U\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}+\lambda_{2}\left|\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}\right]  \tag{3.15}\\
\leq & \lambda_{3} \mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]+o\left(\mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]\right) .
\end{align*}
$$

Here $o(\varepsilon)$ means it vanishes faster than $\varepsilon$ as $\varepsilon \rightarrow 0$.
(ii) When $\lambda_{0}=\lambda_{1},(3.10)$ is equivalent to the following integral form: for any $\xi_{1}, \xi_{2} \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{align*}
& \lambda_{0} \mathbb{E}\left[\left\langle\partial_{x} U\left(\xi_{1}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right), \xi_{1}-\xi_{2}\right\rangle\right] \\
+ & \mathbb{E}\left[\left|\partial_{x} U\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}+\lambda_{2}\left|\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}\right]  \tag{3.16}\\
\leq & C \mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]+o\left(\mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]\right) .
\end{align*}
$$

(iii) In general, (3.10) is equivalent to the following integral form: for any $\xi_{1}, \xi_{2} \in \mathbb{L}^{2}\left(\mathcal{F}_{T}^{1}\right)$,

$$
\begin{align*}
& \mathbb{E}\left[\lambda_{0}\left\langle\partial_{x} U\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right), \xi_{1}-\xi_{2}\right\rangle+\lambda_{1}\left\langle\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right), \xi_{1}-\xi_{2}\right\rangle\right] \\
+ & \mathbb{E}\left[\left|\partial_{x} U\left(\xi_{1}, \mathcal{L}_{\xi_{2}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}+\lambda_{2}\left|\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{1}}\right)-\partial_{x} U\left(\xi_{2}, \mathcal{L}_{\xi_{2}}\right)\right|^{2}\right]  \tag{3.17}\\
\leq & \lambda_{3} \mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]+o\left(\mathbb{E}\left[\left|\xi_{1}-\xi_{2}\right|^{2}\right]\right) .
\end{align*}
$$

Assumption 3.12 (i) $G$ satisfies Assumption 3.2-(i) and is $\vec{\lambda}$-anti-monotone for some $\vec{\lambda} \in D_{4}$;
(ii) $H$ satisfies Assumption 3.1-(i) and there exist constants $\underline{L}_{x p}^{H}>0, \underline{L}_{x x}^{H}>0, \bar{\gamma}>\underline{\gamma}>0$ s.t.

$$
\begin{gather*}
\underline{\kappa}\left(\partial_{x p} H\right) \geq \underline{L}_{x p}^{H}, \quad \underline{\kappa}\left(\partial_{x x} H\right) \geq \underline{L}_{x x}^{H},  \tag{3.18}\\
\underline{\gamma} \underline{L}_{x p}^{H} \leq \underline{L}_{x x}^{H} \leq \bar{L}_{x x}^{H} \leq \bar{\gamma} \underline{L}_{x p}^{H}, \quad \bar{L}_{x p}^{H} \leq \bar{\gamma} \underline{L}_{x p}^{H} \tag{3.19}
\end{gather*}
$$

Note that we do not require structural conditions on $\partial_{x \mu} H$ here, and $\partial_{p p} H$ can be degenerate.

## 4 Propagation of anti-monotonicity

In this section we show that any classical solution $V$ to the master equation (1.1) could propagate the anti-monotonicity under appropriate conditions.

Theorem 4.1 Let Assumption 3.12 hold and $V$ be a classical solution of the master equation (1.1) such that

$$
\partial_{x x} V(t, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right), \quad \partial_{x \mu} V(t, \cdot, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)
$$

and all the second and higher order derivatives of $V$ involved above are also continuous in the time variable and are uniformly bounded. Assume further that there exist a constant $L_{x x}^{V}>0$ such that

$$
\begin{gather*}
\left|\partial_{x x} V\right| \leq L_{x x}^{V}  \tag{4.1}\\
\text { and } \quad \lambda_{0}>\frac{\bar{\gamma}^{2}\left[1+L_{x x}^{V}\right]^{2}-8 \lambda_{3}}{4 \underline{\gamma}} \text { so that } \theta_{1}:=\frac{\bar{\gamma}\left[1+L_{x x}^{V}\right]}{\sqrt{4\left(\underline{\gamma} \lambda_{0}+2 \lambda_{3}\right)}}<1 \text {. } \tag{4.2}
\end{gather*}
$$

Introduce the following symmetric matrices, which depend only on $\underline{\gamma}, \bar{\gamma}, \vec{\lambda}$, and $L_{x x}^{V}$ :

$$
\begin{gather*}
A_{1}:=\left[\begin{array}{ccc}
4\left[1-\theta_{1}\right] & 0 & 0 \\
0 & 2 \lambda_{2} & 0 \\
0 & 0 & {\left[1-\theta_{1}\right]\left[\lambda_{0} \underline{\gamma}+2 \lambda_{3}\right]}
\end{array}\right] \\
A_{2}:=\left[\begin{array}{ccc}
\lambda_{0} & \lambda_{0} & \left|\lambda_{0}-\frac{1}{2} \lambda_{1}\right|+\lambda_{3} \\
\lambda_{0} & \left|\lambda_{1}\right| & \frac{1}{2}\left|\lambda_{1}\right|+\lambda_{2}+\lambda_{3} \\
\left|\lambda_{0}-\frac{1}{2} \lambda_{1}\right|+\lambda_{3} & \frac{1}{2}\left|\lambda_{1}\right|+\lambda_{2}+\lambda_{3} & \left|\lambda_{1}\right|+2 \lambda_{3}
\end{array}\right]+\left[\begin{array}{ccc}
0 & 1 & 1 \\
1 & \lambda_{2} & \lambda_{2} \\
1 & \lambda_{2} & 0
\end{array}\right] L_{x x}^{V} . \tag{4.3}
\end{gather*}
$$

Then, whenever

$$
\begin{equation*}
\underline{L}_{x p}^{H} \geq \underline{\kappa}\left(A_{1}^{-1} A_{2}\right) L_{2}^{H} \tag{4.4}
\end{equation*}
$$

$V(t, \cdot)$ is $\vec{\lambda}$-anti-monotone in the sense of (3.10) for all $t \in[0, T]$.
Proof. Without loss of generality, we shall prove the theorem only for $t_{0}=0$.
Fix $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{0}\right)$ and $\eta \in \mathbb{L}^{2}\left(\mathcal{F}_{0}\right)$. Given the desired regularity of $V$ and $H$, the following system of McKean-Vlasov SDEs has a unique solution $(X, \delta X)$ :

$$
\begin{align*}
X_{t}=\xi-\int_{0}^{t} & \partial_{p} H\left(X_{s}, \mu_{s}, \partial_{x} V\left(s, X_{s}, \mu_{s}\right)\right) d s+B_{t}+\beta B_{t}^{0}, \quad \mu_{t}:=\mathcal{L}_{X_{t} \mid \mathcal{F}_{t}^{0}} \\
\delta X_{t}=\eta-\int_{0}^{t} & {\left[\partial_{p x} H\left(X_{s}, \mu_{s}, \partial_{x} V\left(s, X_{s}, \mu_{s}\right)\right) \delta X_{s}+\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\partial_{p \mu} H\left(X_{s}, \mu_{s}, \partial_{x} V\left(s, X_{s}, \mu_{s}\right), \tilde{X}_{s}\right) \delta \tilde{X}_{s}\right]\right.}  \tag{4.5}\\
& \left.\quad+\partial_{p p} H\left(X_{s}, \mu_{s}, \partial_{x} V\left(s, X_{s}, \mu_{s}\right)\right)\left[\Upsilon_{s}+\bar{\Upsilon}_{s}\right]\right] d s
\end{align*}
$$

where $\quad \Upsilon_{t}:=\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{x \mu} V\left(t, X_{t}, \mu_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \quad \bar{\Upsilon}_{t}:=\partial_{x x} V\left(t, X_{t}, \mu_{t}\right) \delta X_{t}$.
In the sequel, for simplicity of notation, we omit the variables $\left(t, \mu_{t}\right)$ as well as the dependence on $\partial_{x} V$, and denote

$$
H_{p}\left(X_{t}\right):=\partial_{p} H\left(X_{t}, \mu_{t}, \partial_{x} V\left(t, X_{t}, \mu_{t}\right)\right), \quad H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right):=\partial_{p \mu} H\left(X_{t}, \mu_{t}, \tilde{X}_{t}, \partial_{x} V\left(t, X_{t}, \mu_{t}\right)\right)
$$

and similarly for $H_{x p}, H_{p p}, H_{x \mu}, \partial_{x x} V, \partial_{x \mu} V$, etc. We remark that, $\left(\tilde{X}_{t}, \delta \tilde{X}_{t}\right)$ is a conditionally independent copy of ( $X_{t}, \delta X_{t}$ ) and $\mu_{t}$ is $\mathcal{F}_{t}^{0}$-measurable.

Recall (4.5) and introduce:

$$
\begin{gather*}
I_{t}:=\mathbb{E}\left[\left\langle\Upsilon_{t}, \delta X_{t}\right\rangle\right], \quad \bar{I}_{t}:=\mathbb{E}\left[\left\langle\bar{\Upsilon}_{t}, \delta X_{t}\right\rangle\right] ;  \tag{4.6}\\
\Gamma_{t}:=(\text { AntiMon })_{X_{t}}^{\vec{X}} V(t, \cdot)\left(\delta X_{t}, \delta X_{t}\right)=\lambda_{0} \bar{I}_{t}+\lambda_{1} I_{t}+\mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}+\lambda_{2}\left|\Upsilon_{t}\right|^{2}-\lambda_{3}\left|\delta X_{t}\right|^{2}\right] .
\end{gather*}
$$

By the calculation in [32, Theorem 4.1] we have

$$
\begin{align*}
\frac{d}{d t} I(t)= & \mathbb{E}\left[-\left\langle H_{p p}\left(X_{t}\right) \Upsilon_{t}, \Upsilon_{t}\right\rangle-\left\langle\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \Upsilon_{t}-\bar{\Upsilon}_{t}\right\rangle\right. \\
& \left.+\left\langle\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{x \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \delta X_{t}\right\rangle\right] ; \\
\frac{d}{d t} \bar{I}(t)= & \mathbb{E}\left[-\left\langle H_{p p}\left(X_{t}\right) \bar{\Upsilon}_{t}, \bar{\Upsilon}_{t}\right\rangle-2\left\langle H_{p p}\left(X_{t}\right) \bar{\Upsilon}_{t}, \Upsilon_{t}\right\rangle\right.  \tag{4.7}\\
& \left.-2\left\langle\bar{\Upsilon}_{t}, \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]\right\rangle+\left\langle H_{x x}\left(X_{t}\right) \delta X_{t}, \delta X_{t}\right\rangle\right],
\end{align*}
$$

and, by the calculation in [32, Theorem 5.1] we have

$$
\begin{align*}
d \Upsilon_{t} & =\left(d B_{t}\right)^{\top} K_{1}(t)+\beta\left(d B_{t}^{0}\right)^{\top} K_{2}(t)+\left[K_{3}(t) \Upsilon_{t}+K_{4}(t)\right] d t  \tag{4.8}\\
d \bar{\Upsilon}_{t} & =\left(d B_{t}\right)^{\top} \bar{K}_{1}(t)+\beta\left(d B_{t}^{0}\right)^{\top} \bar{K}_{2}(t)+\left[2 H_{x p}\left(X_{t}\right) \bar{\Upsilon}_{t}-\partial_{x x} V\left(X_{t}\right) H_{p p}\left(X_{t}\right) \Upsilon_{t}+\bar{K}_{3}(t)\right] d t,
\end{align*}
$$

where $\left(K_{5}(t)\right.$ and $K_{6}(t)$ in [32] turn to $K_{3}(t)$ and $K_{4}(t)$ respectively here)

$$
\begin{align*}
& K_{1}(t):=\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{x x \mu} V\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \\
& K_{2}(t):=K_{1}(t)+\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left[\left(\partial_{\mu x \mu} V\right)\left(X_{t}, \bar{X}_{t}, \tilde{X}_{t}\right)+\partial_{\tilde{x} x \mu} V\left(X_{t}, \tilde{X}_{t}\right)\right] \delta \tilde{X}_{t}\right], \\
& K_{3}(t):=H_{x p}\left(X_{t}\right)+\partial_{x x} V\left(X_{t}\right) H_{p p}\left(X_{t}\right), \\
& K_{4}(t):=\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left[H_{x \mu}\left(X_{t}, \tilde{X}_{t}\right)+\partial_{x x} V\left(X_{t}\right) H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right)\right] \delta \tilde{X}_{t}\right],  \tag{4.9}\\
& \bar{K}_{1}(t):=\partial_{x x x} V\left(X_{t}\right) \delta X_{t}, \\
& \bar{K}_{2}(t):=\bar{K}_{1}(t)+\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left(\partial_{\mu x x} V\right)\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \\
& \bar{K}_{3}(t):=\left[H_{x x}\left(X_{t}\right)-\partial_{x x} V\left(X_{t}\right) H_{p x}\left(X_{t}\right)\right] \delta X_{t}-\partial_{x x} V\left(X_{t}\right) \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right] .
\end{align*}
$$

In particular, this implies that

$$
\begin{align*}
& \frac{d}{d t} \mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right] \geq 2 \mathbb{E}\left[\left\langle\Upsilon_{t}, K_{3}(t) \Upsilon_{t}+K_{4}(t)\right\rangle\right] \\
& \frac{d}{d t} \mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right] \geq 2 \mathbb{E}\left[\left\langle\bar{\Upsilon}_{t}, 2 H_{x p}\left(X_{t}\right) \bar{\Upsilon}_{t}-\partial_{x x} V\left(X_{t}\right) H_{p p}\left(X_{t}\right) \Upsilon_{t}+\bar{K}_{3}(t)\right\rangle\right] \tag{4.10}
\end{align*}
$$

Moreover, by (4.5) we have

$$
\begin{equation*}
\frac{d}{d t} \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]=-2 \mathbb{E}\left[\left\langle H_{p x}\left(X_{t}\right) \delta X_{t}+\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]+H_{p p}\left(X_{t}\right)\left[\Upsilon_{t}+\bar{\Upsilon}_{t}\right], \delta X_{t}\right\rangle\right] . \tag{4.11}
\end{equation*}
$$

Thus, by (4.7), (4.10), and (4.11), we have

$$
\begin{aligned}
& \frac{d}{d t} \Gamma_{t} \geq \lambda_{0} \mathbb{E}\left[-\left\langle H_{p p}\left(X_{t}\right) \bar{\Upsilon}_{t}, \bar{\Upsilon}_{t}\right\rangle-2\left\langle H_{p p}\left(X_{t}\right) \bar{\Upsilon}_{t}, \Upsilon_{t}\right\rangle\right. \\
&\left.-2\left\langle\bar{\Upsilon}_{t}, \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]\right\rangle+\left\langle H_{x x}\left(X_{t}\right) \delta X_{t}, \delta X_{t}\right\rangle\right] \\
&+ \lambda_{1} \mathbb{E}\left[-\left\langle H_{p p}\left(X_{t}\right) \Upsilon_{t}, \Upsilon_{t}\right\rangle-\left\langle\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \Upsilon_{t}-\bar{\Upsilon}_{t}\right\rangle\right. \\
&\left.+\left\langle\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{x \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \delta X_{t}\right\rangle\right] \\
&+2 \mathbb{E}\left[\left\langle\bar{\Upsilon}_{t},\left[2 H_{x p}\left(X_{t}\right) \bar{\Upsilon}_{t}-\partial_{x x} V\left(X_{t}\right) H_{p p}\left(X_{t}\right) \Upsilon_{t}+\bar{K}_{3}(t)\right]\right\rangle+\lambda_{2}\left\langle\Upsilon_{t},\left[K_{3}(t) \Upsilon_{t}+K_{4}(t)\right]\right\rangle\right] \\
&+2 \lambda_{3} \mathbb{E}\left[\left\langle H_{p x}\left(X_{t}\right) \delta X_{t}+\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]+H_{p p}\left(X_{t}\right)\left[\Upsilon_{t}+\bar{\Upsilon}_{t}\right], \delta X_{t}\right\rangle\right] \\
&=\mathbb{E}[\langle {\left.\left[-\lambda_{0} H_{p p}\left(X_{t}\right)+4 H_{x p}\left(X_{t}\right)\right] \bar{\Upsilon}_{t}, \bar{\Upsilon}_{t}\right\rangle+\left\langle\left[-\lambda_{1} H_{p p}\left(X_{t}\right)+2 \lambda_{2} K_{3}(t)\right] \Upsilon_{t}, \Upsilon_{t}\right\rangle } \\
&+\left\langle\left[\lambda_{0} H_{x x}\left(X_{t}\right)+2 \lambda_{3} H_{p x}\left(X_{t}\right)\right] \delta X_{t}, \delta X_{t}\right\rangle \\
&+\left\langle\lambda_{1} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{x \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]+2 \lambda_{3} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right], \delta X_{t}\right\rangle \\
&-\left\langle 2\left[\lambda_{0} H_{p p}\left(X_{t}\right)+\partial_{x x} V\left(X_{t}\right) H_{p p}\left(X_{t}\right)\right] \Upsilon_{t}, \bar{\Upsilon}_{t}\right\rangle \\
&+\left\langle\left[-2 \lambda_{0}+\lambda_{1}\right] \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]+2 \bar{K}_{3}(t)+2 \lambda_{3} H_{p p}\left(X_{t}\right) \delta X_{t}, \bar{\Upsilon}_{t}\right\rangle \\
&\left.\quad+\left\langle-\lambda_{1} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[H_{p \mu}\left(X_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]+2 \lambda_{2} K_{4}(t)+2 \lambda_{3} H_{p p}\left(X_{t}\right) \delta X_{t}, \Upsilon_{t}\right\rangle\right] .
\end{aligned}
$$

Next, by Assumptions 3.1-(i) and 3.12-(ii), and (3.19) we have

$$
\begin{aligned}
\frac{d}{d t} & \Gamma_{t} \geq\left[4 \underline{L}_{x p}^{H}-\lambda_{0} L_{2}^{H}\right] \mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]+\left[2 \lambda_{2} \underline{L}_{x p}^{H}-\left[\left|\lambda_{1}\right|+\lambda_{2} L_{x x}^{V}\right] L_{2}^{H}\right] \mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right] \\
& +\left[\lambda_{0} \underline{L}_{x x}^{H}+2 \lambda_{3} \underline{L}_{x p}^{H}-\left[\left|\lambda_{1}\right|+2 \lambda_{3}\right] L_{2}^{H}\right] \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \\
& -2 L_{2}^{H}\left[\lambda_{0}+L_{x x}^{V}\right] \mathbb{E}\left[\left|\Upsilon_{t}\right|\left|\bar{\Upsilon}_{t}\right|\right] \\
& \left.-\left[\left|\lambda_{1}-2 \lambda_{0}\right| L_{2}^{H}+2 \bar{\gamma}\left[1+L_{x x}^{V}\right] \underline{L}_{x p}^{H}+2 L_{x x}^{V} L_{2}^{H}+2 \lambda_{3} L_{2}^{H}\right]\right]\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]\right)^{\frac{1}{2}} \\
& -L_{2}^{H}\left[\left|\lambda_{1}\right|+2 \lambda_{2}\left[1+L_{x x}^{V}\right]+2 \lambda_{3}\right]\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right]\right)^{\frac{1}{2}} .
\end{aligned}
$$

Note that, recalling the $\theta_{1}$ in (4.2),

$$
4 \theta_{1} \mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]+2 \bar{\gamma}\left[1+L_{x x}^{V}\right]\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]\right)^{\frac{1}{2}}+\theta_{1}\left[\lambda_{0} \underline{\gamma}+2 \lambda_{3}\right] \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \geq 0
$$

Then, recalling (4.3) and denoting $a:=\left[\left(\mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]\right)^{\frac{1}{2}},\left(\mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right]\right)^{\frac{1}{2}}, \quad\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\right]$,

$$
\begin{aligned}
& \frac{d}{d t} \Gamma_{t} \geq\left[4\left[1-\theta_{1}\right] \underline{L}_{x p}^{H}-\lambda_{0} L_{2}^{H}\right] \mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]+\left[2 \lambda_{2} \underline{L}_{x p}^{H}-\left[\left|\lambda_{1}\right|+\lambda_{2} L_{x x}^{V}\right] L_{2}^{H}\right] \mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right] \\
& \quad+\left[\left[1-\theta_{1}\right]\left[\lambda_{0} \underline{\gamma}+2 \lambda_{3}\right] \underline{L}_{x p}^{H}-\left[\left|\lambda_{1}\right|+2 \lambda_{3}\right] L_{2}^{H}\right] \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \\
& \quad-2 L_{2}^{H}\left[\lambda_{0}+L_{x x}^{V}\right] \mathbb{E}\left[\left|\Upsilon_{t}\right|\left|\widehat{\Upsilon}_{t}\right|\right] \\
& \quad-L_{2}^{H}\left[\left|\lambda_{1}-2 \lambda_{0}\right|+2 L_{x x}^{V}+2 \lambda_{3}\right]\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\bar{\Upsilon}_{t}\right|^{2}\right]\right)^{\frac{1}{2}} \\
& \quad-L_{2}^{H}\left[\left|\lambda_{1}\right|+2 \lambda_{2}\left[1+L_{x x}^{V}\right]+2 \lambda_{3}\right]\left(\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right]\right)^{\frac{1}{2}} \\
& \quad=a\left[A_{1} \underline{L}_{x p}^{H}-A_{2} L_{2}^{H}\right] a^{\top} \geq 0,
\end{aligned}
$$

where the last inequality thanks to (4.4) and the fact that $A_{1} \geq 0$. Thus

$$
(\text { AntiMon })_{\xi}^{\vec{\lambda}} V(0, \eta, \eta)=\Gamma_{0} \leq \Gamma_{T}=(\text { AntiMon })_{X_{T}}^{\vec{\lambda}^{2}} G\left(\delta X_{T}, \delta X_{T}\right) \leq 0
$$

That is, $V(0, \cdot, \cdot)$ is $\vec{\lambda}$-anti-monotone.

## 5 The Lipschitz continuity

We first show that the anti-monotonicity of $V$ implies the uniformly Lipschitz continuity of $\partial_{x} V$ in $\mu$ under $W_{2}$. Unlike in [32], since we do not require the first order derivatives of $G, H$ to be bounded, here we do not expect the Lipschitz continuity of $V$ itself.

Theorem 5.1 Let Assumptions 3.1-(i), 3.2-(i) hold and $V$ be a classical solution of the master equation (1.1) such that

$$
\partial_{x x} V(t, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right), \quad \partial_{x \mu} V(t, \cdot, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)
$$

and all the second and higher order derivatives of $V$ involved above are also continuous in the time variable and are uniformly bounded. Assume further that $V(t, \cdot, \cdot)$ is $\vec{\lambda}$-anti-monotone in the sense of (3.10) for all $t \in[0, T]$. Then $\partial_{x} V$ is uniformly Lipschitz continuous in $\mu$ under $W_{2}$, with a Lipschitz constant $C_{2}^{\mu}$ depending only on $\vec{\lambda}$, the parameters in (3.3) and (3.4), and $L_{x x}^{V}$.

Proof. In this proof, $C>0$ denotes a generic constant depending only on quantities mentioned in the statement of the theorem. As in the proof of Theorem 4.1, without loss of generality we show the theorem only for $t_{0}=0$. First, by (3.10) we have, for any $\xi, \eta \in \mathbb{L}^{2}\left(\mathcal{F}_{t}^{1}\right)$,

$$
\begin{equation*}
\mathbb{E}\left[\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} V\left(t, \xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}\right] \leq C\left|\tilde{\mathbb{E}}\left[\left\langle\partial_{x \mu} V\left(t, \xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}, \eta\right\rangle\right]\right|+C \mathbb{E}\left[|\eta|^{2}\right] . \tag{5.1}
\end{equation*}
$$

Next, applying Hölder's inequality to (5.1) we have

$$
\begin{equation*}
\mathbb{E}\left[\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}^{1}}\left[\partial_{x \mu} V\left(t, \xi, \mathcal{L}_{\xi}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}\right] \leq C \mathbb{E}\left[|\eta|^{2}\right] \tag{5.2}
\end{equation*}
$$

From now on we fix $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{0}\right)$ and $\eta \in \mathbb{L}^{2}\left(\mathcal{F}_{0}\right)$ and continue to use the notation as in the proof of Theorem 4.1. In particular, $X, \delta X, \mu_{t}, \Upsilon, \bar{\Upsilon}$ are defined by (4.5). Applying (5.2) by replacing $\mathbb{E}$ with $\mathbb{E}_{\mathcal{F}_{t}^{0}}$ and noting that $X_{t}$ is $\mathcal{F}_{t}$-measurable, we have

$$
\begin{equation*}
\mathbb{E}\left[\left|\Upsilon_{t}\right|^{2}\right]=\mathbb{E}\left[\mathbb{E}_{\mathcal{F}_{t}^{0}}\left[\left|\tilde{\mathbb{E}}_{\mathcal{F}_{T}}\left[\partial_{x \mu} V\left(t, X_{t}, \mu_{t}, \tilde{X}_{t}\right) \delta \tilde{X}_{t}\right]\right|^{2}\right]\right] \leq C \mathbb{E}\left[\mathbb{E}_{\mathcal{F}_{t}^{0}}\left[\left|\delta X_{t}\right|^{2}\right]\right] \leq C \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \tag{5.3}
\end{equation*}
$$

Using Hölder's inequality on (4.5) and noting in particular $\left|\bar{\Upsilon}_{t}\right| \leq L_{x x}^{V}\left|\delta X_{t}\right|$, we obtain

$$
\begin{equation*}
\left|\delta X_{t}\right|^{2} \leq 2|\eta|^{2}+C \int_{0}^{t}\left[\left|\delta X_{s}\right|^{2}+\left|\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\left|\delta \tilde{X}_{s}\right|\right]\right|^{2}+\left|\Upsilon_{s}\right|^{2}\right] d s \tag{5.4}
\end{equation*}
$$

Taking expectation on (5.4) and using (5.3), we derive

$$
\mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \leq 2 \mathbb{E}\left[|\eta|^{2}\right]+C \int_{0}^{t} \mathbb{E}\left[\left|\delta X_{s}\right|^{2}\right] d s
$$

Then it follows from Grönwall's inequality that

$$
\begin{equation*}
\sup _{t \in[0, T]} \mathbb{E}\left[\left|\delta X_{t}\right|^{2}\right] \leq C \mathbb{E}\left[|\eta|^{2}\right] . \tag{5.5}
\end{equation*}
$$

Next, by (4.8), we have

$$
\Upsilon_{t}=\Upsilon_{T}-\int_{t}^{T}\left[K_{3}(s) \Upsilon_{s}+K_{4}(s)\right] d s-\int_{t}^{T}\left(d B_{s}\right)^{\top} K_{1}(s)-\beta \int_{t}^{T}\left(d B_{s}^{0}\right)^{\top} K_{2}(s) .
$$

Take conditional expectation $\tilde{\mathbb{E}}_{\mathcal{F}_{t}}$, we have

$$
\begin{equation*}
\Upsilon_{t}=\tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\partial_{x \mu} G\left(X_{T}, \mu_{T}, \tilde{X}_{T}\right) \delta \tilde{X}_{T}\right]-\int_{t}^{T} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[K_{3}(s) \Upsilon_{s}+K_{4}(s)\right] d s . \tag{5.6}
\end{equation*}
$$

Then by (5.6) and the required regularity of $G, H$ and $V$, we have

$$
\left|\Upsilon_{t}\right|^{2} \leq C \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left|\delta \tilde{X}_{T}\right|^{2}\right]+C \int_{t}^{T} \tilde{\mathbb{E}}_{\mathcal{F}_{t}}\left[\left|\Upsilon_{s}\right|^{2}+\left|\delta \tilde{X}_{s}\right|^{2}\right] d s
$$

Now take conditional expectation $\tilde{\mathbb{E}}_{\mathcal{F}_{0}}$, we get

$$
\tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\Upsilon_{t}\right|^{2}\right] \leq C \tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\delta \tilde{X}_{T}\right|^{2}\right]+C \int_{t}^{T} \tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\Upsilon_{s}\right|^{2}+\left|\delta \tilde{X}_{s}\right|^{2}\right] d s
$$

Thus, by the Grönwall inequality we have

$$
\begin{equation*}
\left|\Upsilon_{0}\right|^{2}=\tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\Upsilon_{0}\right|^{2}\right] \leq C \tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\delta \tilde{X}_{T}\right|^{2}\right]+C \int_{0}^{T} \tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\left|\delta \tilde{X}_{s}\right|^{2}\right] d s \tag{5.7}
\end{equation*}
$$

Note that, recalling the setting in Section 2, $\delta \tilde{X}_{t}$ is measurable with respect to $\mathcal{F}_{t}^{0} \vee \tilde{\mathcal{F}}_{t}^{1}$, which is independent of $\mathcal{F}_{0}$ under $\tilde{\mathbb{P}}$. Then the conditional expectation in the right side of (5.7) is actually an expectation. Plug (5.5) into (5.7), we have

$$
\begin{equation*}
\left|\tilde{\mathbb{E}}_{\mathcal{F}_{0}}\left[\partial_{x \mu} V\left(0, \xi, \mu_{0}, \tilde{\xi}\right) \tilde{\eta}\right]\right|^{2}=\left|\Upsilon_{0}\right|^{2} \leq C \mathbb{E}\left[|\eta|^{2}\right] . \tag{5.8}
\end{equation*}
$$

This implies

$$
\begin{equation*}
\left|\tilde{\mathbb{E}}\left[\partial_{x \mu} V\left(0, x, \mu_{0}, \tilde{\xi}\right) \tilde{\eta}\right]\right| \leq C\left(\mathbb{E}|\eta|^{2}\right)^{\frac{1}{2}}, \quad \mu_{0} \text { - a.e. } x . \tag{5.9}
\end{equation*}
$$

Since $\partial_{\mu} V$ is continuous, then (5.9) actually holds for all $x$. In particular, this implies that there exists a constant $C_{2}^{\mu_{0}}>0$ such that

$$
\begin{equation*}
\left|\partial_{x} V\left(0, x, \mathcal{L}_{\xi+\eta}\right)-\partial_{x} V\left(0, x, \mathcal{L}_{\xi}\right)\right|=\left|\int_{0}^{1} \mathbb{E}\left[\partial_{x \mu} V\left(0, x, \mathcal{L}_{\xi+\theta \eta}, \xi+\theta \eta\right) \eta\right] d \theta\right| \leq C_{2}^{\mu_{0}}\left(\mathbb{E}\left[|\eta|^{2}\right]\right)^{\frac{1}{2}}( \tag{5.10}
\end{equation*}
$$

Now, taking random variables $\xi, \eta$ such that $W_{2}^{2}\left(\mathcal{L}_{\xi+\eta}, \mathcal{L}_{\xi}\right)=\mathbb{E}\left[\left.\eta\right|^{2}\right]$, the above inequality exactly means that $\partial_{x} V(0, x, \cdot)$ is uniformly Lipschitz continuous in $\mu_{0}$ under $W_{2}$ with uniform Lipschitz constant $C_{2}^{\mu_{0}}$.

We emphasize that the above Lipschitz continuity is under $W_{2}$, while the global wellposedness of the master equation requires the $W_{1}$-Lipschitz continuity. As in [32], we shall derive the desired $W_{1}$-Lipschitz continuity from the $W_{2}$-Lipschitz continuity by utilizing the pointwise representation for the Wasserstein derivative developed in [43]. Note again that in Theorem 5.1 we only have the Lipschitz continuity for $\partial_{x} V$, but not for $V$, so at below we shall also consider $\vec{U}(t, x, \mu):=\partial_{x} V(t, x, \mu)$, which formally should satisfy the following vectorial master equation on $[0, T) \times \mathbb{R}^{d} \times \mathcal{P}_{2}\left(\mathbb{R}^{d}\right)$, with terminal condition $\vec{U}(T, x, \mu)=\partial_{x} G(x, \mu)$ :

$$
\begin{gather*}
-\partial_{t} \vec{U}-\frac{\widehat{\beta}^{2}}{2} \operatorname{tr}\left(\partial_{x x} \vec{U}\right)+\partial_{x} H(x, \mu, \vec{U})+\partial_{p} H(x, \mu, \vec{U}) \cdot \partial_{x} \vec{U}-\overrightarrow{\mathcal{N}} \vec{U}=0, \quad \text { where } \\
\overrightarrow{\mathcal{N}} \vec{U}(t, x, \mu):=  \tag{5.11}\\
\operatorname{tr}\left(\tilde { \tilde { \mathbb { E } } } \left[\frac{\widehat{\beta}^{2}}{2} \partial_{\tilde{x}} \partial_{\mu} \vec{U}(t, x, \mu, \tilde{\xi})-\partial_{\mu} \vec{U}(t, x, \mu, \tilde{\xi})\left(\partial_{p} H\right)^{\top}(\tilde{\xi}, \mu, \vec{U}(t, \tilde{\xi}, \mu))\right.\right. \\
+ \\
\left.\left.+\beta^{2} \partial_{x} \partial_{\mu} \vec{U}(t, x, \mu, \tilde{\xi})+\frac{\beta^{2}}{2} \partial_{\mu \mu} \vec{U}(t, x, \mu, \tilde{\xi}, \tilde{\xi})\right]\right)
\end{gather*}
$$

To be precise, fix $t_{0}$, $\xi$, we first consider the following McKean-Vlasov SDE on $\left[t_{0}, T\right]$ :

$$
\begin{align*}
X_{t}^{\xi} & =\xi-\int_{t_{0}}^{t} \partial_{p} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right) d s+B_{t}^{t_{0}}+\beta B_{t}^{0, t_{0}}, \quad \rho_{t}:=\rho_{t}^{\xi}:=\mathcal{L}_{X_{t}^{\xi} \mid \mathcal{F}_{t}^{0}} ; \\
\nabla Y_{t}^{\xi} & =\partial_{x} G\left(X_{T}^{\xi}, \rho_{T}\right)-\int_{t}^{T} \partial_{x} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right) d s-\int_{t}^{T} \nabla Z_{s}^{\xi} \cdot d B_{s}-\int_{t}^{T} \nabla Z_{s}^{0, \xi} \cdot d B_{s}^{0} . \tag{5.12}
\end{align*}
$$

Next, given $\rho$ as above, for fixed $x \in \mathbb{R}^{d}$ and letting ( $e_{1}, \cdots, e_{d}$ ) denote the natural basis of $\mathbb{R}^{d}$, we introduce a series of FBSDEs, possibly McKean-Vlasov type:

$$
\begin{align*}
& \left\{\begin{array}{l}
X_{t}^{\xi, x}=x-\int_{t_{0}}^{t} \partial_{p} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right) d s+B_{t}^{t_{0}}+\beta B_{t}^{0, t_{0}} ; \\
\nabla Y_{t}^{\xi, x}=\partial_{x} G\left(X_{T}^{\xi, x}, \rho_{T}\right)-\int_{t}^{T} \partial_{x} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right) d s-\int_{t}^{T} \nabla Z_{s}^{\xi, x} \cdot d B_{s}-\int_{t}^{T} \nabla Z_{s}^{0, \xi, x} \cdot d B_{s}^{0} ;
\end{array}\right.  \tag{5.13}\\
& \left\{\begin{array}{c}
\nabla_{k} X_{t}^{\xi, x}=e_{k}-\int_{t_{0}}^{t}\left[\left(\nabla_{k} X_{s}^{\xi, x}\right)^{\top} \partial_{x p} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{k}^{2} Y_{s}^{\xi, x}\right)^{\top} \partial_{p p} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)\right] d s ; \\
\nabla_{k}^{2} Y_{t}^{\xi, x}=\left(\nabla_{k} X_{T}^{\xi, x}\right)^{\top} \partial_{x x} G\left(X_{T}^{\xi, x}, \rho_{T}\right)-\int_{t}^{T} \nabla_{k}^{2} Z_{s}^{\xi, x} \cdot d B_{s}^{t_{0}}-\int_{t}^{T} \nabla_{k}^{2} Z_{s}^{0, \xi, x} \cdot d B_{s}^{0, t_{0}} \\
-\int_{t}^{T}\left[\left(\nabla_{k} X_{s}^{\xi, x}\right)^{\top} \partial_{x x} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{k}^{2} Y_{s}^{\xi, x}\right)^{\top} \partial_{p x} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)\right] d s ;
\end{array}\right.  \tag{5.14}\\
& \left\{\begin{array}{l}
\nabla_{k} \mathcal{X}_{t}^{\xi, x}=-\int_{t_{0}}^{t}\left[\left(\nabla_{k} \mathcal{X}_{s}^{\xi, x}\right)^{\top} \partial_{x p} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right)+\left(\nabla_{k}^{2} \mathcal{Y}_{s}^{\xi, x}\right)^{\top} \partial_{p p} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right)\right. \\
\left.\quad+\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\left(\nabla_{k} \tilde{X}_{s}^{\xi, x}\right)^{\top}\left(\partial_{\mu p} H\right)\left(X_{s}^{\xi}, \rho_{s}, \tilde{X}_{s}^{\xi, x}, \nabla Y_{s}^{\xi}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{s}^{\xi, x}\right)^{\top} \partial_{\mu p} H\left(X_{s}^{\xi}, \rho_{s}, \tilde{X}_{s}^{\xi}, \nabla Y_{s}^{\xi}\right)\right]\right] d s ; \\
\nabla_{k}^{2} \mathcal{Y}_{t}^{\xi, x}=\tilde{\mathbb{E}}_{\mathcal{F}_{T}}\left[\left(\nabla_{k} \tilde{X}_{T}^{\xi, x}\right)^{\top} \partial_{\mu x} G\left(X_{T}^{\xi}, \rho_{T}, \tilde{X}_{T}^{\xi, x}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{T}^{\xi, x}\right)^{\top} \partial_{\mu x} G\left(X_{T}^{\xi}, \rho_{T}, \tilde{X}_{T}^{\xi}\right)\right] \\
\quad+\left(\nabla_{k} \mathcal{X}_{T}^{\xi, x}\right)^{\top} \partial_{x x} G\left(X_{T}^{\xi}, \rho_{T}\right)-\int_{t}^{T} \nabla_{k}^{2} \mathcal{Z}_{s}^{\xi, x} \cdot d B_{s}^{t_{0}}-\int_{t}^{T} \nabla_{k}^{2} \mathcal{Z}_{s}^{0, \xi, x} \cdot d B_{s}^{0, t_{0}} \\
\quad-\int_{t}^{T}\left[\left(\nabla_{k} \mathcal{X}_{s}^{\xi, x}\right)^{\top} \partial_{x x} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right)+\left(\nabla_{k}^{2} \mathcal{Y}_{s}^{\xi, x}\right)^{\top} \partial_{p x} H\left(X_{s}^{\xi}, \rho_{s}, \nabla Y_{s}^{\xi}\right)\right. \\
\left.\quad+\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\left(\nabla_{k} \tilde{X}_{s}^{\xi, x}\right)^{\top} \partial_{\mu x} H\left(X_{s}^{\xi}, \rho_{s}, \tilde{X}_{s}^{\xi, x}, \nabla Y_{s}^{\xi}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{s}^{\xi, x}\right)^{\top} \partial_{\mu x} H\left(X_{s}^{\xi}, \rho_{s}, \tilde{X}_{s}^{\xi}, \nabla Y_{s}^{\xi}\right)\right]\right] d s ;
\end{array}\right. \tag{5.15}
\end{align*}
$$

$$
\left\{\begin{array}{l}
\nabla_{\mu_{k}} X_{t}^{x, \xi, \tilde{x}}=-\int_{t_{0}}^{t}\left[\left(\nabla_{\mu_{k}} X_{s}^{x, \xi, \tilde{x}}\right)^{\top} \partial_{x p} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{\mu_{k}}^{2} Y_{s}^{x, \xi, \tilde{x}}\right)^{\top} \partial_{p p} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)\right.  \tag{5.16}\\
\left.\quad+\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\left(\nabla_{k} \tilde{X}_{s}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu p} H\left(X_{s}^{\xi, x}, \rho_{s}, \tilde{X}_{s}^{\xi, \tilde{x}}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{s}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu p} H\left(X_{s}^{\xi, x}, \rho_{s}, \tilde{X}_{s}^{\xi}, \nabla Y_{s}^{\xi, x}\right)\right]\right] d s ; \\
\nabla_{\mu_{k}}^{2} Y_{t}^{x, \xi, \tilde{x}}=\tilde{\mathbb{E}}_{\mathcal{F}_{T}}\left[\left(\nabla_{k} \tilde{X}_{T}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu x} G\left(X_{T}^{\xi, x}, \rho_{T}, \tilde{X}_{T}^{\xi, \tilde{x}}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{T}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu x} G\left(X_{T}^{\xi, x}, \rho_{T}, \tilde{X}_{T}^{\xi}\right)\right] \\
\quad+\left(\nabla_{\mu_{k}} X_{T}^{x, \xi, \tilde{x}}\right)^{\top} \partial_{x x} G\left(X_{T}^{\xi, x}, \rho_{T}\right)-\int_{t}^{T} \nabla_{\mu_{k}}^{2} Z_{s}^{x, \xi, \tilde{x}} \cdot d B_{s}-\int_{t}^{T} \nabla_{\mu_{k}}^{2} Z_{s}^{0, x, \xi, \tilde{x}} \cdot d B_{s}^{0} \\
\quad-\int_{t}^{T}\left[\left(\nabla_{\mu_{k}} X_{s}^{x, \xi, \tilde{x}} \top^{\top} \partial_{x x} H\left(X_{s}^{\xi, x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{\mu_{k}}^{2} Y_{s}^{x, \xi, \tilde{x}}\right)^{\top} \partial_{p x} H\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{\xi, x}\right)\right.\right. \\
\left.\quad+\tilde{\mathbb{E}}_{\mathcal{F}_{s}}\left[\left(\nabla_{k} \tilde{X}_{s}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu x} H\left(X_{s}^{\xi, x}, \rho_{s}, \tilde{X}_{s}^{\xi, \tilde{x}}, \nabla Y_{s}^{\xi, x}\right)+\left(\nabla_{k} \tilde{\mathcal{X}}_{s}^{\xi, \tilde{x}}\right)^{\top} \partial_{\mu x} H\left(X_{s}^{\xi, x}, \rho_{s}, \tilde{X}_{s}^{\xi}, \nabla Y_{s}^{\xi, x}\right)\right]\right] d s .
\end{array}\right.
$$

The following local (in time) result provides the crucial $W_{1}$-Lipschitz continuity of $\vec{U}$.
Proposition 5.2 Let Assumptions 3.1-(i) and 3.2-(i) hold. Recall the constants $\bar{L}_{x x}^{H}, \bar{L}_{x p}^{H}, L_{2}^{H}$ in (3.3), $L_{2}^{G}, \bar{L}_{x x}^{G}$ in (3.4), and $\tilde{L}_{2}^{G}$ in Remark 3.3. Then there exists a constant $\delta>0$, depending only d, $\bar{L}_{x x}^{H}, \bar{L}_{x p}^{H}, L_{2}^{H}, \bar{L}_{x x}^{G}, \tilde{L}_{2}^{G}$, such that whenever $T-t_{0} \leq \delta$, the following hold.
(i) The McKean-Vlasov FBSDEs (5.12), (5.13), (5.14), (5.15), and (5.16) are well-posed on $\left[t_{0}, T\right]$, for any $\mu \in \mathcal{P}_{2}$ and $\xi \in \mathbb{L}^{2}\left(\mathcal{F}_{t_{0}}, \mu\right)$.
(ii) Define $\vec{U}\left(t_{0}, x, \mu\right):=\nabla Y_{t_{0}}^{x, \xi}$. Then we have the pointwise representation:

$$
\begin{equation*}
\partial_{\mu_{k}} \vec{U}\left(t_{0}, x, \mu, \tilde{x}\right)=\nabla_{\mu_{k}}^{2} Y_{t_{0}}^{x, \xi, \tilde{x}} \tag{5.17}
\end{equation*}
$$

Moreover, there exists a constant $C_{1}^{\mu}>0$, depending only on $d, \bar{L}_{x x}^{H}, \bar{L}_{x p}^{H}, L_{2}^{H}, L_{2}^{G}, \bar{L}_{x x}^{G}$ such that

$$
\begin{equation*}
\left|\partial_{\mu} \vec{U}(0, x, \mu, \tilde{x})\right| \leq C_{1}^{\mu} . \tag{5.18}
\end{equation*}
$$

(iii) Assume further that Assumptions 3.1-(ii) and 3.2-(ii) hold true. Then the vectorial master equation (5.11) has a unique classical solution $\vec{U}$. Moreover,

$$
\vec{U}(t, \cdot, \cdot), \quad \partial_{x} \vec{U}(t, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2}\right), \quad \partial_{\mu} \vec{U}(t, \cdot, \cdot, \cdot) \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)
$$

and all their derivatives in the state and probability measure variables are continuous in the time variable and are uniformly bounded.
(iv) The following decoupled McKean-Vlasov FBSDE

$$
\begin{align*}
& X_{t}^{x}=x+B_{t}^{t_{0}}+\beta B_{t}^{0, t_{0}} ; \\
& Y_{t}^{x, \xi}=G\left(X_{T}^{x}, \rho_{T}\right)-\int_{t}^{T} H\left(X_{s}^{x}, \rho_{s}, \vec{U}\left(s, X_{s}^{x}, \rho_{s}\right)\right) d s-\int_{t}^{T} Z_{s}^{x, \xi} \cdot d B_{s}-\int_{t}^{T} Z_{s}^{0, x, \xi} \cdot d B_{s}^{0} \tag{5.19}
\end{align*}
$$

is well-posed on $\left[t_{0}, T\right]$ for any $x \in \mathbb{R}^{d}$. Define $V\left(t_{0}, x, \mu\right):=Y_{t_{0}}^{x, \xi}$. Then $V$ is the unique classical solution of the master equation (1.1) and $\partial_{x} V=\vec{U}$ on $[0, T] \times \mathbb{R}^{d} \times \mathcal{P}_{2}$.

We emphasize that at above $C_{1}^{\mu}$ depends on $L_{2}^{G}$ in (3.4), but the $\delta$ depends only on $\tilde{L}_{2}^{G}$ in Remark 3.3, not on $L_{2}^{G}$.

Proof. The proof of (i)-(iii) is very lengthy, but essentially identical to as that of [32, Proposition 6.2], except that [32] considers both $\partial_{\mu} V$ and $\partial_{x \mu} V=\partial_{\mu} \vec{U}$. Therefore we omit it here.
(iv) By the smoothness of $\vec{U}$ obtained in (iii), clearly the $V$ defined in (iv) is smooth and $Y_{t}^{x, \xi}=V\left(t, X_{t}^{x}, \rho_{t}\right)$. By applying Itô's formula (2.2) we see that $V$ satisfies the PDE:

$$
\begin{align*}
-\partial_{t} V- & \frac{\widehat{\beta}^{2}}{2} \operatorname{tr}\left(\partial_{x x} V\right)+H(x, \mu, \vec{U})-\operatorname{tr}\left(\overline { \tilde { \mathbb { E } } } \left[\frac{\widehat{\beta}^{2}}{2} \partial_{\tilde{x}} \partial_{\mu} V(t, x, \mu, \tilde{\xi})+\frac{\beta^{2}}{2} \partial_{\mu \mu} V(t, x, \mu, \bar{\xi}, \tilde{\xi})\right.\right.  \tag{5.20}\\
& \left.\left.-\partial_{\mu} V(t, x, \mu, \tilde{\xi})\left(\partial_{p} H\right)^{\top}(\tilde{\xi}, \mu, \vec{U}(t, \tilde{\xi}, \mu))+\beta^{2} \partial_{x} \partial_{\mu} V(t, x, \mu, \tilde{\xi})\right]\right)=0 .
\end{align*}
$$

Differentiate it with respect to $x$, we obtain the PDE for $\vec{U}^{\prime}:=\partial_{x} V$ :

$$
\begin{align*}
& -\partial_{t} \overrightarrow{U^{\prime}}-\frac{\widehat{\beta}^{2}}{2} \operatorname{tr}\left(\partial_{x x} \overrightarrow{U^{\prime}}\right)+\partial_{x} H(x, \mu, \vec{U})+\partial_{p} H(x, \mu, \vec{U}) \cdot \partial_{x} \vec{U} \\
& -\operatorname{tr}\left(\underset{\tilde{E}}{\widehat{\mathbb{E}}^{2}} \frac{\partial^{2}}{2} \partial_{\tilde{x}} \partial_{\mu} \overrightarrow{U^{\prime}}(t, x, \mu, \tilde{\xi})+\frac{\beta^{2}}{2} \partial_{\mu \mu} \overrightarrow{U^{\prime}}(t, x, \mu, \bar{\xi}, \tilde{\xi})\right.  \tag{5.21}\\
& \left.\left.\quad-\partial_{\mu} \vec{U}^{\prime}(t, x, \mu, \tilde{\xi})\left(\partial_{p} H\right)^{\top}(\tilde{\xi}, \mu, \vec{U}(t, \tilde{\xi}, \mu))+\beta^{2} \partial_{x} \partial_{\mu} \overrightarrow{U^{\prime}}(t, x, \mu, \tilde{\xi})\right]\right)=0 .
\end{align*}
$$

Compare this with (5.11), we see that $\vec{U}$ also satisfies (5.21). Thus, by the uniqueness we have $\vec{U}=\vec{U}^{\prime}=\partial_{x} V$. Plug this into (5.20) we verify that $V$ satisfies (1.1).

## 6 Uniform estimates of $\partial_{x x} V$

We note that all the above results rely on the bound $L_{x x}^{V}$ of $\partial_{x x} V$ in (4.1). In particular, in Theorem 4.1 the $\underline{L}_{x p}^{H}$ depends on $L_{x x}^{V}$. Then it is crucial to obtain an a priori uniform estimate of $L_{x x}^{V}$ which is independent of $\underline{L}_{x p}^{H}$. Recall (2.5), we have $\partial_{x x} V=\partial_{x x} u$, so it suffices to establish the a priori estimate for the solution $u$ to the backward SPDE in (2.4), for an arbitrarily given $\rho$ (not necessarily satisfying the forward SPDE in (2.4)).

For this purpose we consider a special form of $H$.
Assumption 6.1 $H$ takes the following form:

$$
\begin{equation*}
H(x, \mu, p)=\left\langle A_{0} x, p\right\rangle+H_{0}(x, \mu, p) \tag{6.1}
\end{equation*}
$$

where $A_{0} \in \mathbb{R}^{d \times d}$ is a constant matrix and $H_{0}: \mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ is a function satisfying
(i) $H_{0} \in \mathcal{C}^{2}\left(\mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d}\right)$ and there exist constants $\underline{L}_{x x}^{H_{0}}, \bar{L}_{x x}^{H_{0}}, L_{2}^{H_{0}}>0$ such that

$$
\begin{gather*}
\underline{\kappa}\left(\partial_{x x} H_{0}\right) \geq \underline{L}_{x x}^{H_{0}}, \quad\left|\partial_{x x} H_{0}\right| \leq \bar{L}_{x x}^{H_{0}},  \tag{6.2}\\
\left|\partial_{x p} H_{0}\right|,\left|\partial_{p p} H_{0}\right|,\left|\partial_{x \mu} H_{0}\right|,\left|\partial_{p \mu} H_{0}\right| \leq L_{2}^{H_{0}} \text { on } \mathbb{R}^{d} \times \mathcal{P}_{2} \times \mathbb{R}^{d} . \tag{6.3}
\end{gather*}
$$

(ii) $H_{0}$ satisfies Assumption 3.1-(ii).

Given $A_{0}$, consider its Jordan decomposition:

$$
\begin{equation*}
A_{0}=Q_{0} J_{0} Q_{0}^{-1} \tag{6.4}
\end{equation*}
$$

where $J_{0} \in \mathbb{C}^{d \times d}$ is the Jordan normal form of $A_{0}$ and $Q_{0} \in \mathbb{C}^{d \times d}$ is invertible. Let $\bar{Q}_{0}$ denote the conjugate of $Q_{0}$ and thus $Q_{0} \bar{Q}_{0}^{\top}$ is positive definite. The following estimate will be crucial.

Lemma 6.2 Recall (3.1). For any $t \geq 0$, we have

$$
\begin{equation*}
\left|e^{-A_{0} t}\right| \leq \sqrt{L^{A_{0}}} e^{\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right] t}, \quad \text { where } \quad L^{A_{0}}:=\inf _{Q_{0}} \frac{\bar{\kappa}\left(Q_{0} \bar{Q}_{0}^{\top}\right)}{\underline{\kappa}\left(Q_{0} \bar{Q}_{0}^{\top}\right)} \tag{6.5}
\end{equation*}
$$

Here the infimum is over all $Q_{0}$ satisfying (6.4).
Proof. Fix $J_{0}, Q_{0}$ as in (6.4). It is obvious that $e^{-A_{0} t}=Q_{0} e^{-J_{0} t} Q_{0}^{-1}$. We claim that

$$
\begin{equation*}
\left|\left\langle e^{-J_{0} t} x, y\right\rangle\right| \leq e^{\left[1-\underline{\kappa^{\prime}}\left(A_{0}\right)\right] t}|x||y|, \quad \forall x, y \in \mathbb{C}^{d} \tag{6.6}
\end{equation*}
$$

Then, for any $x, y \in \mathbb{R}^{d}$ with $|x|=|y|=1$, we have

$$
\begin{aligned}
& \left|\left\langle e^{-A_{0} t} x, y\right\rangle\right|=\left|\left\langle e^{-J_{0} t} Q_{0}^{-1} x, Q_{0} \top y\right\rangle\right| \leq e^{\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right] t}\left|Q_{0}^{-1} x\right|\left|Q_{0} \top y\right| \\
& \leq e^{\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right] t} \sqrt{\bar{\kappa}\left(Q_{0}^{-1}\left(\bar{Q}_{0}^{\top}\right)^{-1}\right)} \sqrt{\bar{\kappa}\left(Q_{0} \bar{Q}_{0}^{\top}\right)}=e^{\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right] t} \sqrt{\frac{\bar{\kappa}\left(Q_{0} \bar{Q}_{0}^{\top}\right)}{\underline{\kappa}\left(Q_{0} \bar{Q}_{0}^{\top}\right)}} .
\end{aligned}
$$

Since $Q_{0}$ is arbitrary, this implies (6.5) immediately.
To see (6.6), assume the Jordan normal form $J_{0}=\operatorname{diag}\left(J_{1}, \cdots, J_{k}\right)$. Here $d_{1}+\cdots+d_{k}=d$; $J_{i}=\lambda_{i} I_{d_{i}}+U_{d_{i}} \in \mathbb{R}^{d_{i} \times d_{i}}, i=1, \cdots, k ; \lambda_{1}, \cdots, \lambda_{k}$ are all the eigenvalues of $A_{0}$; and $U_{d_{i}}$ is the matrix whose $(j, j+1)$-component is $1, j=1, \cdots, d_{i}-1$, and all other components are 0 . It is straightforward to see that

$$
e^{-J_{0} t}=\operatorname{diag}\left(e^{-J_{1} t}, \cdots, e^{-J_{k} t}\right)
$$

Note that, for each $i$, since $I_{d_{i}}$ and $U_{d_{i}}$ can commute, and $U_{d_{i}}^{d_{i}}=0$,

$$
e^{-J_{i} t}=e^{-\lambda_{i} t} e^{-U_{d_{i}} t}=e^{-\lambda_{i} t} \sum_{n=0}^{d_{i}-1} \frac{(-t)^{n}}{n!} U_{d_{i}}^{n} .
$$

For any $x^{(i)}, y^{(i)} \in \mathbb{C}^{d_{i}}$, it is clear that

$$
\left|\left\langle U_{d_{i}}^{n} x^{(i)}, y^{(i)}\right\rangle\right| \leq \frac{1}{2}\left[\left|x^{(i)}\right|^{2}+\left|y^{(i)}\right|^{2}\right] .
$$

Then, for $x=\left(x^{(1)}, \cdots, x^{(k)}\right), y=\left(y^{(1)}, \cdots, y^{(k)}\right) \in \mathbb{C}^{d}$ with $|x|=|y|=1$, we have

$$
\begin{aligned}
& \left|\left\langle e^{-J_{0} t} x, y\right\rangle\right|=\left|\sum_{i=1}^{k}\left\langle e^{-J_{i} t} x^{(i)}, y^{(i)}\right\rangle\right| \leq \sum_{i=1}^{k}\left|e^{-\lambda_{i} t}\right| \sum_{n=0}^{d_{i}-1} \frac{t^{n}}{n!}\left|\left\langle U_{d_{i}}^{n} x^{(i)}, y^{(i)}\right\rangle\right| \\
& \leq e^{-\underline{\kappa}^{\prime}\left(A_{0}\right) t} \sum_{i=1}^{k} \sum_{n=0}^{d_{i}-1} \frac{t^{n}}{n!} \frac{1}{2}\left[\left|x^{(i)}\right|^{2}+\left|y^{(i)}\right|^{2}\right] \leq e^{-\underline{\kappa}^{\prime}\left(A_{0}\right) t} \sum_{n=0}^{d-1} \frac{t^{n}}{n!} .
\end{aligned}
$$

This implies (6.6) immediately.
Remark 6.3 (i) The form (6.1) is assumed for the estimate (6.5) and for the property

$$
\begin{equation*}
d e^{-A_{0} t}=-e^{-A_{0} t} A_{0} d t=-A_{0} e^{-A_{0} t} d t \tag{6.7}
\end{equation*}
$$

required in the proof of Theorem 6.4 below. In general $e^{-\int_{0}^{t} \partial_{x p} H d s}$ does not enjoy these properties. When $d=1$, however, $e^{-\int_{0}^{t} \partial_{x p} H d s}$ obviously satisfies similar properties and thus we do not need the special form (6.1). Moreover, we remark that any alternative structures which could ensure a uniform a priori bound for $\partial_{x x} u$ can serve our purpose.
(ii) It is clear that, under (6.1), (6.2), and (6.3), we may set

$$
\begin{equation*}
\underline{L}_{x p}^{H}:=\underline{\kappa}\left(A_{0}\right)-L_{2}^{H_{0}}, \quad \bar{L}_{x p}^{H}:=\left|A_{0}\right|+L_{2}^{H_{0}} ; \quad \underline{L}_{x x}^{H}:=\underline{L}_{x x}^{H_{0}}, \quad \bar{L}_{x x}^{H}:=\bar{L}_{x x}^{H_{0}}, \quad L_{2}^{H}:=L_{2}^{H_{0}} . \tag{6.8}
\end{equation*}
$$

Then (3.3) and (3.18) hold true. We shall remark though that the term $\underline{\kappa}\left(A_{0}\right)$ and the condition $\underline{\kappa}\left(\partial_{x x} H_{0}\right) \geq \underline{L}_{x x}^{H_{0}}$ are not used in Theorem 6.4 below.
(iii) When $A_{0}$ is symmetric, one can easily see that $L^{A_{0}}=1$, and in this case (6.5) can be improved: $\left|e^{-A_{0} t}\right| \leq e^{-\underline{\kappa^{\prime}}}\left(A_{0}\right) t$.

Then we have the following uniform a priori estimate.
Theorem 6.4 Let Assumptions 3.2-(i), 6.1 hold and $\rho:[0, T] \times \Omega \rightarrow \mathcal{P}_{2}$ be $\mathbb{F}^{0}$-progressively measurable with $\sup _{t \in[0, T]} \mathbb{E}\left[\int_{\mathbb{R}^{d}}|x|^{2} \rho_{t}(d x)\right]<+\infty$. Assume $(u, v)$ is a classical solution to the backward SPDE in (2.4) for the given $\rho$ here ( $\rho$ is not necessarily a solution to the forward SPDE in (2.4)) such that $\partial_{x x} u$ is bounded and, for some fixed constant $\bar{L}^{A} \geq 1$,

$$
\begin{gather*}
L^{A_{0}} \leq \bar{L}^{A}, \quad \underline{\kappa}^{\prime}\left(A_{0}\right) \geq \theta_{2}:=\max \left\{\theta_{3}, \frac{\bar{L}_{x 0}^{H_{0}}}{2 \bar{L}_{x x}^{G}}+1\right\}  \tag{6.9}\\
\text { where } \quad \theta_{3}:=1+L_{2}^{H_{0}} \bar{L}^{A}\left[1+\bar{L}_{x x}^{G} \bar{L}^{A}+\sqrt{\left(1+\bar{L}_{x x}^{G} \bar{L}^{A}\right)^{2}-1}\right]
\end{gather*}
$$

Then the following estimate holds:

$$
L_{x x}^{u}(\theta):=\frac{\left|\partial_{x x} u(t, x)\right| \leq L_{x x}^{u}\left(\theta_{3}\right), \quad \forall(t, x), \quad \text { where }}{\theta-1-L_{2}^{H_{0}} \bar{L}^{A}-\sqrt{\left(\theta-1-L_{2}^{H_{0}} \bar{L}^{A}\right)^{2}-2 L_{2}^{H_{0}} \bar{L}_{x x}^{G}\left(\bar{L}^{A}\right)^{2}[\theta-1]}} L_{2}^{H_{0}} \bar{L}^{A} \quad \theta \geq \theta_{3}
$$

We note that (6.9) implies $L_{x x}^{u}(\theta)$ is well-defined for $\theta \geq \theta_{3}$, and we emphasize that the bound $L_{x x}^{u}\left(\theta_{3}\right)$ depends only on $L_{2}^{H_{0}}, \bar{L}_{x x}^{G}$ and $\bar{L}^{A}$, in particular not on $T, \underline{\kappa}^{\prime}\left(A_{0}\right)$, or $\bar{L}_{x x}^{H_{0}}$.

Proof. Fix $\left(t_{0}, x\right)$. First, under our conditions it is clear that the following FBSDE on $\left[t_{0}, T\right]$ has a unique solution $\left(X^{x}, \nabla Y^{x}, \nabla Z^{x}, \nabla Z^{0, x}\right)$ :

$$
\begin{align*}
& X_{t}^{x}=x-\int_{t_{0}}^{t} \partial_{p} H\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right) d s+B_{t}^{t_{0}}+\beta B_{t}^{0, t_{0}} \\
& \nabla Y_{t}^{x}=\partial_{x} G\left(X_{T}^{x}, \rho_{T}\right)-\int_{t}^{T} \partial_{x} H\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right) d s-\int_{t}^{T} \nabla Z_{s}^{x} \cdot d B_{s}-\int_{t}^{T} \nabla Z_{s}^{0, x} \cdot d B_{s}^{0} . \tag{6.11}
\end{align*}
$$

In particular, $\partial_{x} u$ serves as the decoupling field:

$$
\begin{equation*}
\nabla Y_{t}^{x}=\partial_{x} u\left(t, X_{t}^{x}\right), \quad t \in\left[t_{0}, T\right] \tag{6.12}
\end{equation*}
$$

Next, denote $L_{0}:=L_{x x}^{u}\left(\underline{\kappa}^{\prime}\left(A_{0}\right)\right)$, and consider the following BSDE on $\left[t_{0}, T\right]$ :

$$
\begin{align*}
& \nabla^{2} Y_{t}^{x}=\partial_{x x} G\left(X_{T}^{x}, \rho_{T}\right)-\int_{t}^{T} \nabla^{2} Z_{s}^{x} \cdot d B_{s}-\int_{t}^{T} \nabla^{2} Z_{s}^{0, x} \cdot d B_{s}^{0} \\
& \quad-\int_{t}^{T}\left[\nabla^{2} Y_{s}^{x}\left[A_{0}^{\top}+\partial_{p x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\right]+\left[A_{0}+\partial_{x p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\right] \nabla^{2} Y_{s}^{x}\right.  \tag{6.13}\\
& \left.\quad+\partial_{x x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right] \partial_{p p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right]\right] d s
\end{align*}
$$

Here $A \wedge L_{0}:=\left[\left(-L_{0}\right) \vee a_{i j} \wedge L_{0}\right]_{i, j}$ is the truncated matrix. The above BSDE has a Lipschitz continuous driver and thus is well-posed. Recalling (6.7) and applying Itô's formula we have

$$
\begin{aligned}
& e^{-A_{0} t} \nabla^{2} Y_{t}^{x} e^{-A_{0}^{\top} t}=e^{-A_{0} T} \partial_{x x} G\left(X_{T}^{x}, \rho_{T}\right) e^{-A_{0}^{\top} T}-\int_{t}^{T} e^{-A_{0} s}\left[\nabla^{2} Z_{s}^{x} \cdot d B_{s}+\nabla^{2} Z_{s}^{0, x} \cdot d B_{s}^{0}\right] e^{-A_{0} s} \\
& \quad-\int_{t}^{T} e^{-A_{0} s}\left[\nabla^{2} Y_{s}^{x} \partial_{p x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\partial_{x p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right) \nabla^{2} Y_{s}^{x}\right. \\
& \left.\quad+\partial_{x x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right] \partial_{p p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right]\right] e^{-A_{0}^{\top} s} d s .
\end{aligned}
$$

Take conditional expectation $\mathbb{E}_{\mathcal{F}_{t}}$ on both sides, we obtain

$$
\begin{aligned}
& \nabla^{2} Y_{t}^{x}=e^{A_{0}(t-T)} \mathbb{E}_{\mathcal{F}_{t}}\left[\partial_{x x} G\left(X_{T}^{x}, \rho_{T}\right)\right] e^{A_{0}^{\top}(t-T)} \\
& \quad-\int_{t}^{T} e^{A_{0}(t-s)} \mathbb{E}_{\mathcal{F}_{t}}\left[\nabla^{2} Y_{s}^{x} \partial_{p x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\partial_{x p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right) \nabla^{2} Y_{s}^{x}\right. \\
& \left.\quad+\partial_{x x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right] \partial_{p p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\left[\nabla^{2} Y_{s}^{x} \wedge L_{0}\right]\right] e^{A_{0}^{\top}(t-s)} d s .
\end{aligned}
$$

Recall (3.2) and apply Lemma 6.2, we have

$$
\begin{aligned}
& \left|\nabla^{2} Y_{t}^{x}\right| \leq e^{2\left[1-\underline{\underline{\kappa}}^{\prime}\left(A_{0}\right)\right](T-t)} \bar{L}_{x x}^{G} \bar{L}^{A}+\frac{\bar{L}_{x x}^{H_{0}} \bar{L}^{A}}{2\left[\underline{\kappa}^{\prime}\left(A_{0}\right)-1\right]}\left[1-e^{2\left[1-\underline{\underline{\prime}}^{\prime}\left(A_{0}\right)\right](T-t)}\right] \\
& \quad+L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right] \int_{t}^{T} e^{2\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right](s-t)} \mathbb{E}_{\mathcal{F}_{t}}\left[\left|\nabla^{2} Y_{s}^{x}\right|\right] d s .
\end{aligned}
$$

Taking the conditional expectation $\mathbb{E}_{\mathcal{F}_{t_{0}}}$ and noting that $\underline{\kappa}^{\prime}\left(A_{0}\right) \geq \theta_{2} \geq \frac{\bar{L}_{x x}^{H_{0}}}{2 \bar{L}_{x x}^{G}}+1$, we derive

$$
\begin{aligned}
\mathbb{E}_{\mathcal{F}_{t_{0}}}\left[\left|\nabla^{2} Y_{t}^{x}\right|\right] \leq & e^{2\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right](T-t)} \bar{L}_{x x}^{G} \bar{L}^{A}+\bar{L}_{x x}^{G} \bar{L}^{A}\left[1-e^{2\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right](T-t)}\right] \\
& +L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right] \int_{t}^{T} e^{2\left[1-\underline{\kappa}^{\prime}\left(A_{0}\right)\right](s-t)} \mathbb{E}_{\mathcal{F}_{t_{0}}}\left[\left|\nabla^{2} Y_{s}^{x}\right|\right] d s \\
\leq & \bar{L}_{x x}^{G} \bar{L}^{A}+L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right] \int_{t}^{T} e^{2\left[1-\underline{k}^{\prime}\left(A_{0}\right)\right](s-t)} \mathbb{E}_{\mathcal{F}_{t_{0}}}\left[\left|\nabla^{2} Y_{s}^{x}\right|\right] d s .
\end{aligned}
$$

Then by Grönwall's inequality we have

$$
\begin{gather*}
\mathbb{E}_{\mathcal{F}_{t_{0}}}\left[\left|\nabla^{2} Y_{t}^{x}\right|\right] \leq \bar{L}_{x x}^{G} \bar{L}^{A}+\frac{\bar{L}_{x x}^{G} L_{2}^{H_{0}}\left|\bar{L}^{A}\right|^{2}\left[2+L_{0}\right]}{2\left[\underline{\kappa}^{\prime}\left(A_{0}\right)-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right]} \times  \tag{6.14}\\
{\left[1-e^{-\left[2\left[\underline{\underline{\kappa}}^{\prime}\left(A_{0}\right)-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right]\right][T-t]}\right]}
\end{gather*}
$$

Recall (6.10), one can check straightforwardly that

$$
\begin{equation*}
\frac{d}{d \theta} L_{x x}^{u}(\theta)=\frac{1}{L_{2}^{H_{0}} \bar{L}^{A}}\left[1-\frac{\left(\theta-1-L_{2}^{H_{0}} \bar{L}^{A}\right)-L_{2}^{H_{0}} \bar{L}_{x x}^{G}\left(\bar{L}^{A}\right)^{2}}{\sqrt{\left(\theta-1-L_{2}^{H_{0}} \bar{L}^{A}\right)^{2}-2 L_{2}^{H_{0}} \bar{L}_{x x}^{G}\left(\bar{L}^{A}\right)^{2}[\theta-1]}}\right]<0, \quad \forall \theta \geq \theta_{3} . \tag{6.15}
\end{equation*}
$$

Then, since $\underline{\kappa}^{\prime}\left(A_{0}\right) \geq \theta_{2} \geq \theta_{3}$ and $L_{0}=L_{x x}^{u}\left(\underline{\kappa}^{\prime}\left(A_{0}\right)\right)$, by (6.9) and (6.10) we have

$$
2\left[\underline{\kappa}^{\prime}\left(A_{0}\right)-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right] \geq 2\left[\theta_{3}-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{x x}^{u}\left(\theta_{3}\right)\right] \geq 0
$$

Thus (6.14) implies

$$
\begin{aligned}
\mathbb{E}_{\mathcal{F}_{t_{0}}}\left[\left|\nabla^{2} Y_{t}^{x}\right|\right] & \leq \bar{L}_{x x}^{G} \bar{L}^{A}+\frac{\bar{L}_{x x}^{G} L_{2}^{H_{0}}\left|\bar{L}^{A}\right|^{2}\left[2+L_{0}\right]}{2\left[\underline{\kappa}^{\prime}\left(A_{0}\right)-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right]} \\
& =\frac{2 \bar{L}_{x x}^{G} \bar{L}^{A}\left[\underline{\underline{\kappa}}^{\prime}\left(A_{0}\right)-1\right]}{2\left[\underline{\kappa}^{\prime}\left(A_{0}\right)-1\right]-L_{2}^{H_{0}} \bar{L}^{A}\left[2+L_{0}\right]}=L_{0},
\end{aligned}
$$

where the last equality is due to the straightforward calculation. In particular, by setting $t=t_{0}$, we have $\left|\nabla^{2} Y_{t_{0}}^{x}\right| \leq L_{0}$. Similarly we can show $\left|\nabla^{2} Y_{t}^{x}\right| \leq L_{0}$ for all $t \in\left[t_{0}, T\right]$. Then $\nabla^{2} Y_{s}^{x} \wedge L_{0}=$ $\nabla^{2} Y_{s}^{x}$ and thus (6.13) becomes

$$
\begin{align*}
& \nabla^{2} Y_{t}^{x}=\partial_{x x} G\left(X_{T}^{x}, \rho_{T}\right)-\int_{t}^{T} \nabla^{2} Z_{s}^{x} \cdot d B_{s}-\int_{t}^{T} \nabla^{2} Z_{s}^{0, x} \cdot d B_{s}^{0} \\
& \quad-\int_{t}^{T}\left[\nabla^{2} Y_{s}^{x}\left[A_{0}^{\top}+\partial_{p x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\right]+\left[A_{0}+\partial_{x p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)\right] \nabla^{2} Y_{s}^{x}\right.  \tag{6.16}\\
& \left.\quad+\partial_{x x} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right)+\nabla^{2} Y_{s}^{x} \partial_{p p} H_{0}\left(X_{s}^{x}, \rho_{s}, \nabla Y_{s}^{x}\right) \nabla^{2} Y_{s}^{x}\right] d s
\end{align*}
$$

By considering the equation for $\partial_{x x} u$ derived from the BSPDE in (2.4), one can readily see from (6.11), (6.12), and (6.16) that $\nabla^{2} Y_{t}^{x}=\partial_{x x} u\left(t, X_{t}^{x}\right)$. In particular, $\left|\partial_{x x} u\left(t_{0}, x\right)\right|=\left|\nabla^{2} Y_{t_{0}}^{x}\right| \leq$ $L_{0}$. Since $\left(t_{0}, x\right)$ is arbitrary, we have $\left|\partial_{x x} u(t, x)\right| \leq L_{0}=L_{x x}^{u}\left(\underline{\kappa}^{\prime}\left(A_{0}\right)\right)$ for all $(t, x)$. This, together with (6.15), implies (6.10).

## 7 Global well-posedness

In this section we establish the global well-posedness of the master equation. We shall first construct the global well-posedness of the vectorial master equation (5.11). Following the idea in [23, 22, 43, 32], the key is to extend a local classical solution to a global one through an a priori uniform Lipschitz continuity estimate of the solution in $\mu$. We note that Theorem 6.4 implies the uniform a priori bound of $\partial_{x x} V$. Then, by applying Theorem 4.1 and 5.1, we obtain the uniform a priori Lipschitz continuity of $\vec{U}=\partial_{x} V$ with respect to $\mu$ under $W_{2}$. Moreover, by Proposition 5.2 we derive the desired uniform a priori Lipschitz continuity of $\vec{U}$ with respect to $\mu$ under $W_{1}$.

We now present the main well-posedness result. Note that the dependence on the parameters is quite subtle, so we will introduce them carefully. Following the order of the assumptions below, one can easily construct a class of $G$ and $H$ satisfying all of them, see e.g. Example 7.2. In particular, in light of Lemma 6.3 (iii), we may set $\bar{L}^{A}=1$ and consider symmetric $A_{0}$.

Theorem 7.1 Let Assumption 3.2 with $\bar{L}_{x x}^{G}, L_{2}^{G}$ and Assumption 3.12 (i) with $\vec{\lambda} \in D_{4}$ hold true, and $H$ takes the form (6.1) such that Assumption 6.1 (ii) holds and there exists $L_{2}^{H_{0}}$ satisfying the requirements in (6.3). Fix an arbitrary $\bar{L}^{A} \geq 1$ and set $\theta_{3}$ as in (6.9) and $L_{x x}^{V}:=L_{x x}^{u}\left(\theta_{3}\right)$ as in (6.10). Assume further the following hold true.
(i) There exist $0<\underline{\gamma}<\bar{\gamma}$ such that $\underline{\gamma} \leq \bar{L}_{x x}^{G}, \bar{\gamma}>1$, and (4.2) holds true.
(ii) Set $A_{1}, A_{2}$ as in (4.3). The matrix $A_{0}$ satisfies:

$$
\begin{equation*}
L^{A_{0}} \leq \bar{L}^{A}, \quad \underline{\kappa}\left(A_{0}\right) \geq\left[1+\underline{\kappa}\left(A_{1}^{-1} A_{2}\right)\right] L_{2}^{H_{0}}, \quad \underline{\kappa}^{\prime}\left(A_{0}\right) \geq \theta_{3}, \quad\left|A_{0}\right|+L_{2}^{H_{0}} \leq \bar{\gamma}\left[\underline{\kappa}\left(A_{0}\right)-L_{2}^{H_{0}}\right] . \tag{7.1}
\end{equation*}
$$

(iii) There exist $0<\underline{L}_{x x}^{H_{0}} \leq \bar{L}_{x x}^{H_{0}}$ satisfying (6.2) and

$$
\begin{equation*}
\underline{\gamma}\left[\underline{\kappa}\left(A_{0}\right)-L_{2}^{H_{0}}\right] \leq \underline{L}_{x x}^{H_{0}} \leq \bar{L}_{x x}^{H_{0}} \leq\left[\bar{\gamma}\left[\underline{\kappa}\left(A_{0}\right)-L_{2}^{H_{0}}\right]\right] \wedge\left[2 \bar{L}_{x x}^{G}\left[\underline{\boldsymbol{\kappa}}^{\prime}\left(A_{0}\right)-1\right]\right] . \tag{7.2}
\end{equation*}
$$

Then the master equation (1.1) on $[0, T]$ admits a unique classical solution $V$ with bounded $\partial_{x} V$, $\partial_{x x} V$ and $\partial_{x \mu} V$.

Furthermore, the McKean-Vlasov FBSDEs (5.12), (5.13), (5.14), (5.15), (5.16) and (5.19) are also well-posed on $[0, T]$ and the representation formula (5.17) remains true on $[0, T]$.

Proof. The uniqueness as well as the wellposedness of the involved FBSDEs and the representation formula (5.17) follow exactly the same arguments as in [32, Theorem 6.3]. Thus we shall only prove the existence.

Set $\underline{L}_{x p}^{H}, \bar{L}_{x p}^{H}, \underline{L}_{x x}^{H}, \bar{L}_{x x}^{H}, L_{2}^{H}$ as in (6.8). Then clearly Assumptions 3.1 and 3.12 hold true. By (7.1) and (7.2) we see that (6.9) holds true and thus we have the a priori estimate (6.10). Moreover, by (7.1) we have $\underline{L}_{x p}^{H} \geq \underline{\kappa}\left(A_{1}^{-1} A_{2}\right) L_{2}^{H}$, and thus the result of Theorem 4.1 holds true.

We now let $C_{2}^{\mu}$ be the a priori (global) uniform Lipschitz estimate of $\partial_{x} V$ with respect to $\mu$ under $W_{2}$, as established by Theorems 4.1 and 5.1. Let $\delta>0$ be the constant in Proposition 5.2 , but with $\bar{L}_{x x}^{G}$ replaced with $L_{x x}^{V}$ and $L_{2}^{G}$ replaced with $C_{2}^{\mu}$. Let $0=T_{0}<\cdots<T_{n}=T$ be a partition such that $T_{i+1}-T_{i} \leq \frac{\delta}{2}, i=0, \cdots, n-1$.

First, since $T_{n}-T_{n-2} \leq \delta$, by Proposition 5.2 the master equation (1.1) on $\left[T_{n-2}, T_{n}\right]$ with terminal condition $G$ has a unique classical solution $V$. For each $t \in\left[T_{n-2}, T_{n}\right]$, applying Theorem 6.4 we have $\left|\partial_{x x} V\left(T_{n-1}, \cdot, \cdot\right)\right| \leq L_{x x}^{V}$. Note that by Proposition 5.2-(iii)(iv) $V(t, \cdot, \cdot)$ has further regularities, this enables us to apply Theorems 4.1 and 5.1 and obtain that $\partial_{x} V(t, \cdot, \cdot)$ is uniform Lipschitz continuous in $\mu$ under $W_{2}$ with Lipschitz constant $C_{2}^{\mu}$. Moreover, by Proposition 5.2-(ii) $\partial_{x} V\left(T_{n-1}, \cdot, \cdot\right)$ is also uniformly Lipschitz continuous in $\mu$ under $W_{1}$.

We next consider the master equation (1.1) on $\left[T_{n-3}, T_{n-1}\right]$ with terminal condition $V\left(T_{n-1}, \cdot, \cdot\right)$. We emphasize that $\partial_{x} V\left(T_{n-1}, \cdot, \cdot\right)$ has the above uniform regularity with the same constants $L_{x x}^{V}, C_{2}^{\mu}$, then we may apply Proposition 5.2 with the same $\delta$ and obtain a classical solution $V$ on $\left[T_{n-3}, T_{n-1}\right]$ with the additional regularities specified in Proposition 5.2-(iii)(iv). Clearly this extends the classical solution of the master equation to $\left[T_{n-3}, T_{n}\right]$. We emphasize again that, while the bound of $\partial_{x \mu} V(t, \cdot)$ may become larger for $t \in\left[T_{n-3}, T_{n-2}\right]$ because the $C_{1}^{\mu}$ in (5.18) now depends on $\left\|\partial_{x \mu} V\left(T_{n-1}, \cdot\right)\right\|_{L^{\infty}}$ instead of $\left\|\partial_{x \mu} V\left(T_{n}, \cdot\right)\right\|_{L^{\infty}}$, by the global a priori estimates in Theorems 4.1 and 5.1 we see that $\partial_{x} V(t, \cdot)$ corresponds to the same $L_{x x}^{V}$ and $C_{2}^{\mu}$ for all $t \in\left[T_{n-3}, T_{n}\right]$. This enables us to consider the master equation (1.1) on $\left[T_{n-4}, T_{n-2}\right]$ with terminal condition $V\left(T_{n-2}, \cdot, \cdot\right)$, and then we obtain a classical solution on $\left[T_{n-4}, T_{n}\right]$ with the desired uniform estimates and additional regularities.

Repeat the arguments backwardly in time, we may construct a classical solution $V$ for the original master equation (1.1) on $[0, T]$ with terminal condition $G$. Moreover, since this procedure is repeated only $n$ times, by applying (5.18) repeatedly we see that (5.18) indeed holds true on $[0, T]$.

We conclude the paper by providing an example which satisfies all the assumptions in Theorem 7.1. We emphasize that there is no smallness assumption imposed here.

Example 7.2 For simplicity let $d=1$. Fix positive constants $0<\underline{\alpha} \leq \bar{\alpha}$ and $0<\underline{\gamma}<\bar{\gamma}$ with $\bar{\gamma}>1$, and fix $\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$ satisfying the requirements in (3.9). Set $\bar{L}^{A}:=1$ and let $M_{0}$ be a large number which will be specified later. Assume
(i) $G$ satisfies Assumption 3.2 with

$$
\begin{equation*}
-\bar{\alpha} M_{0} \leq \partial_{x x} G(x, \mu) \leq-\underline{\alpha} M_{0} \quad \text { on } \quad \mathbb{R} \times \mathcal{P}_{2}(\mathbb{R}) ; \tag{7.3}
\end{equation*}
$$

(ii) $H$ satisfies Assumption 6.1 with $A_{0}:=M_{0}^{3}>L_{2}^{H_{0}}$ in (6.1), and

$$
\begin{equation*}
\underline{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right] \leq \partial_{x x} H_{0}(x, \mu, p) \leq \bar{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right] \quad \text { on } \quad \mathbb{R} \times \mathcal{P}_{2}(\mathbb{R}) \times \mathbb{R} . \tag{7.4}
\end{equation*}
$$

Then, for $M_{0}$ large enough, which may depend on $\underline{\alpha}, \bar{\alpha}, \underline{\gamma}, \bar{\gamma},\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right)$, and $L_{2}^{G}$, $L_{2}^{H_{0}}$, one can choose appropriate $\lambda_{0}$ such that all the conditions in Theorem 7.1 hold true.

Proof. We first emphasize that (7.3) and (7.4) involve only $\partial_{x x} G$ and $\partial_{x x} H_{0}$. Note that the parameters $L_{2}^{G}, L_{2}^{H_{0}}$, which $M_{0}$ will depend on, do not involve these derivatives. So it is rather easy to construct $G$ and $H_{0}$ satisfy both Assumptions 3.2, 6.1, and (7.3), (7.4) with arbitrarily large $M_{0}$. Moreover, recall (3.4) and (6.2), by (7.3) and (7.4) it is clear that

$$
\begin{equation*}
\bar{L}_{x x}^{G}=\bar{\alpha} M_{0}, \quad \underline{L}_{x x}^{H_{0}}=\underline{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right], \quad \bar{L}_{x x}^{H_{0}}=\bar{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right] . \tag{7.5}
\end{equation*}
$$

Then the $\theta_{3}$ in (6.9) and $L_{x x}^{u}\left(\theta_{3}\right)$ in (6.10) become: recalling $\bar{L}^{A}=1$,

$$
\begin{align*}
\theta_{3} & :=1+L_{2}^{H_{0}}\left[1+\bar{\alpha} M_{0}+\sqrt{\left(1+\bar{\alpha} M_{0}\right)^{2}-1}\right] \\
L_{x x}^{V}=L_{x x}^{u}\left(\theta_{3}\right): & : \frac{2 \bar{\alpha} M_{0}\left(\theta_{3}-1\right)}{\theta_{3}-1-L_{2}^{H_{0}}+\sqrt{\left(\theta_{3}-1-L_{2}^{H_{0}}\right)^{2}-2 L_{2}^{H_{0}} \bar{\alpha} M_{0}\left(\theta_{3}-1\right)}} . \tag{7.6}
\end{align*}
$$

We now show that the following $\lambda_{0}$ satisfies all the requirements:

$$
\begin{equation*}
\lambda_{0}=\frac{\bar{\gamma}^{2}\left[1+L_{x x}^{V}\right]^{2}-8 \lambda_{3}}{4 \underline{\gamma}}+1 . \tag{7.7}
\end{equation*}
$$

First, by the choice of $\lambda_{0}$, it is obvious that $\lambda_{0}>\frac{\bar{\gamma}^{2}\left[1+L_{x x}^{V}\right]^{2}-8 \lambda_{3}}{4 \underline{\gamma}}$, which verifies (4.2).
Next, let $O(M)$ denote a generic positive function of $M$ such that $\frac{O(M)}{M}$ is bounded both from above and away from 0 . Then we see that

$$
\begin{equation*}
\theta_{3}=O\left(M_{0}\right), \quad L_{x x}^{V}=O\left(M_{0}\right), \quad \lambda_{0}=O\left(M_{0}^{2}\right) . \tag{7.8}
\end{equation*}
$$

By (3.10) we have

$$
(\text { AntiMon })_{\xi}^{\vec{\lambda}} U(\eta, \eta) \leq\left[-\lambda_{0} \underline{\alpha} M_{0}+\bar{\alpha}^{2} M_{0}^{2}-\lambda_{3}\right] \mathbb{E}\left[|\eta|^{2}\right]+\left[\left|\lambda_{1}\right| L_{2}^{G}+\lambda_{2}\left|L_{2}^{G}\right|^{2}\right]|\mathbb{E}[\eta]|^{2} .
$$

Since $\lambda_{0} M_{0}=O\left(M_{0}^{3}\right)$, it is clear that $G$ is $\vec{\lambda}$-anti-monotone when $M_{0}$ is large enough.
Moreover, since $d=1$, we have $\underline{\kappa}\left(A_{0}\right)=\bar{\kappa}\left(A_{0}\right)=\underline{\kappa}^{\prime}\left(A_{0}\right)=A_{0}$ and $L^{A_{0}}=1 \leq \bar{L}^{A}=1$. Recall (4.2) and (4.3). When $M_{0}$ is large, it is clear that $1-\theta_{1}$ is uniformly away from 0 and then it follows from (7.8) that $\underline{\kappa}\left(A_{1}^{-1} A_{2}\right)=O\left(M_{0}^{2}\right)$. Thus, since $A_{0}=M_{0}^{3}$ and $\bar{\gamma}>1$, for $M_{0}$ sufficiently large we have the following inequalities which verify (7.1):

$$
A_{0} \geq 1+\underline{\kappa}\left(A_{1}^{-1} A_{2}\right) L_{2}^{H_{0}}, \quad A_{0} \geq \theta_{3} \quad \text { and } \quad A_{0}+L_{2}^{H_{0}} \leq \bar{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right] .
$$

Finally, since $\bar{L}_{x x}^{G}=\bar{\alpha} M_{0}$, it is clear that $2 \bar{L}_{x x}^{G}\left[A_{0}-1\right] \geq \bar{\gamma}\left[A_{0}-L_{2}^{H_{0}}\right]$ for $M_{0}$ large enough. Then (7.4) implies (7.2).

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